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## TOPOLOGICAL CRITERIA FOR SCHLICHTNESS

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*Abstract* We give two sufficient criteria for schlichtness of envelopes of holomorphy in terms of topology. These are weakened converses of results of Kerner and Royden. Our first criterion generalizes a result of Hammond in dimension 2. Along the way, we also prove a generalization of Royden's theorem.

*Keywords:* envelope of holomorphy; schlichtness; covering space

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Let  $\Omega \subseteq \mathbb{C}^n$  be a domain. The *envelope of holomorphy* of  $\Omega$  is a pair  $(\tilde{\Omega}, \pi)$  consisting of a connected Stein manifold  $\tilde{\Omega}$  and a locally biholomorphic map  $\pi: \tilde{\Omega} \rightarrow \mathbb{C}^n$ , together with a holomorphic inclusion  $\alpha: \Omega \rightarrow \tilde{\Omega}$ , characterized by the following properties:  $\pi \circ \alpha$  is the identity, and each holomorphic function  $f$  on  $\Omega$  has a unique holomorphic extension  $F_f$  on  $\tilde{\Omega}$ , with  $f = F_f \circ \alpha$ . Let  $\Omega' = \pi(\tilde{\Omega})$  and let  $i = \pi \circ \alpha: \Omega \rightarrow \Omega'$ . The envelope of holomorphy  $(\tilde{\Omega}, \pi)$  is *schlicht* if  $\pi: \tilde{\Omega} \rightarrow \Omega'$  is biholomorphic. One would like to give conditions on  $\Omega$  to have a schlicht envelope of holomorphy.

Two results of Kerner and Royden lead to necessary conditions. Kerner [5] has shown that  $\alpha_*: \pi_1(\Omega) \rightarrow \pi_1(\tilde{\Omega})$  is surjective. Royden [8] has shown that  $\alpha^*: H^1(\tilde{\Omega}; \mathbb{Z}) \rightarrow H^1(\Omega; \mathbb{Z})$  is injective. It follows trivially that if  $(\tilde{\Omega}, \pi)$  is schlicht, so  $\tilde{\Omega} = \Omega'$ , then  $i_*: \pi_1(\Omega) \rightarrow \pi_1(\Omega')$  is surjective and  $i^*: H^1(\Omega'; \mathbb{Z}) \rightarrow H^1(\Omega; \mathbb{Z})$  is injective.

Neither of these conditions is sufficient, by a result of Fornæss and Zame [1] (see [2, § 3]). Following an idea of Hammond [2], one may seek sufficient conditions by adjoining to the surjectivity of  $i_*$  (or injectivity of  $i^*$ ) the assumption that  $\pi: \tilde{\Omega} \rightarrow \Omega'$  is a covering space. This strong assumption is still reasonable, as covering maps certainly occur among envelopes of holomorphy; indeed, Fornæss and Zame show in [1] that for any covering map  $\pi: \tilde{\Omega} \rightarrow \Omega'$  there is a domain  $\Omega \subseteq \Omega'$  with envelope of holomorphy  $(\tilde{\Omega}, \pi)$ .

Specifically, Hammond has shown that, in dimension  $n = 2$ , if  $i_*: \pi_1(\Omega) \rightarrow \pi_1(\Omega')$  is surjective and  $\pi: \tilde{\Omega} \rightarrow \Omega'$  is a covering map, then  $(\tilde{\Omega}, \pi)$  is schlicht. We give an elementary proof of Hammond's theorem in all dimensions  $n \geq 2$ . In addition, we give a sufficient condition for schlichtness in terms of the injectivity of  $i^*$  on cohomology, again assuming  $\pi$  is a covering map. Along the way, we give an alternative proof of Royden's theorem, which also extends it to coefficient groups other than  $\mathbb{Z}$ .

**Theorem 1.** *If  $\pi$  is a covering map and  $i_*: \pi_1(\Omega) \rightarrow \pi_1(\Omega')$  is surjective, then  $(\tilde{\Omega}, \pi)$  is schlicht.*

This extends the theorem of Hammond for dimension  $n = 2$ . Hammond's proof relies on a result of Jupiter [4], which is special to dimension 2.

**Proof.** The number of sheets of the covering map  $\pi$  is equal to the index of  $\pi_*(\pi_1(\tilde{\Omega}))$  in  $\pi_1(\Omega')$  (see, for example, [3, Proposition 1.32]). The surjectivity of  $i_* = \pi_* \circ \alpha_*$  implies that  $\pi_*$  is surjective. Hence, the index of the image subgroup is 1, so  $\pi: \tilde{\Omega} \rightarrow \Omega'$  is 1-sheeted, i.e. a homeomorphism. Since  $\pi$  is a holomorphic homeomorphism, it is biholomorphic and so  $\tilde{\Omega}$  is schlicht.  $\square$

Compare with the more technical proof in [2].

The cohomology in Royden's result is Čech cohomology with coefficients in the sheaf of locally constant  $\mathbb{Z}$ -valued functions. Since our spaces are manifolds, Čech cohomology coincides with singular cohomology (with coefficients in  $\mathbb{Z}$ ); see, for example, [6, Theorem 73.2]. Recall also that by the universal coefficient theorem,  $H^1(X; G) = \text{Hom}(\pi_1(X), G)$  for a path-connected space  $X$  and abelian coefficient group  $G$  [3, p. 198].

Before we go on, observe that this proves Royden's theorem as a consequence of Kerner's theorem and extends it to other coefficient groups.

**Theorem 2 (Royden).** *For any abelian group  $G$ ,  $\alpha^*: H^1(\Omega; G) \rightarrow H^1(\tilde{\Omega}; G)$  is injective.*

**Proof.** Since  $\alpha_*: \pi_1(\Omega) \rightarrow \pi_1(\tilde{\Omega})$  is surjective,

$$\alpha^*: \text{Hom}(\pi_1(\Omega), G) \rightarrow \text{Hom}(\pi_1(\tilde{\Omega}), G)$$

is injective and these Hom groups coincide with  $H^1(\Omega; G)$ ,  $H^1(\tilde{\Omega}; G)$ .  $\square$

Royden proves this for  $G = \mathbb{Z}$  using Čech cohomology, in particular the exponential short exact sequence (hence the restriction to  $G = \mathbb{Z}$ ). No such result holds for higher cohomology groups [1, Theorem 4].

Now, we aim to give a sufficient criterion for schlichtness by assuming  $i^*: H^1(\Omega'; G) \rightarrow H^1(\Omega; G)$  is injective for every abelian group  $G$ , and that  $\pi$  is a covering map. Our strategy is to deduce that  $\pi_*: \pi_1(\tilde{\Omega}) \rightarrow \pi_1(\Omega')$  is surjective, as in the proof of Theorem 1. This would follow if we could deduce that  $i_*: \pi_1(\Omega) \rightarrow \pi_1(\Omega')$  is surjective, but, in general, injectivity of  $\text{Hom}(A, G) \rightarrow \text{Hom}(B, G)$  does not imply surjectivity of  $B \rightarrow A$ . The problem is that if the image of  $B$  is a proper subgroup which is not contained in any proper normal subgroup, then there is no non-zero  $f: A \rightarrow G$  vanishing on the image of  $B$ . For example, let  $\mathfrak{S}_3$  be the symmetric group on three letters and let  $B = \mathbb{Z}/2\mathbb{Z}$  be the subgroup generated by a transposition. If  $f: \mathfrak{S}_3 \rightarrow G$  is any group homomorphism such that the restriction  $f|_B$  is zero, then  $f$  itself is zero.

We must solve this problem by adjoining a hypothesis to ensure that every proper subgroup of  $\pi_1(\Omega')$  is contained in a proper normal subgroup. However, this alone is not enough. For, suppose that  $B \subset \pi_1(\Omega')$  is a proper subgroup, contained in a proper normal

subgroup  $N$ . We get a non-zero homomorphism  $f: \pi_1(\Omega') \rightarrow G = \pi_1(\Omega')/N$ , namely the quotient map, whose restriction to  $B \subseteq N$  is zero, so  $\text{Hom}(\pi_1(\Omega'), G) \rightarrow \text{Hom}(B, G)$  is not injective. This will prove the theorem we want, but only if  $G$  is abelian, so we can identify these Hom groups with singular cohomology.

So we need to know that every proper subgroup of  $\pi_1(\Omega')$  is not only contained in a proper normal subgroup, but in one such subgroup  $N$  whose quotient  $G = \pi_1(\Omega')/N$  is abelian.

Fortunately, this condition is more natural than it sounds. It holds if  $\pi_1(\Omega')$  is nilpotent, as in that case every maximal proper subgroup is normal and has prime index (see [7, Theorem 5.40]).

We get the following.

**Theorem 3.** *If  $\pi$  is a covering map,  $\pi_1(\Omega')$  is nilpotent and  $i^*: H^1(\Omega'; G) \rightarrow H^1(\Omega; G)$  is injective for every abelian group  $G$ , then  $(\tilde{\Omega}, \pi)$  is schlicht.*

**Proof.** Since  $i^* = \alpha^* \circ \pi^*$  is injective,  $\pi^*$  is injective as well. Via  $\pi_*$ , we regard  $\pi_1(\tilde{\Omega})$  as a subgroup of  $\pi_1(\Omega')$ . Recall that if  $H$  is any nilpotent group, then every maximal proper subgroup  $N$  of  $H$  is normal and has prime index, and, in particular,  $H/N$  is abelian. If  $\pi_1(\tilde{\Omega}) \not\subseteq \pi_1(\Omega')$ , there exists a maximal subgroup  $\pi_1(\tilde{\Omega}) \subseteq N \subsetneq \pi_1(\Omega')$  and hence a surjection  $\pi_1(\Omega') \rightarrow G = \pi_1(\Omega')/N$  to an abelian group with  $\pi_1(\tilde{\Omega})$  mapping to zero. This surjection is non-zero and lies in the kernel of

$$\pi^*: H^1(\Omega'; G) = \text{Hom}(\pi_1(\Omega'), G) \rightarrow \text{Hom}(\pi_1(\tilde{\Omega}), G) = H^1(\tilde{\Omega}; G)$$

for the abelian group  $G = \pi_1(\Omega')/N$ , contradicting the injectivity of  $\pi^*$ .

It follows that  $\pi_1(\tilde{\Omega}) = \pi_1(\Omega')$ . As before, this implies that  $\pi$  is a degree 1 covering map, and hence a biholomorphism.  $\square$

Solvability would not be enough, as shown by the example of  $\mathbb{Z}/2\mathbb{Z} \subset \mathfrak{S}_3$ . This would not only obstruct the proof given above, but would actually lead to a counter-example to the version of the statement, with solvable in place of nilpotent.

**Example 4.** Recall that Artin's braid group on three strands, denoted  $B_3$ , is the fundamental group of the complement of the braid arrangement  $A_2$  in  $\mathbb{C}^3$ , the union of the three hyperplanes defined by  $(y-x)(z-x)(z-y) = 0$ . Quotienting by the small diagonal, the line  $x = y = z = 0$ ,  $B_3$  is the fundamental group of  $\Omega' \subset \mathbb{C}^2$ , the complement of the union of three lines through the origin in  $\mathbb{C}^2$ . Let  $B_2 \subset B_3$  be a subgroup corresponding to two of the strands, so  $B_2 \cong \mathbb{Z}$  has index 3 in  $B_3$  and is not normal. There exists a covering space  $\tilde{\Omega} \rightarrow \Omega'$  such that  $\pi_1(X) = B_2 \subset B_3$ . Since  $U$  is a Stein manifold, so is  $\tilde{\Omega}$  [9]. By [1, Theorem 5], there exists a domain  $\Omega \subset \Omega'$  with envelope of holomorphy  $\tilde{\Omega}$ . This is not schlicht, but for every abelian group  $G$ ,  $\text{Hom}(B_3, G) \rightarrow \text{Hom}(B_2, G)$  is injective.

More generally, let  $H$  be any finitely presented group.  $H$  is the fundamental group of a 2-complex, which may be embedded in  $\mathbb{R}^5$ , or, for that matter,  $\mathbb{C}^3$ ; then, a tubular neighbourhood  $\Omega'$  of this complex (in  $\mathbb{C}^3$ ) still has  $\pi_1(\Omega') = H$ . Any subgroup  $K \subset H$

occurs as the fundamental group of a covering space  $\tilde{\Omega} \rightarrow \Omega'$ . Again,  $\tilde{\Omega}$  is Stein since  $\Omega'$  is, and there exists a domain  $\Omega \subset \Omega'$  with envelope of holomorphy  $\tilde{\Omega}$ .

It is not necessary to assume that  $i^*$  is injective when coefficients are taken in any abelian group  $G$ . It would be enough to assume that  $i^*$  is injective when coefficients are taken in any finite cyclic group, in any abelian quotient  $G$  of  $\pi_1(\Omega')$  or even just in a single abelian quotient  $G = \pi_1(\Omega')/N$  for some proper normal subgroup  $N$  containing  $\pi_1(\tilde{\Omega})$ .

If, in addition,  $\pi: \tilde{\Omega} \rightarrow \Omega'$  is a normal covering space, then  $\pi_1(\tilde{\Omega}) \subseteq \pi_1(\Omega')$  is a normal subgroup and we can take  $G$  to be an abelian quotient of  $\pi_1(\Omega')/\pi_1(\tilde{\Omega})$ , which is the group of deck transformations.

**Corollary 5.** *Suppose  $\pi$  is a normal covering map with deck transformation group  $H$ . If there exists a non-zero abelian quotient  $G$  of  $H$  such that  $i^*: H^1(\Omega'; G) \rightarrow H^1(\Omega; G)$  is injective, then  $(\tilde{\Omega}, \pi)$  is schlicht.*

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