A STOCHASTIC PARAMETER REGRESSION APPROACH FOR TIME-VARYING RELATIONSHIP BETWEEN GOLD AND SILVER PRICES

by

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A thesis

submitted in partial fulfillment

of the requirements for the degree of

Master of Science in Mathematics

Boise State University

August 2012

BOISE STATE UNIVERSITY GRADUATE COLLEGE

DEFENSE COMMITTEE AND FINAL READING APPROVALS

of the thesis submitted by

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Thesis Title: A Stochastic Parameter Regression Approach for Time-Varying Relationship between Gold and Silver Prices

Date of Final Oral Examination: 11 May 2012

The following individuals read and discussed the thesis submitted by student Birsen Canan-McGlone, and they evaluated her presentation and response to questions during the final oral examination. They found that the student passed the final oral examination.

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The final reading approval of the thesis was granted by Jaechoul Lee, Ph.D., Chair, Supervisory Committee. The thesis was approved for the Graduate College by John R. Pelton, Ph.D., Dean of the Graduate College.

ACKNOWLEDGMENTS

The author wishes to express her gratitude to her advisor, Dr. Jaechoul Lee, who offered invaluable assistance, support, and guidance.

She also wishes to express her love and gratitude to her beloved husband, Dave McGlone for his understanding and endless support during her studies.

ABSTRACT

In this thesis, we studied the gold and silver relationship using stochastic-parameter regression models. We formulated their time-varying relationship as a state-space model and used the Kalman filter algorithm to estimate the stochastic regression parameters for gold and silver prices. The data set used in this thesis covers 31 years using the London fix prices between January 1969 and December 2000. The start date was selected as the first full year silver prices were included in the London fix prices. Our stochastic parameter regression model explained well the time-varying relationship between gold and silver prices. As a special case of the stochastic parameter regression model, we also fitted the random walk, the random walk with drift model and random coefficient model. The random walk with drift model appeared to have the closest fit with 12-month forecast errors minimal among those four models considered in this thesis.

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CHAPTER 1

INTRODUCTION

Throughout history, gold and silver have been used as currency. In the last two centuries, governments of many countries backed their printed money with gold or silver, or both. This is called gold or silver, or bi-metallic standard. However, during the second half of the last century, most countries abandoned gold and silver standards and stopped using gold and silver in their currencies. Since then, gold and silver have become commodities traded on general commodities markets. Still, gold and silver hold a special place in the minds of investors who would like a hedge against inflation. In addition, given recent changing economic conditions along with a growing distrust of the monetary system, many states in the U.S. have chosen to legalize gold and silver as currency: Idaho, Utah, and Washington are a few of them.

From an investment point of view, there are additional incentives for investors in gold and silver. Both are used in the jewelry markets. Recently demand has increased for silver in industrial uses (such as the medical field, food preparation, and contaminant remediation). Gold has similar industrial uses, but there is a significant difference in their prices. Gold is extensively used in electronics manufacturing, more than silver, because it has good electrical properties and is not prone to oxidation as silver is.

There have been a number of studies in economics on the time-varying relationship between gold and silver prices. Chan and Mountain [2] analyzed weekly data and interest rates for the early 1980s and developed time series models to test for the causality between the price of gold, the price of silver and interest rates by using an arbitrage model that takes advantage of a price difference between two or more markets. They concluded that there is a causal feedback relationship between the price of gold and the price of silver.

Akgiray et al. [1] investigated daily returns for gold and silver for the period between 1975 and 1986, where the returns were the natural logarithms of the ratio of the two successive daily spot prices. They found no forecastibility in the way of returns. Because the variance of the returns was not constant, they modeled this variance as a GARCH process (Generalized Auto Regression Conditional Heteroscedasticity).

Escribano and Granger [4] focused on gold and silver price during 1971-1990. They found that cointegration occurred during certain periods, especially during the bubble period from September of 1979 to March of 1980. To establish a linear relationship for the entire data set between gold and silver prices, they used dummy variables for intercept terms. They claimed their model performed better than the random walk model for available data. However, their model failed for the out-of-sample predictability. They concluded that a dependency between gold and silver prices decreased after 1990, indicating that the two markets were separating.

Lee and Lin [7] used the AR(1)-GJR-GARCH(1,1) model and three copula functions to analyze the dynamic relationship of gold and silver futures in TOCOM and COMEX markets before and during uptrend. First, they applied the Chow test to separate the sample period prior to and during uptrend and before applying the AR(1)-GJR-GARCH(1,1) model. They did this to investigate the returns and volatility of the two commodities in both market. Then, they used three copula functions to fit the marginal and joint probability density function (pdf), resulting in a better model. They found that silver returns were higher than gold in both markets during this period.

The analysis performed in this work is different from previous works, because we studied the historical relationship between gold and silver prices and the future direction of their relationship. We analyzed the relationship of gold and silver London Fix prices between 1969 and 2001 using a state-space model. We investigated this relationship using four models: first order autoregression coefficient, random walk with drift, random walk without drift, and random coefficient models. The results of these analysis are compared based on their forecasting ability.

The state-space model (SSM) formulation is considered to be a powerful tool that is applicable to a wide range of time series models [8]. Once a problem was configured in a state-space form, the Kalman filter was applied in conjunction with the Newton-Raphson maximum likelihood estimator algorithm to estimate the fixed parameters of the state-space model. Then the fixed parameters were used to estimate regression parameters for smoothing and prediction.

CHAPTER 2

BACKGROUND

2.1 Gold and Silver

2.1.1 A Brief History of Gold and Silver

Many historical facts about gold and silver are likely of not much use to the analysis in this work. However, they provide insight on the close historical relationship between gold and silver. As far back as 3100 B.C., there is evidence of a gold/silver value ratio set by the founder of the first Egyptian dynasty as 10/25 [12]. This is the earliest set relationship between gold and silver. In 1700, Sir Isaac Newton in his capacity of Master of the Mint, fixes the price of gold in Britain at 84 shilling, 11 pence per troy ounce. During this time, the royal commission recalls all gold currencies and fixes the gold silver ratio as 16/1. This legal ratio lasted over 200 years [12]. Throughout history, gold and silver have been used as currency and their prices were controlled by the governments. Starting around the second half of the 20th century, governments of the world started to loosen their control on the price of gold and silver.

We found it useful to provide a list of some of the more significant historical facts and dates that is pertinent to our work [10], [12]:

- 1961: owning gold is forbidden for Americans abroad as well as at home. At the same time the central banks of Belgium, Italy, The Netherlands, Switzerland, West Germany, United Kingdom, and United States form the London Gold pool and agree to buy and sell gold at $35.0875 per ounce.

- 1964: The U.S. is taken off the silver standard. The issuance of silver certificates is stopped and the redemption of them is suspended after 1968.

- 1968: Governors of the central banks in the gold pool announce they will no longer buy gold and sell gold in the private market. A two-tier pricing system starts: Official transactions between monetary authorities are to be conducted at an unchanged price $35 per troy ounce, and other transactions are to be conducted at a fluctuating free-market price. Gold backing of Federal Reserve notes is eliminated. In technology side, Intel introduces a microchip with 1024 transistors interconnected with gold circuit. More new uses of gold in electronics and medical fields are being discovered.

- 1969: The U.S. Mint presses its last silver coin.

- 1971: The U.S. terminates all gold sales or purchases, thereby ending conversion of foreign officially held dollars into gold. Under the Smithsonian agreement, the U.S. dollar is devalued by raising the value of gold to $38 per troy ounce.

- 1973: The U.S. devalues the dollar again and announces it will raise the official dollar price of gold to $42.22 per troy ounce. All currencies allowed to freely float without regard to gold prices. In June, the gold price rises more than $120 in London market. Japan lifts the prohibition on import of gold.

- 1974: Americans allowed to own gold other forms than just jewelry.

- 1975: U.S. treasury and IMF start selling its gold. Trading in gold for future delivery begins on New York's Commodity exchange and on Chicago's International Monetary Market and Board of Trade. The Krugerrand is launched on to U.S. markets.

- 1978: The IMF and the U.S. abolish the official price of gold. Member governments can buy or sell gold in private markets. The U.S. Congress passes the American Arts Gold Medallion Act, representing the first official issue of a gold piece for sale to individuals. Japan lifts its ban on gold exports.

- 1979: Canada introduces 1 ounce Maple leaf.

- 1979/June-1980/March silver bubble caused by the Hunt brothers of Texas. This resulted substantial changes in market trading rules [5].

- 1980: IMF sells 1/3 of its gold to IMF members. The U.S. sells 15.8 million troy ounce of gold to strengthen its trade balance. A weakening U.S. dollars raises interest in gold, assumed to be aided by historic events such as the U.S. recognition of Communist China, events in Iran and Sino-Vietnam border disturbances. Gold reaches historic high of $870 and drops to $591 at the year end.

- 1981: The U.S Treasury forms a gold commission to assess and make recommendations with regard to the policy of the U.S. government concerning the role of gold in domestic and international monetary system. Gold is used in coating of the first space shuttle's liquid impellers.

- 1986: The American Eagle gold Bullion coin is introduced by the U.S. Mint. Treasury starts purchasing newly minted gold. On the technological side, gold coated compact discs are introduced, which provides perfect reflective surfaces, eliminates pinholes and eliminating all possibility of oxidative deterioration of the surfaces.

- 1987: British Royal Mint introduces the Britannia Gold Bullion coin. World stock market crashes on October; in commodities market shows increase in gold activity. The world gold council is established to sustain and develop demand for the end uses of gold.

- 1988: Japan purchases huge amounts of gold to mint a commemorative gold coin to celebrate 60th anniversary of Emperor Hirohito's reign.

- 1989: Austria introduces the Philharmonic bullion coin.

- 1993: Germany lifts its tax restriction on financial gold, causing a increase in private demand of gold. India and Turkey free their gold markets.

- 1994: Russia formally establishes gold market.

- 1996: The Mars Global surveyor is launched with an on board gold coated parabolic telescope-mirror.

- 1997: The U.S. Congress passes a bill allowing U.S. individual retirement account holders to buy gold bullion coins and bars for their accounts as long as they are 99.5% purity of gold.

- 1999: The Euro is introduced backed by a new European Central Bank holding 15% of its reserves in gold.

- 2000: Gold coated mirrors is used in space observatories.

The post-1968 data is necessary to interpret the shock to the price of gold and silver. For that reason, we will be referring to these points in the later sections while interpreting our model results.

2.1.2 Industrial Uses of Gold and Silver

Throughout history, gold and silver were mainly used as currency and jewelry. However, recent changes in their status as currency do not diminish their value. They are two of the best conductors of electricity and heat, most reflective of all metals, are powerful anti-bacterial agents, and are easily worked since they are malleable and ductile. These properties make them two of the most sought out and revered metals [13], [11].

The most important industrial use of gold is in electronics manufacturing. A small amount of gold is used in almost every electronic device. Silver is the best conductor of electricity so it finds many uses in electronics as well. Even though gold is a number-3 conductor. One of gold's advantages is that it does not oxidize. Chemical reactions use silver as a catalyst. More than 700 tons of silver are used each year to produce ethylene oxide and formaldehyde, both of which are essential to the plastics industry. Silver oxide-zinc batteries are being used in portable electronic devices, like watches, cameras, and other small electronic devices. As one of the most reflective metals next to gold, silver is used in specialized optical devices, automobile windshields, and both commercial and household mirrors [11], [13].

Another use of gold is in health care. Gold is known to have been used in dentistry as early as 700 B.C. Gold is also used as a drug to treat rheumatoid arthritis. Radio-active gold has been used as a medical diagnostic tool and in cancer treatments. Gold is also used in surgical instruments [13]. Ancient people knew that silver kept the freshness and prevented spoilage of oil, wine, and water. Wealthy ancient Greeks and Romans stored these in silver jugs. Silver compound shows a toxic effect on bacteria, viruses, algae, and fungi. Silver also has wide range of use in health and medical applications. These include dressing and ointments for burns and wounds, anti-bacterial pharmaceutical, and coating of surgical instruments. Silver is being used for water purification and treatment containing radioactive and biological contaminants [11]. Even though gold has similar antibacterial properties, using gold instead of silver would be a very expensive alternative. Recently clothing is being manufactured with silver-impregnated fabric to kill bacteria and fungi in order to reduce disease and odor. Recent research shows that silver also promotes the production of new cell growth, speeding the healing process of wounds and bones [11].

Gold is too expensive to use randomly. It is used purposefully and only when less expensive alternatives cannot be found. Most of the ways gold is used in industry have been developed only during the past three decades. This trend will most likely continue. As the use of sophisticated electronics increase, the use of industrial gold will increase. The need for silver is also expected to keep increasing [11].

There is very little gold/silver recovery from industrial-use and that use is increasing. The combination of increased demand and limited supply of both gold and silver will increase their value and importance over time [13].

2.2 Regression Models with Stochastic Parameters

2.2.1 The State-Space Model

The state-space model or dynamic linear model (DLM), in its basic form, employs an order one autoregression (AR(1)) as the state equation:

$$\beta_t - \beta = \Phi(\beta_{t-1} - \beta) + a_t \tag{2.1}$$

where the state equation determines the rule for the generation of the $K \times 1$ state vector β_t from the past $K \times 1$ state β_{t-1}, for time points $t = 1, \ldots, n$. The $K \times 1$ stochastic parameter β_t has constant mean β, and Φ is a $K \times K$ matrix of fixed parameters. We assume a_t are $K \times 1$ independent and identically distributed (IID), zero mean normal vectors with $K \times K$ covariance matrix Q. In the state equation (2.1), we assume the process starts with initial vector β_0 that has mean μ_0 and $K \times K$ covariance matrix Σ_0 (or σ_0 for univariate case). The state-space model adds an additional component to the regression model in assuming we do not observe the state vector β_t directly, but only a linear transformed version of it with noise added:

$$y_t = \alpha + z_t'\beta_t + e_t \tag{2.2}$$

where z_t is $q \times K$ measurement or observations vector, y_t are $q \times 1$ observation vector in (2.2) and e_t are $q \times 1$ IID, zero mean normal vectors with $q \times q$ covariance matrix R, and α is an intercept term. The model in (2.2) is called measurement, or space equation.

The state-space model is a powerful tool which opens the way to handling a wide range of time series models. Once a problem is put in a state-space form, the Kalman filter may be applied and this in turn leads to algorithms for prediction and smoothing. The Kalman filter is a recursive procedure for computing the optimal estimator of the state vector at time t based on all the information available at time t, [8], [3], [6].

The current values of the state vector is of prime interest and Kalman filter enables the estimate of the state vector to be continually updated as new observations (or information) become available. The state vector may not have an economic interpretation but in cases where it does, it is more appropriate to estimate its value at a particular point in time using all the information in the sample, not just a part of it.

In econometrics, the Kalman filter became important, due to its forecasting ability. Another reason for the popularity of the Kalman filter is that when the disturbances and the initial state vector are normally distributed, it enables the likelihood function to be calculated via what is known as the prediction error decomposition. This opens the way for the estimation of any unknown parameters in the model. It also provides the basis for statistical testing and model specification.

The derivation of the Kalman filter given below is based on the assumption that the disturbances and initial state vectors are normally distributed. A standard result on the multivariate normal distribution is then used to show how it is possible to recursively calculate the distribution of β_t, conditional on the information set at time t for all t from 1 to T. These conditional distributions are themselves normal and thus are completely specified by their means and covariance matrix.

After having derived the Kalman filter, it is shown that the mean of the conditional distribution of β_t is an optimal estimator of β_t in the sense that it minimizes the mean square error (MSE) of predictions. When the normality assumption is dropped there is no longer any guarantee that the Kalman filter will give the conditional mean of the state vector. However, it is still an optimal estimator in the sense that it minimizes the MSE within the class of all linear estimators [8].

2.2.2 Kalman Filter

Let us assume we have T observations of $y_1, \ldots y_T$ on a dependent variable, and correspondingly, observations $A_{i1}, \ldots A_{iT}$ where $i = 1, \ldots, K$. Assuming linear relation between y_t and A_{it}, we can write:

$$y_t = \beta_{1t}A_{1t} + \beta_{2t}A_{2t} + \cdots + \beta_{Kt}A_{Kt} + e_t \tag{2.3}$$

or equivalently,

$$y_t = \alpha + A_t\beta_t + e_t \tag{2.4}$$

$$\beta_t - \beta = \Phi(\beta_{t-1} - \beta) + a_t \tag{2.5}$$

where $A_t = (A_{1t} \ldots A_{Kt})'$, We assume a_t and e_t are independent white noise and are uncorrelated. A_{it} are either fixed observations or random variables, independent of both a_t and e_t. The model described by (2.4) and (2.5) is a special case of a general class of models called state-space models. This model has been extensively presented in [3]. We will use this state-space model in Chapter 3 as a base model for analyzing silver and gold prices. Specifically, the state-space model equations are rearranged to obtain a random walk and random coefficient model.

In this work, we assume that the joint distribution of y_t and β_t for given $y_1, \ldots, y_{t-1}$ is normal. Therefore, this distribution is multivariate normal. Thus, this distribution can be expressed in terms of its mean and covariance matrix. The objective of the Kalman filter algorithm is then to provide a convenient way of computing the expected value and covariance matrix of the stochastic parameter β_t, given information available at time t. At this point, it would be useful to introduce some further notation.

First, consider the distribution of β_t, given information up to time $t-1$, $y_1, \ldots, y_{t-1}$. We will use $\beta(t|n)$ to denote expected value (mean) of this distribution, and $P(t|n)$ to denote its $K \times K$ covariance matrix. These can be written as

$\beta(t|n) = E[\beta_t|y_1 \ldots y_n]$; $P(t|n) = Var[\beta_t|y_1 \ldots y_n]$.

We also need to consider the distribution of y_t, given information up to and including time $t-1$. The mean and variance of this distribution can be written as

$y(t|n) = E[y_t|y_1 \ldots y_n]$; $h_t = Var[y_t|y_1 \ldots y_n]$.

We already assumed that β_t and y_t are normally distributed. Now it is necessary to find the joint distribution of β_t and y_t given the observation $y_1, \ldots, y_{t-1}$. As a consequence of the normality assumption, it follows that this joint distribution will be multivariate normal. Using the well-known results on the multivariate normal distribution, we can find the mean and the covariance of this distribution. For further details, see Appendix of [8] and [6].

Derivation of the expected value and covariance matrix of β_t and y_t given $y_1, \ldots, y_{t-1}$, can be found in [6] and [8]. These expected values and covariance matrices comprise the Kalman algorithm. Here, we will only list the following pertinent equations of the Kalman algorithm.

$$\beta(t|t-1) = \Phi\beta(t-1|t-1) + (I+\Phi)\beta \tag{2.6}$$

$$P(t|t-1) = \Phi P(t-1|t-1)\Phi' + Q \tag{2.7}$$

$$h_t = A_t' P(t|t-1) A_t + R \tag{2.8}$$

$$\beta(t|t) = \Phi\beta(t-1|t-1) + (I-\Phi)\beta + P(t|t-1)A_t h_t^{-1}[y_t - A_t'\beta(t|t-1)] \tag{2.9}$$

$$P(t|t) = P(t|t-1) - P(t|t-1)A_t h_t^{-1} A_t' P(t|t-1) \tag{2.10}$$

To find expected values for β_t and y_t, given all the previous values of y_t, we use Kalman algorithm equations (2.6)-(2.10). Here $\beta(t|t-1)$ is the conditional expected value of β_t, P is the conditional covariance matrix of β_t, and h_t is the covariance of y_t.

The Kalman algorithm provides a computationally efficient framework to solve the problem of estimating the fixed parameters $(\Phi, \alpha, \beta, Q, R)$ of the regression model and forecasting the future values of y_t. These parameters will be estimated by the maximum likelihood method (ML) utilizing Newton-Raphson method.

In order to utilize the Kalman filter algorithm, initial values, namely $\beta(0|0)$ and $P(0|0)$, are needed. These are mean and covariance of β_0 given no previous observation. These starting values can be substituted into the Kalman filter equations (2.6)-(2.10) to compute recursively the means and the variances of the dependent variables and stochastic parameters in any time period, given information available in the previous period and the fixed coefficient of the model.

In actuality, this algorithm does not give us the fixed parameters, but provides us a basis for the maximum likelihood estimation of the fixed parameters. Then, we could estimate the stochastic parameters over the same period. The details of the maximum likelihood estimation are not the main subject of this thesis. Shumway and Stoffer [8] give a practical approach for calculating maximum likelihood estimates numerically using Newton-Raphson method in detail in Chapter 6, along with a computational algorithm in R. The estimation of the fixed parameters of the state-space model is very involved. Here, we give a short summary of the maximum likelihood estimation in our model.

2.2.3 Estimation of Fixed Coefficients: Maximum Likelihood Estimation

Let $\Theta = (\mu_0, \Sigma_0, \Phi, \alpha, \beta, Q, R)$ represent the vector of the unknown parameters in model (2.1) and (2.2) containing the initial mean and covariance of β_0, denoted by μ_0, Σ_0, the transition matrix Φ, and Q and R are the state and space observation covariance matrix, respectively. The likelihood function can be evaluated under the initial assumption that the initial β_0 is normal with mean μ_0, and variance Σ_0, and errors $a_1, \ldots, a_T$, and $e_1, \ldots, e_T$ are jointly normal and $\{a_t\}$ and $\{e_t\}$ are uncorrelated.

The likelihood is computed using the innovations $\epsilon_1, \ldots, \epsilon_t$ defined as $\epsilon_t = y_t - A_t\beta(t|t-1) - \alpha$. The innovation form of the likelihood is given in Shumway and Stoffer [8], Newbold and Bos [6], and references therein.

Given observations taken over t time points, we want to obtain estimates of the fixed parameters. In doing so, we need to derive the likelihood function which is the joint distribution of $y_1, \ldots, y_t$ as a function of fixed parameters of our model. The joint probability density function of $y_1, \ldots, y_t$ can be expressed as the product of the conditional density of y_t given $y_1, \ldots, y_{t-1}$, that is $f(y_1, \ldots, y_t) = f(y_1) \cdot f(y_2|y_1) \cdot f(y_3|y_1, y_2) \ldots f(y_t|y_1, \ldots, y_{t-1})$. We also know that the distribution of y_t given all the observations $y_1, \ldots, y_{t-1}$ is normal with expected value given by $y(t|t-1) = \alpha + A_t'\beta(t|t-1)$ and covariance h_t given as $h_t = A_t'P(t|t-1)A_t + \sigma^2$. The conditional probability density function, then, can be written as:

$$f(y_t|y_1, y_2, \ldots, y_{t-1}) = \frac{1}{\sqrt{2\pi h_t}} e^{-(y_t - A_t'\beta(t|t-1))^2/2h_t} \tag{2.11}$$

Following the conditional distribution function of y_t, we can express the likelihood function, that is the joint distribution function, as

$$L = (2\pi)^{-n/2} \prod_{t=1}^{n} h_t^{-1/2} e^{-\sum_{t=1}^{n}(y_t - A_t'\beta(t|t-1))^2/(2h_t)} \tag{2.12}$$

Taking the natural logarithms yields the log-likelihood function, which is actually easier to work with than the likelihood function, as follows:

$$lnL = \frac{n}{2}\ln(2\pi) + \frac{1}{2}\sum_{t=1}^{n} \ln h_t + \sum_{t=1}^{n}\left(\frac{y_t - A_t'\beta(t|t-1))^2}{h_t}\right) \tag{2.13}$$

The log-likelihood equation (2.13) provides the function that must be maximized to obtain maximum likelihood estimates of the fixed parameters. In reality, it is taken as $-lnL$, and it must be minimized. The equation (2.13) includes the observation y_t and A_t, along with conditional variance h_t of the dependent variable and conditional expectation of stochastic parameter $\beta(t|t-1)$. The conditional variance h_t and the expectation $\beta(t|t-1)$ are functions of the fixed parameters defined in the Kalman filter algorithm. We use the Kalman algorithm to calculate them recursively for the given fixed parameters. To find the fixed parameters, we need to solve the log-likelihood function.

Because h_t and $\beta(t|t-1)$ are complicated functions, it is not possible to solve this log-likelihood function analytically. Therefore, numerical optimization algorithms must be employed with respect to fixed parameters of the state-space model. Today, most statistical softwares include optimization packages. In this thesis, we used *optim()* function, which comes standard with R-package. The only thing this function requires is a definition of the functions to be optimized, which are given in the Kalman filter algorithm. The numerical algorithm used in the *optim()* is Newton-Raphson method. This method requires only the derivatives of the function to be minimized. Then, the derivatives are evaluated numerically.

Another advantage of the numerical maximization of the likelihood function is that it provides the standard errors associated with estimators along with point estimates of the fixed parameters. This is accomplished through the information matrix. Let Θ be the vector of all the fixed parameters and L be the likelihood function, then the information matrix is the expectation of the second derivatives of $\ln L$ with respect to each parameters:

$$I(\Theta) = \frac{\partial^2 \ln L}{\partial \Theta \partial \Theta'} \tag{2.14}$$

It can be assumed that this approximation is valid for large sample sizes. Then, the covariance matrix $V(\Theta)$ of the maximum likelihood estimators of Θ is $V(\Theta) = I(\Theta)^{-1}$.

Newton-Raphson Estimation Procedure

The steps involved in performing a Newton-Raphson estimation procedure are as follows.

1. Choose: Initial values of mean and variance of β_t: μ_0, Σ_0, and initial fixed parameters of state-space model:
   $\Theta^{(0)} = (\Phi, \alpha, \beta, Q, R)$

2. Run the Kalman filter algorithm (2.6)-(2.10) to obtain a set of error values $\epsilon_t^{(0)} = y_t - \alpha - A_t \cdot \beta_t$ and covariance $\Sigma_t^{(0)}$; $t = 1, \ldots, n$.

3. Run Newton-Raphson algorithm with $-lnL_Y(\Theta)$ as the criterion function to obtain a new set of $\Theta^{(1)}$.

4. At the iteration j, repeat Step 2 to obtain new set of $\epsilon_t^{(j)} = y_t - \alpha - A_t \cdot \beta_t$ and $\Sigma_t^{(j)}$. Then, repeat Step 3 to obtain a new set of $\Theta^{(j)}$, stop when MLE stabilize; for example, stop when the values of $\Theta^{(j+1)}$ differ from $\Theta^{(j)}$, or when $L_Y(\Theta^{(j+1)})$ differs from $L_Y(\Theta^{(j)})$, by some pre-determined small amount [8].

The only time inputs in this procedure are for the initial values for $\Theta = (\mu_0, \Sigma_0, \Phi, \alpha, \beta, Q, R)$. This is a trial and error process, and one can only make educated guesses for these inputs. However, this can be a very arduous process. In this study, we assumed linear relation between y_t and A_t, in that the intercept and slope provided initial values for α and μ_0. We also assumed initial β is to be equal to μ_0. For the rest of the fixed parameters, we made educated guess. The algorithm we used in this study can be found in A.

2.2.4 Prediction of Future Values

We start with set of observed data, namely y_t and A_{it}. Both data set cover the same period in time. From these data, we find the fixed parameters of the state-space model and stochastic regression coefficients. Then, we would be able to describe the relationship between these two variables.

The primary aim of the state-space model approach is to produce estimators for the underlying unobserved signal β_t. Then, we will assumed that y_t are generated by this stochastic regression model. We will further assume that this model continues to hold true in the future.

The problem we encounter here is that, to predict future values of y, we have to have the foreknowledge of A_i due to the fact that A_i is the independent variable and y is the dependent variable of the stochastic regression model. Thus, the predictions of future values of y will be *a conditional forecast*, that is, we derive prediction of the values that the dependent variable will take, given specific future values of the independent variable A_{it}.

The stochastic parameter model only allows us to predict the future values of non-observable β_t. The best forecast for β_{n+l}, given information for time period n, is the conditional expectation of β_{n+l}. We can also compute the covariance matrix of β_{n+l} and y_{t+l} as follows

$$\beta(n+l|n) = \Phi\beta(n+l-1|n) + (I-\Phi)\beta \tag{2.15}$$

$$P(n+l|n) = \Phi P(n+l-1|n)\Phi' + Q \tag{2.16}$$

$$h(n+l|n) = Var(y_{n+l}|y_1,\dots y_n) \tag{2.17}$$

Assuming normal distribution, the 95% prediction interval for y_{n+l} is given by

$$y(n+l|n) \pm 1.96\sqrt{h(n+l|n)} \tag{2.18}$$

where $y(n+l|n) = \alpha + A_{n+1}\beta(n+l|n)$. We will use this $y(n+l|n)$ to calculate 12-month ahead prediction of gold price and compare these conditional predictions with the real gold prices of 2001 in Section 3.5.

2.2.5 Estimation of the Sample Period Stochastic Parameter

Before we use the model parameters estimated with the Kalman filter for forecasting, we would like to know how well these estimates can trace the real observed data. Later on, for example, it may be of interest to economic historians to know how the gold and silver price relationship has evolved over time. Because we assume causality, that is, past values are future independent, we need to find the estimates of β_t, over the period $t = 1 \dots n$. These estimates should be based on all the available data. At the end of Kalman algorithm, we will have $\beta(n|n)$ and $P(n|n)$, the mean and covariance of β_t at the last point in time. As Shumway and Stoffer [8] nicely stated, the purpose of the state-space model is to produce estimators for the underlying unobserved signal β_t, given the data $y_1, \dots, y_s$. When $s < t$, the problem is called forecasting, when $s > t$, the problem is called smoothing.

If we believe that a stochastic parameter regression model might provide a good description of the data set, then it is natural to expect the estimates could trace the observations as closely as possible, through the time period given. In terms of space or measurement equation (2.2), we can estimate the varying parameters β_t over the time period $t = 1, \dots, n$. These estimates should be based on all available data, so that the optimal estimation of β_t is its conditional expectation given the entire data, namely $\beta(t|n) = E[\beta_t|y_1, \dots, y_n]$. In order to find the standard errors associated with these estimators, so that we could derive interval estimates, we need the conditional covariance matrices $P(t|n) = Var(\beta_t|y_1, \dots, y_n)$.

Here, the Kalman filter algorithm does not provide these estimates we require. It only generates, the mean and the covariance of the stochastic regression coefficients β_t, given only information available up to time t. Thus, the only estimates we have that utilize all the observed data are $\beta(n|n)$ and $P(n|n)$. These are the mean and the covariance matrix of the stochastic parameter vector at the last point in the sampling period. In our work, we used the Kalman smoother given by Shumway and Stoffer [8] to apply the fixed interval smoother. For the state-space model specified in (2.4) through (2.5), $\beta(n|n)$ and $P(n|n)$ were obtain through the Kalman filter algorithm (2.6)-(2.10), for $t = n, \dots, 1$,

$$\beta(t-1|n) = \beta(t-1|t-1) + J_{t-1}(\beta(t|n) - \beta(t|t-1)) \tag{2.19}$$

$$P(t-1|n) = P(t-1|t-1) + J_{t-1}(P(t|n) - P(t|t-1))J'_{t-1} \tag{2.20}$$

where $J_{t-1} = P(t-1|t-1)\Phi'[P(t-1|t)]^{-1}$. The proof of (2.19) and (2.20) can be found in Shumway and Stoffer [8].

The Kalman smoothing algorithm starts by setting $t = n-1$ in (2.19) and (2.20), recursively producing the mean and covariance matrix of the conditional distribution for β_{n-1}. This process only requires information generated by the Kalman filter algorithm. The equations (2.19) and (2.20) are reapplied by setting $t = n-2, n-3, \ldots$ to obtain point estimates of $\beta(t|n)$ of all the stochastic parameters over the sample period along with the associated covariance matrices $P(t|n)$.

2.2.6 Summary

In this chapter, we have presented the mathematical background of estimation and prediction in a regression model with stochastic parameters. The Equations (2.4) and (2.5), which we adapted for our study, require that all the regression coefficients be stochastic, obeying a vector first-order autoregressive process, AR(1). In the next chapter, we will apply this model to analyze the historical gold and silver prices as a stochastic regression process setting the silver prices as independent variable and the gold prices as dependent variable. We also extend our study in to three other models that assume stochastic parameters to be other than AR(1). In these models, we will assume the regression coefficients can be random, random walk with drift or random walk process.

CHAPTER 3

STOCHASTIC PARAMETER MODEL FOR GOLD AND SILVER

In this chapter, we are analyzing time-varying relationships between gold and silver from a stochastic regression point of view. The gold price y_t is modeled by using the silver price z_t as co-variate with a time-varying coefficient β_t in Equation (2.4). The resulting model sets $p = 1$ and $K = 1$ in the state-space model expressions (2.4) and (2.5), simplifying to:

$$y_t = \alpha + \beta_t z_t + e_t \tag{3.1}$$

where α is a fixed intercept constant, β_t is time-varying regression coefficient, and e_t is white noise with mean zero and variance σ_e^2. We consider the time-varying regression coefficient β_t to be a first-order autoregression (AR):

$$\beta_t - \beta = \Phi(\beta_{t-1} - \beta) + a_t \tag{3.2}$$

where β is the constant mean of β_t and a_t is white noise with mean zero and variance σ_a^2. We assumed that the noise processes $\{a_t\}$ and $\{e_t\}$ are uncorrelated. The AR parameter Φ is assumed to be constant $|\Phi| < 1$ for causality. The AR(1) stochastic parameter model (3.2) can be reexpressed as

$$\beta_t = \Phi\beta_{t-1} + (1-\Phi)\beta + a_t. \tag{3.3}$$

The fixed parameter vector in the state-space model (3.1) and (3.2) is $\Theta = (\Phi, \alpha, \beta, \sigma_a, \sigma_e)'$. We use Kfilter2 and Ksmooth2, given by Shumway and Stoffer [8], for the Kalman filter and Kalman Smoothing algorithms. We fitted four different model: random coefficient, first-order autoregression coefficient, random walk with drift, and random walk models.

The models were fitted with the data displayed in Figure 3.1. This figure shows the monthly average of London Fix gold prices denoted by y_t, and the monthly average of London Fix silver prices, denoted z_t from January of 1969 through December of 2000, for a total of $n = 384$ observations. The data set is taken from The Pert Mint web site [9]. This web site provides all up to date prices for gold and silver in monthly average and daily average forms.

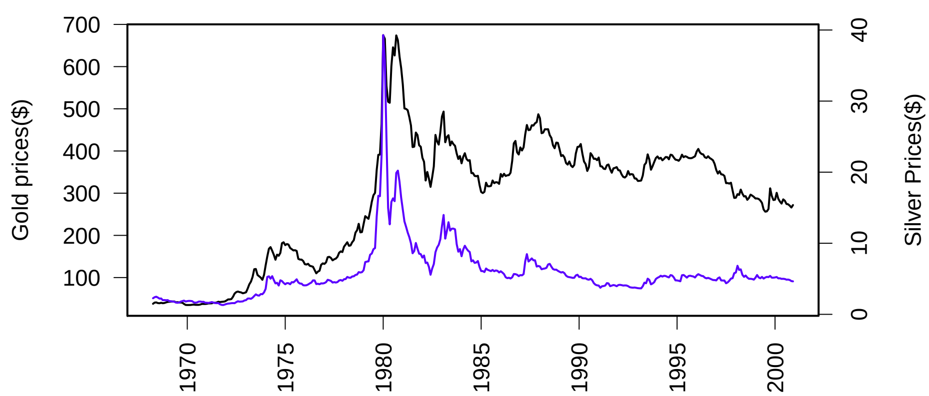

Figure 3.1: London Fix monthly average gold (black) and silver (blue) prices between January 1969 and December 2000.

3.1 Random Coefficient Model

When we set $\Phi = 0$ in (3.3), the state equation takes the following form:

$$\beta_t = \beta + a_t \tag{3.4}$$

and this implies that if $\Phi = 0$, then the stochastic parameter model in (3.3) becomes a random coefficient model (3.4) with mean β and variance σ_a^2. Additionally, if σ_a is small relative to β, the state system becomes nearly deterministic, ($\beta_t \approx \beta$), simplifying to a classical linear model.

The Newton-Raphson estimation procedures were applied to the gold and silver price data as described in Section 2.2.3. Table 3.1 summarizes the estimates of the fixed parameters vector Θ and the standard errors (SE) associated with each fixed parameter. The results summarized in Table 3.1 indicate that the gold-silver relationship is definitely not deterministic. The time-varying coefficient $\{\beta_t\}$ in this model is white noise with constant mean β.

Table 3.1: Fixed Parameters of Random Coefficient Model

	α	β	σ_a	σ_e
Estimate	-43.417	62.315	19.185	17.805
SE	8.833	2.186	0.812	10.358

On the other hand, smoothed β_t are not white noise. The Kalman smoothing procedure allows us to find the estimates of β_t for given observations of the silver and gold prices, $\{(z_1, y_1), \ldots, (z_n, y_n)\}$, so that we can trace back the smoothed $\{\beta_t\}$. These $\{\beta_t\}$ actually represent the gold silver price ratio for this model. Figure 3.2 displays the estimated trace of β_t based on all the available data used.

Another result of this model is that the mean of the residual error of prediction is -28.5. We assumed that the prediction error is normally distributed with mean zero. Figure 3.3 shows that the random coefficient model does not support this assumption, therefore, we can conclude that the random coefficient model works poorly and we need to consider another models. It can also be seen from the residuals of this model in Figure 3.3 that the impact of 1980's silver bubble is very pronounced in this model. The random coefficient model could not handle this unusual occurrence in silver price speculation, which given in Section 2.1.1.

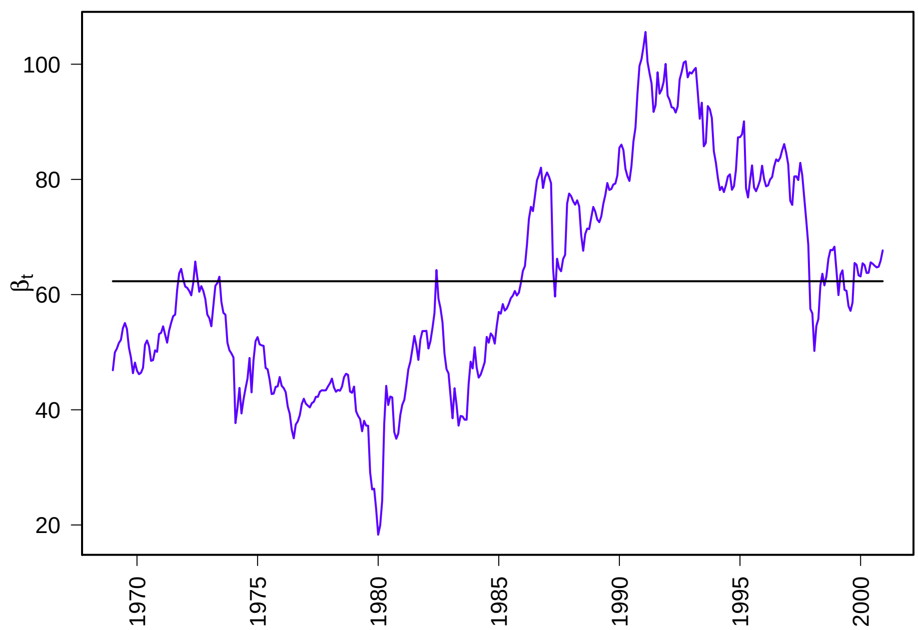

Figure 3.2: Random coefficient model's estimates for one step-ahead prediction coefficients β_t is black, estimates for smoothed β_t is blue. These parameters represents the gold/silver ratio.

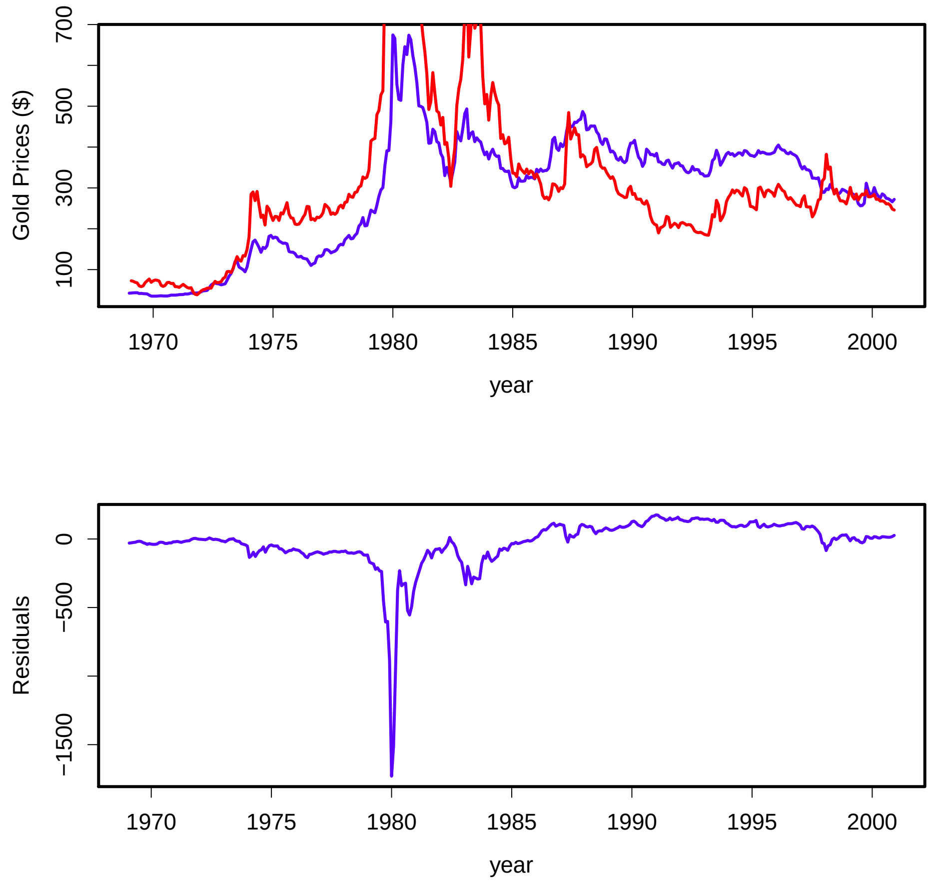

Figure 3.3: Gold prices (blue) random coefficient regression model fit (red) and residuals.

3.2 First-Order Autoregression Coefficient Model

When we don't set $\Phi = 0$, the state equation (2.5) represents a first-order autoregressive process, AR(1). The model becomes a stochastic parameter regression model, and the fixed parameter vector for this model is $\Theta = (\Phi, \alpha, \beta, \sigma_a^2, \sigma_e^2)'$, where σ_a^2 is variance of a_t and σ_e^2 is variance of e_t. The Newton-Raphson estimates of the fixed parameters are listed in Table 3.2, along with the standard errors (SE) corresponding to each fixed parameter, Section 2.2.3. The ML estimate of ϕ is very close to 1. This indicates possible violation of causality assumption of AR(1) process. In Figure 3.4, we see that β_t is closer to a path of a random walk process than AR(1) [8].

Figure 3.5 shows gold prices (black), model fit (red), and the corresponding residuals. Due to the fact that the model followed the real gold prices very closely and the difference between the price of gold in January of 1696 and December of 2000 was so large, this makes it very difficult to separate gold prices from model fit. This is only a scaling issue. The famous silver bubble of 1980 is clearly shown in residuals. This model could not account for this drastic change in silver prices caused by only a few people; refer to Section 2.1.1.

Figure 3.6 shows that the β_t for prediction and for smoothed are very close. If we use the smoothed estimates of β_t as our stochastic regression coefficient, what we obtain is a representation of historic gold-silver price ratio, while estimates of prediction β_t are giving us one step-ahead predictions. There is one interesting feature here about the gold silver ratio. Contrary to some earlier suggestions that the gold silver ratio should be between 16 and 20, we can see that before 1990 this ratio was much higher than that and has an upward trend. After 1990, the ratio started decreasing and has a downward trend. Both β_t for prediction and smoothing closely follow the gold silver ratio. Further discussion on this subject will be given in Section 3.5 along with economic implications.

Table 3.2: Fixed parameters of first-order autoregression Coefficient model

	Φ	α	β	σ_a	σ_e
Estimate	0.991	33.6444	50.572	2.467	1.115
SE	0.0054	4.381	14.048	0.091	0.554

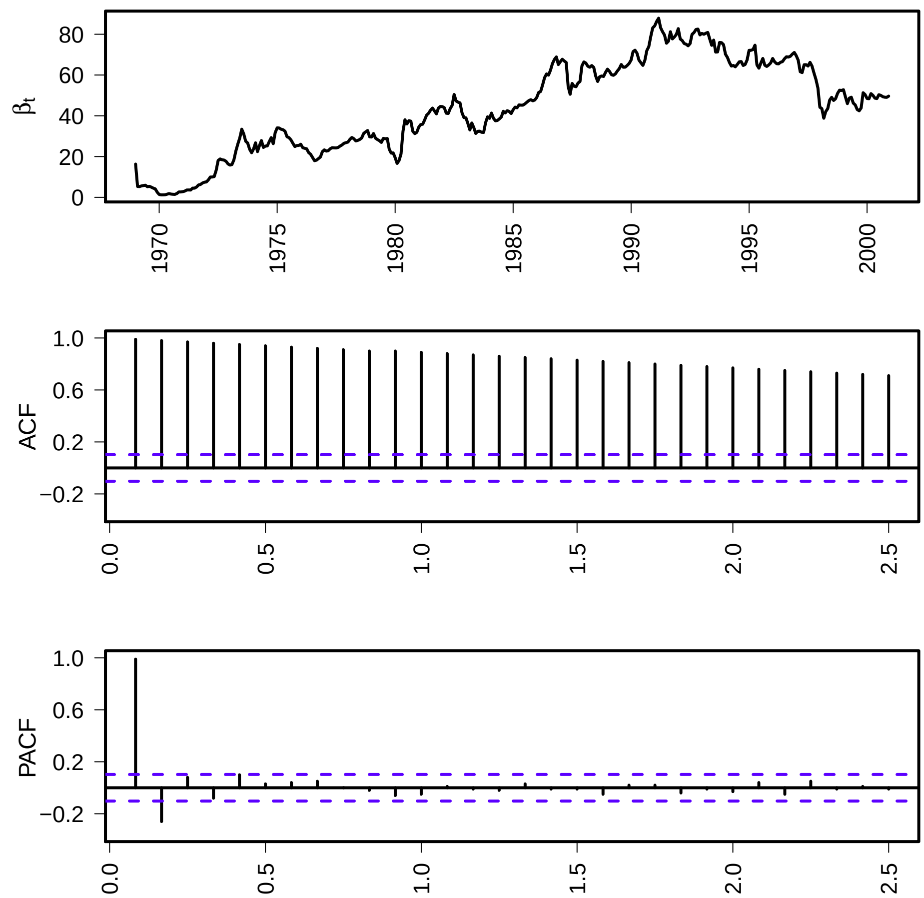

Figure 3.4: First-order autoregression Coefficient model's estimates for one step-ahead prediction coefficient β_t and its ACF and PACF.

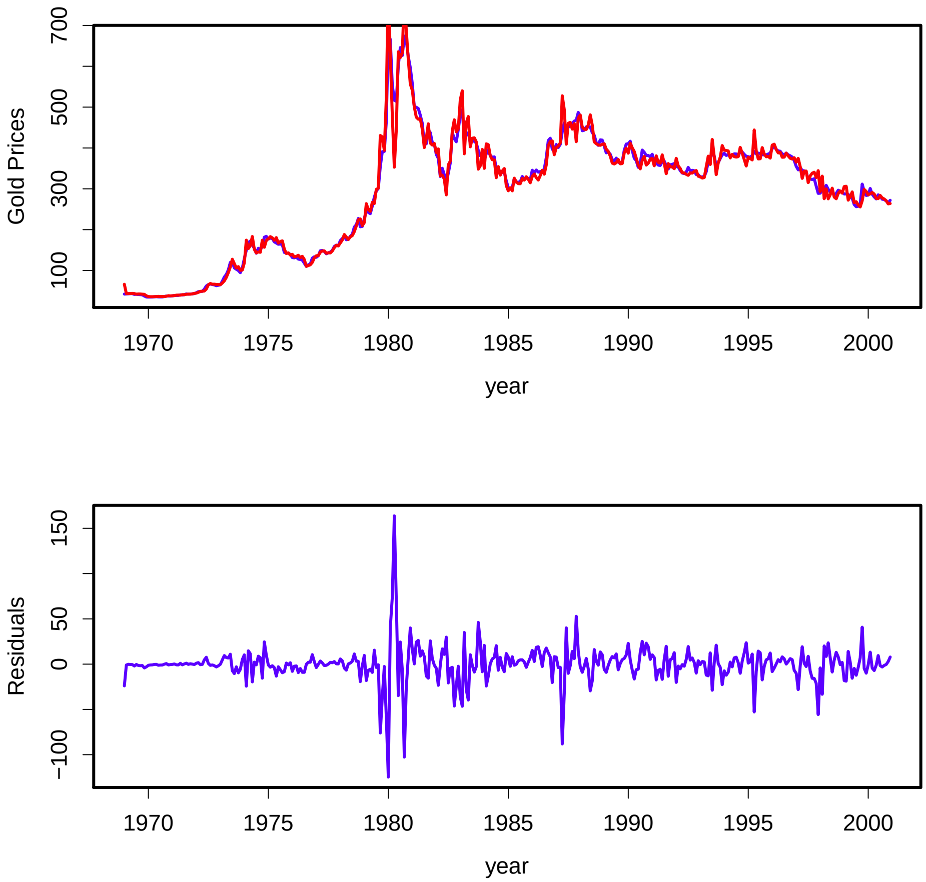

Figure 3.5: Gold prices (black) first-order autoregression Coefficient model fit(red) and its residuals.

Figure 3.6: First-order autoregression Coefficient model's estimates for one step-ahead prediction coefficient β_t is black, estimates for smoothed β_t is blue. These parameters represents the gold silver ratio.

3.3 random walk with Drift Model

The previous model showed us that the time varying parameters of the stochastic regression model was actually closer to the random walk process than the AR(1) process. This is the reason we investigated the random walk with drift model.

In state equation (3.3), if we set $\Phi = 1$ and keep a constant d on the right side of the equation as a drift constant, the first-order autoregression coefficient model becomes a random walk with drift model,

$$\beta_t = \beta_{t-1} + d + a_t \tag{3.5}$$

The fixed parameter vector for this model is $\Theta = (\alpha, d, \sigma_a, \sigma_e)'$. These parameters are estimated by the Newton-Raphson algorithm procedure and these estimates are listed in Table 3.3. The standard error corresponding to each fixed parameter is also included in the table. In the previous model, we had $(1-\Phi)\beta$ term in (3.2) and its value is 0.45 even though β is 50.575. This is due to $\Phi \approx 1$; see Table 3.2. In this model, instead of $(1-\Phi)\beta$, we have drift constant d, which is 0.186. In the limiting case when $\Phi \to 1$, $(1-\Phi)\beta$ can be considered as a drift constant. Therefore, we can say that previous model had larger drift influence on the model.

Table 3.3: Fixed Parameters of random walk with Drift Model

	α	d	σ_a	σ_e
Estimate	50.108	0.186	2.323	0.861
SE	4.286	0.117	0.084	0.481

The Kalman filter algorithm was utilized to find the estimate for one step-ahead prediction β_t and the Kalman smoothing algorithm for smoothed β_t. Both are shown in Figure 3.7 and closely follow each other. Figure 3.8 shows the estimates of one

Figure 3.7: Random walk with drift model's estimates for one step-ahead prediction coefficients β_t is black and smoothed β_t is blue.

step-ahead predictions, β_t, along with its ACF and PACF. This clearly shows that β_t's path is a typical random walk with drift model. When we compare this with Figure 3.4, it can be seen that the stochastic parameter β_t is more like a random walk than an AR(1) process.

Close inspection of Figure 3.9 indicates that this model also traces the real gold prices very closely. Similar to the stochastic parameter regression model residuals, 1980's silver bubble is very pronounce in this model's residuals as well.

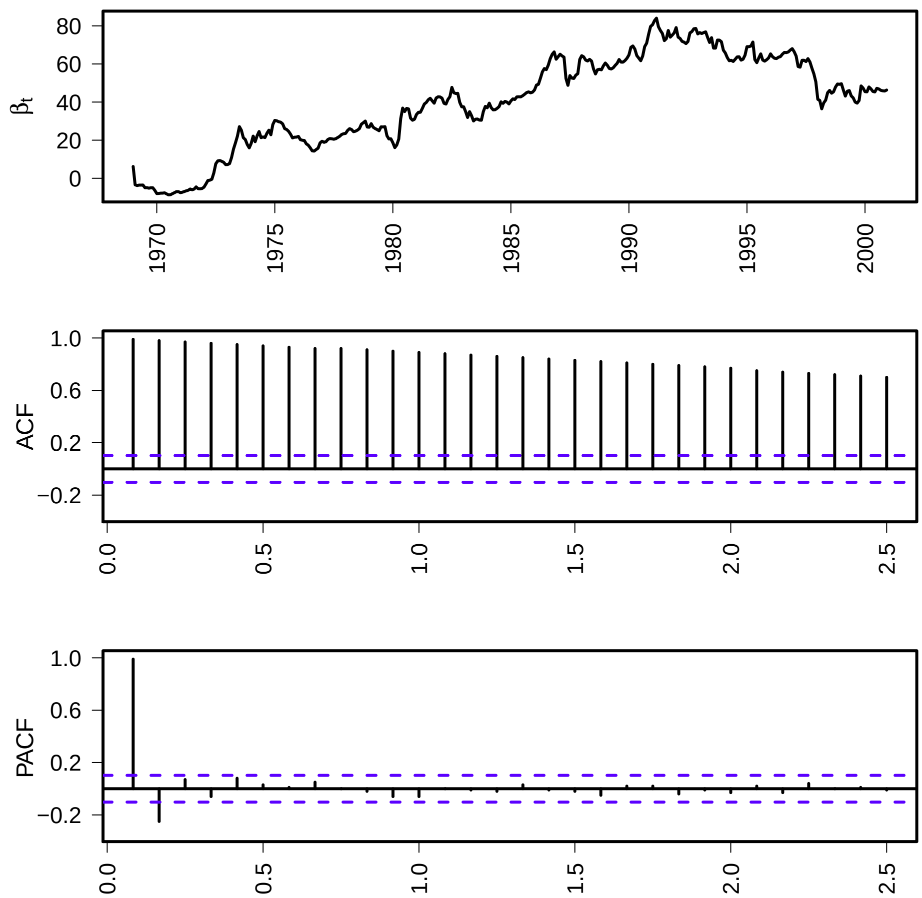

Figure 3.8: Random walk with drift model's estimates of one step-ahead prediction coefficients β_t and its ACF and PACF.

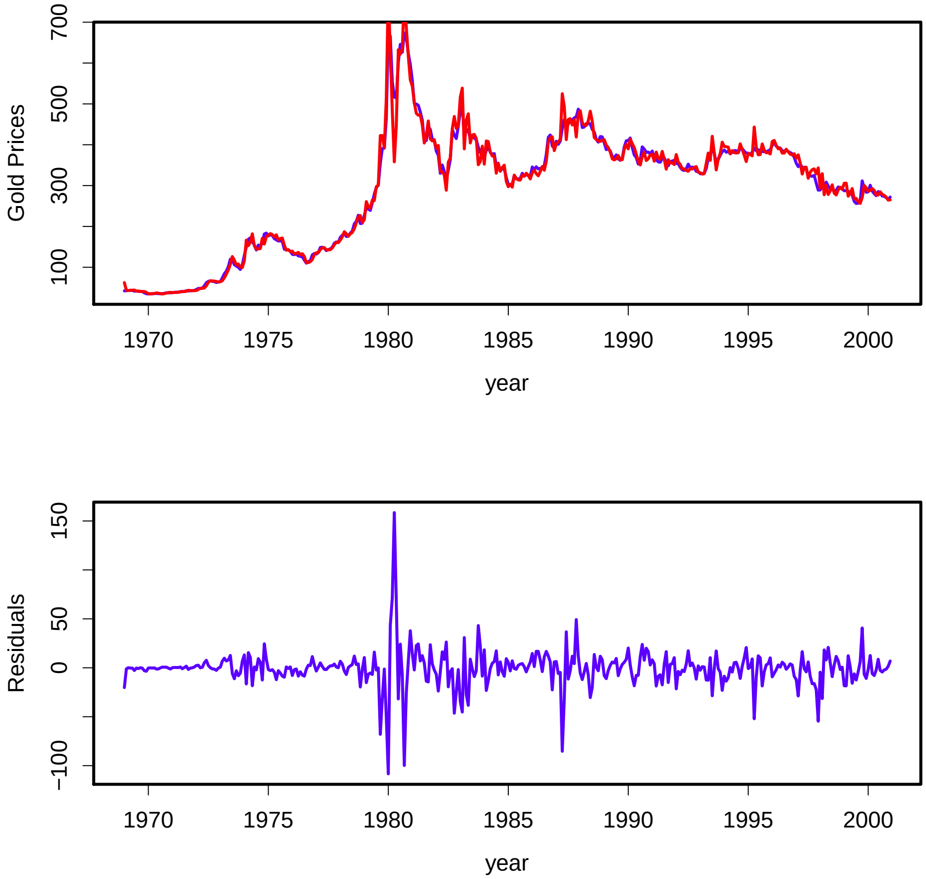

Figure 3.9: Gold prices (black) random walk with drift model fit(red) and residuals.

3.4 random walk without Drift Model

When we set $\phi = 1$, in (3.3), this model becomes a random walk without drift, $\beta_t = \beta_{t-1} + a_t$. When the stochastic regression model's fixed coefficient estimates for Φ is nearly 1, this models turns into a random walk model, $\lim_{\Phi \to 1}(\Phi - 1) \approx 0$. . It is no surprise that Figure (3.10) of the first-Order autoregression coefficient model and Figure (3.6) of the random walk model are almost identical. The fixed parameter vector for this model is $\Theta = (\alpha, \sigma_a, \sigma_e)'$. These parameters are estimated by the Newton-Raphson algorithm procedure and are are listed in Table 3.4. The standard error corresponding each fixed parameter is also included in the table. The difference between these parameters and those of the random walk with drift model are not significant. It appears that the major difference is the drift constant. The standard

Table 3.4: Fixed Parameters of random walk Model

	α	σ_a	σ_e
Estimate	46.526	2.262	0.874
SE	3.968	0.077	0.503

error corresponding each fixed parameter is also included in the table. The difference between these parameters and those of the random walk with drift model are not significant. It appears that the major difference is the drift constant. Figure 3.11 shows the estimates of one step-ahead predictions, β_t, along with its ACF and PACF. This clearly shows that β_t's path is a typical random walk model. When we compare this with Figure 3.4, it can be seen that the stochastic parameter β_t is more like a random walk than an AR(1) process.

Figure 3.10: Random walk without drift model's estimates for one step-ahead prediction's β_t is black, estimates for smoothed β_t is blue.

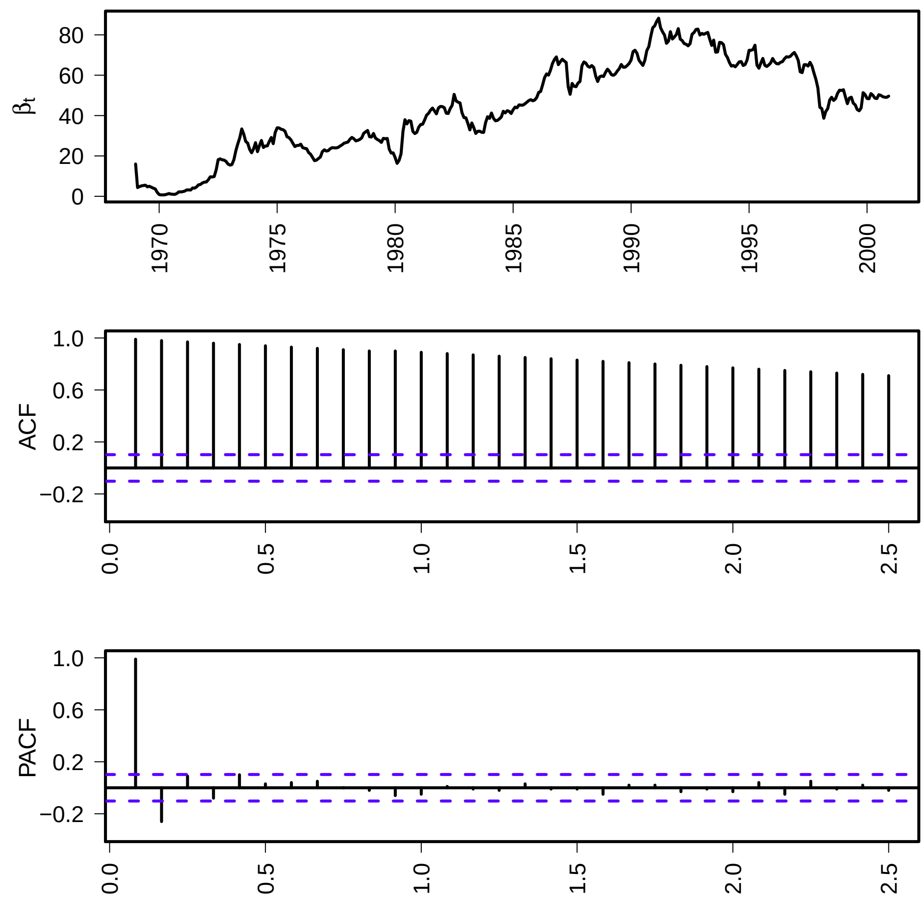

Figure 3.11: random walk Without Drift model's estimates of one step-ahead prediction coefficients β_t and its ACF and PACF.

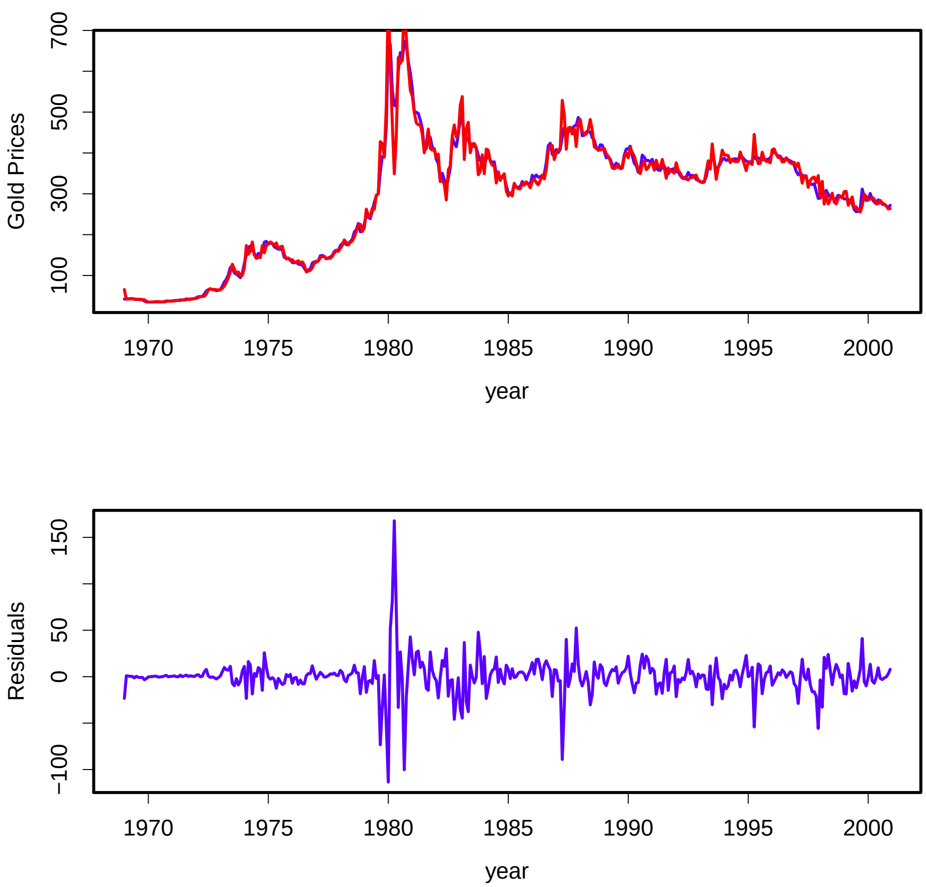

Figure 3.12: Gold prices (black) random walk without drift model fit (red) and residuals.

3.5 Prediction

Figure 3.13 compares one-step ahead gold price predictions from the four models (random coefficient, first-order autoregression, random walk with drift, and random walk without drift) described in Sections 3.1-3.4, along with gold and silver prices between January of 1969 and December of 2000. Excluding the random coefficient model, all three models closely trace the gold prices. Figure 3.14 displays all four models' one-step ahead prediction for β_t. After the first five years, all three models' (except the random coefficient model) β_t are close to each other.

We compared all four models' forecastability, as described in Section 2.2.4. Figure 3.15 shows all four models' prediction of 12 months into the future. We see that all four models under predicted gold prices for 2001; the random walk with drift gave the best predictions, and the random coefficient model produced the worst predictions.

While silver closely followed gold trends most of the years between 1969 and 2000, after 2001 this is not true. It appears that silver lost some of its appeal for a short period of time, although there were yet known reasons for this. However, in the political side, a new administration took over the presidential office around 2001. During this time of the political change, gold prices started to rise; silver, on the other hand, was not so quick to respond to this change. We do not know what caused the discrepancy between the gold price forecasts and actual gold prices. However, because the forecasts were close to actual gold prices up to 2000, we conjecture that the predicted values of the gold prices obtained from the random walk with drift model would have been closer to gold prices, if such a change in the political environment was not made.

Table 3.5 lists the sum of squares and sum of absolute values of for prediction

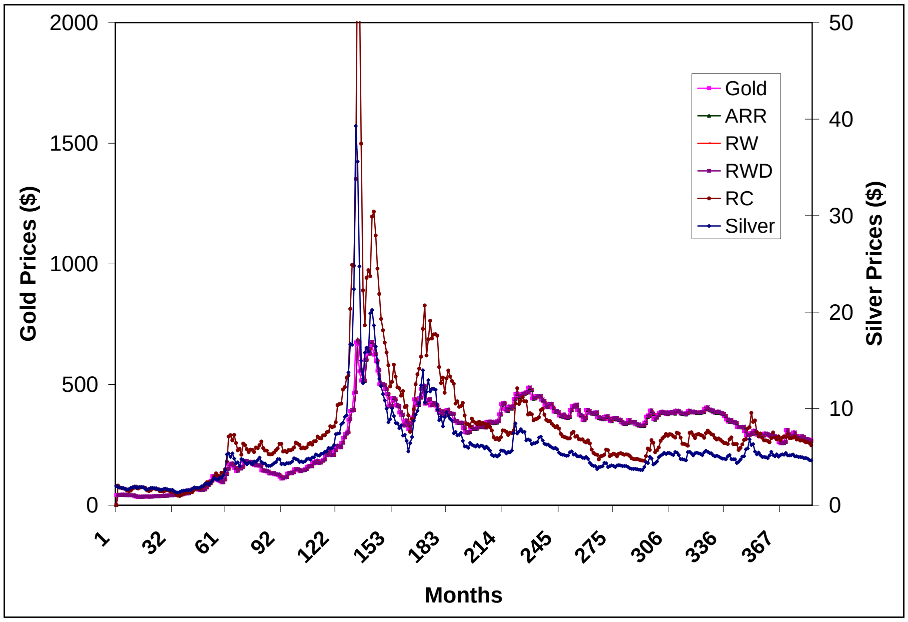

Figure 3.13: Gold, silver prices and model fit of gold for four models: AR(1) coefficient model (ARR), random walk with drift (RWD), random walk (RD), and random coefficient (RC)

errors (SSPE and SAPE) of all four models. It is clear from these error values that random walk with drift is the best model compared to others even though all four models under estimate future gold price with given silver price.

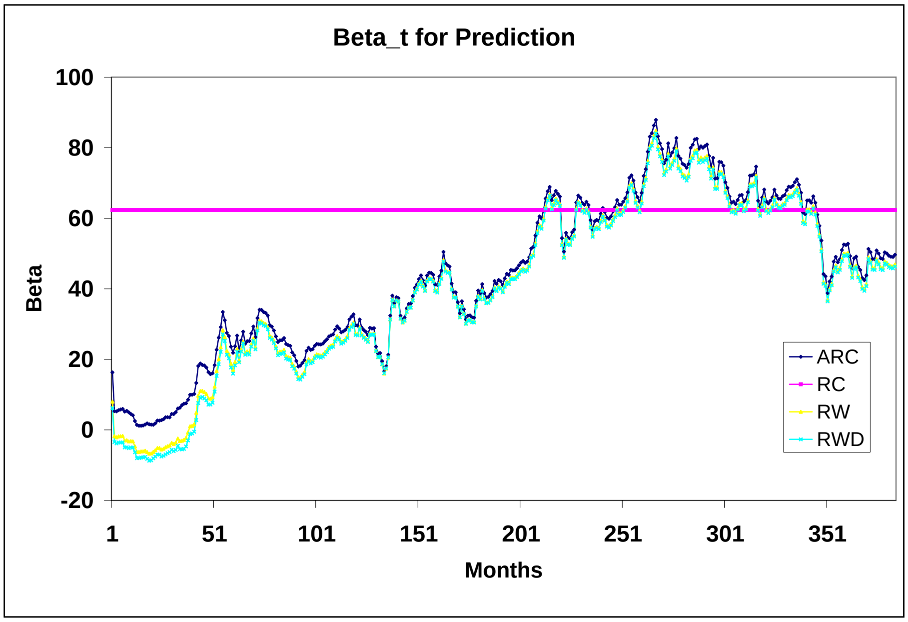

Figure 3.14: β_t of one step ahead predictions of four models between January of 1969 and December of 2000

Table 3.5: (SSPE) and (SAPE) of four models' 2001, 12 months predictions

	RC	ARC	RWD	RW
SSE	23327	6524	3094	6183
SAE	505	243	164	236

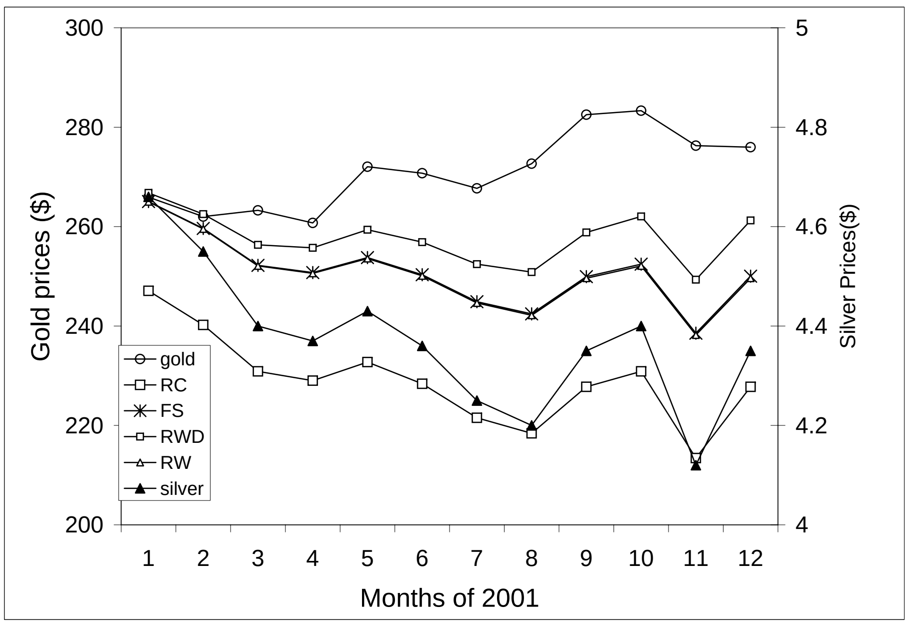

Figure 3.15: Prediction for random coefficient (RC), random walk (RW), random walk with drift (RWD), AR(1) coefficient model (FS) between January of 1969 and December of 2000

CHAPTER 4

CONCLUSIONS

In Section 2.1.1, we listed each historical decisions or events between 1969 and 2000 that caused similar effects in gold and silver prices. During these effects, if there was a spike in the gold prices, there was a spike in the silver prices as well. Based on this historical data, we assumed gold and silver prices were dictated by the same fundamentals (i.e., what affected gold affected silver as well). This lead us to examine the dynamic relationship between gold and silver prices between 1969 and 2000.

The relationship between gold and silver was assumed to follow a stochastic parameter regression model and stochastic parameters were assumed to be a first-order autoregression process, AR(1). We formulated this relationship into a state-space form, and we used the Kalman filter and smoother algorithms to estimate both the fixed parameters and the stochastic regression coefficients. We determined that the stochastic parameters were closer to a random walk process than the AR(1) process. Therefore, we applied random walk process with a drift, random walk without a drift, and random coefficient models. When it comes to forecastability of the four models, the random walk with drift model seems to be the best model even though all four models under estimated gold prices with given silver prices.

In the 1980s there was a huge increase in silver prices due to famous silver bubble caused by the Hunt brothers of Texas [5]. Around the same time IMF sold 1/3 of its gold reserve and the U.S. sold around 16 million troy ounces of gold [12]. These last two events caused a bubble-like increase in gold price. Even though the reasons for the increase in gold and silver prices were not related, it appeared that gold and silver followed the same trend.

In addition, all four models showed large residuals with the largest occurring in the random coefficient model. It appears that those events around 1980 are outlier. This kind of outlier are future unpredictable events that are hard to predict. Therefore, it is our opinion that this kind of analysis is best to study historical relationship between gold and silver and is less useful for long term investment purposes. With this being said, however, our stochastic parameter regression approach allows others to predict future gold prices with only one variable, silver price, with grate accuracy for the short term. This approach might be powerful for future predictions if no extreme events occur for the period of interest.

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APPENDIX A

R-PROGRAMS

This program calculates the fixed parameters of the state space model and the time varying coefficients of the stochastic regression model. It is a modification of the example 6.13 from Shumway and Stoffer [8], and uses **tsa3.rda** which can be downloaded from the web site provided in [8]. This program can be adapted to other three models by only changing the initial parameters and fixed parameter vector.

```
# Stochastic Regression Model with
# AR(1) Stochastic parameter Model
# Kfilter2 and Ksmoother2 given Shumway and Stoffer

load("tsa3.rda")
GoldSilverTBill<-read.csv(
            file="GoldSilverLondonfix.csv",
            head=TRUE,sep=",")
n<-nrow(GoldSilverTBill)
m<-ncol(GoldSilverTBill)
acf2=function(series,max.lag=NULL){
  num=length(series)
```

```
  if (is.null(max.lag)) max.lag=ceiling(10+sqrt(num))
  if (max.lag > (num-1)) stop("Number of lags
              exceeds number of observations")
  ACF=acf(series, max.lag, plot=FALSE)$acf[-1]
  PACF=pacf(series, max.lag, plot=FALSE)$acf
  LAG=1:max.lag/frequency(series)
  minA=min(ACF)
  minP=min(PACF)
  U=2/sqrt(num)
  L=-U
  minu=min(minA,minP,L)-.01
  ACF<-round(ACF,2); PACF<-round(PACF,2)
  return(cbind(LAG, ACF, PACF) )
  }
gold<-ts(as.numeric(data.matrix(GoldSilverTBill$GoldLondonFixAMAverage)),
    start=c(1968,4),end=c(2012,1),frequency=12)       # Gold prices Bid Average
silver<-ts(as.numeric(data.matrix(GoldSilverTBill$SilverLondonFixAverage)),
    start=c(1968,4),end=c(2012,1),frequency=12)   # Silver Prices Bid Average

#TBill<-ts(as.numeric(data.matrix(GoldSilverTBill$TBilRatio_3month)),
    start=c(1968,4),end=c(2012,1),frequency=12)
# Here we change the window max-min values
#cbind(gold,silver)
#dev.new()
#plot(cbind(lgold,lsilver))
```

```
#dev.new()
#plot(cbind(as.numeric(lgold),as.numeric(lsilver)))
yearMin=1969
yearMax=2000
monthMin=1
monthMax=12
y<-window(gold,c(yearMin,monthMin),c(yearMax,monthMax))
z<-window(silver,c(yearMin,monthMin),c(yearMax,monthMax))
# summary(lm(lgold~lsilver))
mu0=16; Sigma0=0.004; phi=0.991860026; alpha=33.644412
b=50.57212; cQ=2.46836910; cR=1.123121
init.par<-c(phi,alpha,b,cQ,cR)      # initial parameters
num<-length(y)
A<-array(z,dim=c(1,1,num))
input<-matrix(1,num,1)
# Function to calculate likelihood
Linn<-function(para) {
   phi<-para[1]
   alpha<-para[2]
   b<-para[3]
   Ups<-(1-phi)*b
   cQ<-para[4]
   cR<-para[5]
  #kf<-Kfilter2(num,y,A,mu0,Sigma0,phi,Ups,alpha,theta,cQ,cR,0,input)
   kf<-Kfilter2(num,y,A,mu0,Sigma0,phi,Ups,alpha,1,cQ,cR,0,input)
```

```
   return(kf$like)
}
tol<-0.0001
est<-optim(init.par,Linn,NULL,method="BFGS",hessian=TRUE,
        control=list(trace=1,REPORT=1,reltol=tol))
SE<-sqrt(diag(solve(est$hessian)))
phi<-est$par[1]
alpha<-est$par[2]
b<-est$par[3]
Ups<-(1-phi)*b
cQ<-est$par[4]
cR<-est$par[5]
u<-rbind(estimate=est$par,SE)
colnames(u)<-c("phi","alpha","b","sig_w","sig_v")
u
#ks=Ksmooth2(num,y,A,mu0,Sigma0,phi,Ups,alpha,tetha,cQ,cR,0,input)
ks=Ksmooth2(num,y,A,mu0,Sigma0,phi,Ups,alpha,1,cQ,cR,0,input)
SE_bsmooth<-(ks$Ps)
SE_bprediction<-(ks$Pp)
SE_bfilter<-(ks$Pf)
################################################################
#log(gold)~log(silver)  Model
################################################################
Bsmooth <-ts(as.vector(ks$xs),start=c(yearMin,monthMin),
                end=c(yearMax,monthMax), frequency=12)
```

```
Bprediction<-ts(as.vector(ks$xp),start=c(yearMin,monthMin),
                    end=c(yearMax,monthMax), frequency=12)
Bfilter<-ts(as.vector(ks$xf),start=c(yearMin,monthMin),
                    end=c(yearMax,monthMax), frequency=12)
sil <-ts(as.vector(z), start=c(yearMin,monthMin),
                    end=c(yearMax,monthMax), frequency=12)
gol<-ts(as.vector(y), start=c(yearMin,monthMin),
                    end=c(yearMax,monthMax), frequency=12)
SE_bs<-ts(as.vector(SE_bsmooth),start=c(yearMin,monthMin),
                    end=c(yearMax,monthMax), frequency=12)
SE_bp<-ts(as.vector(SE_bprediction),start=c(yearMin,monthMin),
                    end=c(yearMax,monthMax), frequency=12)
SE_bf<-ts(as.vector(SE_bfilter),start=c(yearMin,monthMin),
                    end=c(yearMax,monthMax), frequency=12)
Bp<-window(Bprediction,c(yearMin,monthMin),c(yearMax,monthMax))
Bs<-window(Bsmooth,c(yearMin,monthMin),c(yearMax,monthMax))
Bf<-window(Bfilter,c(yearMin,monthMin),c(yearMax,monthMax))
postscript(file="FullStochasticBeta.eps",
paper="special",
width=10,
height=7,
horizontal=FALSE)
par(mfrow=c(1,1), mex=0.8)
par(mar=c(4,4,2,2),mfrow=c(1,1), mex=0.8,cex=1.4,lwd=2)
plot(Bp, col="black",ann=FALSE,lwd=2,las=2)
```

```
mtext(bquote(beta[t]),line=2,side=2, cex=1.5)
lines(Bs, col="blue",lwd=2)
#lines(Bf, col="red",lwd=2)
dev.off()
gol_hat <- sil*Bp+alpha
res_gold<- gol-gol_hat
postscript(file="FullStochCoefModel.eps",
paper="special",
width=10,
height=10,
horizontal=FALSE)
par(mfrow=c(2,1), mex=0.8,cex=1.4,lwd=3)
plot(gol, xlab="year", ylab="Gold Prices",col="blue",cex=1.4,lwd=3)
lines(gol_hat,  col="red",lwd=3)
plot(res_gold, xlab="year", ylab="Residuals",col="blue",cex=1.4,lwd=3)
dev.off()
x11()
par(mfrow=c(2,1), mex=0.8,cex=1.4,lwd=3)
plot(gol, xlab="year", ylab="Gold Prices",col="blue",cex=1.4,lwd=3)
lines(gol_hat,  col="red",lwd=3)
plot(res_gold, xlab="year", ylab="Residuals",col="blue",cex=1.4,lwd=3)
postscript(file="FSacfBp.eps",
paper="special",
width=10,
height=10,
```

```
horizontal=FALSE)
ACFPCAF<-acf2(Bp)
xx<-dim(ACFPCAF)
LAG<-ACFPCAF[,1]
ACF<-ACFPCAF[,2]
PACF<-ACFPCAF[,3]
minA=min(ACF)
minP=min(PACF)
U=2/sqrt(num)
L=-U
minu=min(minA, minP, L)-0.1
par(mfrow=c(3,1), mex=0.8,cex=1.4, lwd=3)
par(mar = c(4,6,2,0.8))   #, oma = c(1,1.2,1,1), mgp = c(1.5,0.6,0))
  plot.ts(Bp,ann=FALSE,las=2)
mtext(bquote(beta[p]), side=2, line=3,cex=1.5)

  plot(LAG, ACF, type="h",ylim=c(minu,1),ann=FALSE,las=2,yaxt="n")
mtext("ACF", side=2, line=3,cex=1.5)
axis(side=2, at=c(-0.2, 0.2, 0.6, 1),las=2)
    abline(h=c(0,L,U), lty=c(1,2,2), col=c(1,4,4),yaxt="n")
  plot(LAG, PACF, type="h",ylim=c(minu,1),ann=FALSE,las=2,yaxt="n")
mtext("PACF", side=2, line=3,cex=1.5)
axis(side=2, at=c(-0.2, 0.2, 0.6, 1),las=2)
    abline(h=c(0,L,U), lty=c(1,2,2), col=c(1,4,4))
dev.off()
```

```
postscript(file="FSacfBs.eps",
paper="special",
width=10,
height=10,
horizontal=FALSE)
ACFPCAF<-acf2(Bs)
xx<-dim(ACFPCAF)
LAG<-ACFPCAF[,1]
ACF<-ACFPCAF[,2]
PACF<-ACFPCAF[,3]
minA=min(ACF)
minP=min(PACF)
U=2/sqrt(num)
L=-U
minu=min(minA, minP, L)-0.1
par(mfrow=c(3,1), mex=0.8,cex=1.4, lwd=3)
par(mar = c(4,6,2,0.8))   #, oma = c(1,1.2,1,1), mgp = c(1.5,0.6,0))
  plot.ts(Bs,ann=FALSE,las=2)
mtext(bquote(beta[s]), side=2, line=3,cex=1.5)


  plot(LAG, ACF, type="h",ylim=c(minu,1),ann=FALSE,las=2,yaxt="n")
mtext("ACF", side=2, line=3,cex=1.5)
axis(side=2, at=c(-0.2, 0.2, 0.6, 1),las=2)
    abline(h=c(0,L,U), lty=c(1,2,2), col=c(1,4,4),yaxt="n")
  plot(LAG, PACF, type="h",ylim=c(minu,1),ann=FALSE,las=2,yaxt="n")
```

```
mtext("PACF", side=2, line=3,cex=1.5)
axis(side=2, at=c(-0.2, 0.2, 0.6, 1),las=2)
    abline(h=c(0,L,U), lty=c(1,2,2), col=c(1,4,4))
dev.off()
postscript(file="FSacfBf.eps",
paper="special",
width=10,
height=10,
horizontal=FALSE)
ACFPCAF<-acf2(Bf)
xx<-dim(ACFPCAF)
LAG<-ACFPCAF[,1]
ACF<-ACFPCAF[,2]
PACF<-ACFPCAF[,3]
minA=min(ACF)
minP=min(PACF)
U=2/sqrt(num)
L=-U
minu=min(minA, minP, L)-0.1
par(mfrow=c(3,1), mex=0.8,cex=1.4, lwd=3)
par(mar = c(4,6,2,0.8))   #, oma = c(1,1.2,1,1), mgp = c(1.5,0.6,0))
  plot.ts(Bf,ann=FALSE,las=2)
mtext(bquote(beta[f]), side=2, line=3,cex=1.5)

  plot(LAG, ACF, type="h",ylim=c(minu,1),ann=FALSE,las=2,yaxt="n")
```

```
mtext("ACF", side=2, line=3,cex=1.5)
axis(side=2, at=c(-0.2, 0.2, 0.6, 1),las=2)
    abline(h=c(0,L,U), lty=c(1,2,2), col=c(1,4,4),yaxt="n")
  plot(LAG, PACF, type="h",ylim=c(minu,1),ann=FALSE,las=2,yaxt="n")
mtext("PACF", side=2, line=3,cex=1.5)
axis(side=2, at=c(-0.2, 0.2, 0.6, 1),las=2)
    abline(h=c(0,L,U), lty=c(1,2,2), col=c(1,4,4))
dev.off()
# This part for data file
sil <-ts(as.vector(z), start=c(yearMin,monthMin),
            end=c(yearMax,monthMax), frequency=12)
gol<-ts(as.vector(y), start=c(yearMin,monthMin),
            end=c(yearMax,monthMax), frequency=12)
SE_bs<-ts(as.vector(SE_bsmooth),start=c(yearMin,monthMin),
                    end=c(yearMax,monthMax), frequency=12)
SE_bp<-ts(as.vector(SE_bprediction),start=c(yearMin,monthMin),
                        end=c(yearMax,monthMax), frequency=12)
SE_bf<-ts(as.vector(SE_bfilter),start=c(yearMin,monthMin),
                        end=c(yearMax,monthMax), frequency=12)
Bp<-window(Bprediction,c(yearMin,monthMin),c(yearMax,monthMax))
Bs<-window(Bsmooth,c(yearMin,monthMin),c(yearMax,monthMax))
Bf<-window(Bfilter,c(yearMin,monthMin),c(yearMax,monthMax))
NEW_Data<-cbind(sil, gol, gol_hat,  res_gold, Bsmooth,
                        SE_bs,Bprediction,SE_bp,Bfilter, SE_bf);
write.csv(NEW_Data, file="GoldSilverKalman_FS.csv",na="NA")
```

```
# Silver and Gold between 2001-January and 2001-December
y2001<-window(gold,c(2001,1),c(2001,12))
z2001<-window(silver,c(2001,1),c(2001,12))
y2001<-as.vector(as.numeric(y2001))
z2001<-as.vector(as.numeric(z2001))
betaT<-rep(0,12)
y_hat<-rep(0,12)
xp=length(Bp)
xs=length(Bs)
xf=length(Bf)
xpsf<-cbind(Bp, Bs,Bf)
beta<-as.vector(as.numeric(Bp))
# phi, alpha, b, cQ, cR Fixed parameters
betaT0=beta[xp]
betaT[1]=b+phi*(betaT0-b) #+rnorm(1,0, cQ)
y_hat[1]=alpha+betaT[1]*z2001[1] #+rnorm(1,0, cR)
for (i in 2:12){
betaT[i]=b+phi*(betaT[i-1]-b) #+rnorm(1,0, cQ)
y_hat[i]=alpha+betaT[i]*z2001[i] #+rnorm(1,0, cR)
  }
SSE=sum( (y_hat-y2001)^2)
SE=sum(abs(y_hat-y2001))
y2001<-ts(as.numeric(y2001),start=c(2001,1),end=c(2001,12),frequency=12)
y_hat<-ts(as.numeric(y_hat),start=c(2001,1),end=c(2001,12),frequency=12)
FS_prediction<-y_hat
```

```
write.csv(FS_prediction, file="Prediction_FS.csv",na="NA")
postscript(file="FSPrediction.eps",
paper="special",
width=10,
height=7,
horizontal=FALSE)
par(mar = c(5,6,2,0.8), mfrow=c(1,1), mex=0.8,cex=1.2,lwd=3)
plot(y2001,xlab="Year 2001", ylim=c(min(y2001, y_hat),
                       max(y2001, y_hat)),ylab="Gold Prices ($)",
                   col="blue",lwd=3)
lines(y_hat,col="red",lwd=3)
dev.off()
gol_1<-append(gol, y2001,after=length(gol))
gol_hat_1<-append(gol_hat, y_hat, after=length(gol_hat))

gol_1<-ts(gol_1,start=c(1969,2),end=c(2001,12),frequency=12)
gol_hat_1<-ts(gol_hat_1,start=c(1969,2),end=c(2001,12),frequency=12)
res_gold_1<- gol_1-gol_hat_1
colnames(u)<-c("phi","alpha","b","sig_w","sig_v")
u
SSE
SE
```