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Boundary value problems of degenerate integro-differential equations on the boundary of the region with application in filtration

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Abstract. Boundary value problems for ordinary and integro-differential equations degenerating on the boundary of the region are considered. The existence of solutions is proven and a priori estimates for the solutions of the problems are obtained. The method of semidiscretization is applied for approximate solution, and the convergence of the approximate method is proven. Keywords: boundary, degenerate, integro-differential, factorization, maximum, ordinary, filtration, semidiscretization.

1 Introduction

The study of initial-boundary value problems for parabolic equations is crucial both theoretically and practically. Scholars like V.D. Ilyin, A.S. Kalashnikov, O.D. Oleinik, O.A. Ladyzhenskaya, Zh.M. Liberman, Zh.L. Lionis, N. Babushka, and others have significantly advanced our understanding of the solvability of these problems. Their contributions have paved the way for exploring various theoretical aspects and practical applications. Furthermore, the successful application of numerical methods in investigating boundary value problems for both ordinary and partial differential equations of parabolic nature underscores the multidisciplinary approach required in this field. By building upon these foundational works, researchers can delve deeper into the behavior and solutions of such problems, thereby driving progress in both theoretical knowledge and real-world applications [1-15].

In the work of S.G. Mikhlin [11], variational mesh approximation for degenerate ordinary differential equations in the form:

$$-\frac{d}{dx}(x^{\alpha}p(x)\frac{du}{dx}) + g(x)u = f(x)$$

subject to

$$u(1) = 0, \qquad 0 < \alpha < 1$$

In the research conducted by R. Merc and V.Ya. Rivkind[13], finite difference methods were employed to approximate solutions for the first boundary value problem concerning a degenerate parabolic equation. Meanwhile, V. Walter[12], through a series of studies utilizing the method of characteristics, developed solutions for the Cauchy problem and

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demonstrated the convergence of nonlinear boundary value problems associated with parabolic equations. Presently, significant attention is directed towards investigating degenerate boundary value problems pertaining to ordinary and integro-differential equations of parabolic type. The interest in these problems stems from their relevance to gas and liquid filtration issues. These boundary value problems find applications in scenarios such as one-dimensional gas filtration. Non-stationary filtrations are typically described by nonlinear parabolic differential equations, wherein the coefficients of porosity and permeability of the formation are often spatially variable, sometimes both horizontally and vertically. Consequently, these coefficients become functions of coordinates. In certain instances, the permeability of the formation at the region's boundary reaches zero, leading to degenerate equations along the boundary. Works dedicated to addressing and solving such problems have been outlined in references [1] and [3].

2 Methodology

Currently, there is a notable influx of publications dedicated to the examination of degenerate equations. In the work of V.V. Bobkov and O.A. Liskovets [15], a boundary value problem for non-degenerate ordinary differential equations, based on the maximum principle, is analyzed, yielding a priori estimates for the solution. Additionally, in this work, boundary value problems concerning degenerate ordinary differential equations on the region's boundary are investigated. Exact estimates for the solution of boundary value problems for ordinary differential equations are acquired through a modification of the differential sweep method [1],. Moreover, boundary value problems for integro-differential parabolic equations, which degenerate on the boundaries of the region, are deliberated. The direct method is employed for the solution, and the convergence of this approach is thoroughly examined [2].. The proof provided is constructive, and under certain assumptions, a priori estimates for the solution are obtained. These advancements, as outlined in references [1] and [2], contribute significantly to the understanding and resolution of degenerate equations in various contexts.

Under certain assumptions regarding the input functions, a priori estimates for the solution are obtained, and the convergence of the direct method is proven [3].

3 Results

The issue of gas filtration is studied, which is described by an integro-differential equation taking into account pressure and velocity relaxation. First, auxiliary problems for ordinary differential equations are considered.

Problem 1. Consider ordinary differential equations degenerating on the boundary of the region

$$\frac{1}{m(x)}\frac{d}{dx}(k(x)\frac{du}{dx}) = a(x)u + c(x)k(x)\frac{du}{dx} + b(x) \qquad (1)$$

satisfying the conditions:

$$\gamma_0 k(x)\frac{du}{dx}\bigg|_{x=0} - \alpha_0 u\big|_{x=0} - \beta_0 = 0 \qquad (2)$$

$$\gamma_1 k(x)\frac{du}{dx}\bigg|_{x=1} - \alpha_1 u\big|_{x=1} - \beta_1 = 0 \qquad (3)$$

$$|\alpha_i| + |\gamma_i| \neq 0, \qquad i = \overline{1,2}$$

Here, $a(x), b(x), c(x), m(x), K(x) -$ are given functions on the interval and $\mathrm{k}(0)=0,$ are $\mathrm{k(x)}, \mathrm{m(x)} -$ positive for $x>0,\ a(x) \geq a_i > 0$. The boundary conditions depend on the convergence of the integral $\int_0^1 \frac{dx}{k(x)}$, if $\int_0^1 \frac{dx}{k(x)} = \infty$, $\int_0^1 \frac{\int_0^x m(\xi)d\xi}{k(x)} < +\infty$, the condition at $x=0$ is replaced by the condition $\left| u(x)\big|_{x=0} \right| < +\infty$.

Lemma 1. If $a(x), b(x), c(x), m(x), k(x) \in C[a,b]$, then there exists a solution to problems (1)-(3), and for the solution, the estimate holds (in case $\int_0^1 \frac{dx}{k(x)} < +\infty$).

$$|u| \leq M$$

Where

$$M = \begin{cases} \max\limits_{0\le x\le 1} \left|\frac{b(x)}{a(x)}\right|, if\ \beta_0 = \beta_1 = 0 \\ \max\limits_{0\le x\le 1} \left\{\frac{|\beta_0|}{\alpha_0}, \frac{|\beta_1|}{\alpha_1}, max\left|\frac{b(x)}{a(x)}\right|\right\}, if\ \alpha_0, \alpha_1 > 0 \end{cases}$$

The proof relies on a modification of the differential sweep method [1].

Problem 2. Consider problems arising in solving gas filtration problems taking into account pressure and velocity relaxation under certain assumptions. In dimensionless form, the filtration equation can be described by integro-differential equations.

$$\frac{1}{m(x)}\frac{\partial}{\partial x}(k(x)\frac{\partial u}{\partial x}) = a(x,t,u)\frac{\partial u}{\partial t} + f(x,t,u) + \int_0^1 R(t,s)u(x,s)ds \quad (4)$$

with the initial condition

$$u(x,0) = \varphi(x) \quad (5)$$

and boundary conditions

$$k(x)\frac{du}{dx}\bigg|_{x=0} = k(x)\frac{\partial u}{\partial x}\bigg|_{x=1} = 0\ ,\ \int_0^1 \frac{\mathrm{dx}}{\mathrm{k(x)}} < 0, \quad (6)$$

If

$$\int_0^1 \frac{\mathrm{dx}}{\mathrm{k(x)}} = +\infty,\ \int_0^1 \frac{\int_0^1 \mathrm{m}(\xi)\mathrm{d}\xi}{\mathrm{k(x)}}\mathrm{dx} < +\infty$$

the condition at x=0 is replaced by the condition $\left| \mathrm{u(x,t)}\big|_{\mathrm{x=0}} \right| < +\infty$.

All input functions are given functions in the domain of their arguments, where $k(0)=0,\ a(x,t,u) \geq a_i > 0$ and $m(x)$ and $k(x)$ are positive for $x>0$.

To solve problems (4)-(6), we will apply the direct method, i.e. $\Omega = \{0 \leq x \leq 1, 0 \leq t \leq \tau\},$ cover with straight lines $t_j = j\tau,\ \tau > 0,\ j = 0,...,N,\ N = \left\lfloor \frac{T}{\tau} \right\rfloor$.

Let $\int_0^1 \frac{dx}{k(x)} < +\infty$. Approximate problem (4)-(6) with the following difference equations

$$\frac{1}{m(x)}\frac{d}{dx}(k(x)\frac{du_j}{dx}) = a(x,t_j,u_{j-1})\delta_{\bar{t}}u_j + f(x,t_j,u_{j-1}) + \sum_{i=0}^{j-1} R_{j,i}u_i\tau, \qquad j=\overline{1,N} \quad (7)$$

$$u_0(x) = \varphi_1(x) \quad (8)$$

$$k(x)\frac{du_j}{dx}\bigg|_{x=0} = k(x)\frac{du_j}{dx} = 0 \quad (9)$$

Problem (7)-(9) is solved sequentially layer by layer starting at $\mathrm{j}=1,\ldots$. Each time, it is linear with respect to $u_j(x),\ j=\overline{1,n}$ and has a unique solution based on the lemma under the assumption that all known smooth functions in the equation.

$$|u_j| \le \left| \frac{\frac{a(x,t_j,u_{j-1})}{\tau}u_{j-1} + c\sum_{j=0}^{j-1} R_{j+i}u_i + f(x,t_j,u_{j-1})}{\frac{M(x,t_j,u_{j-1})}{\tau}} \right| \le (1+c_2T\tau)|u|_{j-1} + c_2\tau$$

$$\left|u_j\right|_j \le (1+c_2T\tau)\|u\|_{j-1} + c_2\tau$$

Introduce a norm,

$$|u|_j = \max_{1\le k\le i}|u_k|, \qquad |\bullet| = \max|\bullet|$$

By induction

$$\|u_j\| \le \|\varphi\| e^{C_2\tau^2} + \frac{c_1}{Tc_2}(e^{c_2T^2}-1), \quad j=1,\ldots N$$

where $c_1, c_2 -$ are some constants. Linearly extend $\mathrm{u}^\sigma(\mathrm{x},\mathrm{t})$, so that at $t\in[t_{j-1},t_j]$, the function

$$\mathrm{u}^\tau(\mathrm{x},\mathrm{t}) = \frac{\mathrm{t}-\mathrm{t}_{j-1}}{\tau}\mathrm{u}_j(\mathrm{x}) + \frac{\mathrm{t}_j-\mathrm{t}}{\tau}\mathrm{u}_{j-1}(\mathrm{x})$$

If $\lim_{x\to0}\frac{\int_0^x \mathrm{m}(\xi)\mathrm{d}\xi}{\mathrm{k}(\mathrm{x})}$ is finite, it is proven that the family of functions $\mathrm{u}^\tau(\mathrm{x},\mathrm{t}),\ \mathrm{k}(\mathrm{x})\frac{\partial \mathrm{u}^\tau(\mathrm{x},\mathrm{t})}{\partial \mathrm{x}},\ \frac{\partial \mathrm{u}^\tau(\mathrm{x},\mathrm{t})}{\partial \mathrm{x}},\ \frac{\partial}{\partial \mathrm{t}}\left(\mathrm{k}(\mathrm{x})\frac{\partial \mathrm{u}^\tau(\mathrm{x},\mathrm{t})}{\partial \mathrm{x}}\right)$ is compact (in terms of uniform convergence), and there exists a unique solution to problem (1)-(2).

If $\int_0^1 \frac{\mathrm{dx}}{\mathrm{k}(\mathrm{x})} = +\infty,\quad \int_0^1 \frac{\int_0^x \mathrm{m}(\xi)\mathrm{d}\xi}{\mathrm{k}(\mathrm{x})}\mathrm{dx} < \infty$ then considering as $\sigma(\mathrm{z}) = \int_0^z \frac{\int_0^x \mathrm{m}(\xi)\mathrm{d}\xi}{\mathrm{k}(\mathrm{x})}\mathrm{dx}$ an absolutely continuous function, an estimate holds

$$\left|\mathrm{u}^\tau(\mathrm{x},\mathrm{t}'') - \mathrm{u}^\tau(\mathrm{x},\mathrm{t}')\right| \le \mu_1\left|\sigma(\mathrm{x}'') - \sigma(\mathrm{x}')\right| + \mu_0\left|\mathrm{t}''-\mathrm{t}'\right|$$

Where μ_1, μ_2 , are some constants. Thus, $u^\tau(x,t)$ is uniformly continuous function and $u_t^\tau(x,t)$ is also a continuous function. In this case, there also exists a unique solution to problems (1),(2).

The approximate solution constructed by the direct method converges to the exact solution with a rate of $0(\tau)$. τ - - time step.

4 Conclusion

The theorems of existence and uniqueness of the considered initial boundary value problems for degenerate integro-differential parabolic equations have been proven. The proof of the existence theorem is constructive and relies on the use of the direct method combined with the differential sweep method. It has been established that the approximate solution converges to the exact solution with a rate of - time step.

A priori estimates for the solutions of boundary value problems for ordinary differential equations have been obtained, which have been used to derive estimates for the solutions of integro-differential equations. The results obtained can be applied to the study of gas filtration problems.

References

1. Abdurazakov, A., Mahmudova, N., Mirzamakhmudova, N. (2020). Universum: Technical Sciences, **7-1(76).**
2. Abdujabbor, A., Nasiba, M., Nilufar, M. (2022). Eurasian Journal of Physics, Chemistry and Mathematics, **6,** 18-21.
3. Abdurazakov, A., Mahmudova, N., Mirzamakhmudova, N. (2021). Universum: Technical Sciences, **10-1(91).**
4. Lieberman, G. M. (2003). Second Order Parabolic Differential Equations. Hong Kong: World Scientific.
5. Kalashnikov, A. S. (1993). Differential Equations, **29(6),** 999-1009.
6. Oleinik, O. A., Shubin, M. A. (1982). Russian Mathematical Surveys, **37(6),** 261-285.
7. Ladyzhenskaya, O. A. (2006). Olga Alexandrovna Ladyzhenskaya, 1922–2004.
8. Lionis, J. L. (1972). Some Methods for Solving Nonlinear Problems.
9. Friedman, A. (1968). Equations of Parabolic Type.
10. Mikhlin, S. G. (1974). Variational-Grid Approximation.
11. Walter, W. (1972). Some New Aspects of the Line Method for Parabolic Differential Equations.
12. Mertz, R., Rivkin, V. N. (1967). Finite-Difference Method for Solving Degenerate Elementary Parabolic Equations.
13. Babushka, N., Vitasik, D., Prager, I. (1967). Numerical Processes for Solving Differential Equations.
14. Rasulov, R., Mahkamova, D. (2024). AIP Conference Proceedings **3004**, 1
15. Hayotov, A., Rasulov, R. (2021). AIP Conference Proceedings **2365**, 1