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Decompositions of Ideals of Minors Meeting a Submatrix

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DECOMPOSITIONS OF IDEALS OF MINORS MEETING A SUBMATRIX

KENT M. NEUERBURG AND ZACH TEITLER

Abstract. We compute the primary decomposition of certain ideals generated by subsets of minors in a generic matrix or in a generic symmetric matrix, or subsets of Pfaffians in a generic skew-symmetric matrix. Specifically, the ideals we consider are generated by minors that have at least some given number of rows and columns in certain submatrices.

1. Introduction

The paper [1] concerns ideals of minors fixing a submatrix, meaning the set of minors in an $m \times n$ matrix that involve all r of the first r columns of the matrix. One of the main results of that paper, Theorem A, gives the primary decomposition of the ideal generated by this set of minors. We generalize this to consider minors that involve at least r of the first a columns:

Theorem 1.1. *Let k be a field, let X be a* generic *$m \times n$ matrix, that is $X = (x_{i,j})$ for $1 \leq i \leq m$, $1 \leq j \leq n$, and let $R = k[X] = k[x_{i,j}]$. Regard X as a block matrix, $X = (AB)$, where A has size $m \times a$ and B has size $m \times (n-a)$. Let J be the ideal generated by the set of t-minors of X that involve at least r columns of A, let $I_t(X)$ be the ideal generated by the t-minors of X, and similarly let $I_r(A)$ be the ideal generated by the r-minors of A. Then $J = I_t(X) \cap I_r(A)$.*

We generalize further than this, to allow several blocks as well as restrictions on both rows and columns. We also give similar statements for ideals generated by sets of minors of a generic symmetric matrix, requiring some number of rows or columns in certain submatrices. Before we give these statements, we consider one possible application in the setting of the two-block theorem above.

It is sometimes useful to consider, for a homogeneous ideal I, the ideal $I_{\leq d}$ generated by the forms in I of degree $\leq d$. For example, in resolving the singularities of the affine cone $V(I) \subset \mathbb{A}^n$, upon blowing up the origin, the total transform of I may have embedded components supported along the projective variety $V(I_{\leq d})$, lying in the exceptional divisor $\cong \mathbb{P}^{n-1}$, for various d; see [10]. When I is a determinantal ideal, generated by minors of a matrix whose entries are homogeneous forms, then $I_{\leq d}$ is generated by just some of the minors of the matrix.

Corollary 1.2. *Let $X = (x_{i,j})$ be a generic $m \times n$ matrix, regarded as consisting of two blocks, $X = (AB)$, where A has size $m \times a$ and B has size $m \times (n-a)$. Fix the ring*

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$R = k[X] = k[x_{i,j}]$ where every entry in A has degree p and every entry in B has degree $q > p$; that is, $\deg(x_{i,j}) = p$ if $1 \le j \le a$, $\deg(x_{i,j}) = q$ otherwise. Fix t and d. Let $I_t(X)_{\le d}$ be the ideal generated by those t-minors of X of degree less than or equal to d, and for each r let $I_r(A)$ be the ideal generated by the r-minors of A. Then $I_t(X)_{\le d} = I_t(X) \cap I_r(A)$ for $r = \lceil \frac{tq-d}{q-p} \rceil$.

Indeed, a $t \times t$ minor M with r columns in A and $t - r$ columns in B will have degree $\deg M = pr + q(t - r)$; the value of r in the statement is the least integral solution to $\deg M \le d$.

Corollary 1.3. *Consider the vector bundles $F = \mathcal{O}^m_{\mathbb{P}^N}$, $G = \mathcal{O}_{\mathbb{P}^N}(p)^{\oplus a} \oplus \mathcal{O}_{\mathbb{P}^N}(q)^{\oplus n-a}$. Let $f : F \to G$ be a general map, let $f' : F \to \mathcal{O}_{\mathbb{P}^N}(p)^{\oplus a}$ be the induced map, let $\Delta_t(f)$ be the degeneracy locus, $\Delta_t(f) = \{x \in \mathbb{P}^N \mid \operatorname{rank} f_x < t\}$, and let $\Delta_t(f)_{\le d}$ be the locus defined by the ideal $I(\Delta_t(f))_{\le d}$. Then $\Delta_t(f)_{\le d} = \Delta_t(f) \cup \Delta_r(f')$ for $r = \lceil \frac{tq-d}{q-p} \rceil$.*

This is similar to [11], which dealt with $\mathbb{P}^2$ and had $n = m + 1$ in order to obtain general Hilbert-Burch matrices of a given type. (In particular [11, Prop. 3.4] simply recreated a special case of [1, Thm. A].)

Sections 2 and 3 review some background of posets, dosets, algebras with straightening law, and doset algebras with straightening law. Then we give our results for minors in generic matrices (Section 4), minors in generic symmetric matrices (Section 5), and Pfaffians in generic skew-symmetric matrices (Section 6).

Throughout, all rings are commutative with unity.

2. Orders and straightening

A **poset** (partially ordered set) is a set together with a transitive, reflexive, antisymmetric relation $\le$.

Definition 2.1 ([2, Definition 1.0.3])**.** A **doset** of a poset P is a subset $D \subset P \times P$ such that

(1) $(a, a) \in D$ for all $a \in P$,
(2) if $(a, b) \in D$ then $a \le b$, and
(3) if $a \le b \le c \in P$, then $(a, c) \in D$ if and only if $(a, b) \in D$ and $(b, c) \in D$.

Example 2.2. (1) Let $[n] = \{1, \dots, n\}$ and let $P_n = 2^{[n]}$, the power set of $[n]$. We order P_n as follows. For $A = \{a_1 < \cdots < a_s\} \subset [n]$ and $B = \{b_1 < \cdots < b_t\} \subset [n]$, $A \le B$ if and only if $s \ge t$ and $a_i \le b_i$ for $i = 1, \dots, t$. This makes P_n a poset.

Note, $A \le B$ if and only if in the diagram

a_1	a_2	$\cdots$	a_t	$\cdots$	a_s
b_1	b_2	$\cdots$	b_t		

the first row is at least as long as the second and the entries are weakly increasing down each column.

(2) Fix m and n. Let $P_{m,n} \subset P_m \times P_n$ consist of pairs of subsets (A, B) such that $|A| = |B|$, with $(A, B) \le (A', B')$ if and only if $A \le A'$ and $B \le B'$.

(3) Let $D_n \subset P_n \times P_n$ consist of pairs (A, B) such that $|A| = |B|$ and $A \le B$. Then D_n is a doset.

Example 2.3. Here are some key examples of posets and dosets of minors in matrices.

(1) $P_{m,n}$ is the **poset of minors** (of an $m \times n$ matrix): $(A, B) \in P_{m,n}$ corresponds to the minor with rows indexed by A and columns indexed by B. This poset is usually denoted $\Delta(X)$, where X is an $m \times n$ matrix.
(2) D_n is the **doset of minors** of a symmetric $n \times n$ matrix: $(A, B) \in D_n$ corresponds to the minor with rows indexed by A and columns indexed by B. An element of D_n is called a **doset minor**. We denote this poset $\Delta^s(Y)$, where Y is a symmetric $n \times n$ matrix. (It is denoted $\Delta(Y)$ in [2]; we adjoin the s for "symmetric" in order to avoid ambiguity.)

    The condition $A \leq B$ means that a minor is a doset minor if and only if the main diagonal of the minor lies in the upper triangle of the matrix (including the diagonal).
(3) Let $P_n(2)$ be the subset of $A \in P_n$ such that $|A|$ is even. Then $P_n(2)$ is the **poset of Pfaffians** of a skew-symmetric $n \times n$ matrix: $A \in P_n(2)$ corresponds to the Pfaffian of the submatrix with rows and columns indexed by A. Following [2] we denote this poset $\Pi(Z)$, where Z is a skew-symmetric $n \times n$ matrix.

Definition 2.4 ([3, §4.A],[2, Definition 1.0.1])**.** Let A be a B-algebra and $P \subset A$ a subset with a partial order $\leq$. Then A is a **graded algebra with straightening law** (abbreviated ASL) on P over B if

(1) $A = \bigoplus_{i \geq 0} A_i$ is a graded B-algebra such that $A_0 = B$, P consists of homogeneous elements of positive degree, and P generates A as a B-algebra.
(2) A is a free B-module with a basis given by products $\xi_1 \cdots \xi_m$, $m \geq 0$, $\xi_i \in P$, such that $\xi_1 \leq \cdots \leq \xi_m$. These products are called **standard monomials**.
(3) For all incomparable $\xi, \nu \in P$, the product $\xi\nu$ can be written as a combination of standard monomials

$$\xi\nu = \sum a_\mu \mu, \qquad a_\mu \in B, a_\mu \neq 0, \qquad \mu \text{ standard monomial,}$$

in which every μ contains a factor $\zeta \in P$ such that $\zeta \leq \xi$ and $\zeta \leq \nu$. These are called **straightening relations**.

Definition 2.5 ([2, Definition 1.0.4])**.** Let A be a B-algebra and $D \subset A$ a subset such that D is a doset of a poset P. Then A is a **graded doset algebra with straightening law** (abbreviated DASL) on D over B if

(1) $A = \bigoplus_{i \geq 0} A_i$ is a graded B-algebra such that $A_0 = B$, D consists of homogeneous elements of positive degree, and D generates A as a B-algebra.
(2) A is a free B-module with a basis given by products $(\alpha_1, \alpha_2) \cdots (\alpha_{2k-1}, \alpha_{2k})$, $k \geq 1$, $(\alpha_{2i-1}, \alpha_{2i}) \in D$, $\alpha_1 \leq \cdots \leq \alpha_{2k}$. These products are called **standard monomials**.
(3) Suppose $M = (\alpha_1, \alpha_2) \cdots (\alpha_{2k-1}, \alpha_{2k})$, with **standard representation** $M = \sum \lambda_N N$, $0 \neq \lambda_N \in B$, each N a standard monomial. Let $N = (\beta_1, \beta_2) \cdots (\beta_{2\ell-1}, \beta_{2\ell})$ be one of the standard monomials appearing in the standard representation of M. Then for every permutation σ of $\{1, \ldots, 2k\}$, the sequence $\{\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(2k)}\}$ is lexicographically greater than or equal to the sequence $(\beta_1, \ldots, \beta_{2\ell})$.
(4) In the notation above, if there is a permutation σ such that $\alpha_{\sigma(1)} \leq \cdots \leq \alpha_{\sigma(2k)}$ then the standard monomial $(\alpha_{\sigma(1)}, \alpha_{\sigma(2)}) \cdots (\alpha_{\sigma(2k-1)}, \alpha_{\sigma(2k)})$ must appear in the standard representation of M with coefficient ± 1.

Example 2.6. Fix an arbitrary commutative ring B with unity.

(1) Let X be an $m \times n$ **generic matrix**, that is $X = (x_{i,j})$, $1 \le i \le m$, $1 \le j \le n$, the $x_{i,j}$ variables over B. Then $A = B[X] = B[x_{i,j}]$ is a graded ASL on $\Delta(X)$ over B [3, Chap. 4],[2, Thm. 1.0.5].
(2) Let Y be an $n \times n$ **generic symmetric matrix**, that is $Y = (y_{i,j})$, $1 \le i, j \le n$, $y_{i,j} = y_{j,i}$. Then $A = B[Y]$ is a graded DASL on $\Delta^s(Y)$ over B [2, Thm. 1.0.10].
(3) Let Z be an $n \times n$ **generic skew-symmetric matrix**, that is $Z = (z_{i,j})$, $1 \le i, j \le n$, $z_{i,j} = -z_{j,i}$, $z_{i,i} = 0$. Then $A = B[Z]$ is a graded ASL on $\Pi(Z)$ over B [2, Thm. 1.0.14].

3. Order ideals

We use ASLs and DASLs entirely for the following properties.

Definition 3.1. Let P be a poset. An **order ideal** is a subset $I \subset P$ such that if $\alpha \in I$ and $\beta \le \alpha$ then $\beta \in I$. The order ideal **generated by** $S \subset P$ is the smallest order ideal containing S, that is, $\{\alpha \in P \mid \alpha \le s \text{ for some } s \in S\}$. The order ideal **cogenerated by** $S \subset P$ is the largest order ideal disjoint from S, that is, $\{\alpha \in P \mid \alpha \not\le s \text{ for all } s \in S\}$.

When A is an ASL on P and $I \subset P$, we write AI for the (ring) ideal generated by I.

Lemma 3.2 ([3, Prop. 5.2])**.** *Let A be an ASL on P and let $I, J \subset P$ be ideals. Then $AI \cap AJ = A(I \cap J)$.*

We will prove a similar lemma for DASLs. First, we introduce a partial order for dosets.

Definition 3.3. Let D be a doset of P. Then D is a poset with the partial order $(a, b) \le_1 (c, d)$ if and only if $a \le c$ in P. A **doset order ideal** is an order ideal in the poset $(D, \le_1)$. As before, the ideal generated by $S \subset D$ is the smallest ideal containing S and the ideal cogenerated by $S \subset D$ is the largest ideal disjoint from S.

A DASL on D is not necessarily an ASL on $(D, \le_1)$. Again when A is a DASL on D and $I \subset D$ is a doset order ideal, we write AI for the ring ideal generated by I.

Lemma 3.4. *Let A be a DASL on D over B and let $I \subset D$ be a doset order ideal. Then AI is spanned over B by the standard monomials $N = (\beta_1, \beta_2) \cdots (\beta_{2\ell-1}, \beta_{2\ell})$ such that $(\beta_1, \beta_2) \in I$.*

Proof. Let $(\alpha_1, \alpha_2) \in I$, $f \in A$, and let $N = (\beta_1, \beta_2) \cdots (\beta_{2\ell-1}, \beta_{2\ell})$ be one of the standard monomials appearing in the standard representation of $(\alpha_1, \alpha_2) f$. The sequence $(\beta_1, \beta_2, \ldots, \beta_{2\ell})$ is lexicographically less than or equal to (α_1, α_2), so in particular $\beta_1 \le \alpha_1$. Hence $(\beta_1, \beta_2) \le_1 (\alpha_1, \alpha_2)$ and hence $(\beta_1, \beta_2) \in I$. Thus every standard monomial appearing in every element of AI has a factor in I. ☐

Lemma 3.5. *Let A be a DASL on D and let $I, J \subset D$ be ideals. Then $AI \cap AJ = A(I \cap J)$.*

Proof. A standard monomial $N = (\beta_1, \beta_2) \cdots (\beta_{2\ell-1}, \beta_{2\ell})$ appearing in the standard representation of an element of $AI \cap AJ$ has $(\beta_1, \beta_2) \in I$ and $\in J$, hence in $I \cap J$. This shows $AI \cap AJ \subset A(I \cap J)$ and the reverse inclusion is obvious. ☐

Finally we recall the following results.

Proposition 3.6 ([3, Thm. 6.3])**.** *Let B be a domain, X a generic matrix, $A = B[X]$, and $\delta \in \Delta(X)$, the poset of minors (see Example 2.3(1)). Let $I(X, \delta)$ be the ideal in A generated by the order ideal cogenerated by δ. Then $I(X, \delta)$ is a prime ideal.*

Proposition 3.7 ([8, Theorem 1], [5, Remark 2.5(a)]). *Let B be a domain, Y a generic symmetric matrix, $A = B[Y]$, and $\delta \in \Delta^s(Y)$, the doset of minors (see Example 2.3(2)). Let $I(Y, \delta)$ be the ideal in A generated by the doset order ideal cogenerated by δ. Then $I(Y, \delta)$ is a prime ideal.*

Proposition 3.8 ([2, Thm. 2.1.12]). *Let B be a domain, Z a generic skew-symmetric matrix, $A = B[Z]$, and $\delta \in \Pi(Z)$, the poset of Pfaffians (see Example 2.3(3)). Let $I(Z, \delta)$ be the ideal in A generated by the order ideal cogenerated by δ. Then $I(Z, \delta)$ is a prime ideal.*

4. Minors

We are interested in ideals generated by certain sets of t-minors in a generic matrix X. Specifically, we will require the generating minors to have at least r_1 rows in the first R_1 rows of X, at least r_2 rows contained in the first R_2 rows of X, and so on; and similarly for columns.

Let X be a generic $m \times n$ matrix, $X = (x_{i,j})$ for $1 \le i \le m$, $1 \le j \le n$, and fix $A = B[X] = B[\{x_{i,j}\}]$ for a commutative ring B with unity. For $1 \le t \le \min(m, n)$, a t-minor may be specified by listing its rows and columns; we write $[a_1, \dots, a_t \mid b_1, \dots, b_t]$, where $1 \le a_1 < \cdots < a_t \le m$ and $1 \le b_1 < \cdots < b_t \le n$, for the minor with rows $a_1, \dots, a_t$ and columns $b_1, \dots, b_t$.

Fix sequences $1 \le R_1 \le \cdots \le R_p \le m$ and $1 \le C_1 \le \cdots \le C_q \le n$ where $p, q \ge 0$. The sequences $R = (R_1, \dots, R_p)$ and $C = (C_1, \dots, C_q)$ (possibly empty if $p = 0$ or $q = 0$) describe the division of X into row and column blocks, respectively. Specifically, let X_{R_i} be the submatrix of X consisting of the first R_i rows and let X^{C_j} be the submatrix consisting of the first C_j columns. Fix also sequences $r = (r_1, \dots, r_p)$ and $c = (c_1, \dots, c_q)$.

We are interested in the t-minors that have at least r_i rows contained in X_{R_i} and at least c_j columns contained in X^{C_j}, for each i, j (with no restriction if $p = 0$ or $q = 0$).

Theorem 4.1. *Let B be a ring and $A = B[X]$. Let $J = J(X, t, R, C, r, c)$ be the ideal generated by t-minors of X that have at least r_i rows contained in X_{R_i} for each $1 \le i \le p$ (no restriction if $p = 0$) and at least c_j columns contained in X^{C_j} for each $1 \le j \le q$ (no restriction if $q = 0$). Then*

$$J = I_t(X) \cap I_{r_1}(X_{R_1}) \cap \cdots \cap I_{r_p}(X_{R_p}) \cap I_{c_1}(X^{C_1}) \cap \cdots \cap I_{c_q}(X^{C_q}). \tag{1}$$

Example 4.2. When $p = q = 0$, $J = I_t(X)$.

When $p = 0$ and $q = 1$, we are in the two-block setting of the Introduction. If also $c_1 = C_1$, we recover [1, Thm. A].

Remark 4.3. We are essentially working with the special case of Mohammadi's block adjacent simplicial complexes [9] in which each block is contained in the previous one and they all have the last column of the matrix as a common endpoint (in Mohammadi's indexing; for us, we take blocks to start at the first column or row). Unlike Mohammadi, we allow non-maximal minors, we allow restrictions on both the rows and columns appearing in the minor, and we allow the overlaps between "consecutive" blocks to be arbitrarily large.

Proof. Each of the following sets of minors is an order ideal in $\Delta(X)$:

(1) The set of minors of size $\ge t$, the generating set of $I_t(X)$, is the order ideal generated by $[m - t + 1, \dots, m \mid n - t + 1, \dots, n]$, or cogenerated by $[1, \dots, t - 1 \mid 1, \dots, t - 1]$.
(2) The set of ($\ge r_i$)-minors of X_{R_i} is the order ideal generated by $[R_i - r_i + 1, \dots, R_i \mid n - r_i + 1, \dots, n]$, or cogenerated by $[1, \dots, r_i - 1, R_i + 1, \dots, n \mid 1, \dots, n - R_i + r_i - 1]$.

(3) Similarly, the set of ($\geq c_j$)-minors of X^{C_j} is the order ideal generated by $[m - c_j + 1, \ldots, m \mid C_j - c_j + 1, \ldots, C_j]$, or cogenerated by $[1, \ldots, n - C_j + c_j - 1 \mid 1, \ldots, c_j - 1, C_j + 1, \ldots, n]$.

By Lemma 3.2, the intersection of the ideals generated by these sets is equal to the ideal generated by the intersection of the sets. □

Note, if B is a domain this gives J as an intersection of prime ideals. However it may fail to be a primary decomposition of J, as redundancies may arise in the following ways. For example, if $r_j > R_j - R_i$ then every minor containing at least r_j rows of X_{R_j} must contain at least $r_j - (R_j - R_i)$ rows of X_{R_i}; now if $r_i \leq r_j - R_j + R_i$ then the condition imposed by r_i is implied by the r_j condition and the prime ideal $I_{r_i}(X_{R_i})$ is redundant. Or if $t - (m - R_i) \geq r_i$ then every t-minor has at least r_i rows in X_{R_i}. Finally there are a few trivial situations: if $r_i > R_i$ the whole thing is zero; if $R_i = R_j$ or $r_i = r_j$ then one condition is obviously redundant. These are the only possible redundancies as the following proposition shows.

Proposition 4.4. *Suppose*

(1) $R_1 < \cdots < R_p$ *and* $C_1 < \cdots < C_q$,
(2) $r_1 < \cdots < r_p < t$ *and* $c_1 < \cdots < c_q < t$,
(3) $0 \leq r_i \leq R_i$ *for each* i *and* $0 \leq c_j \leq C_j$ *for each* j,
(4) $R_1 - r_1 < \cdots < R_p - r_p < m - t$ *and* $C_1 - c_1 < \cdots < C_q - c_q < n - t$.

Then the intersection (1) *is irredundant.*

Proof. First, fix $1 \leq i \leq p$. Consider the t-minor

$$m = [1, \ldots, r_i - 1, R_i + 1, \ldots, t + R_i - r_i + 1 \mid 1, \ldots, t].$$

We use $R_i - r_i < m - t$ to verify $t + R_i - r_i + 1 \leq m$, so this is a permissible t-minor in an $m \times n$ matrix. For each $j < i$, m has exactly $\min(r_i - 1, R_j)$ rows in X_{R_j}, and this is $\geq r_j$, so $m \in I_{r_j}(X_{R_j})$. For each $j > i$, the number of rows of m in X_{R_j} is either t, if $t + R_i - r_i + 1 \leq R_j$, or else $R_j - R_i + r_i - 1$, if $R_i + 1 \leq R_j \leq t + R_i - r_i + 1$. In the first case $t \geq r_j$ and in the second case $R_j - r_j > R_i - r_i$, so $R_j - R_i + r_i - 1 \geq r_j$; therefore $m \in I_{r_j}(X_{R_j})$. And clearly m has only $r_i - 1$ rows in X_{R_i}. This shows that

$$m \in I_t(X) \cap I_{r_1}(X_{R_1}) \cap \cdots \cap I_{r_{i-1}}(X_{R_{i-1}}) \cap I_{r_{i+1}}(X_{R_{i+1}}) \cap \cdots \cap I_{r_p}(X_{R_p})$$

but $m \notin I_{r_i}(X_{R_i})$. Clearly $m \in \bigcap I_{c_j}(X^{C_j})$. So the term $I_{r_i}(X_{R_i})$ is irredundant for each i. The same argument shows that each $I_{c_j}(X^{C_j})$ is irredundant.

Finally consider the $(t-1)$-minor

$$m' = [1, \ldots, t - 1 \mid 1, \ldots, t - 1].$$

Since each $r_i < t$, m' has at least r_i rows in each X_{R_i} and similarly at least c_j columns in each X^{C_j}. This shows that the term $I_t(X)$ is irredundant. □

5. Minors of symmetric matrices

Now let $Y = (y_{i,j})$ be a generic symmetric $n \times n$ matrix, $y_{i,j} = y_{j,i}$. Fix sequences $R = (R_1, \ldots, R_p)$ with $1 \leq R_1 \leq \cdots \leq R_p \leq n$ and $r = (r_1, \ldots, r_p)$. Let Y_{R_i} be the submatrix consisting of the first R_i rows of Y. We are interested in the t-minors that have at least r_i rows in Y_{R_i} for each i. Note, at this point we allow all minors, not only doset minors.

Theorem 5.1. *Let B be a ring and $A = B[Y]$. Let $J = J(Y, t, R, r)$ be the ideal generated by t-minors of Y that have at least r_i rows contained in Y_{R_i} for each $1 \le i \le p$. Then*

$$J = I_t(Y) \cap I_{r_1}(Y_{R_1}) \cap \cdots \cap I_{r_p}(Y_{R_p}). \tag{2}$$

If B is a domain then each $I_{r_i}(Y_{R_i})$ is a prime ideal.

Proof. First, by [5, Lemma 2.3], every t-minor $[a \mid b]$ is a linear combination of doset t-minors $[c \mid d]$ with $c \le a$. Thus we can take J to be generated by the doset t-minors meeting the row conditions.

Next, each of the following sets is a doset order ideal in $\Delta^s(Y)$:

(1) The set of doset minors of size $\ge t$ is the doset order ideal generated by $[n - t + 1, \ldots, n \mid n - t + 1, \ldots, n]$.
(2) The set of doset ($\ge r_i$)-minors of Y_{R_i} is the doset order ideal generated by $[R_i - r_i + 1, \ldots, R_i \mid n - r_i + 1, \ldots, n]$. If $[a \mid b] \le_1 [R_i - r_i + 1, \ldots, R_i \mid n - r_i + 1, \ldots, n]$ then $[a \mid b]$ involves at least r_i rows of Y_{R_i}; by Laplace expansion and [5, Lemma 2.3], $[a \mid b]$ is a linear combination of doset r_i-minors of Y_{R_i}.

This shows that J is the indicated intersection.

The set of doset ($\ge t$)-minors of Y is cogenerated by $[1, \ldots, t-1 \mid 1, \ldots, t-1]$. The set of doset ($\ge r_i$)-minors of Y_{R_i} is cogenerated by $m = [1, \ldots, r_i-1, R_i+1, \ldots, n \mid 1, \ldots, r_i-1, R_i+1, \ldots, n]$ [5, Remark 2.5(c)]. Indeed, $[a \mid b] \not\ge_1 m$ if and only if $a \not\ge (1, \ldots, r_i-1, R_i+1, \ldots, n)$, if and only if $|a| \ge r_i$ and $a_{r_i} \le R_i$; so $[a \mid b]$ involves at most r_i rows of Y_{R_i}. This shows that each of the ideals being intersected is cogenerated by a single doset element. Therefore if B is a domain then each of them is a prime ideal by Proposition 3.7. □

If B is a domain then once again this writes J as an intersection of prime ideals, but as before it may fail to be a primary decomposition because of redundancy.

Proposition 5.2. *Suppose*

(1) $R_1 < \cdots < R_p$,
(2) $r_1 < \cdots < r_p < t$,
(3) $0 \le r_i \le R_i$ *for each* i,
(4) $R_1 - r_1 < \cdots < R_p - r_p < n - t$.

Then the intersection (2) *is irredundant.*

The proof is the same as before.

6. Pfaffians

Let $Z = (z_{i,j})$ be an $n \times n$ generic skew-symmetric matrix, so that $z_{i,j} = -z_{j,i}$ and $z_{i,i} = 0$, and let $A = B[Z] = B[\{z_{i,j}\}]$. The Pfaffian of Z, denoted $\mathrm{Pf}(Z)$, is a certain polynomial in the entries of Z, with the property that $\mathrm{Pf}(Z)^2 = \det(Z)$. When n is odd, $\mathrm{Pf}(Z) = \det(Z) = 0$; for $n = 2, 4$ we have

$$\mathrm{Pf}\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} = a, \qquad \mathrm{Pf}\begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix} = af - be + cd.$$

In general, for n even,

$$\mathrm{Pf}(Z) = \sum \mathrm{sgn}(\sigma) z_{\sigma(1),\sigma(2)} \cdots z_{\sigma(n-1),\sigma(n)},$$

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where the sum is over all permutations $\sigma \in S_n$ such that $\sigma(2i-1) < \sigma(2i)$ for all i and $\sigma(1) < \sigma(3) < \cdots < \sigma(2n-1)$. Equivalently, the sum is over all unordered partitions of $\{1, \dots, 2n\}$ into pairs; the restrictions on σ simply amount to choosing one representative ordering for each partition. There is a Laplace-like expansion: for each j, $1 \le j \le n$,

$$\mathrm{Pf}(Z) = \sum_{i<j} (-1)^{i+j+1} z_{i,j} \,\mathrm{Pf}(Z^{i,j}) + \sum_{i>j} (-1)^{i+j} z_{i,j} \,\mathrm{Pf}(Z^{i,j}),$$

where $Z^{i,j}$ is the matrix obtained by deleting the ith and jth rows and columns of Z. See [4, 6, 7].

A t-Pfaffian of Z is given by a list of t rows and the same columns; we write briefly $[a_1, \dots, a_t]$, where $1 \le a_1 < \cdots < a_t \le n$, for the Pfaffian of the skew-symmetric submatrix given by the rows $a_1, \dots, a_t$ and the same columns. Of course this is zero if t is odd.

The ideal generated by the size t Pfaffians of Z is denoted $P_t(Z)$. If n is odd then $P_{n-1}(Z)$ is a prime ideal of height 3. More generally, $P_{2p}(Z)$ is a prime ideal of height $\mu(p, n) = (n - 2p + 1)(n - 2p + 2)/2$, see [7].

We are interested in the ideal generated by the subset of Pfaffians with at least r_1 rows in the first R_1 rows of Z, at least r_2 rows in the first R_2 rows of Z, and so on; the row condition implies that these Pfaffians meet the corresponding column conditions as well, i.e., at least r_1 columns in the first R_1 columns of Z, and so on.

Fix a sequence $1 \le R_1 \le \cdots \le R_p \le n$, $R = (R_1, \dots, R_p)$, and another sequence $r = (r_1, \dots, r_p)$ of the same length. Let Z_{R_i} be the submatrix of Z consisting of the first R_i rows and let $Z_{R_i}^{R_i}$ be the $R_i \times R_i$ submatrix of Z in the upper left corner, consisting of the first R_i rows and the first R_i columns. We will also need, for each $R_i + 1 \le k \le n$, the $(R_i + 1) \times (R_i + 1)$ submatrix given by the first R_i rows and columns plus the kth row and column, that is, the set of rows (and columns) corresponding to the set $\{1, \dots, R_i, k\}$. Recall the common notation $[R_i] = \{1, \dots, R_i\}$, so we may write $[R_i] \cup \{k\}$ for the set we want. To simplify notation, we write $Z([R_i])$ for $Z_{R_i}^{R_i}$ and we write $Z([R_i] \cup \{k\})$ for the $(R_i + 1) \times (R_i + 1)$ skew-symmetric submatrix of Z given by the rows (and columns) corresponding to the set $\{1, \dots, R_i, k\}$. Since confusion seems unlikely we will drop the brackets and braces and simply write $Z(R_i)$ and $Z(R_i \cup k)$. Thus for example

$$Z(3 \cup 5) = \begin{pmatrix} 0 & z_{1,2} & z_{1,3} & z_{1,5} \\ -z_{1,2} & 0 & z_{2,3} & z_{2,5} \\ -z_{1,3} & -z_{2,3} & 0 & z_{3,5} \\ -z_{1,5} & -z_{2,5} & -z_{3,5} & 0 \end{pmatrix},$$

with rows and columns given by the set $3 \cup 5 = [3] \cup \{5\} = \{1, 2, 3, 5\}$.

Theorem 6.1. *Let B be a ring and $A = B[Z]$. Let $J = J(Z, 2t, R, r)$ be the ideal generated by $2t$-Pfaffians of Z that have at least r_i rows in Z_{R_i} for $1 \le i \le p$. For each i, if r_i is even, let $J_i = P_{r_i}(Z(R_i))$, and if r_i is odd, let $J_i = \sum_{k=R_i+1}^{n} P_{r_i+1}(Z(R_i \cup k))$. Then*

$$J = P_{2t}(Z) \cap J_1 \cap \cdots \cap J_p. \tag{3}$$

If B is a domain then $P_{2t}(Z)$ is prime and each J_i is a prime ideal.

Proof. Each of the following is an order ideal in $\Pi(Z)$:

(1) The set of $2t$-Pfaffians is the order ideal generated by $[n - 2t + 1, \dots, n]$.

(2) The set of Pfaffians (of all sizes) with at least r_i rows contained in Z_{R_i}. If r_i is even, this is the order ideal generated by $[R_i - r_i + 1, \ldots, R_i]$. If r_i is odd, this is the order ideal generated by $[R_i - r_i + 1, \ldots, R_i, n]$.

So, by Lemma 3.2, J is equal to the intersection of the ideals $P_{2t}(Z)$ and, for each i, the ideal generated by the Pfaffians (of any size) having at least r_i rows in Z_{R_i}.

If r_i is even then the ideal generated by Pfaffians with at least r_i rows in Z_{R_i} is $P_{r_i}(Z(R_i))$. Indeed, if P is any Pfaffian with at least r_i rows in Z_{R_i} then P can be expanded as a combination of r_i-Pfaffians involving those rows.

If r_i is odd and P is any Pfaffian with at least r_i rows in Z_{R_i}, then either P actually has at least $r_i + 1$ rows in Z_{R_i} or else P involves at least one more row, say the kth row, with $k > R_i$. Either way, P can be expanded as a combination of $(r_i + 1)$-Pfaffians in $Z(R_i \cup k)$. So P lies in the sum given in the statement. Conversely, every Pfaffian generator of the sum in the statement must have at least r_i rows in Z_{R_i}.

Now suppose B is a domain. The set of $(\geq 2t)$-Pfaffians of Z is cogenerated by $[1, \ldots, 2t-2]$. This shows $P_{2t}(Z)$ is prime. (Of course $P_{2t}(Z)$ is already well-known to be prime.)

To see that each J_i is prime, note that the order ideal of Pfaffians generating J_i is cogenerated by either $m = [1, \ldots, r_i - 1, R_i + 1, \ldots, n]$ or $m' = [1, \ldots, r_i - 1, R_i + 1, \ldots, n - 1]$, whichever has even length (regardless of whether r_i is even or odd). Let us verify this. For simplicity, suppose that m has even length. We must show that $\alpha \not\geq m$ if and only if α has at least r_i rows in Z_{R_i}, equivalently $\alpha \geq m$ if and only if α has $r_i - 1$ or fewer rows in Z_{R_i}; note that this is the criterion whether r_i is even or odd. Now $\alpha \geq m$ if and only if $|\alpha| \leq r_i - 1$ or $|\alpha| \geq r_i$ and $\alpha_{r_i} \geq R_i + 1$. The forward direction is obvious; conversely, under these conditions, $\alpha_{r_i+t} \geq \alpha_{r_i} + t \geq R_i + 1 + t = m_{r_i+t}$ for all $0 \leq t \leq |\alpha| - r_i$, so each entry of α is at least as great as the corresponding entry of m; and in particular since every entry of α is at most n, $|\alpha| \leq |m|$. This shows that $\alpha \geq m$. So indeed $\alpha \geq m$ if and only if α has $r_i - 1$ or fewer rows in Z_{R_i}.

The argument in case $|m'|$ is even is similar. Note that m' is as long as possible for a member of $\Pi(Z)$ with $R_i + 1$ in the r_i position; so if $\alpha_{r_i} \geq R_i + 1$ then $|\alpha| \leq |m'|$. □

Once again this writes J as a possibly redundant intersection of prime ideals, if B is a domain.

Proposition 6.2. *Suppose*

(1) $R_1 < \cdots < R_p$,
(2) $r_1 < \cdots < r_p < 2t$,
(3) $0 \leq r_i \leq R_i$ *for each* i,
(4) $R_1 - r_1 < \cdots < R_p - r_p < n - 2t$.

Then the intersection (3) *is irredundant.*

The proof is the same as before.

Corollary 6.3. *Let* $Z = (z_{i,j})$ *be a generic skew-symmetric matrix, let* $0 < p < q$, *and fix* $A = B[Z]$ *with degree* $\deg z_{i,j} = 2p$ *if* $i, j \leq R$, $\deg z_{i,j} = p + q$ *if* $i \leq R < j$, *and* $\deg z_{i,j} = 2q$ *if* $i, j > R$. *Fix* t *and* d. *Let* $r = \left\lceil \frac{2tq-d}{q-p} \right\rceil$. *Then* $P_{2t}(Z)_{\leq d} = P_{2t}(Z) \cap I_r$ *where* I_r *is the ideal generated by Pfaffians with at least* r *rows in* Z_R. *If* r *is even,* $I_r = P_r(Z(R))$. *If* r *is odd,* $I_r = \sum_{k=R+1}^{n} P_{r+1}(Z(R \cup k))$ *where* $Z(R \cup k)$ *is the* $(R+1) \times R + 1$ *skew-symmetric submatrix of* Z *given by the rows (and columns) corresponding to the set* $\{1, \ldots, R, k\}$. *If* B *is a domain then* $P_{2t}(Z)$ *and* I_r *are prime.*

Indeed, a $2t$-Pfaffian P with r rows in Z_R has degree $pr + q(2t - r)$; the value of r in the statement is the least integral solution to $\deg P \leq d$.

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