



## FOOTFALL-INDUCED VIBRATION: PREDICTION AND CONTROL STRATEGIES

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### ABSTRACT

Footfalls can produce vibrations that are irritating to occupants, or disruptive to vibration-sensitive spaces in hospitals and research laboratories. This paper describes, using case studies, techniques that can be used to predict vibrations, and mitigation strategies to control them. Firstly, pedestrian loading on footbridges is described. Stochastic simulations are conducted in which a number of pedestrians with random weights, walking speeds, and walking paths cross a bridge with random spacing. The resulting vibration response of the bridge is determined to estimate the peak acceleration of the bridge. For the case considered, tuned mass dampers (TMDs) were subsequently installed on the bridge to increase the effective damping of two modes. Next, crowd loading on stadia and ballroom structures is considered. For these structures, the crowd-structure interaction must be considered since the mass of the crowd is a significant fraction of the mass of the structure, which alters the dynamic response of the system. In the case presented, TMDs are considered to reduce the vibrations associated with crowd excitation. Lastly, the vibrations associated with a pedestrian walking in a corridor adjacent to a vibration-sensitive room in a hospital or research facility are described. Maintaining low vibrations in these hospitals and laboratories is critical as the operation of equipment may be adversely affected by vibrations that are well below the threshold of human perception. Typical mitigation strategies for these vibrations include positioning sensitive equipment near columns or increasing the stiffness or mass of the floor.

Keywords: pedestrian-induced vibration, crowd loading, structural control, sensitive equipment

### 1. INTRODUCTION

As structures become lighter, and more structurally efficient, they have become increasingly susceptible to excessive dynamic motion due to external force excitation. For many structures, the governing source of dynamic excitation is occupant footfalls. This dynamic excitation may range from a single pedestrian walking in a corridor near a laboratory, to a group of walkers or runners on a pedestrian bridge, to a large crowd moving synchronously to a musical beat in a grandstand or ballroom. While in each case the structure is typically represented using modal analysis, there are considerable differences in terms of how the pedestrian loading is applied, and how the pedestrians may interact with the structure. This study will discuss the unique treatment of the analysis for each type of structure, including non-linear lock-in behaviour, and crowd-structure interaction.

In recent years, there has been an increase in the amount of research focusing on human-induced vibration in structures. Detailed reviews of much of this research are found in Zivanovic et al. (2005) and Racic et al. (2009). The reason for this increased interest may be partially due to high-profile structures that have been perceived to be “lively”. In addition, advances in imaging technology have resulted in equipment that is sensitive to imperceptible floor vibrations that produce image distortions. As vibration-sensitive facilities continue to be constructed, and as

structures become more lightweight and flexible, the need to understand the impact of vibration on structures will increase.

Many researchers have measured footfall forces during walking, running, bobbing, and jumping activities using force plates (Racic et al. 2009). For each type of activity, dynamic load factors are empirically determined as the Fourier coefficients of the measured footfall forces. The dynamic load factors are usually dependent upon the step frequency, and may be modified in the case of crowds to account for the correlation between individuals (Joint Working Group 2008). The footfall forces can be applied to the structure using either time-domain or frequency-domain analysis techniques.

In this study, methods to predict the human-induced vibration of bridges, stadia grandstands and ballrooms, and sensitive floors are described. A detailed review of the literature is not the purpose of this paper; rather, its purpose is to demonstrate why different types of structures require unique analysis methodologies. Relevant vibration criteria are discussed for each type of structure. Since structures that do not achieve the relevant criteria must be modified, common mitigation techniques for each type of structure are described. These mitigation approaches range from increasing the mass or stiffness of the structure, to incorporating tuned mass damper (TMD) systems. Case studies are used as examples of the analysis and mitigation techniques for each type of structure.

## **2. PEDESTRIAN BRIDGES**

### **4.1 Background**

Pedestrian bridges have several properties that make them highly susceptible to many forms of dynamic excitation. These structures are typically very lightweight, with many having deck masses of only a few tonnes per metre length. In addition, there are often numerous natural frequencies that are less than 4 Hz, which correspond to the frequencies at which people will naturally walk, run, or jump (Matsumoto 1974). Lastly, bridges often possess very low inherent damping – it is common to see damping ratios less than 1% - which results in an extreme amplification of dynamic loading if applied near a resonant frequency. These structural properties ensure that many cable-stayed pedestrian bridges are easily excited by users.

The modal properties of the bridge may be used to predict the dynamic response. It is necessary to determine which modes are most susceptible to pedestrian excitation. Only modes whose shapes are conducive to pedestrian excitation need to be considered for pedestrian comfort. Therefore, modes that do not involve motion of the bridge deck, but are instead limited to motion of the towers, piers, or cables may be neglected. Secondly, deck modes with natural frequencies outside the frequency range of human walking, running, and jumping activities (or their higher harmonics) can be discarded. While people generally walk at frequencies between 1.5 Hz and 2.5 Hz (Matsumoto 1974), they can run or jump at frequencies approaching 4 Hz (Rainer et al. 1988). The literature suggests that natural frequencies as high as 10 Hz may be excited by the higher harmonics of the footfall loading, although it is unlikely that higher harmonics can be well correlated (Rainer et al. 1988, Stoyanoff et al. 2007).

Stochastic simulations are conducted to represent the variability of the individual walkers. To accomplish this, a crowd of walkers is generated, where the properties of the individuals follow a Gaussian distribution with a predefined mean and standard deviation. Tens of thousands of walkers are generated with random weights, walking frequencies, step phases, and step lengths. These individuals in this long queue then step onto the bridge at random time intervals and at random distances from the centre line of the bridge.

A nonlinear feedback mechanism, called “pedestrian lock-in”, has been observed on dynamically active bridges, most famously on the Millenium Bridge in London, UK. As individuals walk, they shift their weight from foot to foot. This shifting of weight creates a lateral force on the bridge deck. If the bridge has a lateral frequency that is half the stepping frequency, then the bridge will begin to sway laterally. Above a threshold of approximately 1%g (Joint Research Centre 2009), pedestrians must adjust their gait to maintain their balance. As multiple pedestrians alter their locomotion, they become highly correlated, and they “lock-in” to the motion of the structure, driving the response still further through a positive feedback phenomenon. The resulting lateral motion is considerably amplified due to this nonlinear lock-in mechanism. Lock-in has been primarily observed for lateral motion, however some researchers have suggested that vertical lock-in is also possible. This research suggests vertical lock-in can occur at a much higher

displacement of at least 10 mm, which is approximately 15%g for a natural frequency of 2 Hz (Joint Research Centre 2009).

There are no universally accepted vibration criteria for footbridges. For this reason, the design team and owner must agree upon a reasonable criterion with consideration to the costs that might be associated with achieving a high level of user comfort. As shown in Table 1, SETRA (SETRA 2006) provides several ranges of comfort criteria that can be discussed with the owner. Several other sources recommend frequency-dependent criteria, where higher accelerations are permitted at higher frequencies (Zivanovic et al. 2005).

Table 1: SETRA footbridge comfort levels (SETRA 2006)

Comfort Level	Lateral (%g)	Vertical (%g)
Maximum	<1.5	<5
Mean	<3	<10
Minimum	<8	<25
Unacceptable	>8	>25

The selection of appropriate pedestrian loading scenarios to be considered is as critical as the criteria selection. The anticipated density of pedestrians walking or running on the bridge must be taken into consideration. For example, if the bridge may be part of a marathon route, very high densities of runners may be expected, particularly if the bridge is located near the start line. Similarly, large numbers of walkers may be anticipated if the bridge is an exit route for a sporting event or fireworks show. If an event occurs infrequently (as is typically the case for fireworks), the owner may choose to apply a more relaxed criterion for these extreme loading scenarios.

#### 4.2 Case Study – Pedestrian Bridge

A light-weight pedestrian bridge located in Ontario is considered. The bridge is cable-supported, and consists of two spans with a total length of approximately 75 m. The engineering design was conducted by Parsons. There are two vertical deck modes with frequencies close to common walking and running frequencies. Since the inherent damping ratio of the bridge was uncertain, two levels of damping were considered: one represents a conservative level (0.3%), and the second represents a typical level (0.7%). Simulations conducted using 0.3% damping predicted that as few as four runners could produce acceleration amplitudes that exceeded SETRA’s minimum acceptable criteria for vertical vibrations. If 0.7% damping was assumed, approximately 15 runners were required to exceed SETRA’s minimum criteria. The project team decided that a supplemental damping system should be incorporated into the deck, since this would otherwise have been a very lively bridge.

A set of TMDs was employed to increase the effective damping of the bridge, and thereby reduce the vibration levels. The webs of the bridge girders were modified to include bolt patterns to which the TMDs could be installed. As shown in Figure 1, the TMD consisted of two steel masses that are connected via leaf springs to a plate that is mounted to the bridge girder. Another TMD unit was mounted to the other side of the bridge to target a second vibration mode. The leaf springs allow the masses to oscillate vertically, while a viscous damping device (VDD) is attached to each mass to increase energy dissipation.

Prior to the TMDs being installed, vibration measurements were conducted on the nearly-completed bridge to determine the as-built frequencies and mode shapes of the bridge deck. These measurements enabled the stiffness and damping of the TMDs to be optimized in the shop before installation. The TMD stiffness was adjusted by varying the distance between the mass and fixed support. The TMD damping was adjusted by changing the orientation angle of the VDDs.

During the frequency measurements, two bridge modes were identified as being easily excited by pedestrians. The first mode was approximately 2 Hz, and could easily be excited by walking; the second mode was approximately 3.4 Hz, and could be excited by running. Therefore, one TMD was tuned to the 2 Hz mode, and installed on the bridge close to its position of maximum modal displacement, which was close to the mid-span of the bridge. The TMD on the opposite side of the bridge was tuned to the 3.4 Hz vibration mode, and was installed on the bridge close to its position of maximum modal displacement, which was closer to the bridge tower.

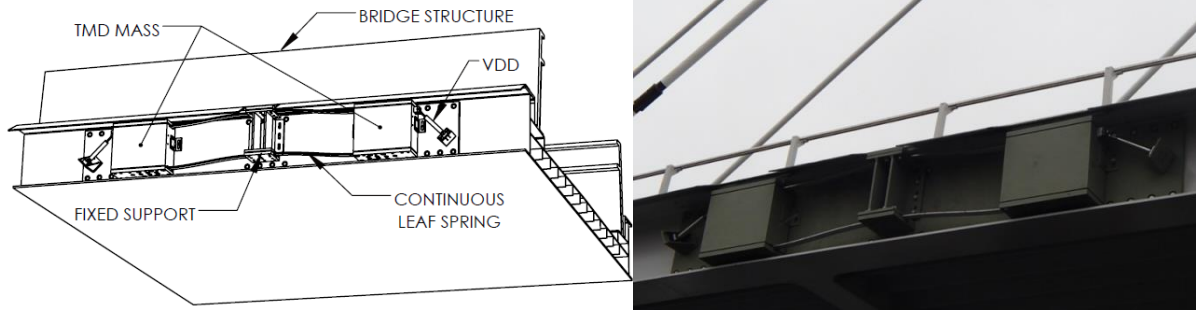


Figure 1: TMD installed on bridge girder

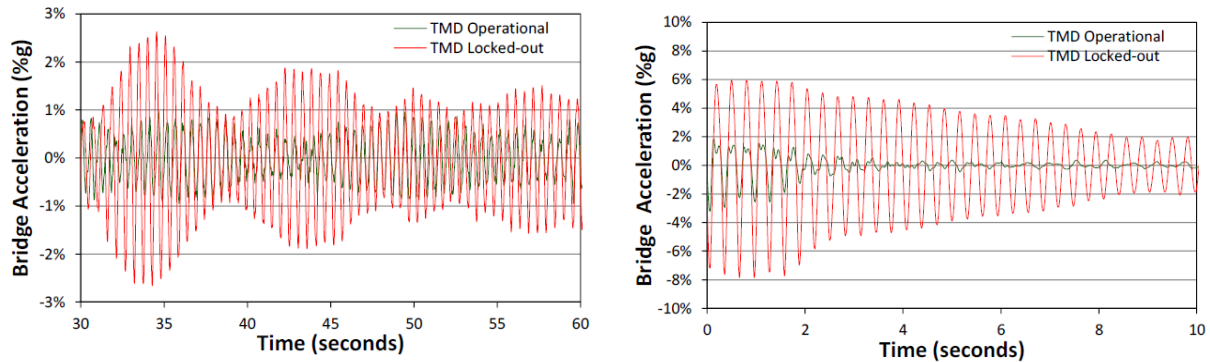


Figure 2: Bridge acceleration with and without TMDs operations, 2 Hz (left) and 3.4 Hz (right)

To verify performance, vibration measurements were conducted with the TMDs operational and inactive. Several running, walking, and jumping scenarios were conducted, using a metronome to target specific frequencies. Figure 2 shows a comparison of the vibration measurements with and without the TMDs operational for the two modes targeted. The response amplitude of both modes was reduced by approximately 50%.

### 3. CROWD LOADING

#### 4.1 Background

Stadia, grandstands and ballrooms require modified vibration analysis since the mass of the crowd is often significant relative to the mass of the structure. As a result, the dynamic interaction between the crowd and structure must be considered, which necessitates that the crowd-structure system is modelled as a coupled multi-degree of freedom system (Dougill et al. 2006, Jones et al. 2011, Joint Working Group 2008). It is often useful to represent the crowd as two degrees of freedom: one representing active participants, and the other representing passive members who do not participate in the rhythmic activity (Joint Working Group 2008). If a single structural vibration mode is considered, the crowd-structure system is represented as the three-degree-of-freedom system as shown in Figure 3, where  $m$ ,  $k$ , and  $c$  represent the effective mass, stiffness, and damping constant of each degree of freedom, respectively. For the active crowd, an internal force pair,  $P(t)$  is applied between the active crowd mass and the structural mass, which represents the dynamic force produced by the crowd.

The natural frequency and damping properties of the crowd have been found to be relatively invariant with group size (Dougill et al. 2006, Joint Working Group 2008). The active and passive (seated) crowd natural frequencies are 2.3 Hz, and 5 Hz, respectively, while the active and passive crowd damping ratios are 25% and 40% of critical, respectively. As a result, when crowd and structural natural frequencies are similar, the crowd-structure interaction dissipates a large amount of energy that is unaccounted for by simpler analysis techniques.

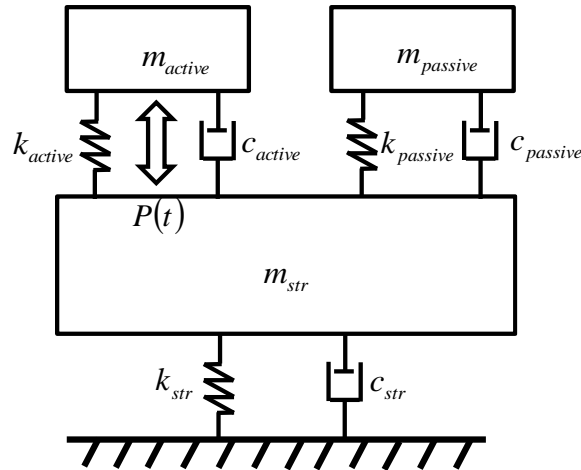


Figure 3: Model of crowd-structure interaction

The structural response is determined by modelling the crowd-structure system either numerically using time domain simulations, or analytically by solving the three degree-of-freedom system directly if the structural modes can be evaluated independently. When the structural frequency is close to the crowd frequencies, there is considerable interaction between the crowd and structure. Since the crowd has high damping, the effective damping of the structure may be significantly increased by this interaction. Measurements of grandstands have suggested that the accelerations predicted using this method are more accurate compared to the conservative predictions obtained if the simplified analysis procedures given in design guides are used (Pavic and Reynolds 2008, Jones et al. 2011). Simplified procedures often resort to representing the structure as a single-degree-of-freedom with unrealistically high damping (often 6%) to accommodate for the missing energy dissipation of the participants (Murray et al. 1997).

#### 4.2 Case Study – Ballroom

This case study considers a large ballroom located in the southern United States, which may be occupied by as many as 4500 people at one time. The ballroom is 73 m long, and spans approximately 36 m using 4.2 m deep trusses. Finite element modelling predicted that the fundamental frequency would be between 4.0 and 4.5 Hz. Therefore, the second loading harmonic of lower frequency motion may excite the structure at its resonant frequency. The structural damping ratio was assumed to be 1%.

The generalized mass of the fundamental mode was estimated to be ~150 tonnes, when the mode shape was normalized to unity at the centre of the ballroom. Since a crowd of 4500 people will have a total mass of a few hundred tonnes, crowd-structure interaction must be considered. The applicable vibration criterion was to limit the peak vertical acceleration to no more than 5%g (Murray et al. 1997, NBCC 2011). A number of different loading scenarios were conducted to determine the sensitivity of the floor to crowd-induced vibration. Therefore, assorted crowd sizes were considered, and the percentage of the crowd that was active was varied. As shown in Table 2, both the size of the crowd, and the percentage of the crowd that is active had a significant effect on the predicted floor acceleration. Cells highlighted in red indicate that the 5%g criterion is exceeded for that scenario.

To reduce the floor vibration, a pair of 5 tonne TMDs was investigated. The TMDs were optimally tuned to the bare structure (that is, the structure without any people on it), and located near the position of maximum modal displacement. Table 3 shows the predicted peak floor acceleration for various loading scenarios when the TMD system is present. The greatest acceleration reductions occur when the crowd is small. As the crowd becomes large (1000 people or more), the crowd-structure interaction begins to modify the effective properties of the structure, such that it is no longer lightly damped, and its frequency is no longer 4.3 Hz, which diminishes the TMD efficacy. The lower occupancy scenarios were deemed to be more important, since a crowd of thousands of people actively engaged in a coordinated rhythmic activity was expected to occur infrequently. Therefore, since the TMD system improved the performance of the floor for small crowds, provisions were made in the floor structure to facilitate a post-construction TMD installation if there were complaints of excessive floor vibration.

Table 2: Peak floor accelerations (%g) for various crowd loading scenarios

		Percentage of Crowd Active				
		10%	25%	50%	75%	100%
		Total Crowd Size	10	0.2%	0.5%	1.0%
50	0.6%		1.4%	3.0%	4.8%	6.9%
100	0.8%		2.0%	4.2%	6.7%	9.8%
250	1.2%		2.9%	5.8%	9.0%	13.5%
500	1.7%		3.9%	7.1%	10.5%	15.2%
1000	2.3%		4.9%	8.4%	11.7%	15.8%
2000	2.5%		4.9%	8.8%	12.6%	15.9%
4500	1.3%		3.5%	8.0%	13.0%	16.2%

Table 3: Peak floor accelerations (%g) for various crowd loading scenarios with TMD

		Percentage of Crowd Active				
		10%	25%	50%	75%	100%
		Total Crowd Size	10	0.0%	0.1%	0.1%
50	0.1%		0.3%	0.6%	0.8%	1.1%
100	0.2%		0.6%	1.1%	1.6%	2.1%
250	0.6%		1.5%	2.7%	3.6%	4.3%
500	1.2%		2.7%	4.6%	5.9%	6.8%
1000	2.0%		4.2%	6.7%	8.4%	9.5%
2000	2.3%		4.5%	7.7%	10.4%	11.9%
4500	1.2%		3.3%	7.4%	11.7%	14.0%

## 4. HOSPITALS AND RESEARCH OCCUPANCIES

### 4.1 Background

In hospitals and research facilities, low vibration environments must be maintained to ensure that medical procedures and sensitive equipment are not adversely affected. For elevated floors, the governing source of vibration is often pedestrians walking in corridors, or moving about within the vibration-sensitive room. If low-vibration environments are not maintained, equipment may experience poor performance such as blurred images at high magnifications.

The performance of vibration-sensitive research environments is often characterized using generic vibration criteria (VC) curves as shown in Figure 4 (Bachmann and Ammann 1987). To determine the VC level, the vibration signal is processed into its 1/3-octave RMS velocity components and overlaid with the VC-curves. The VC level of the floor is the lowest curve that is not exceeded. The ISO Operating Theatre criterion is the threshold of human perception to whole-body vibration; therefore it is often the criteria applied to standard operating rooms. As the surgery scale decreases (e.g. eye surgery, or neurosurgery), a more stringent vibration criterion, typically VC-B, is applied. Similarly, the VC classification becomes more stringent as the resolution of imaging equipment increases; an optical microscope with 400 X magnification typically requires VC-A, while a scanning electron microscope with 30,000 X magnification typically requires VC-D.

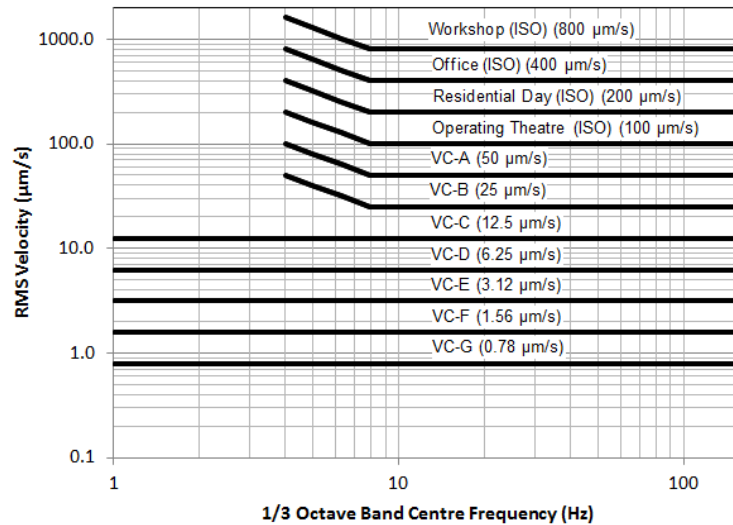


Figure 4: Vibration criteria (VC) curves

VC curves are employed when the manufacturer's equipment-specific vibration criteria are unavailable, or the equipment has not been selected at the time the structural design is conducted. When equipment-specific criteria are available, it is often expressed as an FFT spectrum, or as absolute vibration levels (e.g. displacement of 0.1 mm) that the floor motion must not exceed. Therefore, the method for predicting floor vibration must be capable of expressing the result in either 1/3 octaves, narrowband FFT, or peak-to-peak frames of reference.

Finite element modelling is employed to predict the dynamic properties of the floor structure. This modelling enables the mode shapes and frequencies of floors composed of complicated structural bays to be predicted. Modal analysis is then conducted to determine the floor response to footfall loading. Since the total floor response is composed of the superposition of many vibration modes, the analysis is conducted in the time-domain. The architectural layout of the floor will define where the footfall loading is applied. Therefore, locating corridors away from sensitive occupancies can be an effective vibration mitigation strategy.

Since the sensitive areas will often span several structural bays, it is useful to map predicted vibrations over a large portion of the floor. This mapping can be achieved by determining the worst-case response of a particular area of the floor, and then creating a contour map showing the vibration levels, and overlaying it with the architectural drawings. The resulting map can be used to locate sensitive occupancies or equipment in locations experiencing the least vibration.

#### 4.2 Case Study – College Laboratory

An elevated floor housing vibration-sensitive laboratories at a college in the United States is considered. This laboratory floor will house both teaching laboratories, as well as a nuclear magnetic resonance machine (NMR), which possesses equipment-specific vibration criteria. The floor structure consists of a composite steel deck with typical structural bays spanning approximately 9 m by 9 m. The architectural layout of the floor, and the fast and slow walking paths that were considered in the vibration analysis are shown in Figure 5. The architectural and structural designs were conducted by Ballinger AE.

The predicted vibration performance of the floor is shown in Figure 6, where the colour contours specify the vibration criterion achieved for each point on the floor. From Figure 6 it is observed that the laboratories (located in the central portion of the building) are experiencing ISO Operating Theatre vibration levels, which was the target criterion. Higher vibration levels are predicted in the non-vibration-sensitive offices and corridors that are located near the perimeter of the building. The NMR room is generally achieving the ISO Operating Theatre criterion, however it must be compared to the equipment-specific vibration criteria provided by the manufacturer. Figure 7 shows the vibration predicted in the NMR room compared to the equipment-specific criteria, which is given in narrowband displacement FFT. The NMR room achieves the criteria.

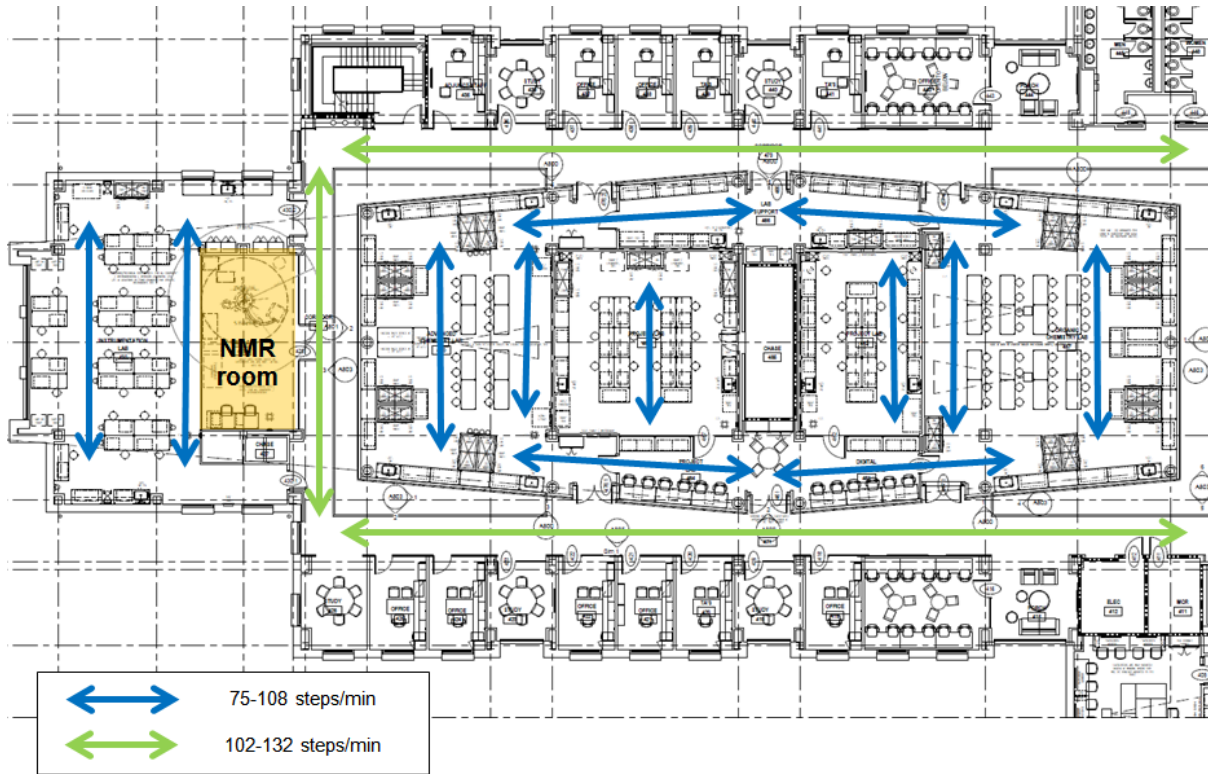


Figure 5: Architectural layout and walking paths of floor

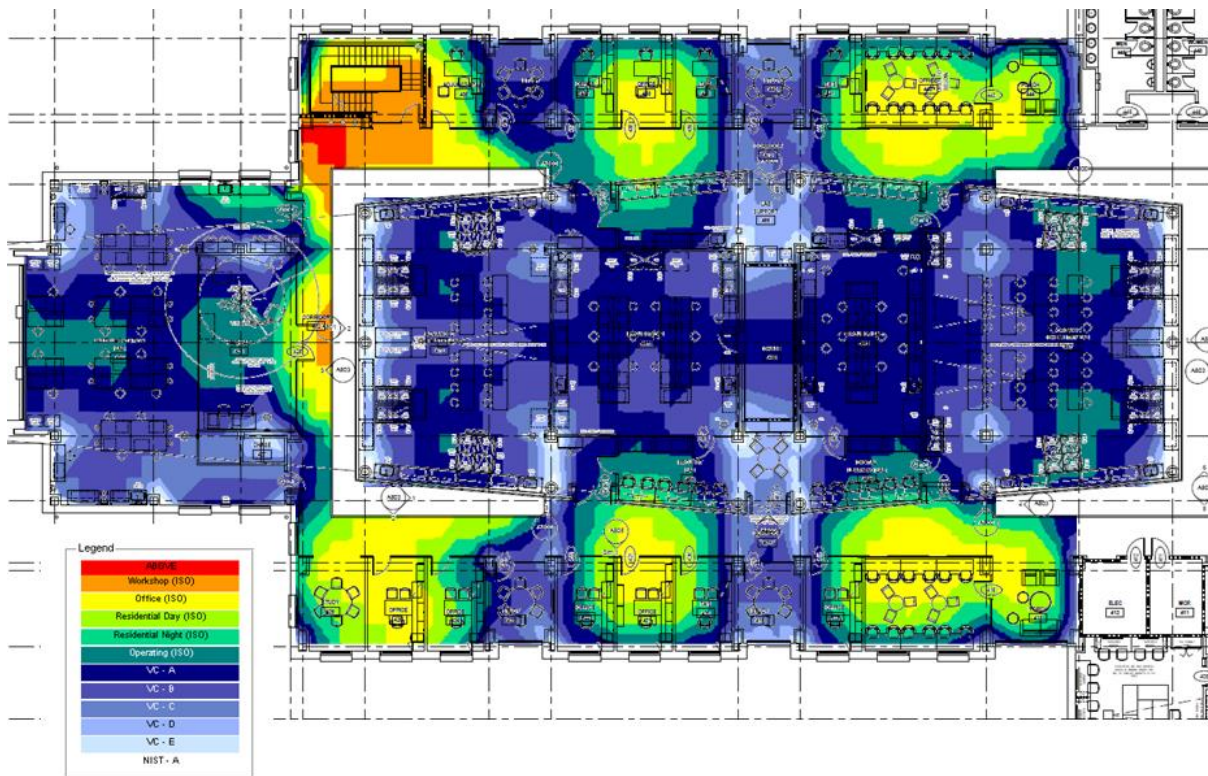


Figure 6: Contour plot of floor vibration response to fast walking in corridors



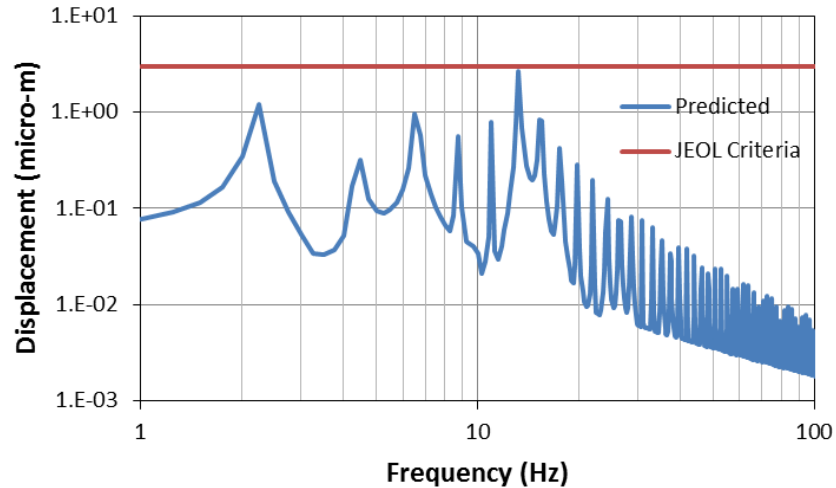


Figure 7: Comparison of floor vibration predicted in NMR room to manufacturer's vibration criteria

When a vibration-sensitive floor does not achieve the relevant vibration criterion, there are several common forms of mitigation available. Sometimes, vibration-sensitive equipment can be relocated away from areas experiencing high vibration levels to areas experiencing lower levels. Contour plots of the vibration levels, such as that shown in Figure 6, can be useful to optimize equipment positioning.

Additionally, structural modifications are typically employed to reduce vibration levels. These modifications often consist of increasing the mass or stiffness of the floor by changing from light-weight to normal-weight concrete, increasing the thickness of the concrete, or increasing the size of the structural beams. In some situations, it may be possible to utilize interstitial columns or posts to connect two or more floors together to significantly increase the mass of the floor without requiring substantially more construction materials (Wesolowsky et al. 2016). The addition of supplementary damping systems is generally not currently employed due to the uncertainty associated with obtaining damping from materials at the very low levels of floor motion required for vibration-sensitive facilities.

## 5. CONCLUSIONS

Many types of structures are susceptible to human-induced vibrations. These range from structures whose vibration levels must satisfy requirements for human comfort, to structures that must maintain imperceptible vibration environments to ensure the proper functioning of vibration-sensitive equipment.

This study has considered lightweight pedestrian bridges, heavily loaded ballrooms and stadia, and vibration-sensitive floors in hospitals and research facilities. The unique loading to which each type of structure is subjected necessitates that appropriate analysis techniques are employed. Vibration criteria are discussed and it is shown that a criterion that is applicable to one type of structure may be significantly different from that which is applicable to a different type of structure. Mitigation techniques are presented for projects in which the predicted vibration exceeded the accepted criteria. Case studies are shared to provide real-world examples of the analysis results.

Through the use of suitable vibration analysis techniques, structures can be designed to have a satisfactory vibration performance.

## REFERENCES

- Amick, H., Gendreau, M., Busch, T., and Gordon, C. 2005 *Evolving criteria for research facilities: I - Vibration. Buildings for Nanoscale Research and Beyond*, Society of Photographic Instrumentation Engineers. San Diego.
- Bachmann, H., and Ammann, W. 1987. *Vibration in Structures - Induced by Man and Machines*. Structural Engineering Documents, Vol. 3e, International Association of Bridge and Structural Engineering, Zurich.

- Dougill, J.W., Wright, J.R., Parkhouse, J.G., and Harrison, R.E. 2006. Human structure interaction during rhythmic bobbing. *The Structural Engineer*: 32-39.
- Joint Research Centre 2009. *Design of Lightweight Footbridges for Human Induced Vibrations*, Institute for the Protection and Security of the Citizen, Luxembourg.
- Joint Working Group, 2008. *Dynamic performance requirements for permanent grandstands subject to crowd action*, The Institution of Structural Engineers, London, UK.
- Jones, C.A., Pavic, A., Reynolds, P., and Harrison, R.E. 2011. Verification of equivalent mass-spring-damper models for crowd-structure vibration response prediction. *Canadian Journal of Civil Engineering* 38: 1122-1135.
- Matsumoto, Y. 1974. A study on dynamic design of pedestrian over-bridges. *Transactions of JSCE* 4: 50-51.
- Parkhouse, J.G., and Ewins, D.J. 2006. Crowd-induced rhythmic loading. *Structures & Buildings* 159 (SB5): 247-259.
- Pavic, A., and Reynolds, P. 2008. Experimental verification of novel 3DOF model for grandstand crowd-structure dynamic interaction. *International Modal Analysis Conference XXVI*. Orlando, USA.
- Rainer, J.H., Pernica, G., and Allen, D.E. 1988. Dynamic loading and response of footbridges. *Canadian Journal of Civil Engineering* 15: 66-71.
- SETRA (Service d'Etudes Techniques des Routes et Autoroutes). 2006. *Footbridges: Assessment of vibrational behaviour of footbridges under pedestrian loading*, Ministry of Transport and Infrastructure, Paris, France.
- Smith, A.L., Hicks, S.J., and Devine, P.J. 2007. *SCI P354: Design of Floors for Vibration: A New Approach*, The Steel Construction Institute, Berkshire, UK.
- Stoyanoff, S., Haskett, T., Pridham, B., Hunter, M., and Zoli, T. 2007. Pedestrian-induced vibrations on footbridges: advanced response analysis. *Bridge Structures* 3(3-4): 229-245.
- NBCC (National Building Code of Canada) 2011. *User's Guide - NBC 2010 Structural Commentaries*. Ottawa: National Research Council of Canada, Ottawa, Canada.
- Wesolowsky, M.J., Love, J.S., Busch, T.A., Tallavo, F.J., and Swallow, J.C. 2016. Improving Floor Vibration Performance Using Interstitial Columns. *International Modal Analysis Conference XXXIV*. Orlando, USA.
- Zivanovic, S., Pavic, A., and Reynolds, P. 2005. Vibration serviceability of footbridges under human-induced excitation: a literature review. *Journal of Sound and Vibration* 279: 1-74.