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On Constructions Of Generalized Skein Modules

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Abstract

Jozef Przytycki introduced skein modules of 3-manifolds and skein deformation initiating algebraic topology based on knots. We discuss the generalized skein modules of Walker, defined by fields and local relations. Some results by Przytycki are proven in a more general setting of fields defined by decorated cell-complexes in manifolds. A construction of skein theory from embedded TQFT-functors is given, and the corresponding background is developed. The possible coloring of fields by elements of TQFT-modules is discussed for those generalized skein modules. Also an approach of defining skein modules from studying compressions of fields is described.

Keywords: generalized skein module, topological quantum field theory

Mathsubject Class.: 57M25, 57M35, 57R42

1 Motivation and questions.

J. Przytycki defined skein modules of 3-manifolds as a kind of universal target of knot invariants satisfying skein relations [P1], [P2]. The skein modules define functors from categories of 3-manifolds and diffeomorphisms (actually codimension-0 embeddings) into module categories. The skein module of a 3-manifold is defined by taking the quotient of a free module with basis the set of isotopy classes of links in the 3-manifold by a submodule generated by *local* relations. The skein modules with the actions by diffeomorphisms thus almost define modular functors in the sense of Turaev [T] (usually missing is the finite type projectivity of skein modules). We define a *skein theory* to be the collection of skein modules with the diffeomorphism actions and gluing structures.

Interesting skein modules are always based on skein relations that *interact* in a non-trivial way with the action of diffeomorphisms on embeddings. Besides classical skein relations like the Kauffman bracket or Homflypt relations, Przytycki considered skein relations he called *deformations* of homotopy and homology via the action of diffeomorphisms on the set of framed links in 3-manifolds. The theory of skein modules and the field of *algebraic topology based on knots* have since then developed into an independent field of study related to important questions in quantum topology like the volume conjecture [L].

Recently K. Walker [W] defined generalizations of skein modules for n-manifolds based on a subtle categorical axiomatics of *fields* in dimensions $\leq n$, *local relations* in dimension n and a sophisticated system of *gluing axioms*. His generalized skein modules are quotients of vector spaces of fields by subspaces generated by local relations. Walkers motivation is to give a geometric construction of (extended) topological quantum field theory (TQFT). He proves that under the assumptions that (i) the definition of fields can be consistently extended to $(n+1)$-manifolds, (ii) there exist natural *positive definite* pairings on skein modules of the n-ball defined by gluing, and (iii) the skein modules of all n-manifolds are finite dimensional, *then* skein theory defines an $(n+1)$-dimensional TQFT. Walker explains how $(n+1)$-dimensional TQFT theory naturally induces skein modules of n-manifolds using supposed properties of a *Feynman path integral* structure on fields. But $(n+1)$-dimensional TQFT also is related with skein relations for $(n+1)$-fields. This consequence of finite dimensionality of TQFT modules has probably first been observed by Witten for $n = 2$.

It is the goal of this paper to review relations between skein modules and TQFT-structures by discussing two general constructions of generalized skein theories. Our hope is that this initiates a study of interesting generalized skein modules on a more conceptual basis. The question of *finite dimensionality* of the TQFT modules, which is essential for TQFT theory, is not of our concern here, see [AU] for related discussions. Most of the classical skein theories a priori do *not* define finite dimensional skein modules for 3-manifolds. According to Walker, $2+1$-dimensional TQFT theory is naturally defined from skein modules of 1-complexes on surfaces, with the skein relations *projected* from skein relations of links embedded in the cylinder over the surface. The power of these natural skein relations is that there are corresponding polynomial invariants of links in S^3. On the other hand skein modules of links in 3-manifolds are naturally associated with $3+1$-dimensional TQFT via Walker's approach. It seems that at this point connections between 3- and 4-dimensional topology are not understood in full detail. It is well-known that this problem is at the heart of Khovanov theory. For a discussion of the categorification of the Jones or Kauffman bracket polynomial see [P3], for a discussion of a categorification of skein modules of I-bundles over surfaces see [APS]. Note that recently, Gaiotto and Witten [GW] have been studying Jones and Khovanov theory for links in 3-manifolds

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M through differential equations for fields on $M \times [0, \infty)$ with boundary conditions on $M \times \{0\}$ defined by links in M. The compression functors in the last section of this paper could possibly be related in some way with this study of physical fields on cylinders over 3-manifolds.

The calculation of skein modules is difficult in general. The questions below indicate some directions in which to study the generalized skein theories of Walker. Recall that for each skein theory in dimension n one can naturally study the skein modules of cylinders over $(n-1)$-manifolds with fixed *boundary fields* on top and bottom. In this way skein modules appear as morphism sets of certain categories generalizing the Jones algebroid of Jones skein theory, see [Wa]. Motivated by the examples in classical skein theory, see [P1], we suggest the following definitions. We will assume throughout that the set of fields is an R-module, for R a commutative unital ring, with basis a collection of base fields.

1.1 Definitions. (i) A skein theory is *consistent* if the skein algebra $\mathfrak{R}$ of the 3-ball with empty boundary field is naturally isomorphic to a subalgebra of a localization of R at a multiplicative set determined by the skein relations, and $\mathfrak{R} \supset R$ with R corresponding to the empty field, see 1.2. Remarks (i) below for an example.

(ii) A skein theory is *strongly consistent* if all morphism modules of the Jones algebroid as above are finitely generated free modules over the algebra $\mathfrak{R}$ (see the Remarks below for an explanation of the $\mathfrak{R}$-module structure.)

(iii) A skein theory is *finitely skein generated* if the skein relations are *generated* by a finite set (in the sense of generating a gluing ideal, for details see Definition 2.9 below and [W].)

1.2 Remarks. (i) If $R = k$ a field (so the set of fields is a k-vector space) consistency reduces to $\mathfrak{R} \cong k$. Consistency requires that the n-ball, which represents trivial n-dimensional manifold topology, is represented in skein theory in an essentially trivial way. On the other hand, non-triviality of skein modules of manifolds should detect non-trivial manifold topology. A typical example of $\mathfrak{R}$ is for the skein module of oriented links in 3-manifolds, defined over $R = \mathbb{Z}[q^{\pm 1}, z, h]$, by

q^{-1}  $- q$  $= w$ 

$$(q^{-1} - q)\emptyset = h\bigcirc$$

with $w = z$ respectively $w = h$ for a crossing of different components respectively for a self-crossing. In this case $\mathfrak{R} = \mathbb{Z}[q^{\pm 1}, z, h, \frac{q^{-1}-q}{h}]$, see [P1] and [K3]. The extension $\mathfrak{R} \supset R$ results from relating unlinks to the empty link, which requires to localize at h. It is a consequence of non-invertibility of ring elements involved in the skein relations that the vacuum, i. e. the empty link, does not generate the skein module of the ball.

(ii) Usually skein modules of n-manifolds are modules over the ring $\mathfrak{R}$ in a natural way. This follows if the inclusion of a punctured manifold (by which we mean the complement of an open ball in the manifold) into the manifold induces an isomorphism of skein modules.

(iii) Strong consistency holds for the usual skein modules of links or surfaces in 3-manifolds, but for quite different reasons. For skein modules of links it follows from consistency of a TQFT respectively the existence of link polynomials like the Jones of Homflypt polynomials in the background of those theories. For skein modules of surfaces in the sense of [K1] it follows from the complete compressibility of surfaces embedded in the 3-ball and the consistency of abstract TQFT for surfaces.

The above definitions suggest the following questions.

1.3 Questions. For which skein relations is a skein theory (strongly) consistent? When is a skein theory finitely skein generated?

It seems interesting to study what general assumptions have the consequence that consistency implies strong consistency, or when finite generation holds in cases when the skein module is not already given by the local relation. Interestingly, if skein modules are directly defined by skein relations, the skein theories are obviously finitely skein generated but it is often not easy to establish consistency (usually this follows from the existence of a topological invariant like a quantum link invariant). On the other hand in section 3 we will define skein theories, which are by construction consistent, but finite skein generation does not seem easy to establish.

In section 2 we review the definition of generalized skein modules following [W] with details referred to [W] and [MW]. We will pay attention to orientations, and reprove some classical observations of Przytycki in a more general setting, emphasizing that many techniques in classical skein theory are based on transversality, which often works for fields defined by decorated embedded complexes in manifolds. Because we are not primarily interested in TQFT we work over a commutative unital ring R not necessarily an algebraically closed field or $\mathbb{C}$. In fact, it is known from classical skein theory that torsion in skein modules often is nicely related with the topology of the manifolds. In section 3 we define consistent n-dimensional skein theories from so called *base field functors* in dimension n and discuss the idea of extending the functor by colorings. (This is related to Question 1.3 above because it is difficult for a skein theory defined in this way to actually determine a generating set of skein relations.) This requires to define a category of base fields on n-cubes resembling Turaev's ribbon tangle category. In section 4 we discuss the definition of skein modules through compression and co-limits, extending the approach in [K2] for the case of Bar-Natan modules.

2 Generalized skein modules

We work in the smooth ($\mathcal{C}^\infty$-) category with corners because our examples are usually smooth embeddings or immersions. Details about straightening corners are not discussed here, see [CF]. We assume that manifolds are equipped with straightening of the corners. Thus for M a smooth manifold, the boundary ∂M is a smooth manifold. We let $\text{int}(M) = M \setminus \partial M$ be the interior of M. We let $\text{Diff}(M)$ respectively $\text{Diff}(M, \partial M)$ denote the group of all diffeomorphisms of M respectively diffeomorphisms of M, which restrict to the identity on ∂M.

Generalized skein modules are defined by extending Przytycki's idea of tangle replacements [P1], see 2.9 Examples below. For more general objects and possibly higher dimensions it becomes important to base the skein theory on objects satisfying strong functoriality and gluing properties.

Following [W] a *system of fields for n-manifolds* is a sequence of symmetric monoidal functors

$$C_j : \mathcal{M}_j \to \mathfrak{S},\ 0 \leq j \leq n,$$

where $\mathfrak{S}$ is a symmetric monoidal category and $\mathcal{M}_j$ is a symmetric monoidal category with objects smooth compact j-dimensional manifolds with corners and morphisms defined by diffeomorphisms. The functor has to satisfy a list of properties with respect to taking boundary and gluing, see [W] or [MW] for a complete discussion and below for an incomplete one. (We will usually assume that $\mathcal{M}_j$ is the category of compact *oriented* manifolds. For an oriented manifold M we let $\overline{M}$ denote the manifold with opposite orientation. The orientation of ∂M is given by the outward normal last convention.)

It turns out that in all of our discussions it will be sufficient to have fields defined for $n-2 \leq j \leq n$. But Walker's assumption is essential because it implies that the generalized skein modules are computable from the relative skein modules of I^n, the gluing properties and locality of fields and skein relations.

In this paper we usually do not work in all the generality of [W]. Throughout we fix a commutative unital ring R and define $C(M) = C_j(M)$ by assigning the free R-module with basis a set of *base fields* $\mathcal{F}(M)$ on M, $0 \leq j \leq n$. We will discuss the corresponding functors $\mathcal{F}_j$ from $\mathcal{M}_j$ into the symmetric category of sets.

2.1 Definition A *system of base fields for n-manifolds* is a sequence of symmetric monoidal functors $\mathcal{F}_j : \mathcal{M}_j \to \mathfrak{S}$ for $0 \leq j \leq n$.

Note that $\mathcal{F}(M \sqcup N) = \mathcal{F}(M) \times \mathcal{F}(N)$ for M, N two j-manifolds and $j \leq n$, and $\mathcal{F}(\emptyset) = \{\emptyset\}$ because the functors are monoidal. We assume that there is a unique empty field, the *vacuum*, $\emptyset \in \mathcal{F}(M)$ for each manifold M. Moreover, a system of base fields has to satisfy the following list of properties.

(i) There is defined a *boundary map*

$$\partial : \mathcal{F}(M) \to \mathcal{F}(\partial M)$$

mapping $\emptyset$ to $\emptyset$.

(ii) We assume $\mathcal{F}(M) = \mathcal{F}(\overline{M})$ in a natural way. Thus for each diffeomorphism $f : M \to N$ (not necessarily orientation preserving) there is defined the induced map $f_* : \mathcal{F}(M) \to \mathcal{F}(N)$ such that $\partial f_* = f_* \partial$, where the second $f_* : \mathcal{F}(\partial M) \to \mathcal{F}(\partial N)$ really is the map induced by the restriction of f to the boundary. If a diffeomorphism $f : M \to M$ is isotopic to the identity diffeomorphism then the field $f_*(c)$ is called *isotopic* to c.

(iii) There is defined an involution $\mathcal{F}(M) \ni c \mapsto \hat{c} \in \mathcal{F}(M)$ such that $\widehat{\partial c} = \partial \hat{c}$ and $f_*(\hat{c}) = \widehat{f_*(c)}$. (This involution can very well be the identity) Moreover $\hat{\emptyset} = \emptyset$

(iv) Let $N \subset M$ be a codimension 0 submanifold. Then for an open dense subset of $f \in \mathrm{Diff}(M)$ there is a restriction map

$$\mathcal{F}(M) \to \mathcal{F}(f(N))$$

(in particular for almost all diffeomorphisms in a neighborhood of the identity). Whenever we consider codimension-0 submanifolds we usually require that restriction of fields to the submanifold is defined.

(v) There is defined a *cylinder functor* assigning to each field on a manifold N of dimension $\leq n-1$ the field $c \times I$ on the manifold $N \times I$. If the field c is closed then it satisfies $\partial(c \times I) = \hat{c} \times \{0\} \sqcup c \times \{1\}$. Note that $\partial N \times I = \overline{N} \times \{0\} \sqcup N \times \{1\}$. Here $\sqcup$ denotes the disjoint union of fields on a manifold or the disjoint union of manifolds, see the following Remark. If the field is not closed then the boundary of $c \times I$ is given by gluing the bottom and top field over the cylinder over ∂c.

2.2 Remarks. (a) We will often use the disjoint union of two disjoint fields on a manifold M: Given c_1, c_2 on M such that there exist disjoint codimension-0 submanifolds M_1, M_2 such that the restriction of c_i to $M \setminus \mathrm{int}(M_i)$ is the empty field. Then it follows easily from Walker's gluing axioms that the field $c_1 \sqcup c_2$ is defined on M.

(b) Note that oriented gluing of fields requires care: Suppose there are given fields on M_i for $i = 1, 2$. If $N_i \subset \partial M_i$ are codimension-0 submanifolds such that the given fields on M_i restrict to fields c_i on N_i then if for an orientation preserving diffeomorphism $h : N_1 \to \overline{N_2}$ we have $h_*(c_1) = \hat{c_2}$ (where we use $\mathcal{F}(N_2) = \mathcal{F}(\overline{N_2})$), there is defined the field $c_1 \cup_h c_2$ on the oriented manifold $M_1 \cup_h M_2$. (This is actually the case of gluing possibly with corners. If $\partial N_i = \emptyset$ then we have the usual gluing without corners.)

(c) Let M be a manifold of dimension $\leq n-1$. Let $P\mathrm{Diff}(M)$ denote the path space of $\mathrm{Diff}(M)$ of paths starting at the identity diffeomorphism, where $\mathrm{Diff}(M)$ is equipped with the usual strong C^∞-topology. It follows from axioms (ii) and (vi) that for each $\alpha \in P\mathrm{Diff}(\mathrm{M})$ and each closed field c on M there is defined a field $\alpha_\sharp(c)$ defined on $M \times I$ by applying the trace diffeomorphism $M \times I \to M \times I$, $(x,t) \mapsto (\alpha_t(x), t)$ to the field $c \times I$ on $M \times I$. Then $\partial \alpha_\sharp(c) = \hat{c} \times \{0\} \sqcup \alpha(1)_*(c)$.

(d) The condition $\mathcal{F}(M) = \mathcal{F}(\overline{M})$ will not always hold for the general definition of fields given by Walker. For example if the set $\mathcal{F}(M)$ is the set of tight positive contact structures then there exist manifolds with $\mathcal{F}(M) \neq \emptyset$ but $\mathcal{F}(\overline{M}) = \emptyset$, see [LS].

(e) Systems of base fields defined by vector spaces of functions *on* M often seem to be related to base fields of complexes *in* M by using *Pontrjagin-Thom* type constructions. It should be interesting to study whether corresponding skein theories can be related in a natural way.

In the following we usually assume that base fields are properly embedded complexes in M of codimension k, possibly *decorated* with orientation or coloring of strata, and satisfying some transversality conditions. Moreover we will also assume that the complex underlying $\hat{c}$ is the same as the complex underlying c. Thus the hat operation is really an operation on decorations. We will not axiomatize the notion of decoration here. The right viewpoint of course is through a functor on a suitable cobordism category of complexes, see [We] for an introduction to the axiomatic of cobordism categories. We will not go into details about this at this point, even though in particular in relation with mapping space topology this is an important point. The consideration of embedded complexes as basic examples of fields has been suggested by Walker. It is justified by important examples like the recent webs and spiders but also because embedded complexes behave very similar to embedded submanifolds when it comes to the action of the diffeomorphisms of the manifolds on the set of embeddings, see [M] and [G]. See also the work by Forman [F] on Witten-Morse theory for cell complexes.

2.3 Definition Two fields c_1, c_2 on M are *cell isotopic* if the underlying cell complexes in M are isotopic relative to the boundary.

2.4 Examples. Let base fields c on M be codimension-k embedded oriented submanifolds. Then usually $\hat{c}$ is defined by changing the orientation of c. For $n = 3$ and $k = 2$ we have oriented tangles in 3-manifolds bounding oriented points in the boundary surfaces. In this case $\hat{c}$ will be the 1-manifold c with all orientations reversed. Thus an invertible knot is an example of a base field c isotopic to the field $\hat{c}$. If decoration is framing without orientation then we have banded tangles in 3-manifolds bounding arcs on surfaces. In this case the involution is trivial. In both cases the corresponding fields on 1-dimensional and 0-dimensional manifolds are empty because of the codimension. More interesting decorations can be defined by letting the complexes underlying base fields be defined by images of immersions in codimension k, and the decoration be given by an actual immersion with image the complex, possibly up to diffeomorphisms of the domain manifold. The study of skein theory in 3-manifolds based on PL-maps of circles into the 3-manifold and their singularties has been initiated by Kalfagianni and Lin, see [K] and [KL]. The choice of map with a given complex as image here can be understood as the decoration.

For $c \in \mathcal{F}(\partial M)$ let $\mathcal{F}(M, c) := \partial^{-1}(c) \subset \mathcal{F}(M)$. This is the set of base fields bounding the field c on ∂M. If $f : M \to N$ is a diffeomorphism then it follows from (ii) that f_* maps $\mathcal{F}(M, c)$ into $\mathcal{F}(N, f_*(c))$. Then $\mathcal{F}(M, \emptyset)$ is the set of closed base fields. The corresponding R-modules of base fields are defined by taking free R-modules. Elements of those R-modules are called fields in [W].

2.5 Remarks. (a) The gluing properties express the *locality of fields*: This means that base fields on n-dimensional manifolds are determined by (i) their restrictions to the handles of a handle-decomposition of the manifold, and (ii) the homomorphisms induced from the gluing diffeomorphisms of the handles. We will see that these properties transfer to skein modules and their calculation.

(b) It is cobordism problem whether boundary operators are onto. Usually there are fields on ∂M that do not bound a field on M. But the empty field $\emptyset$ on ∂M always bounds the empty field on M.

2.6 Examples. (a) Define base fields on a surface by disjoint embeddings of framed oriented points in the interior of the surface. The fields in 3-manifolds then are oriented ribbon tangles, i. e. framed oriented arcs and circles are the basis fields on a 3-manifold (It is also possible to include coupons.) The ribbon tangles can be generalized to ribbon graphs and finally the graphs can be colored by elements of some abstract ribbon category, see [RT] Note there is a unique basis field in 0-dimensional and 1-dimensional manifolds, namely the empty field.

(b) Consider proper oriented codimension-k submanifolds in n-manifolds satisfying obvious transversality with respect to boundary and corners. For $n = 3$ and $k = 2$

this is the classical setting of Przytycki. If we consider possibly non-orientable codimension-k submanifolds and consider $n = 3$ and $k = 1$ with decorations of the surfaces given by coloring the components by elements of a Frobenius algebra then we are in the situation of [K1]. Note that in this case base fields on surfaces are properly 1-manifolds, and base fields on 1-manifolds are just disjoint collections of points.

(c) Another example of fields are uni-trivalent graphs embedded in a surface with trivalent vertices in the interior and univalent vertices in the boundary of the surface defining the fields in 1-manifolds. Again there is only the empty field on 0-manifolds.

Let $\mathcal{R} = (\mathcal{R}(d))_{d\in\mathfrak{B}}$ be sequence of relation subsets $\mathcal{R}(d) \subset C(B^n, d)$, where d runs through a set $\mathfrak{B}$ of representatives of isotopy classes of fields on the boundary ∂B^n of the n-ball B^n. Let B be an n-ball and let $h : B^n \to B$ be an (oriented) diffeomorphism. Then the diffeomorphism induces a homomorphism $C(B^n, d) \to C(B, h_*(d))$, which maps elements $r \in C(B^n, d)$ to *skein relations* in $C(B, h_*(d))$. Next given an n-manifold M and a fixed base field $c \in C(\partial M)$. Let R_M denote the submodule of $C(M, c)$ which is generated by (i) all elements resulting from gluing relations in $C(B, h_*(d))$ to fields on $M \setminus \mathrm{int}B$ with boundary $h_*(\hat{d})$ as above, for arbitrary n-balls $B \subset \mathrm{int}(M)$ and (oriented) diffeomorphisms $h : B^n \to B$, and (ii) relations $b - b'$ for $b, b' \in \mathcal{F}(M, c)$, which are isotopic relative to the boundary. R_M is called the module of skein relations on M relative to c.

2.7 Definition. Let M be an n-manifold and c be a field on ∂M. The *skein module* of (M, c), defined by system of base fields $\mathcal{F}$, the ring R, and relations $\mathcal{R}$ is

$$S_{(\mathcal{F},R,\mathcal{R})}(M, c) := C(M, c)/R_M.$$

In general it is difficult to understand the structure of this module. We often abbreviate notation if the defining structures are given and write $S(M, c)$ only, also $S(M) = S(M, \emptyset)$ is the skein module of M. If $c \neq \emptyset$ then $S(M, c)$ is often called a *relative skein module*. Note that the definition of generalized skein modules is quite technical. Thus *natural* constructions of examples are important.

Note that if M is connected then any two oriented embeddings $h : B^n \to B \subset M$ are isotopic. In this case it is easy to see that it suffices to use a single ball $B \subset M$ in the definition above.

Let $\mathfrak{F}(M, c)$ be the set of *isotopy classes* of base fields in $\mathcal{F}(M, c)$, and correspondingly let $\mathfrak{C}(M, c)$ denote the free R-module generated by isotopy classes of base fields. Then the skein module $S(M, c)$ is also the quotient of $\mathfrak{C}(M, c)$ by skein relations, which are projections of elements of R_M into $\mathfrak{C}(M, c)$. This is the classical definition, emphasizing that skein relations are in fact relations between isotopy classes, even though defined using representatives.

The following is immediate from the definitions and axioms:

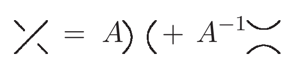
2.8 Proposition *The skein modules defined above come with actions of the diffeomorphisms of the manifolds. Moreover, gluing of fields induces corresponding gluing homomorphisms of skein modules. Thus $\mathcal{R}$ defines a skein theory.*

We denote the action of $h \in \mathrm{Diff}(M)$ also by h_*

Note that the choice of relation sequence $\mathcal{R}$ is highly non-unique. In fact we can always add further relations on balls defined by gluing together relations on balls (this is what Walker calls generating a gluing ideal).

2.9 Definition A skein theory is *finitely skein generated* if it possible to choose the sequence $(\mathcal{R}(d))_d$ such that (i) $\mathcal{R}(d) \neq \emptyset$ for at most finitely many cell isotopy classes of fields d, and (ii) for each isotopy class of field d, if the relation elements are projected to cell-isotopy classes, the resulting sets $\widetilde{\mathcal{R}}(d)$ are finite for all isotopy classes d.

2.10 Examples. (a) The Kauffman bracket relations define a skein theory over $R = \mathbb{Z}[A^{\pm 1}]$ for framed tangles in 3-manifolds. In a projection onto the equatorial disk of an oriented 3-ball the relations are (use blackboard framing with respect to the equatorial disk):



$$\bigcirc D = (-A^2 - A^{-2}) D$$

for D any framed link outside the 3-ball. It is well-known that the skein theory is strongly consistent. It is finitely skein generated by definition.

(b) The skein relations of oriented links in 3-manifolds defined from 2-tangle sequences [P1] and corresponding sequences of elements of R, are basic examples of classical skein theories. They are finitely skein generated by definition. In this case the boundary field is usually the empty field $\emptyset$ or the field given by two positive and two negative points (classical 2-tangles). Note that the boundary field has to be oriented zero-homologous to get a non-empty set of fields. It is usually not easy to decide consistency.

(c) In particular a skein theory of oriented links in 3-manifolds can be defined by choosing a set of elements of the group rings RB_n of the n-strand braid group B_n.

For Przytycki's standard examples these are the powers of the generating braid $\sigma \in B_2$.

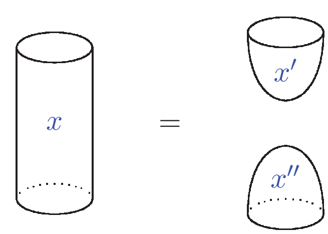
Other choices of skein relations could be for example to define for a fixed n,

$$\sum_{\sigma \in \Sigma_n} (-1)^{\mathrm{sign}(\sigma)} \hat{\sigma},$$

where Σ_n is the symmetric group of order n and $\hat{\sigma}$ is the permutation braid assigned to σ.

(d) The quantum deformation of homology and homotopy are skein theories defined from basis fields given by framed oriented links.

(d) The generalized Bar-Natan modules are defined from the Bar-Natan relations on surfaces in 3-manifolds colored by elements of a Frobenius algebra V over the ring R. Here is a picture of the so called *neck cutting relation* for $x \in V$, where $x' \otimes x'' = \Delta x$ (in Sweedler's notation) and Δ is the co-product of the Frobenius algebra.



Bar-Natan also defines a purely geometric version. His skein relations are natural because they involve a kind of symmetric summing over all simple zero-bordisms of the boundary fields.

Two main principles in the theory of classical skein modules due to J. Przytycki [P1] generalize to the setting above immediately. One that is important for the idea of deformation is the *universal coefficient theorem*, which is proved as in [P1]:

2.11 Proposition. Consider any skein theory over the ring R defined by a set of base fields and local relations $\mathcal{R}$ as above. For a given boundary field c let $S(M, c; R)$ denote the corresponding skein module. Let $\varphi : R \to R'$ be a homomorphism of

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commutative unital rings. Then R' is naturally an R-module. The homomorphism φ induces an epimorphism $C(M,c;R)\otimes_\varphi R' \to C(M,c,R')$ and homomorphisms $\mathcal{R}\to\mathcal{R}'$ and finally $S(M,c;R)\to S(M,c;R')$. Then

$$S(M,c;R') \cong S(M,c;R)\otimes_\varphi R',$$

where we have abbreviated $S(M,c;R') := S_{(\mathcal{F},R',\mathcal{R}')}(M,c)$.

The second important principle in the calculation of classical modules is the handle attachment result from [P1]. The underlying argument depends only on the codimension of the cell complexes in M defining the fields. Let supp(c) denote the embedded complex of M defined by the field c (recall that c could denote an immersion or embedding and carry decoration). The following result follows immediately from transversality for complexes in smooth manifolds.

2.12 Theorem. *Consider a skein theory with base fields defined by embedded codimension k cell complexes. Let M be an n-dimensional manifold and c be a field on M. Let $H = D^j\times D^{n-j}$ be a j-handle attached to $\partial M\setminus supp(c)$ by an embedding $h : S^{j-1}\times D^{n-j}\to\partial M$. Let $M' := M\cup_h H$. Let*

$$i_* : S(M,c)\to S(M',c)$$

be the homomorphism of skein modules induced by the inclusion $M\subset M\cup_h H$. Then i_* is an isomorphism if $k < j-1$ and is onto if $k = j$. If $k = j$ the the kernel of i_* can be described as follows: Let d_h denote the handle slide diffeomorphism of M' defined by the handle. Then elements of the form $\overline{d} - (h_d)_*(\overline{d})$ generate the kernel, where $\overline{d}$ denotes the image of $d\in C(M,c)$ in $S(M,c)$. $\square$

The statement of the theorem requires some explanation. The handle attachment (H,h) defines an isotopy of M', unique up to isotopy, which isotopes the southern hemisphere D^{j-1}_- of the core sphere $S^{j-1}\times\{0\}$across the core disk $D^j\times\{0\}$ and is the identity outside of a neighborhood of the disk in M', and in particularly fixes $\partial M'$ point-wise. Note that for $(k+1)+j<n$, skein balls and the images of isotopies of complexes will miss the co-core of the handle by transversality. But the complement of an open neighborhood of the co-core in M' is naturally diffeomorphic to M. In fact, there is an diffeotopy of this manifold retracting it onto the submanifold M. If $k+j<n$ then a field on M' can be isotoped away from the co-core and defines a field on M. This isotopy is not natural. But any two choices will differ by an application of d_h.

2.13 Example. The result above includes the classical results (i) that for links in a 3-manifold M, the inclusion $M\setminus\mathrm{int}(B)\subset M$ with $B\subset M$ a 3-ball, induces an

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isomorphism of skein modules, and (ii) the 2-handle attachment result of Przytycki [P1].

2.14 Remark. If $k > j$ (e. g. for $k = 2$ and $j = 1$) then the inclusion $M \to M'$ will not induce a surjective homomorphism in general. For $n = 3$, $k = 2$ and $j = 1$ this includes the case of attaching 1-handles to a 3-ball with result a handle-body, often having a rich skein module of links. In general M' is defined from $M \sqcup H$ by gluing the submanifold $S^{j-1} \times D^{n-j} \subset \partial H$ to its image in ∂M under h. Each codimension-k field c on M' can be assumed transversal to the co-core D^{n-j}. The intersection is a codimension-k field c' on the interior of D^{n-j}, such that c is the result of gluing the iterated cylinder field $I^j \times c'$ along the boundary field $\partial(I^j \times c') \subset S^{j-1} \times D^{n-j}$ via the diffeomorphism h to ∂M. It suffices to sum over representatives of all possible isotopy classes of fields in the interior of D^{n-j} to get a surjective homomorphism:

$$\bigoplus_{c'} S(M, c \sqcup (I^j \times c')) \otimes S(D^{n-j}, c') \to S(M', c).$$

Of course it is usually difficult to determine the kernel of this epimorphism.

3 Skein theories from base field functors

Throughout we assume that $n \geq 3$ and a system of base fields $\mathcal{F}$ for n-manifolds is given with $n \geq 3$. We will study base fields on $I^n = I^{n-1} \times I$. Walker [W] points out that in this case 2-categories are defined but we will not discuss those 2-category structures here. Instead we are interested in generalizing some of the ideas of ribbon tangle categories and functors on these, see [RT] and [T].

We want to consider fields on I^n, which restrict to the empty field on $\partial I^{n-1} \times I \subset \partial I^n$. The basic idea of *Morse theory* is to slice the field horizontally into simple pieces. In terms of complexes we would be interested in elementary changes of the topology of the complex in each slice. In the case of embedded submanifolds we could arrange that the projection $I^n \to I$ onto the last coordinate restricts to a Morse function on the submanifold, and correspondingly cut the interval I into subintervals containing only one critical point. Then using isotopy of the fields on the interesting boundaries $I^{n-1} \times \{s\}$ of slices we can arrange to consider elementary fields between a set of representatives in the isotopy classes of fields on I^{n-1}. In principle may restrict to consider single representatives of the isotopy classes of fields on I^{n-1}. But in fact this would obscure some interesting structures that are present.

We begin by constructing the monoidal structure emerging from the product structure of I^{n-1}. Two base fields c_1, c_2 on I^{n-1}, both with empty boundary fields, can be glued together to a field on $I^{n-2} \times [0,2]$, which then can be naturally re-parametrized to I^{n-1}. Let $c_1 \otimes c_2$ denote the resulting field. The product of a field c with the empty field $\emptyset$ is defined by the field c itself. Let $r : I^n \to I^n$ be defined by the reflection $r(x_1, \ldots, x_n) = (x_1, \ldots, x_{n-2}, 1 - x_{n-1}, x_n)$. This restricts to reflections, also denoted r, of $I^{n-1} \times \{1\}$ and $\overline{I^{n-1}} \times \{0\}$ about $I^{n-2} \times \{\frac{1}{2}\}$. Note that $r_*(c_1 \otimes c_2) = r_*(c_2) \otimes r_*(c_1)$. We will also need a reflection $\mathfrak{r} : I^n \to I^n$ defined by reflection about $I^{n-1} \times \{\frac{1}{2}\}$. This reflection restricts to the exchange map on $I^{n-1} \times \{0, 1\}$. Note that if d is a field on I^n such that $\partial d = \hat{c}_0 \sqcup c_1$ then $\partial \mathfrak{r}_*(\hat{d}) = \mathfrak{r}_* \partial \hat{d} = \mathfrak{r}_*(c_0 \sqcup \hat{c_1}) = \hat{c_1} \sqcup c_0$. (In the following we often omit $\times\{j\}$ for $j = 0$ or $j = 1$ if it is obvious.) We let $r \circ \mathfrak{r} = \mathfrak{t}$ denote the *turnover map*. It follows that $\partial \mathfrak{t}_*(\hat{d}) = r_*(\hat{c_1}) \sqcup r_*(c_0)$. Let $d^! := \mathfrak{t}_*(\hat{d})$. We also let $c^! = r_*(\hat{c})$ for fields on I^{n-1}. Since $r \circ r = id$ and $c \mapsto \hat{c}$ is an involution, $c \mapsto c^!$ is an involution.

3.1 Definition. A base field on I^{n-1} with empty boundary is called *simple* if it is not isotopic to a field obtained by gluing two non-empty fields as above.

Now fix a set of representative base fields c, one for each isotopy class of *simple* field on I^{n-1} with empty boundary. This can be done in such a way that we first fix a set of representative complexes and then add decorations in such way that we get a list of non-isotopic simple fields. Thus if two fields differ only by decoration they will be represented in this list by fields only differing by decoration. Next we extend this list of fields by adding for each field c the *companion fields* $\hat{c}$, $r_*(c)$ and $r_*(\hat{c})$. Some of these fields could be isotopic to c or to each other, then our list of *basic simple fields* $\mathcal{F}_{\text{basic}}$ will usually contain isotopic fields. If possible we will choose c such that $r_*(c) = c$ (equality as fields). Note that the set of basic simple fields is closed with respect to the hat or !-operation.

Let $\mathcal{S}^0$ be the set of representatives of fields on I^{n-1} defined by $\otimes$ from the set of basic simple fields. Note that $\otimes$ as above is not a strict monoidal structure. But it is easy to introduce natural coherence structures defined by obvious isotopies of I^{n-1}. But it should be mentioned that $\mathcal{S}^0$ contains a set of representatives for *all* isotopy classes of elements of $\mathcal{F}(I^{n-1})$. Also $c_1 \otimes c_2$ is isotopic to $c_2 \otimes c_1$, even though both are different elements of S^0.

As indicated, $\mathcal{S}^0$ is the set of objects of a monoidal category $\mathcal{S} = \mathcal{S}(\mathcal{F})$. For given $c_0, c_1 \in \mathcal{S}^0$ let the morphisms $\mathrm{Mor}(c_0, c_1)$ be defined by *isotopy classes*, relative to the boundary, of fields d on I^n such that $\partial d = \hat{c_0} \sqcup c_1$. We write $d : c_0 \to c_1$. Composition of morphisms is defined by gluing fields using the Walker's gluing axioms, see [W]. To show that the composition is well-defined the diffeomorphisms defining the isotopies have to be glued. In this way, for each object c the identity morphism $id_c : c \to c$ is defined by the cylinder on c. (This all requires also natural identification of $I^{n-1} \times [0,2] \cong I^n$.)

3.2 Example. Let $n = 4$ and closed base fields on I^3 be defined by oriented links, the involution defined by orientation reversal. Then for an invertible link c the link $\hat{c}$ is isotopic to c. Base fields on I^4 are oriented surfaces properly embedded in I^4 with boundary in $I^3 \times \{0, 1\}$. Note that for a base field c on I^3 defined in this way the field $c^!$ is the concordance inverse, and $(c \otimes c^!) \times \{0\} = \partial d$, where d is a cylinder embedded in I^4 in the obvious way.

Recall that fields are represented by decorated complexes in codimension k. The morphisms form c to c for which the complex underlying the field d is a product can in fact all be realized by isotopies of I^{n-1}, at least up to decoration. This follows from a result of Mazur [M]. Moreover, if $k \geq \frac{n}{2} + 1$ any two such isotopies are isotopic to each other, and thus the morphisms are just the identity.

In general, the situation is of course much more involved. Let $\mathrm{Diff}(n-1)$ denote the space of all diffeomorphisms of I^{n-1}, which are the identity morphism restricted to ∂I^{n-1}. Then for each $\ell \in \pi_1(\mathrm{Diff}(n-1), \mathrm{id})$ and all basic simple fields c there is defined a morphism $\ell_\sharp : c \to c$, by using Remarks (iii). It is called a *twist morphism*, more precisely the twist morphism of c corresponding to ℓ. The homotopy class of the trivial loop is represented by by the the identity morphism $c \to c$. Note that this construction is compatible with other diffeomorphisms of I^{n-1} in the sense that for $f \in \mathrm{Diff}(n)$, $\ell_\sharp(f_*(c)) = (f \times \mathrm{id})_*(\ell_\sharp(c)$. The construction above does also apply to fields, which are not simple but we will not call those morphisms twist morphisms. Note that there there are defined *braid morphisms* defined by exchanging two fields $\beta(c_1 \otimes c_2) = c_2 \otimes c_1$. These can be defined using $I^{n-1} \cong B^{n-1}$ and suitable isotopies of B^{n-1}. But in higher dimensions these braiding morphisms will have order 2, which essentially follows from $\pi_1 SO(n-1) \cong \mathbb{Z}_2$ for $n \geq 4$. The braid morphisms are compatible with twist morphisms in the usual way:

$$\beta(c_1 \otimes c_2) \circ (\mathrm{id}_{c_1} \otimes \ell_\sharp(c_2)) = (\ell_\sharp(c_2) \otimes \mathrm{id}_{c_1}) \circ \beta(c_1 \otimes c_2)$$

For the definition of duality in monoidal categories, see for example [T], 1.3.

3.3. Theorem. *The category $\mathcal{S}$ has duality compatible with twists of its objects.*

Proof. We define the dual object for each basic field c to be the field $c^!$. There are obvious isotopies of I^n (not restricting to the identity on the boundary), which isotope the cylinder field for a simple field c to morphisms $b : c^! \otimes c \to \emptyset$ respectively $\emptyset \to c \otimes c^!$. $\square$

Let $\mathcal{M}_R$ be the category of free finitely generated R-modules, equipped with duality.

3.4 Definition. An n-dimensional *base field functor* is a monoidal duality preserving functor from the category $\mathcal{S}$ into the category $\mathcal{M}_R$ for R a commutative unital ring.

A choice of diffeomorphisms $I^n \cong B^n$ defines a surjective map $\mathcal{S}^0 \to \mathfrak{B}$, which maps d' to d, where $\mathfrak{B}$ is a set of representatives of isotopy classes of fields on ∂B^n, compare 2.7 Definition. For each d pick some element d' in $\mathcal{S}^0$.

3.5 Proposition. *Each n-dimensional base field functor extends to a linear functor from the linearized category $R\mathcal{S}$, which has the set of objects $\mathcal{S}^0$ but with the morphism sets replaced by the free R-modules with bases the morphism sets of $\mathcal{S}$. Let $d' \in \mathcal{S}^0$ be the element chosen for $d \in \mathfrak{B}$ above. Then a generating set of the kernel of the linear morphism defined by the field functor on $Mor(d', \emptyset)$ defines a relation subset $\mathcal{R}(d')$. The resulting sequence $\mathcal{R}(d') \subset C(B^n, d')$ defines a skein theory over R.* $\square$

3.6 Definition. The quotient category of the category $R\mathcal{S}$ with morphism sets defined by the kernels of the field functor F is called the *Jones algebroid* of the functor F. It is determined by the category $\mathcal{S}$ and the skein theory induced from the functor.

Proof of 3.5: There are natural bijections (and induced isomorphisms of the linearized category) $\mathrm{Mor}(c \sqcup d^!, \emptyset) \cong \mathrm{Mor}(c, d)$ and the corresponding isomorphisms $\mathrm{Hom}(V \otimes W^*, R) = (V \otimes W^*)^* \cong V^* \otimes W \cong \mathrm{Hom}(V, W)$. Thus the skein modules determine the corresponding morphism sets. Also note that isotopies between fields induce isomorphisms of the corresponding skein modules. Thus the skein modules of the skein theory determine the morphism sets of the quotient category. $\square$

3.7 Theorem. *The skein theory induced by a base field functor is strongly consistent.*

Proof. This is immediate from the definitions. The skein modules are isomorphic to submodules of free R-modules and thus are free R-modules. The monoidality of the functor implies that the empty field morphism between empty fields maps to the identity of $\mathrm{Hom}(R, R) \cong R$ and thus the corresponding skein module for empty boundary fields is isomorphic to R. $\square$.

3.8 Examples. (a) In [K1] the author discussed the above construction for the case of codimension 1 embedded manifolds and $n = 3$. The base field functor in this case has been defined by forgetting the embeddings and application of an *abstract* $(1 + 1)$-dimensional TQFT. It is shown in [K1] that the base field functor extends to a category with the components of the surfaces, which are the morphisms, colored by elements of the Frobenius algebra defining the TQFT. These elements are interpreted as R-homomorphisms of the Frobenius algebra defined by multiplication.

This extension is possible because all TQFT-morphisms not changing connectedness commute with the multiplication morphisms.

(b) The *colored* ribbon tangle category of Turaev has morphisms defined by oriented framed 1-manifolds embedded in the 3-ball $I^2 \times I$ with boundaries properly embedded in $I^2 \times \{0,1\}$. The 1-manifolds are colored by elements of an abstract ribbon category $\mathfrak{V}$ with duality and compatible twist and braiding. Objects are *standardly* framed arcs in I^2 colored by elements of $\mathfrak{V}$. Moreover the objects also can contain embedded *coupons* colored by homomorphisms of $\mathfrak{V}$ with input and output arcs. The functor of Reshetikhin and Turaev into $\mathfrak{V}$ is an example of a base field functor if the abstract ribbon category $\mathfrak{V}$ is a category of finite dimensional R-modules. But the more interesting procedure here might be to extend the objects by colorings. In Turaev's case this is based on forming parallels of the 1-manifolds using the framings. In general, if base fields are defined by embedded framed submanifolds such extensions should be possible.

We briefly discuss some idea how to extend (a) above to the more general case of base field functors. The idea is to replace homomorphisms $F(d)$ for topologically *complicated* fields d by more simple fields using colorings by homomorphisms. In this way extensions could be helpful in simplifying the skein relations induced by the base field functor. This could open the way to construct finitely skein generated theories like in the Bar-Natan case [K1].

3.9 Definition. (a) A *bulb* in the field c on M is an oriented ball $B \subset \mathrm{int}(M)$ with a base point $* \in \partial B$ and a choice of element $x \in \mathrm{Hom}(F(d))$, where d is the field defined by restricting c to ∂B. We assume that d coincides with the empty field on a compact $(n-1)$-ball containing $*$, and d is simple. The element x is called the bulb color.

(b) An F-colored field is a field with a finite number of disjoint bulbs. Isotopy of F-colored fields is defined in the obvious way by applying the isotopies to the bulbs.

We define a category $\mathcal{S}_F$ with the same objects as $\mathcal{S}$ but with the morphism sets defined by R-linear combinations of isotopy classes of F-colored fields on I^n. We will introduce the following skein equivalence between F-colored fields: Suppose that for two bulbs B_1, B_2 there exists an oriented embedded *connecting handle* $H = B^{n-1} \times I$, where B^{n-1} is the $(n-1)$-ball intersecting the first bulb in $B^{n-1} \times \{1\} \subset \partial B_1$ and the second bulb in $B^{n-1} \times \{0\} \subset \partial B_2$. Let c_0, c_1, c_2 be the restrictions of c to H, B_1, B_2. Let $F(\partial c_0) = V$. We assume that the fields ∂c_i on ∂B_i are empty outside of their intersections with H for $i = 1, 2$, the field c restricts to the empty field on $\partial H \setminus (B_1 \cup B_2)$, and the field c_0 is a *cylinder field* under the natural diffeomorphism $H \cong I^{n-1} \times I$. In particular $d_2 = \hat{d}_1$ for the fields in ∂B_i. In this case we replace the F-colored field by an F-colored field with two fewer bulbs by setting the field c' to be the empty field on the ball $B = B_1 \cup H \cup B_2$, and equal to c outside of $\mathrm{int}(B)$. We multiply c' by the element of R, which is the image of the unit under the homomorphism

$$R \to V \otimes V^* \xrightarrow{x_1 \otimes x_2} V \otimes V^* \xrightarrow{F(c_1) \otimes F(c_2)} R \otimes R \cong R,$$

where x_i are the bulb colors of the two bulbs, the first homomorphism is given by applying the functor F to the duality of $\mathcal{S}$ given by $d_1 \otimes d_1^! \to \emptyset$. Also, bulbs colored by the identity can just be omitted.

3.10 Example. The above deletion of pairs of bulbs in particular applies when the field c restricts to the empty field on ∂B. In this case the bulb color is a homomorphism from R to R. But $R\mathrm{Mor}(\emptyset, \emptyset) \to R$ is onto anyway since the empty field $\emptyset \to \emptyset$ maps to the identity homomorphism $R \to R$. The interesting cases are when c does *not* restrict to the empty field.

3.11 Theorem. *The functor F extends to a monoidal duality preserving functor on the category $\mathcal{S}_F$.*

Proof. We can isotope the bulb balls B_i such that the outer normals to ∂B_i at $*_i$ are parallel to $\{0\} \times I$. Then we apply the functor F to the field on I^n but replace for each bulb the corresponding morphism $F(c_i) : F(d_i) \to R$, where d_i the restriction of c to ∂B_i and c_i is the restriction of c to B_i, by $F(c_i) \circ x_i$ where x_i is the color of the i-th bulb. □

4 Skein theories from compression functors

We assume now that there is given a system of base field for $(n+1)$-manifolds, i. e. sets $\mathcal{F}_j(N)$ for N smooth manifolds of dimension $j \leq n+1$, as in section 2. Throughout this section we say just field for a base field. Moreover for each base field c on the n-manifold M consider the set $\mathcal{F}_c(M)$ of fields which restrict to $\hat{c}$ on $M \times \{0\}$ and to the cylinder field $\partial c \times I$ on $M \times I$. Then the set of *compression fields for c*, $\mathcal{F}_{\mathrm{comp}}(M, c) \subset \mathcal{F}_c(M)$, is a subset satisfying certain conditions with respect to gluing, which are described below following 4.1. We assume that cylinder fields $c \times I$ on $M \times I$ and more general traces of isotopies of fields c are compression fields.

The group $\mathrm{Diff}(M, \partial M)$ acts on the set $\mathcal{F}_c(M)$. We define the group of *compression diffeomorphisms* of c, denoted $\mathrm{Diff}_{\mathrm{comp}}(M, c)$, to be set-wise stabilizer subgroup of $\mathcal{F}_{\mathrm{comp}}(M, c)$, i. e. the group of those diffeomorphisms mapping compression fields to compression fields. We say that two fields d_0 and d_1 in $\mathcal{F}_{\mathrm{comp}}(M, c)$ are *compression isotopic* if there is a path f_t in $\mathrm{Diff}_{\mathrm{comp}}(M, c)$ with $f_0 = \mathrm{id}$ such that $f_1 \circ d_0 = d_1$.

Note that $f_t \circ d_0$ is a *path* in $\mathcal{F}_{\text{comp}}(M, c)$ with respect to any reasonable topology on the spaces of fields. Note that compression isotopy keeps $\partial(M \times I)$ point-wise fixed.

Next, for each field α on ∂M a category $\mathcal{C}(M, \alpha)$ is defined as follows: The objects of the category $\mathcal{C}(M, \alpha)$ are representatives of isotopy classes of elements of $\partial c = \alpha$. The morphisms $c_0 \to c_1$ are compression isotopy classes of fields in $\mathcal{F}_{\text{comp}}(M, c_0)$, which restrict to c_1 on $M \times \{1\}$.

Note that we can form disjoint unions $(M, \alpha) \sqcup (N, \beta) = (M \sqcup N, \alpha \sqcup \beta)$. But we can also glue fields on manifolds using diffeomorphisms h of submanifolds of the boundaries, see [W] for details. We will assume that the locality of compression fields includes that there are defined functors:

$$\mathcal{C}(M, \alpha) \times \mathcal{C}(N, \beta) \to \mathcal{C}(M \cup_h N, \gamma),$$

where γ is the result of gluing the fields α and β using h.

Next assume that for each isotopy class of fields $\alpha \subset F$ and F a diffeomorphism class of $(n-2)$-manifolds, there is given a category of R-modules $\mathcal{R}_{\alpha \subset F}$. If $N = S^{n-2}$ we only write $\mathcal{R}_\alpha$.

4.1 Definition. A *compression functor* G is a collection of functors, parametrized by isotopy classes of fields α on $S^{n-2} = \partial B^{n-1}$,

$$G_{B^{n-1},\alpha} : \mathcal{C}(B^{n-1}, \alpha) \to \mathcal{R}_\alpha,$$

extending to functors

$$G_{M,\beta} : \mathcal{C}(M, \beta) \to \mathcal{R}_{\beta \subset \partial M},$$

compatible with gluing.

We need to explain what it means to *extend compatible with gluing.* First we require the existence of an *algebraic gluing functor* for the categories of modules such that there are commutative diagrams of functors:

$$\begin{array}{ccc}
\mathcal{C}(M, \alpha) \times \mathcal{C}(N, \beta) & \longrightarrow & \mathcal{C}(M \cup_h N, \gamma) \\
{\scriptstyle G_{M,\alpha} \times G_{N,\beta}} \big\downarrow & & {\scriptstyle G_{M \cup_h N,\gamma}} \big\downarrow \\
\mathcal{R}_{\alpha \subset \partial M} \times \mathcal{R}_{\beta \subset \partial N} & \longrightarrow & \mathcal{R}_{\gamma \subset \partial(M \cup_h N)}
\end{array}$$

It follows from field axioms [W] that a field d on $M \times I$ is isotopic relative to the boundary to a product of fields $d_n \circ d_{n-1} \circ \ldots \circ d_2 \circ d_1$, with each field cylindrical except on some ball $B_j \subset M$ for $j = 1, \ldots, n$. Here the composition is the categorical composition in a category of morphism of *all fields* on $M \times I$ being cylindrical over $\partial M \times I$. We require that, by the very definition of compression fields, such a composition by gluing is defined, and the value of the functor $G_{M,\alpha}$ on a field d on $M \times I$ is determined by the values of $G_{B^{n-1},\beta}$ on the fields on $B^{n-1} \times I$ for suitable β.

4.2 Example. In [K2] fields are defined by embedded surfaces in 3-manifolds and the compressions take place in cylinders over the 3-manifolds. The compression condition is defined by requiring that the 3-manifolds embedded in $M \times I$ are defined from the surfaces by attaching handles of only index 2 and 3. The categories $\mathcal{C}(M, c)$ are called *Bar-Natan categories* in this case. The compression functors are called *Bar-Natan functors*. The Bar-Natan functor assigns roughly to a surface F a tensor product $V^{|F|}$, where $|F|$ is the number of components of F, equipped with a certain module structure over $V^{|\partial F|}$. The functor is defined on isotopy classes of compression fields as follows: It is defined by the coproduct morphism $V \to V \otimes V$ respectively the handle-operator $V \to V$ for 2-handle attachments, which are separating respectively non-separating. To a 3-handle attachment the functor assigns the counit $V \to R$.

4.3 Remark. The above collection of compression functors G could be described by a 2-functor on a 2-category. But we wanted to avoid the technical language of higher category theory in this paper.

Given G as above define for each (M, α) a collection of fields $\mathcal{F}_G(M, \alpha)$ by the collection of pairs (c, v) where c is a field on M with $\partial c = \alpha$ and $v \in G(M, c)$, where $G(M, c)$ is an object of the category $\mathcal{R}_{\alpha \subset \partial M}$. The element v should be considered an additional decoration. Now consider or each compression field $d \in \mathcal{F}_{\text{comp}}(M, c_0)$ the associated homomorphism $G(d) : G(B^{n-1}, c_0) \to G(B^{n-1}, c_1)$. Note that $G(d)$ induces homomorphisms $G(\widetilde{d}) : G(M, \widetilde{c}_0) \to G(M, \widetilde{c}_1)$ by the above gluing diagram, where the c_i are fields on M that differ only inside a ball B^{n-1} by changing c_0 to c_1. Now define a skein module $S(M, \alpha)$ as follows: Take the free R-module generated by the isotopy classes of the elements in $\mathcal{F}_G(M, \alpha)$ and take the quotient by the submodule generated by the following two types of elements:

- (i) $(c, r_1 v_1 + r_2 v_2) - r_1(c, v_1) - r_2(c, v_2)$ for all $r_1, r_2 \in R$ and $v_1, v_2 \in G_{M,\alpha}(c)$.
- (ii) $(\widetilde{c}_0, v) - (\widetilde{c}_1, G(\widetilde{d})v)$ for all $v \in G(\widetilde{c}_0, v)$.

This is a generalized skein module $S_G(M, \alpha)$ as defined in section 2.

Recall that the colimit of a functor F from a category $\mathcal{C}$ with set of objects $\mathcal{C}^0$ and set of morphisms $\mathcal{C}^1$ into a category of R-modules is defined by taking the quotient of

$$\bigoplus_{x\in\mathcal{C}^0} F(x)$$

by the submodule generated by all relations $v - F(u)v$ for all $(u : x \to y) \in \mathcal{C}^1$ and $v \in F(x)$.

4.4 Theorem. *The colimit of the functor $G_{M,\alpha}$ is isomorphic to $S_G(M,\alpha)$.*

Proof. Let $S(M,\alpha) = S_G(M,\alpha)$ and let $\widetilde{S}(M,\alpha)$ denote the colimit module of the functor $G_{M,\alpha}$. Thus $\widetilde{S}(M,\alpha)$ is a quotient of $W = \oplus_c G_{M,\alpha}(c)$, where c runs through the set of objects of the category $\mathcal{F}_{\text{comp}}(M,\alpha)$, i. e. representatives of the isotopy classes of fields c on M with $\partial c = \alpha$. There is defined a homomorphism

$$\eta : W \to \widetilde{S}(M,\alpha)$$

by assigning to $v \in G_{M,\alpha}(c) \subset W$ the skein equivalence class $[c,v]$ of the element (c,v). By definition of the R-module structure on W and the definition of η we have that $\eta(r_1v_1 + r_2v_2) = [c, r_1v_1 + r_2v_2] = r_1[c,v_1] + r_2[c,v_2] = r_1\eta(v_1) + r_2(v_2)$ for $v_1, v_2 \in G_{M,\alpha}(c)$. If $v_i \in G_{M,\alpha}(c_i)$ for $i = 1,2$ and $c_1 \neq c_2$ then $\eta(v_1 \oplus v_2) = \eta(v_1)+\eta(v_2) = [c_1,v_1]+[c_2,v_2]$ by definition. Moreover, it follows from the definitions above that all elements $v - G_{M,\alpha}(u)v$ for $u : c_1 \to c_2$ a morphism in $\mathcal{F}_{\text{comp}}(M,\alpha)$ and $v \in G_{M,\alpha}(c_1)$ will map to linear combinations of relations (ii) abve. Thus η descends to a homomorphism

$$\widetilde{S}(M,\alpha) \to S(M,\alpha).$$

Conversely it is not hard to see that assigning to $[c,v] \in S(M,\alpha)$ the image of the vector $v \in G_{M,\alpha}(c) \subset W$ in the colimit module defines a homomorphism ρ, and ρ is an inverse homomorphism for η. $\square$

Let $\mathrm{Diff}(M,c)$ denote the group of diffeomorphisms of $M \times I$ fixing $(M \times \{0\}) \cup (\partial(M \times I))$ point-wise, and let $\mathrm{Diff}'_{\text{comp}}(M,c) \supset \mathrm{Diff}_{\text{comp}}(M,c)$ denote the set-wise stabilizer subgroup of $\mathcal{F}_{\text{comp}}(M,c)$. Then by construction $\mathrm{Diff}'_{\text{comp}}(M,c)$ acts on $\mathcal{F}_{\text{comp}}(M,c)$.

4.5 Definition. A field c on M is called *incompressible* if the set $\mathcal{F}_{\text{comp}}(M,c)$ is the $\mathrm{Diff}'_{\text{comp}}(M,c)$-orbit of $c \times I$.

Thus a field c is incompressible if the only fields that can be compressed from c can be compressed by the action of compression diffeomorphisms on the cylinder field $c \times I$ on $M \times I$.

4.6 Example Consider the case described in [K2]: the fields on M are surfaces $c \subset M$ and compression fields are defined by attaching embedded 2-handles or 3-handles to $c \times [0, \varepsilon] \subset M \times I$. In the case that M is aspherical the incompressibility of the field defined in 4.5 coincides with the usual notion of incompressible in 3-manifold topology. If M is not aspherical an incompressible surface in the sense of 3-manifold topology might not be incompressible in the sense of 4.5 because we can possibly attach a 2-handle along a trivial curve on c_0 to define a surface c_1 containing a non-trivial 2-sphere component. Note that the incompressible surfaces, which are not connected sums with homotopically non-trivial 2-spheres are incompressible in the sense of 4.5. Of course those surfaces, colored with elements of a Frobenius algebra, generate the corresponding Bar-Natan skein module.

We briefly discuss a possible generalization of [K2] along the notions of this section. Let fields be defined by j-dimensional submanifolds in n-manifolds. Define compression fields of a field c by $j+1$-dimensional properly embedded cobordisms $d \subset M \times I$ of c, which are defined from c by attaching only embedded handles of index $k \geq j_0$ for some fixed number $j_0 \geq \frac{j+1}{2}$. In order to define a compression functor we have to assign R-modules to j-manifolds, and morphisms between corresponding R-modules for each compression $c_0 \to c_1$. Suppose that both the modules and homomorphisms determined by G only depend on (i) the indices, and possibly additionally (ii) a finite list of homology or homotopy data of the attachments (i. e. in particular do not depend on embeddings in M respectively $M \times I$). We call such a compression functor *free*. In the case of Bar-Natan theory for a commutative Frobenius algebra V over R in [K2], we assign $V^{|c|}$ to a surface c with $|c|$ components. Then to a separating 2-handle attachment we associate the co-product Δ, and to a non-separating attachment we assign the handle-operator, i. e. multiplication by $\mu\Delta(1)$, where μ is multiplication of the Frobenius algebra. Finally to a 3-handle attachment we associate the co-unit $\varepsilon : V \to R$, see [K2] for further details, in particular with respect to ordering of the tensor product factors. In general we want to find compression functors G to associate morphisms to k-handle attachments $d : c_0 \to c_1$ compatible with compression diffeomorphisms. It is this last requirement which requires the Frobenius algebra structure, i. e. properties of product and co-product, in the Bar-Natan case. It seems an interesting direction of study to detect the algebra necessary for this. At this point we may just note that Bar-Natan theory can be generalized to j-manifolds in n-manifolds with $j_0 := j$. This is not surprising because of the Frobenius structure present in any TQFT, see [TT]. Of course compressions in this case are very restricted and there does not seem to be any interesting resulting theory. But the case $j_0 < j$ seems to be interesting and could be studied for $j = 4$. At this point we just note the following consequence of the definition of a free compression functor.

4.6 Proposition *The skein theory induced from a free compression functor is finitely skein generated.*

In general it seems difficult to construct consistent skein theories using compression functors. In the Bar-Natan skein theory consistency follows from the fact that embedded orientable surfaces are fully compressible in the 3-ball. It seems to be an interesting problem to study the construction of free compression functors and understand its relation to TQFT. It is the main result of [K2] that each commutative Frobenius algebra defines a free compression functor for $n = 2$ and codimension-1 embedded submanifolds.

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