DEVELOPMENTAL UNDERSTANDING OF THE EQUALS SIGN AND ITS EFFECTS ON SUCCESS IN ALGEBRA

by

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A thesis

submitted in partial fulfillment

of the requirements for the degree of

Master of Science in Mathematics Education

Boise State University

May 2010

BOISE STATE UNIVERSITY GRADUATE COLLEGE

DEFENSE COMMITTEE AND FINAL READING APPROVALS

of the thesis submitted by

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Thesis Title: Development Understanding of the Equals Sign and Its Effects on Success in Algebra

Date of Final Oral Examination: 25 March 2010

The following individuals read and discussed the thesis submitted by student Ryan W. Brown, and they evaluated his presentation and response to questions during the final oral examination. They found that the student passed the final oral examination.

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The final reading approval of the thesis was granted by Sharon B. Walen, Ph.D.., Chair of the Supervisory Committee. The thesis was approved for the Graduate College by John R. Pelton, Ph.D., Dean of the Graduate College.

ACKNOWLEDGMENTS

It is important for me to acknowledge my advisor, Dr. Sharon Walen. Her countless hours of discussion, counsel, instruction, and patience were instrumental in my work, preparation, and completion of this thesis. I would also like to acknowledge the entire mathematics-education department at Boise State University. Their desire for improving the quality of mathematics education, for the benefit of students, is both commendable and contagious.

And also to my wife: for her patience, support, and assistance I am extremely grateful.

ABSTRACT

For some students, the equals symbol is viewed as a directive to carry out a procedure, instead of a symbol expressing mathematical equivalence. The purpose of this study was to develop and to pilot a questionnaire to measure students' understandings of relational equivalence as implied by their interpretations and use of the equals symbol. The results of this questionnaire were compared with student testing data with the goal of determining a correlation between understanding of symbolic equivalence and success in a typical algebra course.

It was found that students who demonstrated an ability to define and articulate an appropriate meaning for the equals symbol scored significantly higher on an end-of-course test and on a state achievement test. However, this study also found that students who can define or articulate an appropriate meaning for the equals symbol may not necessarily be able to demonstrate a working knowledge or understanding of the symbol's appropriate uses. Another significant conclusion found was that quality instructional practices contribute to students performing at a seemingly higher level, with respect to symbolic relational equivalence, than those shown in previous studies.

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Introduction

The development of arithmetic skills and the instructional techniques traditionally used at lower grade levels can lead students to an incomplete or incorrect interpretation of the equals sign. This lack of understanding can develop into struggles with beginning algebra concepts. There has been much discussion as to the reasons for these student deficiencies, as well as the type of teaching techniques that would help students develop proper conceptual understanding of the symbol used for equality. For this researcher, it began with a personal experience.

“What do I do first?” she said, as she stared at a simple two-step algebra problem. “Do I add…do I subtract…how about divide?” She was just guessing, and Jill had been struggling all year in my Algebra 1 class. As was the daily routine, I tried desperately not to show my frustration as I helped her through the problem. I’m a little embarrassed to say that, at this point, I was beginning to give up on her. I would try to patiently walk through the steps, trying to ask good leading questions, but I knew that come tomorrow, she would be completely lost on what “steps” to do first. If she got one right, it was because she guessed right. Jill had no idea what it really meant to solve an equation. There was nothing in her mind that made it clear why, in solving an equation such as $2x+3=11$, one would probably want to start by subtracting 3 from both sides. At this point in the year, almost all of the students had mastered solving simple equations. There was a huge barrier in her mind that was preventing her from making any further progress. What was it? I had tried everything I knew. The barrier, as it turned out, went a lot farther back in her career than I realized. For Jill, her lack of success stemmed directly from her lack of understanding of the equals symbol. She did not connect her understanding of equality in the world around her – the concept of sameness – with the mathematical symbol for equality. She did not have a complete understanding of what the equals symbol meant in the context of a mathematical equation.

One of the most enlightening moments for me came while reading the book Thinking Mathematically (Carpenter, Franke, & Levi, 2003). After seeing one of the classroom videos on equality, included on a CD-ROM with the book, I began to realize how many students really don't understand what the equals symbol means. More specifically, some of the student views lead them to an incorrect view of the equals symbol. Following the video, I decided to ask my class to fill in the blank:

$$8 + 4 = \_\_\_\_ + 7$$

I was confident that, unlike the 5th grade students in the video, my algebra students would easily understand that 5 is the only number that would make the statement true. To my surprise, Jill (and several other students) put 12. From then on, I realized that most of the problems that my students were having were a direct result of a procedural understanding of the equals sign. There had been something engrained in them that the *answer* always followed "=." They gave no thought to the concept that the symbol meant there was equivalence between the two halves of a mathematical sentence.

Students always come to us with some sort of prerequisite knowledge. There are instructional methods that force on students, at times, an incomplete understanding of concepts. When most ideas that we seek to have our students understand are so filled with depth and application, why would we ask our students to perform single tasks and not allow them to investigate multiple ways of exploring and solving problems? The importance of allowing students to express their own ideas, patterns, intuitions, and methods cannot be overstated (Witherspoon, 1999). If a student is only exposed to one type of use of the equals sign (i.e., they learn only to do some arithmetic and then write the answer), the concept of equivalence may never be learned. For my student Jill, once I realized my own shortcomings in seeking to get a grasp on her thought processes, we were able to make some progress. Some of the most effective exercises that helped her were the following: 1) Working with an actual scale to balance objects. We then moved on to drawings, following the same concept to balance number values. Finally, we replaced the scale with the equals symbol and began to balance values in the same way. An example follow-up question that seemed to work would be: “If you removed 5 from this side of the scale, how many would you have to remove from the other side to keep the statement true?” then, “Can you write that operation out mathematically?” 2) Writing about her thought processes. She would describe each step in words, as she worked through various types of balance problems. Often, that alone would help *her* identify areas of flawed logic or reasoning. The writing also gave me an invaluable insight as to how to more specifically meet her learning needs. 3) I also had Jill and the other students come up with their own fill-in-the-blank number sentences and share them with others (or me) to solve (Carpenter et al., 2003). This was a wonderful way to get the students to talk about why they chose the numbers and how they knew that the correct answer was indeed the only number that would make the sentence true.

These activities mentioned above, along with many others, would be appropriate at almost any grade level, and in fact would help to better prepare students for learning how to think algebraically. As educators, the important thing to remember is that in order to design activities that will help students overcome deficiencies, it is imperative that we have a solid grasp of the level of understanding our students possess.

Research Question

Although there are most likely studies that demonstrate better methods of instruction, particularly at the lower grade levels, that might encourage the type of understanding needed for success in algebra, the purpose of this study is:

1. To develop and to pilot a paper and pencil questionnaire to measure students' understandings of relational equivalence as implied by their interpretations and use of the equals symbol. Note that in the development and piloting of this questionnaire, it is hoped that teachers of all grade levels will be able to use such a questionnaire as they design and measure the effectiveness of meaningful equal-symbol classroom experiences. This pilot questionnaire will be modeled after, and adapted from, questions used in previous studies completed by Freiman and Lee (2004), Falkner, Levi, and Carpenter (1999), and Hunter (2007).

2. To examine the relationship between the results obtained from this questionnaire and student success in algebra as measured by End-of-Course (EOC) exam and the Idaho State Achievement Test (ISAT).

CHAPTER 2: REVIEW OF THE LITERATURE

Introduction

Throughout the development of mathematics, symbols have crept their way into the manner in which we write and communicate conceptually and procedurally. These symbols, which have become commonplace, are usually shorthand tools designed to simplify the way mathematics is written. For some, a symbol is a mathematical *object*, a *thing* that can be manipulated in the mind. For others, it signifies a *procedure* to be carried out. Students who concentrate on procedure may capably develop a set of skills, which allow them to do computational arithmetic and succeed in the short term; but in the long term, they may lack the flexibility that will give them ultimate success in learning mathematical concepts (Gray & Tall, 1992). When a student encounters a symbol for the first time and is instructed as to its meaning, the hope is that he connects that symbol with some prior conceptual understanding of what it represents. “A mathematical symbol [should] evoke a particular thought, help to unveil a conceptual object, and point to specific features that belong to the significant field of meaning of such a conceptual object” (Saenz-Ludlow & Walgamuth, 1998, p. 154). Some students see these symbols as computational directives with no connection as to how to use them functionally. They read the symbol but do not connect it to anything practical or relational – literacy without understanding (Bickmore-Brand, 1990). In order to get a student to the point of really understanding what a symbol is representing or communicating, teachers must continually provide meaningful, contextual, and coherent experiences to nurture and develop this progressive understanding. Because of how the brain learns (Dewey called it the principle of continuity of experience), students cannot be expected to immediately conceptualize the depth and different meanings embedded in a symbol (Saenz-Ludlow & Walgamuth, 1998). The concept of equality, with respect to the equals symbol, is a concept that students must have a chance to work with in a variety of settings. Several researchers have written about how students can develop a procedural, “do something” notion of the equals symbol without these experiences provided by their teachers (Carpenter et al., 2003; Rivera, 2006; Falkner et al., 1999; Behr, Erlwanger, & Nichols, 1980). For some, the symbol is a call for action, a directive to write the answer. Instead of a symbol expressing relational equivalence, from these student’s perspectives, it is a symbol that is simply requesting an answer (Cobb, 1987). Furthermore, a student’s misinterpretation of the equals sign is not a result of cognitive limitations or immaturity on the part of the learner. It is, however, a direct result of certain instructional strategies and techniques that force this “do something” or “tell me what the answer is” notion on a student’s interpretation of the symbol. The teachers that Cobb surveyed believed their curriculum was a collection of isolated facts and skills that had to be mastered separately. The teacher’s role, as they saw it, was to train students to use particular skills. In turn, their students learned to just mimic their teacher’s talk and behavior and never sought to understand and make connections within the mathematics, especially with respect to the equals symbol. There are other examples of how the language a teacher uses to instruct can reinforce this notion of procedure versus understanding in mathematics. For example, using the phrase “do the same to both sides of an equation” when solving can be very confusing to a student who thinks all he has to do is remove a number to get a desired result. It becomes “a magic trick” instead of an exercise of relational equivalence (Gray & Tall, 2007). A teacher who believes this way and teaches this way inevitably has students who are not mathematical thinkers, simply because of the lack of classroom experiences they have been given. The concept of equality and the understanding of the equals symbol is not as intuitive for students as many teachers believe it to be. It is a concept that is very difficult for many learners to understand completely, and yet, it is essential that a student grasp it before they will understand any formal algebraic concepts (Falkner et al., 1999; Knuth, Stephens, McNeil, & Alibabi, 2006). The equals sign stands for relational equivalence. However, in students’ minds, it is often used and interpreted in many different ways. If students are not exposed to the appropriate multiple forms and representations of a word or symbol, they will never be able to interpret and use it correctly (Witherspoon, 1999).

Student Misinterpretation of the Equals Sign

One of the consequences of teaching from the perspective that mathematics is a collection of isolated facts and skills that have to be mastered separately is that students seem to decide, at some point, that working for understanding is not necessary–that learning the procedure is appropriate. They search for answers, not for understanding. As students do worksheets similar to Figure 1, they see only one form of an equation and one use for the equals sign. It is not surprising that they should reject alternative forms,

$$2+3=\_\_\_$$
$$4+7=\_\_\_$$
$$3+1=\_\_\_$$
$$5+2=\_\_\_$$

Figure 1.
Traditional Worksheets

$$2+\_\_\_=8$$
$$3+5=\_\_\_+2$$
$$9=4+\_\_\_$$
$$4+\_\_\_=6+\_\_\_$$

Figure 2.
Suggested Practice

similar to those shown in Figure 2 (Cobb, 1987, p.110). Another problem that seems to be commonplace among students taught procedurally is not only an incomplete understanding of the equals symbol, but an incorrect view as well. A typical example of this could be seen in a student's work of the following problem:

> John has three bags with seven cookies in each bag. If someone gave him four cookies, how many would he have all together? $3\times 7=21+4=25$.

The incorrect use of the equals sign demonstrated in this student's work clearly reveals a procedural "and the answer is" type of understanding of the equals sign (Cobb, 1987, p. 110). A very disturbing outcome that should be noted in students who take a mechanistic approach to understanding symbols is that they rarely look for reasonableness of results (Witherspoon, 1999). This is one of the most frustrating behaviors for teachers to overcome, and for this researcher, it has always been a difficult task to focus my students on considering the importance of "reasonableness of their results."

As students progress through school and begin to take algebra courses, it becomes absolutely essential for them to have developed a keen understanding of equality and the various forms in which the equals symbol might appear. When a student begins algebra, if he still has an incomplete understanding of "=," the equations in Figure 2 will not make sense. He will have an extremely tough time understanding various manipulations needed to solve equations. When reading and working with the various types of symbols, letters, numbers, and expressions in algebra, a student who thinks that the equals sign means "'write the answer' is likely to believe that an expression like $x + y$ requires a single term answer" (Boaler & Humphreys, 2005, p. 17). Students become bogged down by their lack of symbol interpretation. In their minds, the expression $x = 6 + 3$ has no solution, because we cannot say with meaning, "something equals six and three" (Steinbring, Bartolini-Bussi, & Sierpinska, 1998, p. 10). Similarly, "a mathematical sentence, such as $2x + 5 = 3(x + 2)$, must seem completely nonsensical to someone who views the equals sign as meaning, 'and the answer is'" (Witherspoon, 1999, p. 1). This type of student is way behind in his understanding and will forever struggle with algebraic concepts, if not given special attention to her specific needs on equality. Studies by Carpenter et al. (2003) have shown numerous examples of students' misconceptions of equality. When asked to fill in the box for the equation $8 + 4 = \square + 5$ to make the equation true, many, demonstrating a need to "write the answer" immediately after the symbol, claimed the answer was 12 instead of 7. Then, when asked to explain further, some students wrote: $8 + 4 = 12 + 5 = 17$. This clearly demonstrates a need to write in the answer immediately following the equals symbol, instead of seeking to form a true equivalence statement. In the same study, students who were quick to verify a statement like $3 + 4 = 7$ was true were also quick to claim that $7 = 3 + 4$ was not true. They either stated it was "backward" and couldn't be written that way or said it simply didn't make any sense. Clearly, these students had developed a procedural understanding of equality.

How Students Should View the Equals Sign

The question now becomes, what is it that students should know about the equals symbol, or more importantly, the relational equivalence that it represents? First and foremost, students must be aware that they are using symbols only to aid in communicating their ideas (Witherspoon, 1999). Once that is realized, students can begin to understand the various representations and notations that are frequently used in mathematics. The fact that their misconceptions revolve around a symbol does not mean their lack of relational understanding is trivial or easy to overcome (Carpenter et al., 2003). Teachers must not treat it as such, but give special attention to their students' conceptions, from primary-grade mathematics through algebra. For, as algebra students encounter various uses of the equals symbol, their understanding of equality must be complete. If not, they will get caught up in the rigors of procedure. For example, these students might always believe an answer, or result, should be written immediately following the equals symbol. Or, an algebra student might always believe that adding or subtracting a constant is the first step in solving a problem, without giving any thought to the necessity of maintaining an equivalent mathematical sentence. The goal, by the time a student reaches algebra, would be for them to understand the different ways the equals symbol is used. In arithmetic, 2 + 5 = 7 means the sum of two and five is seven. In certain algebraic contexts, we might say " $x^2 + 2x + 1 = (x + 1)^2$ " should be interpreted as a mathematical sentence being true for all values of x. Another context might define the mathematical sentence $x^2 + 9x + 8 = 0$, which should be interpreted as being mathematically true for two values of x. Once again, if students have not been exposed to various forms and uses (like that in Figure 2), the previous equations would not make sense. There are many other uses for the equals sign. For example, in geometry, we might write $AB = 2$, meaning the distance between points A and B is two units. In summary, the goal is to have students understand the equals sign to represent 1) the sameness of objects, 2) quantitative sameness, and 3) similarity of two numerical statements (Saenz-Ludlow & Walgamuth, 1998).

Measuring and Improving Student Understanding

As mathematics teachers, the overall goal should be to cultivate students' abilities to reason. In the National Council of Teachers of Mathematics (NCTM) document *Principals and Standards for School Mathematics* (2000), a proper, relational understanding of equality is a concept that students must begin to encounter and understand, even in the lower grades. Teachers need to make it clear to all students that their goal is to make sense of and find ways to solve various types of problems. It is *not* to give the teacher the impression that they are acting in line with some sort of procedural expectations (Cobb, 1987). If both the teacher and student are aware of this goal, it will help set the stage for students to seek conceptual understanding. If teachers do not want to stifle a student's progress mentally, then they must be aware that "cognitive activity is constrained by the context of on-going activity" (Cobb, 1987, p. 116). This fact puts a lot of responsibility on the teacher to design activities that are non-procedural. “The standardization of algorithms is unnecessary. Students can develop their own processes for using the symbols in problem solving as long as those processes are consistent with the underlying meanings of the symbols” (Witherspoon, 1999, p. 3). To rid students of the desire to be given a standard algorithm is tough. It is often met with severe opposition from students (and parents) who have been indoctrinated with procedures. These learned procedures, however, are not consistent across time or cultures. Once students find success in investigation and observation, their potentials becomes limitless. Carpenter et al. (2003) found that placing students in a position to challenge their existing conceptions was productive. Engaging them in discussions in which different conceptions about equality emerge and must be resolved can be helpful to broaden their views. True-False number sentences are easy to introduce and use as a tool for engaging students in discussions.

The importance of allowing students to express their ideas, patterns, intuitions, and methods cannot be overstated. The instructor’s job description should be to facilitate, guide, question, and assess. Blair (2003) comments,

> Teachers do not realize how powerful the patterns and generalizations that students express can be. These expressions should be seen as opportunities for class discussions so that all of the students have access to these ideas. As teachers, it is our job to understand how children think about mathematics when they come to school and build on this informal understanding. (p. 2)

Freiman and Lee (2004) developed a series of questions that teachers or researchers could use to quickly and effectively measure the level of student understanding of mathematical equality. These questions were adapted from their research on how best to measure understanding of equality; their source came from earlier studies done primarily by Falkner et al. (1999). The following fill-in-the-blank type questions are included in the pilot questionnaire used in this thesis:

$a + b = c + \_\_\_$, $a + b = \_\_\_ + d$, $c = a + \_\_\_$, and $a + \_\_\_ = c + d$.

Warren's (2003) study agreed with the results presented by Falkner et al. (1999). In Warren's study, six tasks were given to students involving number sentences and finding missing numbers. This study showed that many students had difficulty relating sums and differences, due in part to "=" being used for mathematical equality. Hunter's (2007) study was also designed to explore students' understandings of the equals sign and equivalence. It involved similar questionnaires and was similar to those listed above, with the addition of number sentences in the form of: $a – b = c – d$. Her study revealed that only 28% of the subjects used an appropriate relational strategy. These results highlighted a lack of understanding of the equals sign among participants in the study.

Summary

Both secondary and elementary teachers need to have a wealth of knowledge and experience before they begin to structure classes and lessons that are designed to guide students through an interactive process of meaningful dialog, investigation, and discovery of a mathematical concept. Often, however, teachers (elementary in particular) are not familiar with, nor have they been given an opportunity to develop their own experiences with a deeper, richer, more connected algebra and understanding of how to build these opportunities for their students (Blair, 2003). Obviously, more teacher awareness is needed, both in mathematical content and instruction. It is appropriate to expect students to justify their claims and repeatedly ask questions like, “Will that always be true?” and “How do you know that is true?” to help them gain skills in presenting mathematical arguments and justifications. The notion of mathematical equality is the idea that two expressions have the same value. This is important for two reasons. One, children who understand mathematical and symbolic equality will have a useful way to communicate and represent arithmetic. If a student is trying to figure out 83 – 29, she might be able to relate that to an easier problem: 80 – 30. To be able to correctly reason, and see that $83-29=(80+3)-(30-1)=(80-30)+(3+1)=50+4=54$ is a powerful computational tool and an essential skill used in the understanding of algebraic concepts. Second, this understanding of relational equivalence is a bridge that allows students to be able to conceptualize and understand the concepts underlying the more abstract principles of algebra. Without a robust understanding of the equals symbol, what chance would a student have in being able to understand why subtracting 27 from both sides of $4x+27=87$ would maintain equality (Falkner et al., 1999)? If students were to just memorize rules that would allow them to solve $4x+27=87,$ what chance would they have of being able to understand what they were doing, much less be able to apply and justify the solution processes?

Understanding precisely what the equals symbol represents, and the many appropriate uses for it, is embedded in every area of mathematics. Teachers must start very early and focus on developing students’ understandings as they move very consistently through each activity and make sure every child is developing a complete and correct interpretation of the equals sign. Traditional methods of memorizing facts and practicing worksheets that only show one use of the equals sign throughout most of a student's education bring about a "do something" understanding of the equals sign. This, in turn, renders algebraic sentences meaningless for most students. Developing a robust understanding of equality and of the multiple ways of representing arithmetic facts in the early grades will establish the groundwork for the future learning of more advanced mathematics.

CHAPTER 3: DESIGN OF THE STUDY

Research Design

This research was a mixed-method, quasi-experimental study to develop and pilot a questionnaire that tests both the level of student understanding of symbolic equivalence and to examine the relationships between the results of this questionnaire and students' understanding and success in algebra as measured by their End-of-Course (EOC) exams and their scores on the Idaho State Achievement Test. (Successful completion of the ISAT is a requirement for high school graduation in Idaho.) This study sought to discover if students with a complete, relational equivalence view of the equals symbol, as measured by the developed questionnaire, score significantly higher on ISAT or EOC exams than those who do not. The questionnaire used was derived and adapted from questionnaires developed by Falkner et al. (1999), Freiman and Lee (2004), Stephens (2006), and Hunter (2007).

In addition, the research design included an in-depth interview with the teacher of the students that participated in the study. The first goal of this interview was to determine how one teacher reflected on his teaching for a robust understanding of the equals symbol as he delivered the content contained in a beginning algebra course. A second goal was to see if the teaching goal was successful as reflected by students' scores on the questionnaire.

Participants

The students used for this study were a sample of convenience: a population of 72 ninth-grade algebra students at a local junior-high school. The school is centrally located within a large urban district and is comprised of students from a wide range of socio-economic backgrounds. Of the total 790 students that attend this school, 29 were identified as English language learners, 179 qualified for free or reduced lunches, and 52 were qualified for special-education services. Students were asked to voluntarily fill out the questionnaires during class time and were told that it was not a test, but simply a way to find out how students solve problems. Parents and students signed consent forms and these were collected by the teacher. Data as to whether or not consent was given was not included in the analysis; student papers without consent were not included in these data. The survey was given during May 2009, and required approximately ten minutes of class time.

The teacher was interviewed via an Internet chat to determine a baseline for what activities, specifically focusing on equality and the equals symbol, were accomplished throughout the year. The teacher was also asked about specific instruction given regarding how the equals symbol should be interpreted, and about how he talked about and used the symbol throughout his instruction, discussion, and activities with his students. The researcher had no contact or interaction with the students.

Data Collection

Questionnaires were collected, scored, and correlated to ISAT and EOC exams to determine if students with high levels of symbolic relational equivalence understanding have more success in algebra. The questionnaires involved two sections (see Appendix A). Part 1 consisted of a twelve-question fill-in-the-blank and true-or-false section. Part 2 involved two free-response questions. Part 1 was scored on a twelve-point scale, with each question being marked correct or incorrect. The goal of Part 1 was to measure the students' procedural understanding of the equals sign; could they simply fill in the blanks with numbers to form true statements? Part 2, which focused on discovering each student's conceptual understanding of the equals sign, was also scored using a twelve-point scale using the rubric groupings shown in Tables 2 and 3. These data were then analyzed to determine possible correlations between student scores on the ISAT and EOC exams. Student responses were also grouped into emerging categories found during qualitative analysis and were used to examine trends in student thinking and reasoning.

The second set of data was obtained from an interview with the master teacher after student data were collected. These data were recorded through the format of an Internet chat and were later analyzed from the perspective of a teacher looking at his students' understandings of the equals symbol. The purpose of this interview was to gather data that would allow the researcher to have the master teacher share his thoughts and reflections on his focus throughout the year as he helped his students develop a robust understanding of the equals symbol. In addition, these data would allow for a better interpretation of student responses on the questionnaire through a triangulation process (Glaser & Strauss, 1967).

Research Instruments

Both the Stephens (2006) and the Freiman and Lee (2004) studies mentioned that many errors in student responses were due to computational errors. In both cases, this was a direct result of the use of questions involving three-digit numbers in their symbolic–equivalence statements. Because the purpose of this study was only to measure equivalence understanding as it related to the equals symbol, all of the number sentences utilized one or two-digit numbers to minimize student errors resulting from arithmetic computation.

Best-fit lines were derived from an appropriate computer algebra system, Microsoft Excel (2007), to determine the strength of correlation between both the survey scores and each student's ISAT score, as well as the survey scores and each student's EOC. Categories of responses from Questions 1 and 2 are summarized in Tables 2 and 3. *Question 1 is "Describe why you put the answer you did for #1 above (8 + 4 = ___ + 5). How did you know the answer was correct?" Question 2 is "A friend missed a lot of class time and was having trouble understanding what "=" symbol meant. How would you describe it to help him/her understand?"*

Qualitative data were gathered and recorded during the master-teacher interview through the use of an Internet chat. The interview began with a question asking the master teacher to reflect on his teaching and continued along lines as determined and directed by the teacher’s conversational comments. These data were analyzed and used to help interpret the results of the student data.

The analysis of the qualitative development of student response categories using questionnaire data, and the statistical analysis of the relationships between these categories and the EOC and ISAT scores will be presented in the next sections.

CHAPTER 4: RESULTS

Data Analysis Overview

The data from the questionnaire were analyzed in several ways. It was found that there was almost no correlation between the procedural and conceptual sections, $R^2 = 0.09$. The quality of student responses to the 12 procedural questions in Part 1 had no bearing on whether or not students responded correctly in Part 2. A strong majority of students (79%), labeled for this study as Group A, were able to answer all of the Part 1 questions correctly. They were able to demonstrate, at the very least, that they had been exposed to equations of this form and could structure true mathematically equivalent statements. This was surprising due to the fact that previous studies had not shown anywhere near this high a level of proficiency. However, this ability did not correlate into an ability to demonstrate, on Part 2, a robust knowledge and understanding of the equals symbol. This lack of correlation required that the procedural and conceptual scores be separated before being compared with the students' testing data. The correlation coefficients between testing data along with the scoring rubrics for the conceptual questions numbered 1 and 2 are shown in Tables 1, 2, 3, and 4, respectively. A strong correlation was not found between students' scores on either the procedural or conceptual sections and their corresponding ISAT or EOC scores.

Table 1

Summary of Correlation Coefficients

Categories Compared	R^2
Procedural with Conceptual	0.087
Procedural with ISAT	0.37
Procedural with EOC	0.14
Conceptual with ISAT	0.10
Conceptual with EOC	0.33
Conceptual Question 1 with Conceptual Question 2	-0.17

The Teacher

The students involved in this study had the benefit of working directly with a highly qualified master teacher named Mike (pseudonym). Through this researcher's interaction with Mike over the last ten years, personal knowledge of Mike's teaching experience and abilities, and through the teacher interview conducted after the data were collected, it was evident that he had worked diligently to help his students achieve a high level of understanding in each of the concepts he taught to his students. His classroom, demeanor, and attitude towards his students is inviting, warm, and focused on providing his students with a comfortable, nurturing environment in which to learn. At the time of this study, Mike was concluding his 13th year as a classroom teacher. During these 13 years, he has taught mathematics courses ranging from beginning algebra to advanced placement courses, teaching students in grades 7-12. He is a widely respected professional among his colleagues and possesses a genuine, heart-felt concern for his students' overall well-being and growth. In regards to the topic of this study, it is important to note that Mike has a correct view of how the equals symbol should be viewed and utilized in mathematics in general. He stated, "The equals symbol is a way to show that two quantities have the same value or that there is the same amount of 'stuff' when comparing two things." He also made it clear how aware he was of the importance of students having an appropriate understanding of the equals symbol. He said, "The idea of equality is a very important concept for students to understand if they are going to be successful in algebra."

Methods of Analysis

A qualitative analysis was performed on Part 2 of the questionnaire to develop a categorical breakdown of the different student-response types that would lead to a better interpretation of student understanding. After the data were collected, common themes, or core theoretical concepts, were identified from the student responses. The most obvious method for determining common themes or concepts in the student responses was to identify similar words or phrases used by the students. Once those were identified, similar patterns in student responses were easy to identify. This qualitative method of gathering data, identifying, and sorting the data based on key "codes" or common themes, and then drawing a conclusion or theory based on how the data presents itself, is referred to as Grounded Theory (Glaser & Strauss, 1967).

As mentioned above, Group A was identified as those students that were able to complete all twelve questions of Part 1 correctly. Consequently, Group B consists of the students that missed at least one of the procedural questions in Part 1. This was determined to be the most logical distinction, due to the fact that most students were able to complete Part 1 with absolute competence.

The sorting of the data for the conceptual part of the questionnaire (Part 2) began with an analysis of each question independently. Each of the questions was sorted based on the commonality of student responses. The number of students in each category is summarized in Tables 2 and 3.

Question 1. Describe Why You Put the Answer You Did For (8 + 4 = ___ + 5).

Each of the student responses fits into one of six categories. The categories were awarded points (1 through 6, respectively) based on an increasing level of demonstrated knowledge of equivalence. The categories were: 1) Used mental math, 2) Used algebra, 3) Said it was "the only one," 4) Incomplete discussion of sameness, 5) Complete discussion of sameness, and 6) High number sense. One student did not answer the question and was awarded no points for question one.

Category 1: Used mental math.

Six of the seventy-two respondents wrote their justification of how to complete the problem as some form of mental arithmetic. The responses included statements like, "I just knew the answer," or "mental math." Although they were able to correctly fill in the blank, they were unable to identify or articulate a solid rationale or explanation as to why their choice must be correct. This was interpreted as having a lack of solid understanding in the fact that the mathematical sentence (8 + 4 = ___ + 5) represented an equivalence statement. These six responses were scored one point.

Category 2: Used algebra.

This category was the second largest category of student responses. It was evident that these students had learned, or had been conditioned to believe that they needed to apply some sort of algebraic procedure, perhaps because this was an *algebra* class. And, although a correct response can definitely be found by using a well-known algebraic procedure, the students were still not able to articulate why the number seven had to be the missing number to make the statement true (mathematically equivalent). Almost every response grouped into category 2 included the word “subtraction.” Most responses stated exactly how they would need to “subtract five” from both sides of the equation, or “subtract five from 12” to discover the correct answer. Clearly this is correct, but does not answer the question of “why your answer is correct” (which would need to include a discussion about equality). It simply states “how” the answer was derived. The 24 respondents in Category 2 were scored two points.

Category 3: The only one.

The three respondents in this category did not give a detailed description of why they knew the answer was correct, but they did begin to cross the line from procedure to reasoning. At first it might seem that a response like, “it’s the only one that fits” should have been placed in Category 1; however, it indirectly shows that the student knows there is a reason why it must be true, even though they might not be able to articulate what that reason is. The other two students responded with, “it made it equal.” Once again, the students are demonstrating that they know there is a reason why, that extends beyond a procedural-step explanation, even though they were not able to articulate it.

Category 4: Incomplete discussion of sameness.

The students in this category began to demonstrate a working knowledge that the statement represented sameness of two values. The only reason why these students were not grouped along with those in Category 5 is because of their inability to communicate their thoughts effectively. It was therefore impossible to determine if there existed a lack of complete understanding or just simply a lack of ability to communicate effectively through written language. The student responses were characterized by statements like, “both are 12,” or “the two have to be 12.” It was inferred that the students meant to say both sides of the equation had to be 12; however, the more complete responses were grouped into Category 5.

Category 5: Complete discussion of sameness.

All of the 26 responses in this category (the largest category) made clear statements demonstrating a realization that the values on each side of the equals symbol had to have the same value. An example of one such response is, “Because 8+4 is 12 and to make both sides of the equation equal, you would need to add 7 to the 5 to make the answer 12 and the statement true.” Some students simply showed two separate equations: “8+4=12, therefore 7+5=12…8+4 = 7+5, and 12=12.”

Because the teacher was aware of how students struggle with solving equations due to a lack of understanding of what an equivalent statement represents, it is not surprising that so many students were able to write accurate descriptions of sameness. Mike mentioned, “I see students struggle with making sure they are ‘fair’ to both sides of an equation.” The teacher’s knowledge of this led to his students being exposed to a variety of activities to help them to better grasp this concept. He said, “When solving equations, I talk about the idea of a scale. The expressions on each side of the equation have the same value, just like a balanced scale.” These discussions, which communicated the need to keep each side of an equation the same, made a positive impact on how his students viewed equivalent statements.

Category 6: High number sense.

Students in this category demonstrated not only a clear understanding of symbolic equivalence, but were also able to clearly articulate a higher than “normal” level of number sense. Without performing an algebraic procedure or computing an arithmetic sum of both sides of the equals sign, these seven students were able to view the entire number sentence as one entity and describe the relationships between the numbers. One student wrote, “since 8 + 4 = ___ + 5, I know that 1 got added to the 4 to make 5. So that means I must subtract 1 from 8 to get 7.” This description was interpreted as an above average understanding of the numbers and values in an equivalence statement. This response was 1 out of 7 in this category, out of the total 72 responses.

In regard to this understanding, Mike commented, “I…emphasize that whatever step is taken to alter the left side of the equation, must be done to the other as well. Also, when simplifying expressions with variables, I like to have students pick several random values…so students can see that the simplified form is always equal to the original expression." These activities that allow students to simplify expressions, work with various forms of expressions, and relate the new expression back to the original expression, clearly gave these students a more in-depth understanding of how to work with, and interpret, mathematically equivalent statements. Mike's awareness of how students can struggle with symbolic equivalence allowed him to weave appropriate uses and activities into his instruction. In this researcher's opinion, his awareness and instruction is the prevailing reason why his students performed so well on this type of questionnaire. Nearly 53% of the students were grouped into categories 4, 5, and 6. The most common student response for question one (with 26 out of 72, or 36% fitting this grouping) made reference to "both sides having to equal 12." This was considered an appropriate understanding of the relational equivalence shown in the number sentence. The next most common response involved the student describing an algebraic process that was needed to solve for the missing number. This was made clear by the student referencing a need to "subtract" from one side to obtain the correct answer. Although this process does yield a correct answer, it demonstrated the student's need to solve something via a known procedure when seeing the equals symbol involved in a number sentence, instead of simply discussing the equivalent relationship shown.

Table 2

Question 1: Describe Why You Put the Answer You Did For (8 + 4 = ___ + 5).

Categories based on common student responses	*N*	Scoring Value Applied
0. No understandable response given	1	0
1. Used mental math	6	1
2. Used algebra	24	2
3. The only one	3	3
4. Incomplete discussion of sameness	5	4
5. Complete discussion of sameness	26	5
6. High number sense	7	6

Question 2: Describe to a Friend the Meaning of the Equals Symbol.

The student responses for the second question in the conceptual part of the questionnaire (Part 2) were also sorted and coded based on the commonality of responses. Once again, the categories were awarded points (1 through 6, respectively) based on an increasing level of demonstrated knowledge of equivalence. Five students did not respond, and one student responded with, "I don't know." These responses were awarded no points in the scoring rubric. The six categories were, 1) The answer equals or is, 2) What goes between the answer and the equation, 3) Double response, 4) Scale and balance, or it means the same or equals *after* you do something to each side, 5) Both sides are the same or equivalent, and 6) Complete discussion of equivalent values.

Category 1: "The answer equals or is."

Students in this first category comprised the second largest grouping from Question 2 (18 out of 72, or 25%). Many of the responses were short and nondescript. Most students made simple statements similar to "equals means is," or "the symbols just means equals." Students were not able to provide a meaningful description or definition. They either believed that the question only required a simple answer–because everyone should know what that symbol represents, or they did not have a deep enough understanding to be able to articulate what it represents. It is possible that some students had developed confusion as to when to use the phrase: "the answer." One of the phrases that Mike commented on using in his instruction was, "the answer to."

> As an example, when we are first dealing with equations that can be solved with one or two steps, I like to play a little game called "What's my number?" On the projector screen, I put something like "I'm thinking of one of my favorite numbers. I multiplied the number by 9 and subtracted 7 and got 29. What is my number?" After students work for a few minutes on figuring out the problem, then we work on representing the problem in an equation format. As we talk about the "got" part, I use words like "the answer is" or "the result was." Then we talk about strategies for finding the answer, which leads to discussion of solving equations using inverse operations.

In the above mentioned activity, the example problem might be represented by the equation $9x - 7 = 29$. A discussion about the "answer he got" being 29, might very well have contributed to students believing that the answer always follows the equals symbol. Although there is nothing mathematically incorrect with the statement, students arriving in his classroom with improper understanding of symbolic equivalence might have easily been led off track or improperly reinforced. As the discussion progressed into finding the solution (solving for *x*), and the class began to use words like, “let’s solve this equation,” or “how can we find the answer,” it might have presented an opportunity for some students to sense a contradiction in the vocabulary being used. A student would have heard, originally, that the “answer” was 29. However, once the “favorite number” was discovered, the “answer” now became 4. This possible contradiction in student understanding might have led to such a high number of students using the word “answer” to describe the equals symbol.

Another possible explanation for students using the word “is” to describe the equals symbol is that they had been exposed to a lot of verbal statements within their curriculum that required students to learn techniques for translating sentences into equations. Mike said, “I encourage students to look for key words that indicate the operations involved and key words for where to place the equals sign.” He also mentioned that he discusses common phrasing and looks for verbs to help students break down the sentences. “Chances are the words: costs, ran, was, were, etc., will indicate ‘=’.” Teachers often train students that the word “is” in a word problem will always tell you where to place the equals symbol. And, most of time, that will work as a strategy for translating words into equations. However, it certainly does not foster an understanding of equivalence as these data indicate.

<u>Category 2: What goes between the answer and the equation</u>.

This category included seven students whose responses were more descriptive and detailed; however, they were not indicative of students with a thorough understanding of equivalence. These students seemed to understand that the equals symbol is just the mathematical sign that is placed between the arithmetic work that is to be done, and the answer to that work. One student stated, "…the equals symbol represents the transition to the answer." Another simply said, "you put it between the question and the answer." This student plainly demonstrated, as Cobb (1987) mentions, a need to "do something" when viewing the equals symbol.

Category 3: Double response.

The two students receiving three points for this category began to bridge the gap between the students that could demonstrate knowledge of a proper understanding of symbolic equivalence and those that could not. These students wrote their answers in two parts. Part one would have fit into Category 1 and the second part would have fit into Category 4. These students, given a specific context, might define the equals symbol as meaning, "the result," and in another situation, understand that it means, "the same." The student that wrote, "a result, or it could be described as the same…" was also able to articulate on Question 1 that, "the two equations had to be equivalent to each other." It appears from her response to Question 2 that her initial reaction to the prompt was to say it represents "a result." However, she was able to further articulate a deeper understanding of symbolic relational equivalence. Perhaps this student had been exposed to only one use of the equals symbol in earlier grades, but was now in the process of cognitively transitioning to a better, more complete and flexible understanding.

Category 4: Scale and balance/Do something first.

The ten students awarded 4 points were placed into two separate sub-categories. The first five students all made some reference to the equals symbol being representative of a "scale" or "balance." While this would not be an approved mathematical definition, it demonstrates that these students had moved past viewing the symbol as a procedural directive for the answer to understanding that it is representing some sort of relationship between two quantities. These activities designed by Mike to cultivate an understanding of the equals symbol might have played a key role in developing this comprehension. When asked, "What are some of the common mistakes you see in students with regards to the equals symbol, and what action would you take to correct it?" Mike responded,

> One issue that I see students struggle with is making sure they are 'fair' to both sides of an equation. I like to use the idea of a scale as I said before. One fun way to do this is using a website called the National Library of Virtual Manipulatives (http://nlvm.usu.edu/en/nav/vLibrary.html). They have a great interactive balance with algebra tiles that you can move around and play with on the screen.

The second sub-category included statements that could be summarized by a need to do something first or to have an end result that eventually showed two things are the same. For example, one student wrote, "it signifies you need to reach a numeric equality on either side of the sign." This student understands that there is a relationship between both sides, but believes there is work that must be done first to show this. In other words, an equation like $2+3 = 5$ would show equivalence once you add the 2 and 3. The others had similar responses including using words like, "it turns out to be the same," and "…when reduced, [both sides] would be the same." It might be inferred that these students were implying that both sides were the same, but in order to show that was the case, one would need to perform the arithmetic or operations shown to arrive at a single value. Although this is certainly true, students' comprehension needs to deepen into an understanding that the equals symbol is a declaration of equivalence, not a suggestion or a test for equivalence.

Category 5: Both sides are the same or equivalent.

This grouping of students comprised 27 out of 72, or 38% of the responses to Question 2. These responses were viewed as an appropriate understanding of symbolic relational equivalence. Some of the classroom discussions that Mike had with his students throughout the year were apparently helpful in developing this view of the equals symbol. Many of his students had learned that the right and left-hand side of a mathematical sentence represented the same value. Mike said, "When we are solving equations, I like to draw a vertical dotted line underneath the equals symbol, and emphasize that whatever step is taken to alter the left side of the equation, must be done to the other as well." This distinction of both parts being separate-but-equals was simply and definitively reinforced by his use of the "vertical dotted-line." Most teachers instruct students to "do the same thing to both sides of the equals sign," but Mike's discussions and instruction seem to go a bit further than just making that statement.

> When we are using substitution to solve systems of equations, students will sometimes try to substitute the new expression right next to the original variable instead of taking the variable out and replacing it with the new expression. I like to use different colors of markers or sticky notes on the board to help students see how this works. I also use the example of a substitute teacher. They only come in when I am gone. That is how it works with the equivalent expressions that are being substituted for one another.

He was able to affect different learning styles by using a variety of approaches, giving students visual, written, and mental images to help scaffold a better understanding of equivalence.

<u>Category 6. Complete discussion of equivalent values</u>.

There were only two students in this category. These two students were able to articulate that the equals symbol represents mathematical equivalence. Once student wrote, "…the stuff on the left side of the = symbol has the same value as the stuff on the right." The other wrote, "…they have the same value. For example if you owned a car, and your friend owned a car of the same value, their price would be the same, or equals (=)." It was refreshing to notice that these students made clear mention of equivalent values. Improper uses of the equals symbol, written in various contexts, are very prevalent in our society. To depict the above mentioned student's example, one might have improperly written: "car 1 = car 2"; however, that would be incorrect. The cars themselves are not equal. The student's statement that their "values" had to be equal is correct. Mathematically, we would write: If the value of car 1 is equal to the value of car 2, then the cars' values are the same. If a represents the value of car 1, and b represents the value of car 2, then $a = b$. This is an example of the quality and level of understanding (and communication) that all students need to achieve to be successful in secondary mathematics.

Table 3

Question 2: How Would You Describe the Equals Symbol to a Friend?

Categories based on common student responses	*N*	Scoring Value Applied
No understandable response given	5	0
Did not know	1	0
The “answer”	9	1
“Equals”	7	1
“Is”	2	1
What goes “between” the equation and the answer	7	2
Double response: “the answer” and “the same”	2	3
“Scale” or “balance”	5	4
It means the same or equals *after* you do something to each side (add, subtract, or reduce)	5	4
Both sides the “same” or “equivalent”	27	5
Discussion of equivalent values and/or amounts	2	6

Summary of Analysis

As shown in Table 4, TTESTs were also calculated to determine if there were significant differences between the mean ISAT and EOC scores based on the questionnaire. Since 79% of the students got all 12 procedural questions correct (Group A), the first grouping was created to separate those students who missed at least one question (Group B). The mean ISAT scores of Groups A and B was 245 and 239, respectively. Although this difference would place Group B students outside the proficient status as defined by the State Board of Education, it was not found to be a statistically significant difference (p = .052). Group A's and Group B's mean scores on the EOC were 77% and 72%, respectively, and were also not statistically significant (p = .22). This implies that the procedural portion of this questionnaire would not be a good predictor of success in the course, as measured by the EOC or on the ISAT. For the students in this study, they were generally able to complete most, if not all, of these questions correctly. Either they had been exposed to equivalent statements of this form before, or they had been given enough experience with equivalent statements and equations to be able to complete them and perform the arithmetic necessary to make true statements. For this group of students, this set of simple fill-in-the-blank questions would not serve as a good baseline for determining student understanding of symbolic equivalence.

The mean comparisons on the conceptual section of the survey did show statistically significant differences in the following areas. Groups C and D were compared based on whether or not, in this researcher's opinion, their responses were correct, albeit not necessarily complete. Group C contains those students who scored at least a total score of seven and Group D all those that scored six or less. The mean ISAT scores of Groups C and D were statistically significantly different at 247 and 241 (p = .03). The same was true of the EOC scores of groups C and D at 80% and 72%, respectively (p = .03). This implies that students who were placed in Category 4 or higher in at least one of conceptual questions showed significantly better overall success in algebra as measured by the EOC and the ISAT.

Table 4

Summary of TTESTs

Categories Compared	*p*
EOC of Groups A and B	0.218
ISAT of Group A and B	0.052
EOC of Groups C and D	0.027
ISAT of Groups C and D	0.028

These groupings in each question seem to follow the same pattern of reasoning and understanding that students have demonstrated in previous studies mentioned in the literature. However, it was not true in this study that students with an appropriate level of understanding, demonstrated by their responses to Question 1, also had an appropriate response to Question 2. The correlation coefficient found between the two questions was actually slightly negative at R^2 = -0.17. Therefore, a student's ability to articulate why a particular number is the solution needed to form an equivalent statement does not necessarily mean that student understands, or can articulate, the meaning of the equals symbol. In fact, fully two-thirds of the students in Category 5, Question 2 (who stated the equals sign means "the same") fell into Category 1 or 2 on Question 1. In other words, they were able to define or describe an appropriate meaning for the equals symbol, but could not articulate a valid reason why a particular number was the only solution to a mathematical sentence that maintained equality. Likewise, 60% of the students who fell into Category 5 on Question 1 (complete discussion of sameness) were unable to articulate an appropriate definition or description for the proper use of the equals symbol and were placed in Categories 1 or 2 on Question 2. In summary, this study has demonstrated that students who can define or articulate an appropriate meaning for the equals symbol may not necessarily be able to demonstrate a working knowledge or understanding of the symbol's appropriate uses.

These students may have simply been imitating the words from the teacher. It is possible that they were using "the same as" without understanding, in the same way that they had previously been using "equals." They may be at a very early stage of developing a flexible understanding of the symbol.

CHAPTER 5: RESULTS

Discussion

The first goal of this study was to discover if success in an algebra course, as measured by an End-of-Course test and the State Achievement Test, could be predicted by a student's level of understanding of symbolic relational equivalence. These data suggests that there is not a correlation between the questionnaire's two sections based on previous research on understanding the equals symbol. However, as a group, those that scored well on the conceptual part of the questionnaire (scored at least in Category 4 or higher on one of the questions) did score significantly higher on both the EOC and the ISAT. Therefore, a conceptual questionnaire similar to the one given in Part 2 would provide a valuable insight to a teacher as to the predicted success of her students. At the very least, it would give teachers a starting point and an invaluable awareness of the level of understanding their students have. This knowledge is required for teachers to be able to design meaningful, individualized activities that meet the needs of each group of students.

The second goal of this study was to determine if the newly developed questionnaire would be a worthwhile tool for teachers to use to help guide them as to their students' level of understanding of this topic. Because of the lack of a strong correlation, further investigation, perhaps with different groups of students, would be needed to make this determination. Student responses to Question 2, which asked students to describe their own definition of the equals symbol, turned out to be the most informative, as it clearly showed which students believe that the symbol is used to request an answer. This question would definitely be a good starting point for teachers to use in their analysis and evaluation of their students' levels of understanding as they prepared for and delivered curricula designed to assist students to develop a robust understanding of the equals symbol.

Contributing Factors and Limitations

This study involved a student population of adequate size; however, all students in this study had the same mathematics teacher. Further studies might benefit by testing a larger population of students with a variety of educational backgrounds and teachers. The teacher in this study was a teacher who, by most every standard would be considered a master teacher, had set a personal goal of teaching for a deeper, more conceptual level of understanding for his students throughout the curriculum. This was evident in the discussions this researcher had regarding his instructional methods and was apparent in the higher percentages of students demonstrating an appropriate conceptual understanding of symbolic relational equivalence than reported in previous studies described in the review of the literature. The important role of a good teacher cannot be overstated, as demonstrated by the high number of students in this study that had a solid understanding of this topic.

From the observations in this study, it is clear that these students had been exposed to an appropriate modeling of how the equals symbol should be used and viewed. This teacher had a keen understanding of the importance of symbolic equivalence comprehension as a necessity for success in algebra. This is further demonstrated by the activities in which his students had participated. Not only were his students involved in enriching activities, specifically designed to facilitate proper understanding, but they were consistently exposed to alternate terms used to express equality, "the same as" and "a balance." Throughout his instructional practices, this teacher seemed to be aware of some of the ways his students struggled and was therefore better able to meet each student's individual learning needs with regards to this area of study.

Conclusion

When faced with the need to interpret the equals symbol, students in many previous studies have demonstrated a desire to write an answer. This, "do something when you see that symbol" understanding can lead to difficulties in understanding more abstract concepts that begin in the formal study of algebra. Students who do not acquire a grasp of the equivalence the symbol represents might never be able to read, perform, manipulate, or solve any number of relational equivalent expressions seen throughout mathematics and science. An example of this type of student is the one that partially motivated this study, and I described her at the beginning of this thesis. This study sought to develop a measurement that might be helpful for teachers to highlight deficiencies in student comprehension that might lead to difficulties in understanding of algebraic concepts and to identify students with these difficulties early in their study of mathematics.

From the students that were surveyed, it was clear that many had an appropriate conceptual understanding of the equals symbol and how it should be properly used. It was also shown that many students could perform computations involving equivalent statements but could not necessarily explain or demonstrate a true and complete understanding of how "=" should be interpreted. Although the goal of creating the survey instrument for this study was to develop a test for teachers to use to assess their students and predict success in algebra, it was not found to be an effective tool exclusively, as no correlations were found between the survey scores and student testing data. However, there are two positive outcomes to be summarized.

First, the student responses to the conceptual part of the survey clearly demonstrated which students did not have a clear understanding of symbolic relational equivalence. Any student who responded to Question 2 with, "it means the answer," would clearly need to be exposed to more activities that continued to enrich and develop their understanding of symbolic equivalence. In this thesis, it was evident that these types of questions can be used by algebra teachers to assess their students' levels of understanding of "=" as they begin to teach more advanced mathematical concepts. Knowledge of this key area of understanding, or mathematical obstacle, would be invaluable for the teacher to know as he sought to design meaningful instructional activities.

The second lesson learned from this study is that of the importance of a quality, knowledgeable instructor. Evidently, Mike had an acute awareness of the knowledge students needed to be successful in understanding mathematics, especially that of the equals symbol. He worked at designing activities that focused on enriching each student's proper understanding of equality. When he was made aware of a student that needed further guidance, he dealt with them appropriately. These quality instructional practices contributed to his students performing at a seemingly higher level, with respect to symbolic relational equivalence, than those shown in previous studies. If all students were presented with such opportunities and shown specifically how the equals symbol should be interpreted and used, as opposed to the many improper uses seen in the world around us (and even sometimes in our classrooms), it is not a stretch to imagine that all students would struggle less and find more success in algebra.

REFERENCES

Behr, M., Erlwanger, S., & Nichols, E. (1980). How children view the equals sign. *Mathematics Teaching, 92,* 13-15.

Bickmore-Brand, J. (1990). *Language in Mathematics*. Portsmouth, NH: Heinemann Publishing.

Blair, L. (2003, December 1). It's elementary: Introducing algebraic thinking before high School. Southwest Educational Development Laboratory, 15, No. 1, Article 5. Retrieved March 10, 2009, from www.sedl.org/pubs/sedl-letter/v15n01/5.html

Boaler, J., & Humphreys, C. (2005). *Connecting mathematical ideas: middle school video cases to support teaching and learning.* Portsmouth, NH: Heinemann Publishing.

Carpenter, T., Franke, M., & Levi, L. (2003). *Thinking mathematically: integrating arithmetic and algebra in elementary school*. Portsmouth, NH: Heinemann Publishing.

Cobb, P. (1987). An investigation of young children's academic arithmetic contexts. *Educational Studies in Mathematics, 18,* 109-124.

Falkner, K., Levi, L. & Carpenter, T. (1999). Children's understanding of equality: a foundation for algebra. *Teaching Children Mathematics, 5,* 232-236

Freiman, V., & Lee, L. (2004). Tracking primary student's understanding of the equality sign. *Proceedings from the 28th Conference of the International Group for the Psychology of Mathematics Education* (pp. 415-422). Kirstenhof, Cape Town, South Africa: International Group for the Psychology of Mathematics Education.

Glaser, Barney G & Strauss, Anselm L. (1967). *The Discovery of Grounded Theory: Strategies for Qualitative Research*. Chicago, IL: Aldine Publishing Company.

Gray, E., & Tall, D. (1992). Success and failure in mathematics: the flexible meaning of symbols as process and concept. *Mathematics Teaching, 142,* 6-10.

Gray, E., & Tall, D. (2007). Abstraction as a natural process. *Mathematics Educational Research Journal, 19,* 23-40.

Hunter, J. (2007). Relational or calculational thinking: students solving open number equivalence problems. *Mathematics: Essential Research, Essential Practice, 1,* 421-429.

Knuth, E., Stephens, A., McNeil, N., & Alibabi, M. (2006). Does understanding the equal sign matter: evidence from solving equations. *Journal for Research in Mathematics Educations, 37,* 297-312.

National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics.* Reston, VA: Author.

Rivera, F. (2006). Changing the face of arithmetic: teaching children algebra. *Teaching Children Mathematics, 12*, 306-311.

Saenz-Ludlow, A. & Walgamuth, C. (1998). Third graders' interpretation of equality and the equal symbol. *Educational Studies in Mathematics, 35,* 153-187.

Steinbring, H., Bartolini-Bussi, M., & Sierpinska, A. (1998). *Language and communication in the mathematics classroom.* Reston, VA: National Council of Teachers of Mathematics.

Stephens, M. (2006). Describing and Exploring the Power of Relational Thinking. In P. Grootenboer, R. Zevenberger & M. Chinnappan (Eds.), *Proceedings of the 29th annual conference of the MERGA* (pp. 479-486). Canberra: MERGA.

Warren, E. (2003). The role of arithmetic structure in the transition from arithmetic to algebra. *Mathematics Educational Research Journal, 15,* 122-137.

Witherspoon, M. (1999). And the answer is…symbolic literacy. *Teaching Children Mathematics, 5,* 396.

APPENDIX

Survey Instrument

Thank you for completing this survey. Please respond to the best of your ability.

Fill in the blank.

1. 8 + 4 = ___ + 5
2. 6 + 3 = 2 + ____
3. 7 = 4 + ___
4. 3 + ___ = 7 + 6
5. 10 = 9 + ___
6. 6 + 8 = ___ + 5
7. 4 + 7 = 5 + ____
8. 9 + ___ = 8 + 3

True or False.

9. 3 + 5 = 8
10. 6 = 4 + 2
11. 26 + 34 = 25 + 35
12. 12 = 12

Please answer the following questions.

Describe why you put the answer you did for question #1. How did you decide on your response?

A friend of yours missed a lot of class time and was having trouble understanding what “=” symbol meant. How would you describe it to help him/her understand?