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Expect the Unexpected When Teaching Probability

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Expect the unexpected when teaching probability

Karen Koellner, Mary Pittman, and Jonathan L Brendefur suggest that probability poses a challenge for both teachers and learners

Introduction

Probability has recently made its way into many textbook series and standards documents (NCTM, 2000; NGA, 2010). When students engage in probability problem solving many unexpected situations can arise due to the counterintuitive nature of probability concepts. These situations can be difficult for students and challenging for teachers to analyse during teaching. Recently, as facilitators of a Mathematics Science Partnership grant workshop on probability, we had the opportunity to engage middle school teachers in professional development workshops as well as in their classrooms. In this article, we discuss a rich probability task used with these teachers along with two scenarios that represent challenging aspects of probability for students, and challenging teaching. In these two scenarios, we explain the underlying probabilistic concepts that proved difficult for students. For each probability challenge, we discuss how the process of analysing student thinking can inform teaching strategies that may guide student conceptual development.

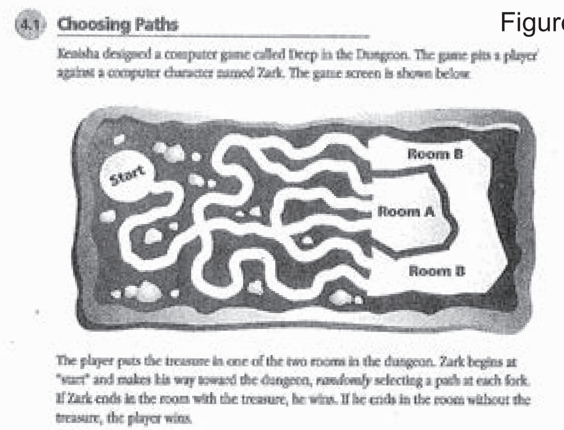
The Problem

We selected the *Maze Problem* to use in our professional development workshop because it is a classic compound event problem that enables exploration in both experimental and theoretical probability. This classic problem can be found in many problem solving books as well as in the *Middle Grades Mathematics Project* (Phillips, Lappan, Winter, & Fitzgerald, 1986) and *Connected Mathematics Project Curriculum* (Lappan, Fey, Fitzgerald, Friel, and Phillips; 1998, p. 41) (see figure 1)

The problem includes a variety of concepts fundamental to probability, including, but not limited to: 1 *experiment*, 2 *trial*, 3 *outcome*, 4 *event*, 5 *randomness*, 6 *expected value*, 7 *the law of large numbers,* 8 *mutual exclusivity,* and, 9 *equiprobability* - See Table 1.

This classic problem is typically presented with a story that has to do with placing a coveted person, or object, in Room A or in Room B. Students are asked to make a conjecture as to which room the person or object are more likely to end up in: Room A or Room B. *The Connected Mathematics Project* (CMP) 7th grade unit *What Do You Expect?* (Lappan, Fey, Fitzgerlad, Friel, and Phillips, 1998 p. 41-43), includes a variation of this problem, see Figure 1.

Figure 1



Experiment	A situation involving chance
Trial	Performance of an experiment
Outcome	A possible result of a probability experiment
Event	An outcome or any combination of outcomes
Randomness	Any trial is random if it is both unpredictable and independent from the outcomes of any prior trail
Expected Value	*Expected value* is the predicted value of an event based on the probability of each outcome. Theoretical expected value is the sum of all possible random events of an experiment multiplied by their probability.
The Law of large numbers	The law of large numbers is a theorem which states; *… as you increase the number of random trials of an experiment the results will approach the theoretical expected value*
Mutually exclusive	Two events are mutually exclusive when they cannot occur at the same time.
Equiprobability	If each outcome of an experiment has the same chance of occurring then the outcomes have equiprobability, i.e., they are equally likely

Table 1. Some fundamental probability skills embedded in the *Maze Problem*

The problem asks students:

1 to decide whether when they were playing the computer game, *Deep in the Dungeon*, in which room would they put the treasure to have the best chance of beating Zark?

2 to find a way to simulate the computer game *Deep in the Dungeon* using chance devices.

3 to play the simulation 20 times with a partner and to record the room that Zark ends and for each trial to record the room in which Zark will be at the end.

4 to determine the experimental probability that Zark ends up in Room A and the probability that Zark will end up in Room B.

A Common Student Solution

One way students engage in this problem is to first determine which room they think the treasure will be in. Possible conjectures might include:

- Room A because it is smaller,
- either room because they both have 3 pathways, or
- Room B because the lowest path goes right to that room.

Because this is a compound event, students need to be able to determine the probability of each pathway going into the two separate rooms. Thus, students can set up an experiment to simulate the *Deep in the Dungeon* game in different ways depending on the chance device(s) they select to use. A chance device is a probability tool used to set up the simulation or experiment to determine the outcome of each event such as a spinner or dice. For example, a group of students might decide to use a spinner at the first junction in the maze, the first event. The spinner would be the chance device that would ensure that the determination at that junction is random and fair. For example, using a spinner, students might say that if they spin a 1, or spin a 2 they will go left, if they spin a 3 or they spin a 4 they will go straight, and if they spin a 5 or a 6 then they will go right. Using this situation, if they spin a 5 they will go right and therefore directly into Room B, see Figure 1.

However, if they spin a 1 or a 2 they will go left. This brings the player to another junction in the maze where there are two options. In creating the simulation students will need to use another chance device to determine the outcome of another event that is both random and fair. Students might decide to use a penny as the chance device to determine if they are going to go right or left. If they flip a heads on the penny, they will go right and a tails they will go left. This will determine which room, A or B, they will end up in for this trial. This provides an example of how students could set up an experiment to simulate *Deep in the Dungeon* and determine experimental probability. Other chance devices in addition to pennies and spinners could be used in a classroom such as dice, or playing cards.

Students then run trials using the experiment they designed with different chance devices at each juncture (event) in the maze so that the experiment is both random and fair. Discussions around the law of large numbers are typical at this point. Students keep track of results and share findings with the class. The combined trials would be used to determine a "whole class" experimental probability.

Implementation of the problem

As part of our professional development, teachers designed lessons to teach the maze problem to their students specifically focused on meeting the needs of the students in their classes. Teachers considered the learning goal, the mathematical focus, how to set up the lesson, the specific needs of their students, the development of the mathematical ideas, and how to close the lesson. Teachers considered a variety of teaching strategies for this lesson. For example, one sixth grade teacher created a maze on the field at her school using chalk, so students could kinaesthetically experience the maze. Students moved through the maze, and at each designated juncture, they stopped and used their chance devices to determine which way they would move next. Several other teachers put students in groups to set up experiments and used different levels of scaffolding and constraints to teach the problem. One strategy was to leave the problem quite open ended and to carefully observe student activity, then to pull the class back together frequently to capitalize on critical moments related to the mathematics and to the challenging aspects of setting up a probability experiment. Other teachers scaffolded the lesson by teaching concepts in the *setup*, *development*, and *closure* phases of the lesson.

Regardless of the teaching strategies employed, student misconceptions tended to appear in most classrooms. Two of these typical situations are presented in the following section along with a discussion of the probabilistic concepts. We also discuss several teaching strategies to support student development in these areas.

Scenario Number 1

A group of students are setting up an experiment for The Maze problem. They decide to label the terminating part of each path with numbers 1 through 6 respectively, and then to use a die to determine which path to travel.

Probability Concepts

The misconception identified in this scenario is that students are identifying each path as being equally likely. LeCoutre (1992) identified this phenomenon in his work as "*misunderstanding of equiprobability*." Students with an *equiprobability* bias assume all outcomes of an event are equally likely regardless of the compound events or junctures throughout the maze. In the scenario above, students believe that each of the six terminating paths is equally likely. The students have used this idea to create an experiment to verify this misconception.

Additionally, the task is a compound event problem, which means that when students set up their maze experiment, they need to set up different random and fair experiments at each juncture. The students in this scenario did not recognise that each juncture required a different number of choices because of their current understanding that each path was equally likely.

Teaching strategies

When teachers recognise that this situation is occurring in their class there are options or teaching strategies to help students make sense of both the related probability concepts of equiprobability and compound events. One option would be to have a discussion about whether each of the terminating paths is equally likely. This discussion should lead students backwards to discuss the difference between a compound event problem and a simple event problem. The teachers we worked with suggested that a conversation about what a real maze looks like might be an effective strategy. They suggested asking students to think of the maze in the problem as an actual corn maze, as some teachers did when they taught the problem. If students imagined they were at the start of the corn maze they would not be able to see the terminating points of each path. Therefore, this conversation scaffolds the understanding of compound versus simple events. Another suggestion is to ask students to consider which path they would take if they rolled a five on the dice. This question encourages students to realise that they do not have a satisfactory answer because they have to figure out whether they would go right, left, or straight ahead from the starting juncture. This conversation should be a good lead into compound events.

Scenario Number 2

In a classroom, a teacher did not specify the number of trials for each group of students to run. One group ran only one trial and determined Room A was most likely. Another group ran twenty trials and determined that Room A and Room B were equally likely.

Probability Concepts

The main probabilistic concepts addressed in this scenario are the law of large numbers and sample size. The conceptual idea is that the more trials you run the closer you will be able to estimate the theoretical probability. The students in this scenario may have the misconception originally identified by Konold (1991; 1993) called *the outcome approach*. Students that only ran one trial, and used this approach to probability, believe they need to predict what will happen on the next trial of an experiment rather than what is likely to occur, or what will occur most often after numerous trials. Students holding this misconception might not realise a need for performing more than one trial. Moreover, Konold's research suggests students with this misconception would feel comfortable making a decision based on only one trial.

Teaching strategies

Appropriate strategies to support student development of the law of large numbers include considering the following questions:

1. Does one trial provide you with enough information to make a decision?
2. Why stop at 20 trials? How many trials should we conduct in order to have a large sample size for an accurate estimation?

These questions appear consistent with the mathematical concepts and can support students developing ideas about sample space. Other problems or activities can also support student understanding of the sample size and the law of large numbers.

Many of the teachers we worked with expressed difficulty with how to address misconceptions surrounding the effects of sample size. This is not surprising as the finding is well established in mathematics education research (e.g., Tversky and Kahneman, 1982). In addition, student misunderstandings regarding sample size are notoriously difficult to modify and it is important not to underestimate the difficulty in helping students to see a need for a large sample size. One way to help support the development of this concept is to use computer software with either a penny or dice simulator. This allows students to conduct hundreds, or thousands, of trials of flipping a penny. A graph of each result is displayed and students can observe the percentages of the number of heads and tails that have occurred. These percentages move toward being equally likely and toward the theoretical probability. These problems provide instant data and help students experiment and determine what happens to the data the more trials they run.

Discussion

Probability can be difficult for students to learn, and to make sense of, due to its counterintuitive nature. Exploring the *Maze problem* provided many opportunities for teachers to engage collaboratively in discussions about student development of probability concepts related to experimental probability, together the unexpected situations that can arise when teaching probability lessons. By exploring the two scenarios of actual challenges that students encountered, teachers in our professional development were able to plan, to anticipate, and to teach a variety of probability lessons with more ease.

The process of working collaboratively with other teachers to plan, teach, and to debrief this problem can open up numerous conversations about the fundamental ideas of probability, the ways students talk and think about the concepts, and the best ways to move students forward. We encourage readers to try the *Maze problem* with a small group of teachers to plan lessons using this rich problem, to anticipate student responses, and to reflect on their teaching.

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