8-17-2016 12:00 AM

On Logarithmic Sobolev Inequality And A Scalar Curvature Formula For Noncommutative Tori

Sajad Sadeghi
The University of Western Ontario

Supervisor
Professor Masoud Khalkhali
The University of Western Ontario

Graduate Program in Mathematics
A thesis submitted in partial fulfillment of the requirements for the degree in Doctor of Philosophy

Abstract

In the first part of this thesis, a noncommutative analogue of Gross' logarithmic Sobolev inequality for the noncommutative 2-torus is investigated. More precisely, Weissler's result on the logarithmic Sobolev inequality for the unit circle is used to propose that the logarithmic Sobolev inequality for a positive element $a = \sum a_{m,n} U^m V^n$ of the noncommutative 2-torus should be of the form

$$\tau(a^2 \log a) \leqslant \sum_{(m,n)\in\mathbb{Z}^2} (|m| + |n|)|a_{m,n}|^2 + \tau(a^2) \log(\tau(a^2))^{1/2},$$

where τ is the normalized positive faithful trace of the noncommutative 2-torus. A possible approach to prove this inequality for arbitrary positive elements will involve a noncommutative multinomial expansion and seems to be exceedingly complicated. In this thesis the above inequality is proved for a particular class of elements of the noncommutative 2-torus.

In the second part of this thesis, the scalar curvature of the curved noncommutative 3-torus is studied. In fact, the standard volume form on the noncommutative 3-torus is conformally perturbed and the corresponding perturbed Laplacian is analyzed. Then using Connes' pseudodifferential calculus for the noncommutative 3-torus, the first three terms of the small time heat kernel expansion for the perturbed Laplacian are derived. Moreover, by using the third term of this expansion and the Cauchy integral formula, the scalar curvature of the curved noncommutative 3-torus is defined. Finally, proving a rearrangement lemma, the scalar curvature is computed and an explicit local formula that describes the curvature in terms of the conformal factor is given.

Keywords: Logarithmic Sobolev inequality, Noncommutative 2-torus, Scalar curvature, Noncommutative 3-torus, Conformal perturbation, Modular operator, Pseudodifferential calculus, Asymptotic expansion, Laplace operator, Spectral triples, Rearrangement.

Co-Authorship

Sajad Sadeghi was supervised by Professor Masoud Khalkhali over the course of this thesis work. The first paper is co-authored by Masoud Khalkhali and Sajad Sadeghi, and the second one is co-authored by Masoud Khalkhali, Ali Moatadelro and Sajad Sadeghi. All authors have participated and contributed significantly in all new results presented in this thesis.

To my beloved parents
for their endless love and sacrifices throughout my life
To my kind wife
for all her support and patience
To my loving brothers
for their affection and moral support

Acknowledgments

I would like to express my deepest gratitude to my advisor Professor Masoud Khalkhali for the continuous support of my Ph.D during the last four years and for suggesting the problems that are the subject of this thesis. I have been always fascinated by his immense knowledge in various fields of mathematics and his ability to simplify difficult concepts for the students. His insightful comments as well as his enthusiasm has been constantly helping me pursue my research.

I would also like to thank my examiner Professor George Elliott, for kindly accepting to review this thesis. I would also like to thank my advisory committee and examination board members Professors Stuart Rankin, David Riley, Tatyana Barron, and John Bell for carefully reading my thesis and for their valuable comments, discussions and suggestions. I am also grateful to Dr. Ali Moatadelro for his helpful discussions and collaboration.

My sincere thanks goes to Professor Masoud Amini, who motivated me to study noncommutative geometry and helped me choose the right path to pursue this field before starting my Ph.D program. He has undoubtedly been one of the most influential persons in my life.

I am deeply grateful to my parents for supporting me spiritually throughout writing this thesis and my life in general. I would never have reached to the present stage without their encouragement, support, and sacrifices. I am indebted to them for my life.

Last but not least, many thanks to my wife, for her affection and tolerance during these years. For me it would have been really tough to complete my Ph.D without her support and patience.

Contents

Chapter 1

Introduction

The field of noncommutative geometry was introduced by Alain Connes in the 1980's [1.1], and since then it has grown very quickly. Noncommutative geometry is based on the idea that one can study the geometric or topological properties of a space by looking at the function algebras (which are commutative) on that space. Exploiting this idea one can relax the commutativity condition and define noncommutative spaces through noncommutative algebras.

In this chapter we will briefly recall some of the ideas of noncommutative geometry, through certain examples. For more details and advanced discussion one can see [1.2], [1.9] and [1.11].

1.1 Noncommutative Topology

The roots of noncommutative geometry can be seen in the seminal work [1.8] of Gelfand and Naimark, where the theory of C*-algebras was born. In this section we explain how that work helps us define noncommutative locally compact topological spaces.

Definition 1.1.1. A complete normed algebra $\mathcal{A}$ over $\mathbb{C}$ is called a *C*-algebra* if
(i) $\|ab\| \leqslant \|a\|\|b\|$ for $a, b \in \mathcal{A}$,
(ii) There is an involution, i.e. a conjugate linear map

$$* : \mathcal{A} \longrightarrow \mathcal{A}, \qquad a \mapsto a^*,$$

such that $(a^*)^* = a$ and $(ab)^* = b^* a^*$,
(iii) $\|a^* a\| = \|a\|^2$ for $a \in \mathcal{A}$.

Let H be a Hilbert space and T be a linear map on H. The *operator norm* of T is defined by

$$\|T\| = \sup \{\|T(x)\| : \ x \in H, \|x\| \leqslant 1\}.$$

If $\|T\| < \infty$, then T is called a *bounded operator*. We denote the space of all bounded linear operators on H by $B(H)$. As an example of a C*-algebra, we can consider $B(H)$ endowed with the operator norm and the involution defined by the adjoints of operators. Indeed, by a famous theorem known as the Gelfand-Naimark-Segal (GNS) theorem we know that every C*-algebra can be embedded in $B(H)$ for some Hilbert space H [1.8].

Let X be a locally compact Hausdorff space. A function f on X is said to be vanishing at infinity if for every $\varepsilon > 0$ there exists a compact set $K \subset X$ such that for $x \in X \setminus K$, $\|f\|_\infty < \varepsilon$, where $\|.\|_\infty$ is the uniform norm, i.e.

$$\|f\|_\infty = \sup\{|f(x)| : \ x \in X\}.$$

The space of all continuous functions on X vanishing at infinity forms an algebra under the pointwise addition and multiplication. This algebra, which is denoted by $C_0(X)$, endowed with the uniform norm and the involution defined by

$$* : C_0(X) \longrightarrow C_0(X), \qquad f^* = \bar{f}$$

is a commutative C*-algebra. We will see soon that these C*-algebras are the only commutative C*-algebras up to isomorphism.

For the C*-algebras $\mathcal{A}$ and $\mathcal{B}$, the homomorphism $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$ is called a *C*-homomorphism*, if it preserves the involution, i.e. for $a \in \mathcal{A}$,

$$\varphi(a^*) = \varphi(a)^*.$$

A *character* of a C*-algebra $\mathcal{A}$ by definition is a nonzero multiplicative linear map $\chi : \mathcal{A} \longrightarrow \mathbb{C}$. The set of all characters of such an algebra is called the *spectrum* of that algebra and is denoted by $\hat{\mathcal{A}}$. For a commutative C*-algebra $\mathcal{A}$, it can be shown that $\hat{\mathcal{A}}$ is a locally compact Hausdorff space with respect to the *weak* topology* which is indeed the topology of pointwise convergence on the continuous dual space of $\mathcal{A}$. The space $\hat{\mathcal{A}}$ is compact if and only if $\mathcal{A}$ is unital (see Theorem 1.3.5 in [1.12]). The next theorem implies that there exists a unique commutative C*-algebra with a given spectrum up to isomorphism.

Theorem 1.1.2. *(Gelfand-Naimark Theorem). Let $\mathcal{A}$ be a commutative C*-algebra and Γ be the Gelfand transform, i.e. the linear map*

$$\Gamma : \mathcal{A} \longrightarrow C_0(\hat{\mathcal{A}}), \qquad a \mapsto \hat{a},$$

where $\hat{a}$ is defined by $\hat{a}(h) = h(a)$ for $h \in \hat{\mathcal{A}}$. Then Γ is an isometric C-isomorphism.*

Proof. See Theorem 1.31 in [1.7]. $\square$

As we stated earlier, for a locally compact Hausdorff space X, $C_0(X)$ is a commutative C*-algebra. For $x \in X$, we define a character χ_x by

$$\chi_x(f) = f(x).$$

The next theorem states that indeed all of the characters of $C_0(X)$ are of this form.

Theorem 1.1.3. *Let X be a locally compact Hausdorff space. Then the map*

$$F : X \longrightarrow \widehat{C_0(X)}, \qquad x \mapsto \chi_x,$$

is a homeomorphism.

Proof. See Proposition 4.5 in [1.15]. □

Using the last two theorems we see that there is an equivalence between the category of commutative C*-algebras and the opposite of the category of locally compact Hausdorff spaces. In fact, the locally compact Hausdorff space associated to the commutative C*-algebra $\mathcal{A}$ is $\hat{\mathcal{A}}$, and the C*-algebra associated to the locally compact Hausdorff space X is $C_0(X)$.

Now we are at the point that we can define a noncommutative locally compact Hausdorff space. Indeed, it suffices to drop the commutativity condition. Therefore, we define a *noncommutative locally compact Hausdorff space* to be a not necessarily commutative C*-algebra. Clearly, if X is a compact space then $C_0(X)$ is a unital C*-algebra and vice versa. Accordingly, one can add more correspondences and form the following table:

locally compact Hausdorff space X	$C_0(X)$
compact Hausdorff space X	unital C*-algebra $C(X)$
one point compactification of X	unitization of $C_0(X)$
Stone-Čech compactification of X	multiplier algebra of $C_0(X)$

Using this table we can also define new noncommutative spaces and processes and create another table:

noncommutative locally compact Hausdorff space	C*-algebra $\mathcal{A}$
noncommutative compact Hausdorff space	unital C*-algebra $\mathcal{A}$
one point compactification	unitization of $\mathcal{A}$
Stone-Čech compactification	multiplier algebra of $\mathcal{A}$

The next step is defining the noncommutative vector bundles. To define them we use Swan's theorem [1.14], which states that there is a one-to-one correspondence between vector bundles over a compact Hausdorff space X and finite projective $C(X)$-modules. Thus a finite projective $\mathcal{A}$-module is called a *noncommutative vector bundle* over the noncommutative compact space $\mathcal{A}$.

1.2 Noncommutative Measure Theory

In this section we will use a theorem that characterizes commutative von Neumann algebras to define noncommutative measure spaces.

Let H be a Hilbert space. The *weak operator topology* on $B(H)$ is the topology generated by the semi norms

$$\|T\|_{x,y} = |\langle Tx, y\rangle| \quad x, y \in H,$$

where $\langle \cdot, \cdot \rangle$ is the inner product on H.

Definition 1.2.1. Let H be a Hilbert space. A *von Neumann algebra* is a $*$-subalgebra of $B(H)$ which is closed in the weak operator topology.

Since the weak operator topology is weaker than the operator norm topology, any von Neumann algebra is a C*-algebra.

Let X be a locally compact space and μ be a positive Radon measure on X. One can show that the following map

$$\pi : L^\infty(X,\mu) \longrightarrow B(L^2(X,\mu)), \quad \varphi \mapsto M_\varphi,$$

is an injective map, where M_φ is the multiplication operator by φ, i.e. for $f \in L^2(X,\mu)$, $M_\varphi(f) = f\varphi$. Moreover, the range of the map π is a von Neumann algebra. The next theorem states that indeed all of commutative von Neumann algebras are of this form.

Theorem 1.2.2. *Let $\mathcal{A}$ be a commutative von Neumann algebra. Then there exist a locally compact space X and a positive Radon measure μ on it such that $\mathcal{A}$, as an algebra, is isomorphic to $L^\infty(X,\mu)$.*

Proof. See Theorem 1.18 in [1.15]. □

The previous theorem leads us to call a not necessarily commutative von Neumann algebra a *noncommutative measure space.*

1.3 The Noncommutative 2-Torus

One of the most famous examples of the noncommutative spaces is the noncommutative 2-torus. In fact, one can think of it as a noncommutative manifold of dimension two. This space is a play ground for testing many interesting ideas, concepts and structures that come to the mind in the noncommutative settings, and fortunately, most of them have been proved to be a justifiable generalization of the commutative case. In this section we shall briefly introduce the noncommutative two torus and see some of its properties. We will start with the definition of the rotation algebras following M. A. Rieffel in [1.13].

Definition 1.3.1. Let $\theta \in \mathbb{R}$. The universal unital C*-algebra generated by two unitaries U, V such that

$$UV = e^{2\pi i\theta} VU,$$

is called a *rotation algebra* and is denoted by A_θ.

By universality we mean that for any C*-algebra $\mathcal{B}$ with unitaries u and v, subject to the relation $uv = e^{2\pi i\theta} vu$, there exists a unique unital C*-morphism from A_θ to $\mathcal{B}$ that sends U to u and V to v.

In the purely algebraic case we know that one can always find an algebra generated by a set of generators and relations, while in the C*-algebra settings we have to also check the C*-indentity. As a result, some times there is no such a universal C*-algebra. Fortunately, in the case of the rotation algebras, there exists a solution to the universal property. Indeed, one can represent A_θ on $L^2(\mathbb{T})$, where $\mathbb{T}$ is the unit circle.

More precisely, we define the unitary operators U and V on $L^2(\mathbb{T})$ by

$$Uf(x) = e^{2\pi ix} f(x), \qquad Vf(x) = f(x+\theta), \qquad f \in L^2(\mathbb{T}).$$

These operators satisfy the relation $UV = e^{2\pi i\theta}VU$. One can show that the C*-algebra generated by U and V in $B(L^2(\mathbb{T}))$ satisfies the universal property (See [1.9], Proposition 12.1).

Let

$$A_\theta^\infty = \left\{ \sum_{(m,n)\in\mathbb{Z}^2} a_{m,n}U^mV^n : a_{m,n} \text{ is rapidly decreasing} \right\}.$$

By rapidly decreasing we mean for all $k \in \mathbb{N}$,

$$\operatorname*{Sup}_{(m,n)\in\mathbb{Z}^2} (1+m^2+n^2)^k |a_{m,n}|^2 < \infty. \tag{1.1}$$

The set A_θ^∞ is a dense subalgebra of A_θ and is called the *noncommutative 2-torus*.

For $\theta = 0$, the commutation relation is $UV = VU$, so A_θ would be commutative. If we set U and V to be functions on the 2-torus $\mathbb{T}^2$, defined by

$$U(\phi_1, \phi_2) = e^{2\pi i\phi_1}, \qquad V(\phi_1, \phi_2) = e^{2\pi i\phi_2},$$

where (ϕ_1, ϕ_2) is the angular coordinate for $\mathbb{T}^2$, then one can see that A_0 is $C(\mathbb{T}^2)$, the algebra of continuous functions on the 2-torus. Moreover, in this case the condition (1.1) is exactly the same condition on the Fourier coefficients of the smooth functions on $\mathbb{T}^2$. Therefore, A_θ and A_θ^∞ are respectively the noncommutative deformations of $C(\mathbb{T}^2)$ and $C^\infty(\mathbb{T}^2)$, and this justifies the name noncommutative 2-torus. In what follows we state some facts about A_θ and A_θ^∞. For the proof of these facts see Sections 12.3 and 12.4 in [1.9].

The algebra A_θ^∞ possesses two *derivations*, i.e. linear maps $\delta_i : A_\theta^\infty \longrightarrow A_\theta^\infty$ for $i = 1, 2$, such that

$$\delta_i(ab) = \delta_i(a)b + a\delta_i(b), \qquad a, b \in A_\theta^\infty.$$

These derivations are defined by the following relations:

$$\delta_1(U) = 2\pi iU, \qquad \delta_1(V) = 0,$$

$$\delta_2(V) = 2\pi iV, \qquad \delta_2(U) = 0.$$

Using the derivation property, one can see that for $a = \sum a_{m,n}U^mV^n \in A_\theta^\infty$,

$$\delta_1(a) = 2\pi i \sum ma_{m,n}U^mV^n, \qquad \delta_2(a) = 2\pi i \sum na_{m,n}U^mV^n.$$

The last formulas are similar to the partial derivatives of a Fourier series on $\mathbb{T}^2$. Therefore, we can think of δ_1 and δ_2 as the *noncommutative partial derivatives* of the elements in A_θ^∞.

For $\theta \in \mathbb{R}\setminus\mathbb{Q}$, the rotation algebra A_θ has a unique *normalized positive faithful trace*, i.e. a linear functional $\tau : A_\theta \longrightarrow \mathbb{C}$ with the following properties:

(i) $\tau(ab) = \tau(ba) \quad a, b \in A_\theta$,

(ii) $\tau(a^*a) > 0 \quad a \neq 0$,

(iii) $\tau(1) = 1$.

This trace extracts the constant term of the elements in A_θ^∞, i.e. $\tau(a) = a_{0,0}$ where

$a = \sum a_{m,n} U^m V^n \in A_\theta^\infty$. Indeed, this trace plays the role of integration and satisfies a kind of integration by parts identity:

$$\tau(\delta_i(a)b) = -\tau(a\delta_i(b)), \qquad a, b \in A_\theta^\infty, \quad i = 1, 2.$$

As we mentioned earlier, many ideas and structures have been tested and proved to be reasonable on the noncommutative 2-torus. In addition to what we have discussed so far, we can also mention a few more items. For example, A. Connes introduced a pseudodifferential calculus on the noncommutative 2-torus in [1.1] which is indeed the cornerstone for the noncommutative geometry. Moreover, A. Connes and P. Tretkoff proved a noncommutative Gauss-Bonnet theorem on the ncommutative 2-torus [1.4]. This theorem was extended to all conformal classes of metrics by M. Khalkhali and F. Fathizadeh [1.5]. It took three decades until the first purely geometric concept in the noncommutative settings was born. In fact, the scalar curvature of the noncommutative 2-torus was introduced and computed by A. Connes and H. Moscovici in [1.3], and independently by M. Khalkhali and F. Fathizadeh in [1.6].

One can also define a *noncommutative n-torus.* Indeed, the approach is the same as the case of 2-torus. In this case we need n unitaries subject to some commutation relations. We will work with the noncommutative 3-torus in Chapter 4.

1.4 Organization of the Thesis

In this thesis we do analysis and geometry on the noncommutative tori. In Chapter 2 we will gather the prerequisites needed in the thesis. Indeed, we will first introduce Sobolev spaces and a family of Sobolev type inequalities. Then the logarithmic Sobolev inequality [1.10] will be discussed. Moreover, the spin structure and the Dirac operator will be introduced. Finally, we shall briefly introduce the notion of a noncommutative Riemannian spin manifold, by defining spectral triples.

In Chapter 3, using a version of the logarithmic Sobolev inequality on the unit circle, introduced by Weissler [1.16], we will state a conjecture concerning a possible logarithmic Sobolev inequality on the noncommutative 2-torus. The conjecture states that for a positive element $a = \sum_{(m,n)\in\mathbb{Z}^2} a_{m,n} U^m V^n$ of the noncommutative 2-torus we have

$$\tau(a^2 \log a) \leqslant \sum_{(m,n)\in\mathbb{Z}^2} (|m| + |n|)|a_{m,n}|^2 + \|a\|_2^2 \log \|a\|_2,$$

where $\|a\|_2^2 = \tau(a^* a)$. We will prove this conjecture for a certain class of elements in the noncommutative 2-torus. Finally, in the last section of the chapter, we will show what we can do towards proving the conjecture for an arbitrary positive element and what the obstructions are.

In Chapter 4 we will use the ideas of [1.3], [1.4], [1.5] and [1.6] to define and compute the scalar curvature of the noncommutative 3-torus with a conformally perturbed metric. This is the first odd dimensional case among the noncommutative tori that have been studied so far. First, we will introduce the noncommutative 3-torus and then we shall study the different classes of conformally perturbed metrics on the noncommutative 3-torus. Moreover, Connes' pseudodifferential calculus on the noncommutative 3-torus will be introduced and using that we will define the scalar curvature of the noncommutative 3-torus. The main part of Chapter 4 will be dedicated to the computation of the scalar curvature. A version of the original rearrangement lemma in [1.3] is needed in the computations. So we shall manipulate the proof of that lemma and will prove a slightly different version using exactly the same method. Finally, we will give the formula for the scalar curvature of the noncommutative 3-torus with a conformally perturbed metric.

Bibliography

[1.1] A. Connes, *C^*-algèbres et géométrie différentielle*. C.R. Acad. Sc. Paris, t. 290, Série A, 599-604, 1980.

[1.2] A. Connes, *Noncommutative geometry*. Academic Press 1994.

[1.3] A. Connes and H. Moscovici, *Modular curvature for noncommutative two-tori*. J. Amer. Math. Soc., 27(3):639-684, 2014.

[1.4] A. Connes, P. Tretkoff, *The Gauss-Bonnet theorem for the noncommutative two torus*, *Noncommutative geometry, arithmetic, and related topics*, 141-158, Johns Hopkins Univ. Press, Baltimore, MD, 2011.

[1.5] F. Fathizadeh, M. Khalkhali, *The Gauss-Bonnet theorem for noncommutative two tori with a general conformal structure*, J. Noncommut. Geom., 6 (2012), no. 3, 457-480.

[1.6] F. Fathizadeh and M. Khalkhali, *Scalar curvature for the noncommutative two torus. J. Noncommut.* Geom., 7(4):1145-1183, 2013.

[1.7] G.B. Folland, *A Course in Abstract Harmonic Analysis*, CRC Press, Boca Raton, Fla. 1995

[1.8] I. M. Gelfand, M. A. Naimark, *On the embedding of normed rings into the ring of operators in Hilbert space*. Mat. Sb. 12, 197-213, 1943.

[1.9] J. M. Gracia-Bondía, J. C. Várilly, and H. Figueroa. *Elements of noncommutative geometry*. Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. Birkhäuser Boston, Inc., Boston, MA, 2001.

[1.10] L. Gross, *Logarithmic Sobolev Inequalities*, Amer. J. Math. 97 (1975), 1061-1083.

[1.11] M. Khalkhali, *Basic Noncommutative Geometry*. Second edition. EMS Series of Lectures in Mathematics. European Mathematical Society (EMS), Zürich, 2013.

[1.12] G. J. Murphy, *C*-algebras and operator theory*, Academic Press, London, 1990.

[1.13] M. A. Rieffel, *C*-algebras associated with irrational rotations*, Pacific J. Math. 93 (1981), no. 2, 415-429. MR623572 (83b:46087)

[1.14] R. Swan, *Vector bundles and projective modules*. Trans. Amer. Math. Soc. 105, 264-277, 1962.

[1.15] M.Takesaki, *Theory of Operator Algebras I*, Springer-Verlag, New York, 2002.

[1.16] F. Weissler, *Logarithmic Sobolev inequalities and hypercontractive estimates on the circle*, J. Funct. Anal. 37 (1980), 218-234.

Chapter 2

Preliminaries

2.1 The Logarithmic Sobolev Inequality

In this section we will first introduce Sobolev spaces and Sobolev inequalities. Then the logarithmic Sobolev inequality will be discussed. The main reference of this section is [2.1]. Throughout this section we shall use the Lebesgue measure on $\mathbb{R}^n$ unless explicitly stated otherwise.

Let $I = (a, b)$ be an open interval in $\mathbb{R}$ and ϕ be a differentiable function on $\mathbb{R}$ such that $\phi(a) = \phi(b) = 0$. Then using integration by parts, for a differentiable function u, we get

$$\int_a^b u(x)\phi'(x)dx = -\int_a^b u(x)'\phi(x)dx.$$

This motivates us to define a wider class of differentiable functions, namely weakly differentiable functions.

Definition 2.1.1. Let $I = (a, b)$ and $1 \leqslant p \leqslant \infty$. The *Sobolev space* $W^{1,p}(I)$ is defined to be

$$W^{1,p}(I) = \left\{ u \in L^p(I) : \exists g \in L^p(I) \text{ such that} \int_I u\phi' = -\int_I g\phi \quad \forall \phi \in C_c^1(I) \right\}.$$

An element $u \in W^{1,p}(I)$ is called *weakly differentiable* with derivative in $L^p(I)$. It can be shown that the function g in the definition is unique a.e.. We call g the *weak derivative* of u. For example, the function $u(x) = |x|$ on $I = (-1, 1)$ is an element of $W^{1,p}(I)$ for $1 \leqslant p \leqslant \infty$ and its weak derivative is $u' = g$, where

$$g(x) = \begin{cases} 1 & 0 < x < 1 \\ -1 & -1 < x < 0 \end{cases}.$$

We can immediately generalize the last definition to the domains in $\mathbb{R}^n$ and also to the higher order derivatives. First we generalize it to the domains in $\mathbb{R}^n$.

Definition 2.1.2. Let $\Omega \subset \mathbb{R}^n$ be an open set and $1 \leqslant p \leqslant \infty$. The Sobolev space $W^{1,p}(\Omega)$ is defined to be

$$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega) : \begin{array}{c} \exists g_1, g_2, \ldots, g_n \in L^p(\Omega) \text{ such that} \\ \int_\Omega u\dfrac{\partial_i \phi}{\partial x_i} = -\int_\Omega g_i\phi \quad \forall \phi \in C_c^\infty(\Omega) \ \forall i = 1, 2, \ldots, n \end{array} \right\}.$$

Moreover, for $u \in W^{1,p}(\Omega)$, we define $\dfrac{\partial u}{\partial x_i} = g_i$. The *weak gradient* of u is also defined by

$$\bigtriangledown u = \text{grad} u = (\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \dots, \frac{\partial u}{\partial x_n}).$$

The Sobolev space $W^{1,p}(\Omega)$ is a normed space with the norm defined by

$$\|u\|_{W^{1,p}} = \|u\|_p + \sum_{i=1}^{n} \|\frac{\partial u}{\partial x_i}\|_p,$$

where $\|.\|_p$ is the L^p norm. Now we define the higher order weak derivatives.

Definition 2.1.3. Let $\Omega \subset \mathbb{R}^n$ be an open set and $1 \leqslant p \leqslant \infty$. For an integer $m \geqslant 2$, the Sobolev space $W^{m,p}(\Omega)$ is defined to be

$$W^{m,p}(\Omega) = \left\{ u \in W^{m-1,p}(\Omega) : \ \frac{\partial u}{\partial x_i} \in W^{m-1,p}(\Omega) \ \ \forall i = 1, 2, \dots, n \right\},$$

or equivalently

$$W^{m,p}(\Omega) = \left\{ u \in L^p(\Omega) : \begin{array}{c} \forall \alpha \text{ with } |\alpha| \leqslant m, \exists g_\alpha \in L^p(\Omega) \text{ such that} \\ \int_\Omega u \mathcal{D}^\alpha \phi = (-1)^{|\alpha|} \int_\Omega g_\alpha \phi \quad \forall \phi \in C_c^\infty(\Omega) \end{array} \right\},$$

where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_m)$ is a *multi index*, i.e. an ordered m-tuple of nonnegative integers, and

$$\mathcal{D}^\alpha \phi = (\frac{\partial_1}{\partial x_1})^{\alpha_1} (\frac{\partial_2}{\partial x_2})^{\alpha_2} \cdots (\frac{\partial_m}{\partial x_m})^{\alpha_m} \phi.$$

Accordingly, we set $\mathcal{D}^\alpha u = g_\alpha$ and define a norm on $W^{m,p}(\Omega)$ by

$$\|u\|_{W^{m,p}} = \sum_{0 \leqslant |\alpha| \leqslant m} \|\mathcal{D}^\alpha u\|_p.$$

Proposition 2.1.1. Let $\Omega \subset \mathbb{R}^n$ be an open set, m be a nonnegative integer and $1 \leqslant p \leqslant \infty$. The Sobolev space $W^{m,p}(\Omega)$ is a Banach space with respect to the norm $\|.\|_{W^{m,p}}$.

Proof. See Proposition 9.1 of [2.1] for $m = 1$. □

Now we turn to the classic *Sobolev inequalities*. In what follows we introduce a family of inequalities known as Sobolev inequalities. A prototype of a Sobolev inequality is the following inequality which is also known as *Sobolev embedding*.

Theorem 2.1.4. *Let $I = (a, b)$ be a open interval in $\mathbb{R}$. There exists a constant C such that for $u \in W^{1,p}(I)$ and for $1 \leqslant p \leqslant \infty$,*

$$\|u\|_{L^\infty(I)} \leqslant C \|u\|_{W^{1,p}(I)}.$$

Proof. See Theorem 8.8 in [2.1]. □

The last theorem implies that one can embed $W^{1,p}(I)$ by a continuous injection in $L^\infty(I)$. Although this result is not valid for the dimensions higher than one, still we can embed $W^{1,p}(\Omega)$ in $L^{p^*}(\Omega)$ for some $p^* \in (p, \infty)$, by a continuous injection. The next inequality which is called *Sobolev, Gagliardo, Nirenberg inequality* does the job.

Theorem 2.1.5. *Let n be a positive integer and $1 \leqslant p < n$. Also let p^* be defined by $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$. There exists a constant $C_{p,n}$ such that for every $u \in W^{1,p}(\mathbb{R}^n)$,*

$$\|u\|_{p^*} \leqslant C_{p,n} \| \bigtriangledown u\|_p.$$

Proof. See Theorem 9.9 in [2.1]. □

To set the stage for the next inequality we need to introduce a new notation. For an open set $\Omega \subset \mathbb{R}^n$ and $1 \leqslant p < \infty$, we denote by $W_0^{1,p}(\Omega)$ the closure in $W^{1,p}(\Omega)$ of $C_c^1(\Omega)$, the space of continuously differentiable functions on Ω with compact supports. The next theorem states the *Poincaré inequality.*

Theorem 2.1.6. *Let Ω be a bounded open set in $\mathbb{R}^n$ and $1 \leqslant p < \infty$. There exists a constant C such that for every $u \in W_0^{1,p}(\Omega)$,*

$$\|u\|_{L^p(\Omega)} \leqslant C \| \bigtriangledown u\|_{L^p(\Omega)}.$$

The constant C depends on p and Ω.

Proof. See Proposition 9.18 and Corollary 9.19 in [2.1]. □

To move toward the logarithmic Sobolev inequality and justify its name we need to introduce Orlicz spaces. The first step to define these spaces is defining N- functions.

Definition 2.1.7. A strictly increasing continuous function $\Psi : [0, \infty) \longrightarrow [0, \infty)$ is called an *N-function* if

(a) Ψ is convex, i.e. for $s, t \geqslant 0$ and $0 < \lambda < 1$,

$$\Psi(\lambda t + (1 - \lambda)s) \leqslant \lambda\Psi(t) + (1 - \lambda)\Psi(s),$$

(b) $\lim_{t\to 0} \frac{\Psi(t)}{t} = 0$ and $\lim_{t\to\infty} \frac{\Psi(t)}{t} = \infty$,

(c) $\frac{\Psi(t)}{t}$ is increasing.

For example, the following functions are N-functions:

$$\Psi(t) = t^p \quad \text{for } 1 < p < \infty,$$

$$\Psi(t) = e^t - t - 1,$$

$$\Psi(t) = t^2 \ln t.$$

Now we can define Orlicz spaces.

Definition 2.1.8. Let Ω be an open set in $\mathbb{R}^n$, μ be a σ-finite measure on Ω and Ψ be an N-function. The linear span of the set of all equivalence classes of measurable functions $f : \Omega :\longrightarrow \mathbb{C}$ modulo equality a.e. such that

$$\int_\Omega \Psi(|f(x)|)\, d\mu(x) < \infty,$$

is denoted by $L_\Psi(\Omega, \mu)$ and is called an *Orlicz space.*

For example, for $1 < p < \infty$, $L^p(\Omega, \lambda)$ is an Orlicz space associated to the function $\Psi(t) = t^p$, where λ represents the Lebesgue measure.

Finally, in the next theorem we have Gross' *logarithmic Sobolev inequality.* For the proof of that see [2.7].

Theorem 2.1.9. *Let ν be the Gaussian measure on $\mathbb{R}^n$, i.e.*

$$d\nu(x) = \frac{1}{(2\pi)^{n/2}} e^{-|x|^2/2} dx,$$

where dx is the Lebesgue measure. Then for a smooth function f,

$$\int_{\mathbb{R}^n} |f(x)|^2 \ln |f(x)|\, d\nu(x) \leq \int_{\mathbb{R}^n} |\bigtriangledown f(x)|^2\, d\nu(x) + \|f\|_2^2 \ln \|f\|_2,$$

where $\|.\|_2$ is the L^2-norm with respect to the Gaussian measure.

The last theorem implies that if f and $\bigtriangledown f$ are in $L^2(\nu)$, then f is in the Orlicz space $L_\Psi(\nu)$, where $\Psi(t) = t^2 \ln t$. So we can think of this inequality as a logarithmic version of the Sobolev inequalities.

2.2 Pseudodifferential Operators and Asymptotic Expansion of the Heat Kernel

In this section we will briefly introduce the theory of pseudodifferential operators and then we shall explain how this theory helps one find an asymptotic expansion of the heat kernel. Indeed, we will gather the necessary material for Chapter 4. The main reference of this section is [2.5].

First we need to fix some notation. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ be a multi index, and $x = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m$. We define

$$|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_m,$$

$$\alpha! = \alpha_1! \alpha_2! \cdots \alpha_m!,$$

and

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_m^{\alpha_m}.$$

We also define the following operators on $C^\infty(\mathbb{R}^m)$, the space of the smooth functions on $\mathbb{R}^m$:

$$d_x^\alpha = (\frac{\partial_1}{\partial x_1})^{\alpha_1}(\frac{\partial_2}{\partial x_2})^{\alpha_2}\cdots(\frac{\partial_m}{\partial x_m})^{\alpha_m},$$

$$D_x^\alpha = (-i)^{|\alpha|} d_x^\alpha.$$

Now with the notations defined above we can define a differential operator on $C^\infty(\mathbb{R}^m)$.

Definition 2.2.1. Let d be a nonnegative integer. A *differential operator* P of order d is defined by

$$P = p(x, D) = \sum_{|\alpha|\leqslant d} a_\alpha(x) D_x^\alpha,$$

where a_α is a smooth function on $\mathbb{R}^m$. The *symbol* $\sigma P = p$ of this operator is also defined to be the polynomial

$$\sigma P = p(x, \xi) = \sum_{|\alpha|\leqslant d} a_\alpha(x) \xi^\alpha,$$

where $\xi \in \mathbb{R}^m$.

The *leading symbol* of P is a homogeneous polynomial in ξ defined by

$$\sigma_L P(x, \xi) = \sum_{|\alpha|=d} a_\alpha(x) \xi^\alpha.$$

Definition 2.2.2. A smooth function f on $\mathbb{R}^m$ is called a *Schwartz class* function if all its derivatives decrease at ∞ faster than the inverse of any polynomial i.e. for any multi indices α and β, there exists a constant $C_{\alpha,\beta}$ such that

$$|x^\alpha D_x^\beta f(x)| \leqslant C_{\alpha,\beta},$$

or equivalently for any multi index α and any nonnegative integer n, there exists a constant $C_{n,\alpha}$ such that

$$|D_x^\alpha f(x)| \leqslant C_{n,\alpha}(1 + |x|)^{-n}.$$

The set of all Schwartz class functions is denoted by $\mathcal{S}$.

Let $f \in \mathcal{S}$ and $\hat{f}$ be the Fourier transform of f, i.e.

$$\hat{f}(\xi) = \int e^{-ix\cdot\xi} f(x) dx.$$

Then one can show that

$$D_\xi^\alpha \hat{f}(\xi) = (-1)^{|\alpha|}\widehat{(x^\alpha f)}(\xi)$$

and

$$\xi^\alpha \hat{f}(\xi) = \widehat{(D_x^\alpha f)}(\xi).$$

Using the latter, if we apply the Fourier inversion formula

$$f(x) = \hat{\hat{f}}(-x) = \int e^{ix\cdot\xi} \hat{f}(\xi) d\xi$$

to Pf, we get

$$Pf(x) = \int e^{ix\cdot\xi} p(x,\xi)\hat{f}(\xi)d\xi = \int e^{i(x-y)\cdot\xi} p(x,\xi) f(y)dyd\xi, \tag{2.1}$$

where P is the differential operator with the symbol

$$p(x,\xi) = \sum_{|\alpha|\leqslant d} a_\alpha(x)\xi^\alpha.$$

In (2.1) even if we replace the polynomial $p(x,\xi)$ with a function possessing some nice properties that control the growth rate of the function and its derivatives, still we get a well defined operator from $\mathcal{S}$ to $\mathcal{S}$. This leads to defining the pseudodifferential operators.

Definition 2.2.3. Let $p(x,\xi)$ be a smooth complex valued function on $\mathbb{R}^m \times \mathbb{R}^m$ that has compact x support and let $d \in \mathbb{R}$. Then $p(x,\xi)$ is called a *symbol of order* d if for all pairs (α,β) of multi indices there exist constants $C_{\alpha,\beta}$ such that

$$|D_x^\alpha D_\xi^\beta p(x,\xi)| \leqslant C_{\alpha,\beta}(1+|\xi|)^{d-|\beta|}.$$

The set of all symbols of order d is denoted by S^d.

We can associate an operator $P(x,D) : \mathcal{S} \longrightarrow \mathcal{S}$ to a symbol $p(x,\xi) \in S^d$. Indeed, a *pseudodifferential operator* of order d with symbol $p(x,\xi)$ is defined by

$$P(x,D)(f)(x) = \int e^{ix\cdot\xi} p(x,\xi)\hat{f}(\xi)d\xi = \int e^{i(x-y)\cdot\xi} p(x,\xi) f(y)dyd\xi.$$

For example, if f is a smooth function with compact support, then

$$p(x,\xi) = f(x)(1+|\xi|^2)^{d/2}$$

is a symbol of order d.

We also define

$$S^{-\infty} = \bigcap_{d\in\mathbb{R}} S^d.$$

The elements of $S^{-\infty}$ are called *smoothing symbols*. The symbols p, q are said to be *equivalent* if $p - q \in S^{-\infty}$. We denote this relation by $p \sim q$. Let d_j be a sequence in $\mathbb{R}$ such that $d_j \longrightarrow -\infty$ and $p_j \in S^{d_j}$, for $j \in \mathbb{N}$. Then we say the symbol p has the *asymptotic expansion* $\sum_{j=1}^\infty p_j$ and we write $p \sim \sum_{j=1}^\infty p_j$, if for each d there exists an integer k_d, such that $p - \sum_{j=1}^k p_j \in S^d$, for all $k \geqslant k_d$. Now in the next proposition we will see that the space of symbols modulo the smoothing symbols is an algebra.

Proposition 2.2.1. Let P, Q be pseudodifferential operators with the symbols p, q respectively. Then PQ is a pseudodifferential operator with a symbol which is asymptotically given by

$$\sigma(PQ) \sim \sum_\alpha d_\xi^\alpha p \cdot D_x^\alpha q/\alpha!$$

Proof. See [2.5]. □

Since we are going to introduce the pseudodifferential operators on compact manifolds, we need to restrict the domain of the pseudodifferential operators. Let U be an open set in $\mathbb{R}^m$ with compact closure. Moreover, assume U includes the x support of $p(x,\xi) \in S^d$. If we restrict the domain of P to $C_c^\infty(U)$, clearly the range is also in $C_c^\infty(U)$, where $C_c^\infty(U)$ is the space of all smooth functions on U with compact support. We define $\Psi_d(U)$ to be the space of all pseudodifferential operators $P : C_c^\infty(U) \longrightarrow C_c^\infty(U)$ of order d. We also define the set of all *pseudodifferential operators* and the set of *smoothing pseudodifferential operators* on U respectively by

$$\Psi(U) = \bigcup_d \Psi_d(U)$$

and

$$\Psi_{-\infty}(U) = \bigcap_d \Psi_d(U).$$

Definition 2.2.4. Let U and U_1 be open sets in $\mathbb{R}^m$ such that $\overline{U}_1 \subset U$. A symbol $p \in S^d(U)$ is called *elliptic* on U_1 if there exists an open set U_2 such that $\overline{U}_1 \subset U_2 \subset \overline{U}_2 \subset U$ and if there exists a symbol $q \in S^{-d}$ such that $pq - 1 \in S^{-\infty}$ over U_2. A pseudodifferential operator is called *elliptic* if its symbol is elliptic.

Now we can extend the definition of pseudodifferential operators to compact manifolds.

Definition 2.2.5. Let M be a compact manifold without boundary and $C^\infty(M)$ be the space of smooth functions on M. A linear operator

$$P : C^\infty(M) \longrightarrow C^\infty(M)$$

is called a *pseudodifferential operator of order* d, if for every open chart U on M and all functions $\phi, \psi \in C_c^\infty(U)$, the operator $\phi P \psi \in \Psi_d(U)$ and it is called *elliptic* if $\phi P \psi$ is elliptic where $\phi\psi \neq 0$.

We denote the set of the pseudodifferential operators of degree d on M by $\Psi_d(M)$, and define the set of all pseudodifferential operators and the set of smoothing pseudodifferential operators on M respectively by

$$\Psi(M) = \bigcup_d \Psi_d(M)$$

and

$$\Psi_{-\infty}(M) = \bigcap_d \Psi_d(M).$$

To define the symbol of a pseudodifferential operator P on M we need to fix a coordinate chart. For a fixed coordinate the symbol $\sigma(P)$ is defined to be the symbol of $\phi P \phi$, where ϕ is a function which is identical to 1 near the point around which we have picked the chart. The symbol is unique modulo $S^{-\infty}$. Of course this symbol depends on the chart. If P, Q are pseudodifferential operators on M, then in every coordinate chart, the asymptotic expansion for PQ in Proposition 2.2.1 remains valid.

Let (M, g) be a Riemannian manifold of dimension n, and $L^2(M, \mathrm{dvol})$ be the space of square integrable functions on M with respect to the volume measure. The Laplace operator $\triangle$ is an elliptic differential operator of order 2 on $L^2(M, \mathrm{dvol})$, which is locally defined by

$$\triangle = -\sum_{j=1}^{n}\sum_{i=1}^{n} \frac{1}{\sqrt{|\det G|}}\partial_i(\sqrt{|\det G|}g^{ij}\partial_j),$$

where $G = (g_{ij})$, and $G^{-1} = (g^{ij})$.

Proposition 2.2.2. Let (M, g) be a closed oriented Riemannian manifold. The Laplace operator is an unbounded formally self adjoint operator on $L^2(M, \mathrm{dvol})$, and its spectrum is contained in the positive part of the real line.

Proof. See [2.2], Chapter 4, Corollaries 18, 19 and 20. □

Now we turn to the *heat operator* $e^{-t\triangle}$ for $t \geqslant 0$. The heat operator is a smoothing integral operator with a smooth kernel $K(t, x, y)$, which is called the *heat kernel.* In fact, the heat kernel is the fundamental solution of the heat equation

$$(\frac{\partial}{dt} + \triangle)f(t, x) = 0,$$

for $t \geqslant 0$ and $f(0, x) = f(x)$. In what follows, using the Cauchy integral formula and the pseudodifferential calculus, we will give an asymptotic expansion of the heat kernel.

Let (M, g) be a closed, oriented Riemannian manifold of dimension n, and $\triangle$ be the Laplace operator acting on $C^\infty(M)$, the algebra of smooth functions on M. Moreover, let γ be a contour going counterclockwise around the nonnegative part of the real axis without touching it. This way it goes around the spectrum of the Laplace operator. Therefore, we have the Cauchy integral formula

$$e^{-t\triangle} = \frac{1}{2\pi i}\int_\gamma e^{-t\lambda}(\triangle - \lambda)^{-1}d\lambda.$$

For $\lambda \notin [0, \infty)$, the operator $(\triangle - \lambda)^{-1}$ is not a pseudodifferential operator, but we shall approximate it by a pseudodifferential operator to find an asymptotic expansion of the heat kernel . To this end, we need to generalize the pseudodifferential theory that we discussed before.

Let $P : C^\infty(M) \longrightarrow C^\infty(M)$ be a self adjoint elliptic differential operator of order d such that its spectrum is contained in $[C, \infty)$ for some real number C and let γ be a contour going counterclockwise around $[C, \infty)$ without touching it. Also let $\mathcal{R}$ be the region in $\mathbb{C}$ consisting of γ plus the component of $\mathbb{C} - \gamma$ that does not contain $[C,\infty)$. For an open set U in $\mathbb{R}^m$ with compact closure and $\lambda \in \mathcal{R}$ we define:

Definition 2.2.6. The smooth function $q = q(x, \xi, \lambda)$ on $\mathbb{R}^m \times \mathbb{R}^m \times \mathcal{R}$ is called a *symbol of order k depending on the complex parameter $\lambda \in \mathcal{R}$* if it satisfies the following conditions:

(a) It has compact x-support in U, and is holomorphic in λ,

(b) For all multi indices α, β, γ, there exist constants $C_{\alpha,\beta,\gamma}$ such that

$$|D_x^\alpha D_\xi^\beta D_\lambda^\gamma q(x, \xi, \lambda)| \leqslant C_{\alpha,\beta,\gamma}(1 + |\xi| + |\lambda|^{1/d})^{k-|\beta|-d|\gamma|}.$$

The set of such symbols is denoted by $S^k(\lambda)(U)$. Moreover, the symbol $q(x, \xi, \lambda)$ is said to be *homogeneous of order k in* (ξ, λ) if for $t \geqslant 1$,

$$q(x, t\xi, t^d\lambda) = t^k q(x, \xi, \lambda).$$

We also denote the set of all operators $Q(\lambda) : C_c^\infty(U) \longrightarrow C_c^\infty(U)$ with symbols $q(x, \xi, \lambda) \in S^k(\lambda)(U)$ by $\Psi_k(\lambda)(U)$. Clearly for each $\lambda \in \mathcal{R}_P$, $Q(\lambda) \in \Psi_k(U)$. Moreover, we set

$$\Psi(\lambda)(U) = \bigcup_k \Psi_k(\lambda)(U).$$

This is the set of all pseudodifferential operators that depend on $\lambda \in \mathcal{R}$ and are defined over the open set U, and obviously depends on the order d that we fixed at the beginning and also the region $\mathcal{R}$.

Similarly, for the symbols q, q_j $(j \in \mathbb{N})$, we say $q \sim \sum_{j=1}^\infty q_j$ if for each $k > 0$ there exists $n(k)$ such that for $n \geqslant n(k)$, $q - \sum_{j=1}^n q_j \in S^{-k}(\lambda)(U)$.

The statement of Proposition 2.2.1 remains valid in the generalized case, i.e. if $P_1 \in \Psi_{k_1}(\lambda)(U)$ and $Q \in \Psi_{k_2}(\lambda)(U)$ have symbols p_1 and q respectively, then $P_1Q \in \Psi_{k_1+k_2}(\lambda)(U)$ and

$$\sigma(PQ) \sim \sum_\alpha d_\xi^\alpha p_1 \cdot D_x^\alpha q/\alpha!. \tag{2.2}$$

Finally we can define the *generalized pseudodifferential operators of order k* on a manifold M, in the way that we defined pseudodifferential operators on it. We denote this class of operators by $\Psi_k(\lambda)(M)$, and we set

$$\Psi(\lambda)(M) = \bigcup_k \Psi_k(\lambda)(M).$$

Recall that $P : C^\infty(M) \longrightarrow C^\infty(M)$ is a self adjoint elliptic differential operator of order d such that its spectrum is contained in $[C, \infty)$ for some real number C and assume that the symbol of P has the decomposition

$$\sigma P = p_d + p_{d-1} + \cdots + p_0,$$

where for $j = 0, 1, \cdots d$, p_j is a homogeneous polynomial of order j in ξ. Although for $\lambda \in \mathcal{R}$, $(P - \lambda)^{-1}$ is not a pseudodifferential operator, we are going to approximate it by a pseudodifferential operator $R(\lambda)$. Indeed, using the formula (2.2), we want to find $R(\lambda)$ such that

$$\sigma(R(\lambda)(P - \lambda)) - 1 \sim 0.$$

We will inductively find $R(\lambda) = r_0 + r_1 + r_2 + \cdots$, such that $r_j \in S^{-d-j}(\lambda)$. For $j < d$, we set $p'_j(x,\xi,\lambda) = p_j(x,\xi)$. We also set $p'_d(x,\xi,\lambda) = p_d(x,\xi) - \lambda$. Then we have

$$\sigma(P-\lambda) = p'_0 + p'_1 + \cdots + p'_d.$$

Applying the formula (2.2) to $R(\lambda)$ and $(P-\lambda)$, we get

$$\sum_\alpha \sum_j \sum_k d_\xi^\alpha r_j \cdot D_x^\alpha p'_k/\alpha! \sim 1.$$

In the above series we can group the homogeneous terms of order $-n$ together and write

$$\sum_n \sum_{|\alpha|+j+d-k=n} d_\xi^\alpha r_j \cdot D_x^\alpha p'_k/\alpha! \sim 1.$$

where $j,k \geqslant 0$ and $k \leqslant d$. In the series we do not have terms with $n < 0$. Considering the term with $n = 0$, we get $r_0 p'_d = 1$. Therefore, $r_0 = (p'_d)^{-1} = (p_d - \lambda)^{-1}$ and also

$$r_n = -r_0 \sum_{\substack{|\alpha|+j+d-k=n \\ j<n}} d_\xi^\alpha r_j \cdot D_x^\alpha p'_k/\alpha!. \tag{2.3}$$

The conditions $|\alpha| + j + d - k = n$ and $j < n$ imply that if $k = d$, then $|\alpha| > 0$. So in the sum in (2.3), $D_x^\alpha p'_k = D_x^\alpha p_k$ and we can write

$$r_n = -r_0 \sum_{\substack{|\alpha|+j+d-k=n \\ j<n}} d_\xi^\alpha r_j \cdot D_x^\alpha p_k/\alpha!. \tag{2.4}$$

We will use a slightly different version of (2.4) in Chapter 4. In fact, we shall use a noncommutative version of that in which r_0 is multiplied from the right.

Now we return to our special case where $P = \triangle$, the Laplace operator, and $d = 2$. In fact, we have all the materials needed to state the asymptotic expansion of the heat kernel except one. In what follows, for a nonnegative integer k we introduce the norm $|.|_{\infty,k}$ on $C^\infty(M)$, where M is a compact manifold of dimension n.

Let U be an open set in $\mathbb{R}^n$. We first define $|.|_{\infty,k}$ for a smooth function f with compact support in U by

$$|f|_{\infty,k} = \sup_{x\in U} \sum_{|\alpha|\leqslant k} |D_x^\alpha f(x)|.$$

Now to define $|.|_{\infty,k}$ on M, we choose a finite number of coordinate charts V_i with diffeomorphisms $h_i : U_i \longrightarrow V_i$, where U_i's are open sets in $\mathbb{R}^n$ with compact closure. For $f \in C_c^\infty(V_i)$ we define

$$|f|^{(i)}_{\infty,k} = |f(h_i)|_{\infty,k}\,.$$

Moreover, let $\{\psi_i\}$ be a partition of unity subordinate to the chosen cover. For $f \in C^\infty(M)$ we define

$$|f|_{\infty,k} = \sum_i |\psi_i f|^{(i)}_{\infty,k}\,.$$

It is known that this is independent of the choice of the cover and also the choice of the partition of unity.

Now we will state the main theorem of this section. For the proof of the next theorem one can see Sections 1.7 and 4.8 in [2.5]. With the notations that we have introduced we have:

Theorem 2.2.7. *Let (M, g) be a closed, oriented Riemannian manifold of dimension n, $\triangle$ be the Laplace operator acting on $C^\infty(M)$ and γ be a contour going counterclockwise around the nonnegative part of the real axis without touching it. Also for $x \in M$ and nonnegative integer m let*

$$a_{2m}(x) = \frac{1}{2\pi i} \iint_\gamma e^{-\lambda} r_{2m}(x, \xi, \lambda) d\lambda d\xi.$$

If $K(t, x, y)$ is the kernel of the heat operator $e^{-t\triangle}$, then $K(t, x, x)$ has the asymptotic expansion

$$K(t, x, x) \sim t^{-n/2} \sum_{m=0}^{\infty} a_{2m}(x) t^m$$

as $t \longrightarrow 0^+$, i.e. for any nonnegative integer k, there exists $m_k \in \mathbb{N}$ and a constant C_k such that

$$\left| K(t, x, x) - t^{-n/2} \sum_{m \leqslant m_k} a_{2m}(x) t^m \right|_{\infty, k} < C_k t^k \qquad \textit{for } \; 0 < t < 1.$$

Moreover, $a_2(x)$ is a constant multiple of the scalar curvature of M at the point $x \in M$.

In Chapter 2, based on the last theorem, and using an analogy we will define the scalar curvature of the noncommutative 3-torus.

2.3 Spin Structure and the Dirac Operator

In this section we will introduce the structure needed to define a Dirac operator on an oriented Riemannian manifold. For the proofs of the facts and theorems that we will state in this section refer to [2.10]. More details can also be found in [2.9].

Throughout this section for a smooth manifold M, by TM and T^*M, we mean, respectively, the tangent bundle and the cotangent bundle of M. Moreover, for a vector bundle E over M, we denote its space of smooth sections by $C^\infty(E)$. We start with the definition of connections.

Definition 2.3.1. Let M be a smooth manifold and E be a vector bundle over M. A connection on E is a linear map

$$\nabla : C^\infty(TM) \otimes C^\infty(E) \longrightarrow C^\infty(E), \qquad (X, Y) \to \nabla_X Y,$$

such that for a smooth function f on M, a smooth vector field X and a smooth section Y of E,

(i) $\nabla_{fX}Y = f\nabla_X Y$

(ii) $\nabla_X fY = f\nabla_X Y + (X \cdot f)Y$, where $X \cdot f$ is the directional derivative of f along X.

Note that we can also think of ∇ as a map from $C^\infty(E)$ to $C^\infty(T^*M \otimes E)$.

Let M be a manifold and ∇ be a connection on TM. Moreover, let $\gamma : [a, b] \longrightarrow M$ be a curve in M. A vector field X is called *parallel along* γ if $\nabla_{\dot\gamma}X = 0$, where $\dot\gamma$ is the derivative of γ. Let $t_0 \in [a, b]$. It is known that for a given vector X_0 in $T_{\gamma(t_0)}M$, the tangent space of M at $\gamma(t_0)$, there exists a unique parallel vector field X along γ such that $X_{t_0} = X_0$.

Let (M, g) be a Riemannian manifold. A connection ∇ on TM is called *torsion free* if for $X, Y \in TM$,

$$\nabla_X Y - \nabla_Y X = [X, Y],$$

where $[X, Y] = XY - YX$. A connection ∇ on TM is called *compatible with the metric* if for $X, Y, Z \in TM$,

$$X \cdot \langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle\,,$$

where $X \cdot \langle Y, Z\rangle$ is the directional derivative of $\langle Y, Z\rangle$ along X.

The fundamental theorem of Riemannian geometry asserts that every Riemannian manifold M has a unique torsion free connection on TM compatible with its metric. This connection is called the *Levi-Civita connection.*

Now we introduce principal bundles.

Definition 2.3.2. Let P and M be smooth manifolds and G be a Lie group that acts freely on P on the right. The manifold P is called a principal G bundle over M if

(i) $M = P/G$, and the canonical projection $\pi : P \longrightarrow M$ is differentiable,

(ii) P is a locally trivial fiber bundle with fiber G over M.

Let P be a principal G bundle over M. For $p \in P$, let T_pP be the tangent space of P at p. We denote the subspace of T_pP consisting of all vectors tangent to the fiber through p by V_p. Indeed, V_p is the kernel of π_{*p}. A *connection* on the principal G bundle P, by definition, is a choice of a subspace H_p of T_pP for each point $p \in P$ such that

(i) $T_pP = H_p \oplus V_p$,

(ii) $H_{pg} = (R_g)_*H_p$ for every $g \in G$, where $(R_g)_*$ is the derivative of the map

$$R_g : P \longrightarrow P, \qquad R_g(a) = ag,$$

(iii) H_p depends differentiably on p.

We call V_p and H_p respectively the *vertical subspace* and the *horizontal subspace* of T_pP. Every vector $X_p \in T_pP$ can be decomposed into $X = vX + hX$ such that $vX \in V_p$ and $hX \in H_p$. Condition (iii) above means that if X is a smooth vector field on P, then vX and hX are so. Since V_p is the kernel of π_{*p}, the restriction of π_{*p} to H_p is an isomorphism and given a vector $X_p \in T_{\pi(p)}$, one can find a unique lifting of X_p to T_pP which is horizontal. We denote this unique lifting by $X_p^{\#}$.

Given an n dimensional vector bundle E over a manifold M, we can construct a principal bundle. In fact, the bundle over M whose fiber at each point p consists of all frames of E_p, the fiber in E over p, is a principal $GL(n)$ bundle, where $GL(n)$ is the group of real $n \times n$ invertible matrices. This bundle is called the *frame bundle of E*.

Conversely, if we have a principal G bundle P over a manifold M, and a representation of G on a finite dimensional vector space V, we can construct a vector bundle over M. Indeed, suppose $\rho : G \longrightarrow GL(V)$ is a representation of G on V. Then we denote by $P \times_\rho V$ the quotient space of $P \times V$ by the action of G defined by

$$(e, v)g = (eg, \rho(g^{-1})v), \qquad e \in E, v \in V, g \in G.$$

The quotient space $P \times_\rho V$ is a vector bundle whose fibers are isomorphic to V and is called the vector bundle *associated to P by the representation ρ*. The space of sections of $P \times_\rho V$ can be identified with the set of all maps $\varphi : P \longrightarrow V$, such that for $p \in P$ and $g \in G$, $\varphi(p \cdot g) = \rho(g^{-1})\varphi(p)$.

Given a principal G bundle P over a manifold M endowed with a connection, and a representation ρ of G on the vector space V, we can define a connection on the vector bundle $P \times_\rho V$. In fact, we define

$$\nabla : C^\infty(TM) \otimes C^\infty(P \times_\rho V) \longrightarrow C^\infty(P \times_\rho V)$$

by $(\nabla_X \varphi)(p) = d_p\varphi(X_p^\#)$, where $p \in P, X \in TM, \varphi \in C^\infty(P \times_\rho V)$ and $X_p^\#$ is the unique lifting of X_p to T_pP which is horizontal. One can see that $\nabla_X \varphi$ is well defined, i.e. $(\nabla_X \varphi)(p \cdot g) = \rho(g^{-1})(\nabla_X \varphi)(p)$.

In order to define Dirac operators we need to define Clifford algebras.

Definition 2.3.3. Let V be a vector space endowed with a symmetric bilinear form $\langle \cdot, \cdot \rangle$. The unital algebra generated by V as a linear subspace, subject to the relations

$$uv + vu = -2 \langle u, v \rangle 1, \qquad u, v \in V,$$

is called the Clifford algebra, and is denoted by $\mathrm{Cl}(V)$.

We denote $\mathrm{Cl}(V)$ for $V = \mathbb{R}^n$ and $V = \mathbb{C}^n$, equipped with the standard Euclidean inner product, respectively by $\mathrm{Cl}(n)$ and $\mathbb{C}\mathrm{l}(n)$. Moreover, it is known that

$$\mathbb{C}\mathrm{l}(n) = \mathrm{Cl}(n) \otimes \mathbb{C}.$$

Let $\mathrm{Cl}(n)^\times$ be the group of invertible elements in $\mathrm{Cl}(n)$. The multiplicative subgroup of $\mathrm{Cl}(n)^\times$ generated by the unit vectors in $\mathbb{R}^n$ is denoted by $\mathrm{Pin}(n)$. We also define the Spin group by

$$\mathrm{Spin}(n) = \{u_1 u_2 \cdots u_{2k} : k \in \mathbb{N}, \text{ and for } j = 1, 2 \cdots 2k,\ u_j \in \mathbb{R}^n, |u_j| = 1\}.$$

Let $\mathrm{O}(n)$ be the group of orthogonal $n \times n$ matrices. We define $\rho : \mathrm{Pin}(n) \longrightarrow \mathrm{O}(n)$ by

$$\rho(u)(v) = uvu^*, \quad u \in \mathrm{Pin}(n), v \in \mathbb{R}^n,$$

where $u^* = x_k x_{k-1} \dots x_1$ if $u = x_1 x_2 \dots x_k$. One can check that ρ is well defined, i.e. $\rho(u) \in \mathrm{O}(n)$.

Let $\mathrm{SO}(n)$ be the group of orthogonal $n \times n$ matrices with determinant 1. It can be shown that $\rho(\mathrm{Spin}(n)) = \mathrm{SO}(n)$, i.e. if we restrict ρ to $\mathrm{Spin}(n)$, we have

$$\rho : \mathrm{Spin}(n) \longrightarrow \mathrm{SO}(n).$$

For $n \geqslant 2$, ρ is a covering map and $\mathrm{Spin}(n)$ is the universal 2-fold covering of $\mathrm{SO}(n)$ (see Proposition 4.7 in [2.10]). If n is even, the Clifford algebra $\mathbb{C}\mathrm{l}(n)$ has a unique irreducible representation whose restriction to $\mathrm{Spin}(n)$ is called the *spin representation*. However, the spin representation is not irreducible and is decomposed into two irreducible representations.

Let (M, g) be an oriented Riemannian manifold. For each $m \in M$, T_mM is equipped with a symmetric bilinear form $\langle \cdot, \cdot \rangle_g$ induced by the metric g. So we can construct $\mathrm{Cl}(T_mM)$. A *bundle of Clifford modules* over M is a bundle S whose fiber S_m over m is a left $\mathrm{Cl}(T_mM) \otimes \mathbb{C}$ module.

Definition 2.3.4. A *Clifford bundle* S over a Riemannian manifold M is a bundle of Clifford modules endowed with a Hermitian metric $\langle \cdot, \cdot \rangle_h$ and a connection ∇^c such that for $X, Y \in TM$, and $S, S_1, S_2 \in C^\infty(S)$,

(i) $X \cdot \langle S_1, S_2 \rangle_h = \langle \nabla^c_X S_1, S_2 \rangle_h + \langle S_1, \nabla^c_X S_2 \rangle_h$, where $X \cdot \langle S_1, S_2 \rangle_h$ is the directional derivative of $\langle S_1, S_2 \rangle_h$ in the direction of X,

(ii) $\langle X \cdot S_1, S_2 \rangle_h + \langle S_1, X \cdot S_2 \rangle_h = 0$,

(iii) $\nabla^c_X(Y \cdot S) = (\nabla_X Y) \cdot S + Y \cdot \nabla^c_X S$, where ∇ is the Levi-Civita connection on TM.

In the previous definition ∇^c is called a *Clifford connection*. Now we can define Dirac operators.

Definition 2.3.5. Let (M, g) be a Riemannian manifold and S be a Clifford bundle over M equipped with the Clifford connection ∇^c. A *generalized Dirac operator* is an operator $D : C^\infty(S) \longrightarrow C^\infty(S)$ which is defined as the composition of the following maps

$$C^\infty(S) \xrightarrow{\nabla^c} C^\infty(T^*M \otimes S) \longrightarrow C^\infty(TM \otimes S) \xrightarrow{c} C^\infty(S),$$

where the second map is induced by identifying TM and T^*M using the metric g, and c denotes the Clifford action.

We just saw that to define the generalized Dirac operators we need a Clifford bundle. The spin structure is what guarantees the existence of such a bundle. Let M be an oriented Riemannian manifold of dimension n, and F be the bundle over M whose fiber over each point $m \in M$ is the set of all oriented orthonormal frames of T_mM. The bundle F is a principal $\mathrm{SO}(n)$ bundle and is called the *bundle of oriented orthonormal frames for the tangent bundle of M*.

Definition 2.3.6. Let (M, g) be an oriented Riemannian manifold of dinension n and F be the bundle of oriented orthonormal frames for the tangent bundle of M. A *spin structure* on M is a principal $\mathrm{Spin}(n)$ bundle $\tilde{F}$ over M which is a double covering of F, and if we restrict the double covering $\tilde{F} \longrightarrow F$ to each fiber, we get the double covering $\rho : \mathrm{Spin}(n) \longrightarrow \mathrm{SO}(n)$. If M has a spin structure, it is called a *spin manifold*.

Let M be an even dimensional spin manifold with spin structure $\tilde{F}$. The vector bundle associated to $\tilde{F}$ by the spin representation is called the *spin bundle of* M. The spin bundle has a natural Hermitian metric (see [2.4], page 24). Our next goal is to construct a connection on the spin bundle of M.

It is known that ∇, the Levi-Civita connection on TM, induces a connection on F, the principal $\mathrm{SO}(n)$ bundle of the orthonormal frames of TM. Now since $\tilde{F}$ is a covering of F, we can lift the connection induced on F by the Levi-Civita connection to a connection on the principal $\mathrm{Spin}(n)$ bundle $\tilde{F}$. Using this lifted connection and the spin representation and applying the argument before Definition 2.3.3 we obtain a connection on the spin bundle. This connection is called the *spin connection* and is denoted by ∇^s. It can be checked that the spin bundle equipped with its Hermitian metric and the spin connection is a Clliford bundle.

Finally we can define the Dirac operator:

Definition 2.3.7. Let M be an even dimensional spin manifold, S be the spin bundle over M and ∇^s be the spin connection on S. The *Dirac operator* $D : C^\infty(S) \longrightarrow C^\infty(S)$ is defined as the composition of the following maps

$$C^\infty(S) \xrightarrow{\nabla^s} C^\infty(T^*M \otimes S) \longrightarrow C^\infty(TM \otimes S) \xrightarrow{c} C^\infty(S),$$

where the second map is induced by identifying TM and T^*M using the metric g, and c denotes the Clifford action.

2.4 Spectral Triples

In this section we shall introduce the notion of a spectral triple, which is indeed the noncommutative analogue of a Riemannian spin manifold.

Definition 2.4.1. Let A be an involutive algebra, H be a Hilbert space, $\pi : A \longrightarrow B(H)$ be a $*$-representation of A. Also let D be an unbounded self adjoint operator on H. The triple (A, H, D) is called a *spectral triple* if

(i) The operator D has a compact resolvent, i.e. for $\lambda \notin \mathbb{R}$, the operator $(D - \lambda)^{-1}$ is a compact operator,

(ii) For $a \in A$, $[D, \pi(a)]$ is a bounded operator, where $[D, \pi(a)] = D\pi(a) - \pi(a)D$.

The following example is a prototype of spectral triples:

Let M be a compact spin manifold, S be the spin bundle of M, and $L^2(S)$ be the Hilbert space of square integrable sections of the spin bundle S. We define

$$\pi : C^\infty(M) \longrightarrow B(L^2(S)), \quad f \mapsto M_f,$$

where M_f is the multiplication operator by f. If D is the Dirac operator, then the triple $(C^\infty(M), L^2(S), D)$ is a spectral triple which is called the *canonical triple*. Moreover, for $p, q \in M$, we have

$$d_g(p, q) = \sup_{f \in C^\infty(M)} \{|f(p) - f(q)| : \ \|[D, \pi(f)]\| \leqslant 1\},$$

where $d_g(p,q)$ is the geodesic distance between p and q (see [2.3], Section VI.1 for the proof). This distance formula means that we can reconstruct a geometric structure using an operator algebraic process.

Bibliography

[2.1] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer, New York, 2011.

[2.2] Y. Canzani, *Analysis on Manifolds via the Laplacian*, Harvard University, 2013.

[2.3] A. Connes, *Noncommutative geometry.* Academic Press (1994).

[2.4] T. Friedrich, *Dirac Operators in Riemannian Geometry*, Graduate Studies in Mathematics No. 25, AMS 2000.

[2.5] P. Gilkey, *Invariance theory, the heat equation, and the Atiyah-Singer index theorem*, Mathematics Lecture Series, 11. Publish or Perish, Inc., Wilmington, DE, 1984.

[2.6] J. M. Gracia-Bondía, J. C. Várilly, and H. Figueroa. *Elements of noncommutative geometry.* Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. Birkhäuser Boston, Inc., Boston, MA, 2001.

[2.7] L. Gross, *Logarithmic Sobolev Inequalities*, Amer. J. Math. 97 (1975), 1061-1083.

[2.8] M. Khalkhali, *Basic Noncommutative Geometry.* Second edition. EMS Series of Lectures in Mathematics. European Mathematical Society (EMS), Zürich, 2013.

[2.9] H. B. Lawson, JR. and M. L. Michelsohn. *Spin Geometry*, volume 38 of Princeton Mathematical Series. Princeton University Press, Princeton, NJ, 1989.

[2.10] J. Roe, *Elliptic Operators, Topology, and Asymptotic Methods*, Second Edition, Chapman & Hall/CRC Research Notes in Mathematics Series, CRC Press, Boca Raton, FL, 1999.

Chapter 3

A Logarithmic Sobolev Inequality for the Noncommutative 2-Torus

3.1 Introduction

The subject of logarithmic Sobolev inequalities has its roots in the paper of E. Nelson [3.11], where he proved the contractivity of the semi-group generated by the Gauss-Dirichlet form operator. Shortly after that, L. Gross introduced a new logarithmic Sobolev inequality in [3.6] and using this gave a different proof of the contractivity of the semi-group generated by the Gauss-Dirichlet form operator.

Let ν be the Gauss measure on $\mathbb{R}^n$ and

$$N : D(N) \subseteq L^2(\mathbb{R}^n, \nu) \longrightarrow L^2(\mathbb{R}^n, \nu),$$

be the Gauss-Dirichlet form operator defined by

$$\langle Nf, g \rangle = \int_{\mathbb{R}^n} \triangledown f(x) \cdot \triangledown g(x) d\nu(x)$$

where $\langle f, g \rangle = \int_{\mathbb{R}^n} f(x)\overline{g(x)} d\nu(x)$, and $\triangledown f$ is the weak gradient of f. Nelson showed that for $1 < q \leq p < \infty$, if

$$e^{-2t} \leq \frac{(q-1)}{(p-1)},$$

then

$$\|e^{-tN}\|_{q\to p} \leq 1,$$

where

$$\|e^{-tN}\|_{q\to p} = \sup\left\{ \|e^{-tN} f\|_p : f \in L^2(\nu) \cap L^q(\nu), \|f\|_q \leqslant 1 \right\}.$$

This means that for

$$t \geqslant \ln \sqrt{\frac{p-1}{q-1}},$$

e^{-tN} is a contraction from $L^q(\mathbb{R}^n, \nu)$ to $L^p(\mathbb{R}^n, \nu)$. He proved more. Indeed, he showed that e^{-tN} is an unbounded operator from $L^q(\mathbb{R}^n, \nu)$ to $L^p(\mathbb{R}^n, \nu)$ if

$$t < \ln \sqrt{\frac{p-1}{q-1}}.$$

The classical Sobolev inequality states that for $f \in C_c^\infty(\mathbb{R}^n)$,

$$\|f\|_{L^q(\mathbb{R}^n,dx)} \leq C_{p,n} \| \, |\bigtriangledown f| \, \|_{L^p(\mathbb{R}^n,dx)} \tag{3.1}$$

where $1 \leq p < \infty$, $\dfrac{1}{q} = \dfrac{1}{p} - \dfrac{1}{n}$, dx is the Lebesgue measure and $C_{p,n}$ is a constant depending only on n and p. So (3.1) implies that if the gradient of the function f is in $L^p(\mathbb{R}^n, dx)$, then f must be in $L^q(\mathbb{R}^n, dx)$. These inequalities strongly depend on the dimension of $\mathbb{R}^n$.

In [3.6], Gross proved the logarithmic Sobolev inequality

$$\int_{\mathbb{R}^n} |f(x)|^2 \ln |f(x)| \, d\nu(x) \leq \int_{\mathbb{R}^n} |\bigtriangledown f(x)|^2 \, d\nu(x) + \|f\|_2^2 \ln \|f\|_2 \tag{3.2}$$

for a smooth function f, and showed that this inequality is equivalent to Nelson's result of contractivity that we just mentioned.

Unlike the classical Sobolev inequality, Gross' logarithmic Sobolev inequality is dimension independent. Using (3.2) we see that if the function f and its gradient are in $L^2(\mathbb{R}^n, \nu)$, then f is in the Orlicz space $L^2 \ln L(\nu)$. This somehow justifies the name logarithmic Sobolev inequality. Gross also derived in [3.6] a weaker version of (3.2), from Nirenberg's form of the classical Sobolev inequality [3.12]. This version, not surprisingly depends on the dimension.

Since then people have given various proofs of logarithmic Sobolev inequalities by different methods: O. Rothaus [3.15] has proved it using Jensen's inequality and the positivity of the lowest eigenfunction for a Sturm-Liouville boundary value problem with Dirichlet boundary conditions. R. A. Adams and F. H. Clarke also have given a simple proof based on the calculus of variations [3.1].

One can also replace $(\mathbb{R}^n, \nu)$ with a probability space (X, μ) and the Gauss-Dirichlet form with a densely defined positive quadratic form on $L^2(X, \mu)$, say $\mathcal{E}$. Then we say the logarithmic Sobolev inequality holds for $\mathcal{E}$ if, for $f \in Dom(\mathcal{E})$,

$$\int_X |f(x)|^2 \ln |f(x)| \, d\mu(x) \leq \mathcal{E}(f, f) + \|f\|_2^2 \ln \|f\|_2.$$

This way one can talk about logarithmic Sobolev inequalites on Riemannian manifolds [3.8].

F. Weissler has proved in [3.17] a logarithmic Sobolev inequality on the circle. Indeed, using Fourier series he has shown that for a positive function f in $L^2(\mathbb{T}, \mu)$, where $\mathbb{T}$ is the unit circle and μ is the normalized Lebesgue measure, if

$$f(\theta) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta},$$

then

$$\int_{\mathbb{T}} f^2 \log f d\mu \leq \sum_{n=-\infty}^{\infty} |n||a_n|^2 + \|f\|_2^2 \log \|f\|_2. \tag{3.3}$$

Since

$$\sum_{n=-\infty}^{\infty} |n||a_n|^2 \leq \sum_{n=-\infty}^{\infty} |n|^2|a_n|^2 = \| \bigtriangledown f\|_2^2 = \int_{\mathbb{T}} |\bigtriangledown f|^2 \, d\mu,$$

Weissler's result is even stronger than the original logarithmic Sobolev inequality

$$\int_{\mathbb{T}} f^2 \log f d\mu \leq \int_{\mathbb{T}} |\bigtriangledown f|^2 \, d\mu + \|f\|_2^2 \log \|f\|_2.$$

There is a useful survey of related topics and applications of logarithmic Sobolev inequalities by Gross in [3.7]. One can also find more references therein.

Since the introduction of noncommutative geometry by Alain Connes in [3.3] (see also [3.4]), noncommutative tori have proved to be an invaluable tool to understand and test many aspects of noncommutative geometry that are not present in the commutative case. The results are simply too many to be cited here. The present paper should be seen as a step in understanding aspects of measure theory and analysis on noncomutative tori that have been largely untouched so far. The combinatorial challenges one faces in extending the logarithmic Sobolev inequality, at least in the form that we understand it, seemed to us as a very interesting problem by itself.

Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$. The universal unital C*-algebra generated by two unitaries U, V such that $UV = e^{2\pi i\theta} VU$, is called an irrational rotation algebra and is denoted by A_θ. It is a simple algebra and has a unique positive faithful normalized trace τ. The C*-algebra A_θ is the noncommutative deformation of $C(\mathbb{T}^2)$, the algebra of continuous functions on the 2-torus. More details about A_θ can be found in [3.4, 3.9, 3.13]. Let

$$A_\theta^\infty = \left\{ \sum_{(m,n)\in\mathbb{Z}^2} a_{m,n} U^m V^n : a_{m,n} \text{ is rapidly decreasing} \right\}.$$

By rapidly decreasing we mean for all $k \in \mathbb{N}$,

$$\operatorname*{Sup}_{(m,n)\in\mathbb{Z}^2} (1 + m^2 + n^2)^k |a_{m,n}|^2 < \infty.$$

The set A_θ^∞ is a dense subalgebra of the irrational rotation algebra and it is the analogue of $C^\infty(\mathbb{T}^2)$, the algebra of smooth functions on the 2-torus. The algebra A_θ^∞ is called the noncommutative 2-torus.

Let $a = \sum_{(m,n)\in\mathbb{Z}^2} a_{m,n} U^m V^n$ be in A_θ^∞. Then

$$\begin{aligned} a^* &= \sum_{(m,n)\in\mathbb{Z}^2} \overline{a_{m,n}} V^{-n} U^{-m} \\ &= \sum_{(m,n)\in\mathbb{Z}^2} \overline{a_{m,n}} e^{-2\pi i mn\theta} U^{-m} V^{-n} \\ &= \sum_{(m,n)\in\mathbb{Z}^2} \overline{a_{-m,-n}} e^{-2\pi i mn\theta} U^m V^n. \end{aligned}$$

So if $a = a^*$, we have

$$a_{m,n} = \overline{a_{-m,-n}} e^{-2\pi i m n \theta}. \tag{3.4}$$

Moreover, if $b = \sum_{(m,n)\in\mathbb{Z}^2} b_{m,n} U^m V^n \in A_\theta^\infty$, we have

$$ab = \sum_{(m,n)\in\mathbb{Z}^2} c_{p,q} U^p V^q,$$

where

$$c_{p,q} = \sum_{(m,n)\in\mathbb{Z}^2} a_{m,n} b_{p-m,q-n} e^{-2\pi i (p-m) n \theta} \tag{3.5}$$

The unique trace τ on A_θ which plays the role of integration in the noncommutative setting, extracts the constant term of the elements of A_θ^∞, i.e. $\tau(a) = a_{0,0}$. This trace can be used to define an L^2-norm on A_θ^∞ by

$$\|a\|_2^2 := \tau(a^* a).$$

Using (3.4) and (3.5) one can show $\|a\|_2^2 = \sum_{(m,n)\in\mathbb{Z}^2} |a_{m,n}|^2$. One can also define Sobolev norms on A_θ^∞. For more details see [3.14], where J. Rosenberg has developed the Sobolev theory on the noncommutative 2-torus.

In this paper we will use Weissler's method [3.17] to prove a logarithmic Sobolev inequality for a class of elements of the noncommutative 2-torus. In Section 2 we will prove some lemmas that we will need later on. In Section 3 we will first state our conjecture about a logarithmic Sobolev inequality for the noncommutative 2-torus and then we will prove that conjecture for a class of elements of the noncommutative 2-torus. This would be the main result of this paper. Although we have not been able to prove the logarithmic Sobolev inequality for arbitrary positive elements, we think the inequality must hold for those elements as well. In Section 4 we will try to generalize the proof of the main result to prove the conjecture, but in the middle of the way we will see that we will face a problem. We hope that we can bypass this problem in a follow-up paper.

3.2 Preliminaries

In this section we will prove some technical lemmas that will be needed later on.

Lemma 3.2.1. *Let G be an analytic function in some complex neighborhood of the interval $[0,1]$. Suppose all the coefficients in the power series expansion of G around $r = 0$ are nonnegative. Then $G(1) \geq 0$.*

Proof. First we show that we can find finitely many points

$$0 = x_0 < x_1 < x_2 < \ldots < x_n = 1$$

in $[0,1]$ and finitely many discs $D_0, D_1, D_2, \ldots, D_n$ such that x_i for $i = 0, 1, \ldots, n$, is the center of D_i, G has a power series expansion around x_i on D_i and $x_i \in D_{i-1}$, for

$i = 1, 2, \ldots, n$. To show this let N be the open set in $\mathbb{C}$ containing $[0, 1]$ on which G is analytic. Define

$$F : [0, 1] \longrightarrow \mathbb{R}^{>0}$$

by sending $r \mapsto \operatorname{dist}(r, N^c)$. The function F is continuous on a compact set, so it attains its minimum. Let δ be the minimum of F and

$$x_0 = 0, x_1 = \frac{\delta}{2}, x_2 = \delta, x_3 = \frac{3\delta}{2}, \ldots, x_{n-1} = \frac{(n-1)\delta}{2}, x_n = 1,$$

where $n = \lfloor \frac{2}{\delta} \rfloor + 1$. For $i = 0, 1, \ldots, n$, let R_i be the radius of convergence of the power series expansion of G around x_i, and D_i be the disc centred at x_i with radius R_i. For $i = 0, 1, \ldots, n$, we have $\frac{\delta}{2} < R_i$. So $x_i \in D_{i-1}$, for $i = 1, 2, \ldots, n$.

Let

$$G(z) = \sum_{k=0}^{\infty} \frac{G^{(k)}(0)}{k!} z^k \tag{3.6}$$

be the power series expansion of G around 0 on D_0. Since $x_1 \in D_0$, we evaluate z as x_1 in (3.6), and since for $k \geq 0$, $G^{(k)}(0) \geq 0$, we have

$$G(x_1) = \sum_{k=0}^{\infty} \frac{G^{(k)}(0)}{k!} x_1^k \geq 0.$$

If we substitute x_1 into the derivative of (3.6), we will get

$$G^{(1)}(x_1) = \sum_{k=1}^{\infty} \frac{G^{(k)}(0)}{(k-1)!} x_1^{k-1}$$

which is non-negative by the same reason. Differentiating (3.6) repeatedly, we can show that the derivatives of G at x_1 which form the coefficients of the power series expansion of G around x_1 on D_1 are nonnegative.

Repeating this argument, we can show that all derivatives of G at each x_i and in particular at $x_n = 1$ are non-negative. So $G(1) \geq 0$. $\square$

We will need the following standard and elementary result of spectral theory in C*-algebras.

Proposition 3.2.1. Let A be a C*-algebra, $x \in A$ and N an open subset of $\mathbb{C}$ containing $\sigma(x)$, the spectrum of x. Then there exists $\delta > 0$, such that for $y \in A$, $\|y - x\| < \delta$ implies $\sigma(y) \subseteq N$.

Proof. See Theorem 10.20 in [3.16]. $\square$

The following proposition will be needed in the proof of the main result of this paper.

Proposition 3.2.2. Let $a = \sum_{(m,n)\in\mathbb{Z}^2} a_{m,n}U^mV^n$ be in A_θ^∞, such that $a > 0$, $a_{0,0} = 1$ and at most finitely many number of $a_{m,n}$'s are nonzero. For $r \in \mathbb{C}$, we put

$$x_r = \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} a_{m,n}r^{(|m|+|n|)}U^mV^n$$

and $P_r(a) = 1 + x_r$. Then there is an open neighborhood W of $[0,1]$ in $\mathbb{C}$, such that for all r in W, $\log P_r(a)$ can be defined.

Proof. Since a is self-adjoint, using (3.4) for real r we have

$$a_{m,n}r^{(|m|+|n|)} = \overline{a_{-m,-n}}r^{(|-m|+|-n|)}e^{-2\pi imn\theta}.$$

So $P_r(a)$ is self-adjoint for real r, and consequently the spectrum of $P_r(a)$ is real for real r. Now we show for $0 \leq r \leq 1$, $P_r(a)$ is a strictly positive element. Suppose for some $0 \leq r \leq 1$, $P_r(a)$ is not strictly positive. Let $[t_0, t_1]$ be the smallest closed interval containing the spectrum of $P_r(a)$. We know that there exists a state ϕ of A_θ, such that $\phi(P_r(a)) = t_0 \leq 0$. Now let

$$M = \{(m,n) \in \mathbb{Z}^2 : \, (m,n) \neq 0, a_{m,n} \neq 0\},$$

$$M_1 = \{(m,n) \in M : \, m \geqslant 0, n \geqslant 0\},$$
$$M_2 = \{(m,n) \in M : \, m > 0, n < 0\}.$$

Then since a is self-adjoint, using (3.4) we have

$$a = 1 + \sum_{(m,n)\in M} a_{m,n}U^mV^n \tag{3.7}$$

$$= 1 + \sum_{(m,n)\in M_1\bigcup M_2} a_{m,n}U^mV^n + \sum_{(-m,-n)\in M_1\bigcup M_2} a_{m,n}U^mV^n$$
$$= 1 + \sum_{(m,n)\in M_1\bigcup M_2} a_{m,n}U^mV^n + \sum_{(m,n)\in M_1\bigcup M_2} a_{-m,-n}U^{-m}V^{-n}$$
$$= 1 + \sum_{(m,n)\in M_1\bigcup M_2} a_{m,n}U^mV^n + \sum_{(m,n)\in M_1\bigcup M_2} \overline{a_{m,n}}e^{-2\pi imn\theta}U^{-m}V^{-n}$$
$$= 1 + \sum_{(m,n)\in M_1\bigcup M_2} a_{m,n}U^mV^n + \sum_{(m,n)\in M_1\bigcup M_2} \overline{a_{m,n}}e^{-2\pi imn\theta}e^{2\pi imn\theta}V^{-n}U^{-m}$$
$$= 1 + \sum_{(m,n)\in M_1\bigcup M_2} a_{m,n}U^mV^n + \sum_{(m,n)\in M_1\bigcup M_2} \overline{a_{m,n}}V^{-n}U^{-m}.$$

By the same reasoning we can show

$$P_r(a) = 1 + \sum_{(m,n)\in M} a_{m,n}r^{(|m|+|n|)}U^mV^n$$

$$= 1 + \sum_{(m,n)\in M_1 \bigcup M_2} a_{m,n} r^{(|m|+|n|)} U^m V^n$$

$$+ \sum_{(m,n)\in M_1 \bigcup M_2} \overline{a_{m,n}} r^{(|m|+|n|)} V^{-n} U^{-m}.$$

Since a is strictly positive, using (3.7) we see that

$$\phi(a) = 1 + \sum_{(m,n)\in M_1 \bigcup M_2} a_{m,n} \phi(U^m V^n)$$

$$+ \sum_{(m,n)\in M_1 \bigcup M_2} \overline{a_{m,n}} \phi(V^{-n} U^{-m}) > 0.$$

Let $h_{m,n} = a_{m,n}\phi(U^m V^n)$. Then regarding the fact that

$$\phi(U^m V^n) = \overline{\phi(V^{-n} U^{-m})},$$

we have

$$\phi(a) = 1 + \sum_{(m,n)\in M_1 \bigcup M_2} (h_{m,n} + \overline{h_{m,n}}) > 0. \tag{3.8}$$

On the other hand,

$$\phi(P_r(a)) = 1 + \sum_{(m,n)\in M_1 \bigcup M_2} a_{m,n} r^{(|m|+|n|)} \phi(U^m V^n)$$

$$+ \sum_{(m,n)\in M_1 \bigcup M_2} \overline{a_{m,n}} r^{(|m|+|n|)} \phi(V^{-n} U^{-m}) = t_0 \leq 0.$$

So

$$\phi(P_r(a)) = 1 + \sum_{(m,n)\in M_1 \bigcup M_2} r^{(|m|+|n|)} (h_{m,n} + \overline{h_{m,n}}) \leq 0. \tag{3.9}$$

Then let

$$r^{(|m_0|+|n_0|)} = \mathrm{Min}\left\{ r^{(|m|+|n|)} : (m,n) \in M \right\}$$

and note that $0 \leq r^{(|m_0|+|n_0|)} \leq 1$. Now we have two cases. Either

$$-1 < \sum_{(m,n)\in M_1 \bigcup M_2} (h_{m,n} + \overline{h_{m,n}}) \leq 0,$$

or

$$\sum_{(m,n)\in M_1 \bigcup M_2} (h_{m,n} + \overline{h_{m,n}}) > 0. \tag{3.10}$$

In the first case, since

$$\sum_{(m,n)\in M_1 \bigcup M_2} (h_{m,n} + \overline{h_{m,n}}) \leq r^{(|m_0|+|n_0|)} \sum_{(m,n)\in M_1 \bigcup M_2} (h_{m,n} + \overline{h_{m,n}}),$$

we have

$$-1 < \sum_{(m,n)\in M_1 \bigcup M_2} r^{(|m_0|+|n_0|)}(h_{m,n} + \overline{h_{m,n}})$$

$$\leq \sum_{(m,n)\in M_1 \bigcup M_2} r^{(|m|+|n|)}(h_{m,n} + \overline{h_{m,n}}).$$

So

$$\sum_{(m,n)\in M_1 \bigcup M_2} r^{(|m|+|n|)}(h_{m,n} + \overline{h_{m,n}}) > -1,$$

which contradicts (3.9). In the second case again we have

$$\sum_{(m,n)\in M_1 \bigcup M_2} r^{(|m_0|+|n_0|)}(h_{m,n} + \overline{h_{m,n}})$$

$$\leq \sum_{(m,n)\in M_1 \bigcup M_2} r^{(|m|+|n|)}(h_{m,n} + \overline{h_{m,n}}) \leq -1,$$

which means

$$\sum_{(m,n)\in M_1 \bigcup M_2} r^{(|m_0|+|n_0|)}(h_{m,n} + \overline{h_{m,n}})$$

is strictly negative. But this contradicts (3.10), for

$$\sum_{(m,n)\in M_1 \bigcup M_2} r^{(|m_0|+|n_0|)}(h_{m,n} + \overline{h_{m,n}}),$$

and

$$\sum_{(m,n)\in M_1 \bigcup M_2} (h_{m,n} + \overline{h_{m,n}})$$

have the same signs.

Then we show there exist $B_1, B_2 > 0$ such that for $0 \leq r \leq 1$, $\sigma(P_r(a)) \subseteq [B_1, B_2]$, where $\sigma(P_r(a))$ is the spectrum of $P_r(a)$. Since for $0 \leq r \leq 1$,

$$\|P_r(a)\| = \|1 + \sum_{(m,n)\in M} a_{m,n} r^{(|m|+|n|)} U^m V^n\|$$

$$\leq 1 + \sum_{(m,n)\in M} |a_{m,n}| r^{(|m|+|n|)} \|U^m V^n\| \leq 1 + \sum_{(m,n)\in M} |a_{m,n}|,$$

and since the spectral radius of $P_r(a)$ is less than $\|P_r(a)\|$, it suffices to put

$$B_2 = 1 + \sum_{(m,n)\in M} |a_{m,n}|.$$

Now suppose there is no such B_1. So for each $n > 0$ there exists $r_n \in [0,1]$, and $\lambda_n \in (0, \frac{1}{n})$, such that $\lambda_n \in \sigma(P_{r_n}(a))$. Obviously $\lim_{n\to\infty} \lambda_n = 0$. Since $\{r_n\}_{n=1}^{\infty}$ is a bounded sequence, it has a convergent subsequence. For simplicity we will call that sequence again $\{r_n\}_{n=1}^{\infty}$. Let $\lim_{n\to\infty} r_n = r_0$. Then $\lim_{n\to\infty} P_{r_n}(a) = P_{r_0}(a)$. Let $\mathrm{Inv}(A_\theta)$ be the set of invertible elements in A_θ. It is an open set, hence its complement is closed. Then for $n > 0$, since $\lambda_n \in \sigma(P_{r_n}(a))$,

$$P_{r_n}(a) - \lambda_n 1 \notin \mathrm{Inv}(A_\theta).$$

Then

$$\lim_{n\to\infty} P_{r_n}(a) - 1\lambda_n = P_{r_0}(a) \notin \mathrm{Inv}(A_\theta),$$

which means $0 \in \sigma(P_{r_0}(a))$. But this is a contradiction, for we have shown for $0 \leq r \leq 1$, $P_r(a)$ is strictly positive.

Now we pick a neighborhood of $[B_1, B_2]$ away from the y-axis. Let

$$N = \left\{ x + iy : \frac{2B_1}{3} \leq x \leq B_2 + 1, -1 \leq y \leq 1 \right\}.$$

Clearly for $0 \leq r \leq 1$, $\sigma(P_r(a)) \subseteq N$. So by Proposition 3.2.1, for $0 \leq r \leq 1$, there exists δ_r such that for $y \in A_\theta$, $\|y - P_r(a)\| < \delta_r$ implies $\sigma(y) \subseteq N$. Since

$$P(a) : \mathbb{C} \longrightarrow A_\theta, \quad r \mapsto P_r(a),$$

is a continuous map, for δ_r there exists $\gamma_r > 0$, such that for $r' \in \mathbb{C}$, $|r' - r| \leq \gamma_r$ implies $\|P_{r'}(a) - P_r(a)\| < \delta_r$. So if $r' \in B_{\gamma_r}(r)$, then $\sigma(P_{r'}(a)) \subseteq N$ where $B_{\gamma_r}(r)$ is the 2-dimensional open ball centred at r with radius γ_r. Now let

$$W = \bigcup_{0 \leq r \leq 1} B_{\gamma_r}(r).$$

Obviously W is a complex open neighborhood of the interval $[0, 1]$ and the way that we have constructed W implies if $r \in W$, then $\sigma(P_r(a) \subseteq N$. Since N is in the right half plane, using the standard branch of the logarithm, for $r \in W$, we can define $\log P_r(a)$. $\square$

3.3 The Main Result

In this section we will first state our conjecture about a logarithmic Sobolev inequality for the noncommutative 2-torus and then we will prove it for certain elements.

Conjecture 3.3.1. *Let $a = \sum_{(m,n)\in\mathbb{Z}^2} a_{m,n} U^m V^n$ be in A_θ^∞ and assume $a > 0$. Then*

$$\tau(a^2 \log a) \leqslant \sum_{(m,n)\in\mathbb{Z}^2} (|m| + |n|)|a_{m,n}|^2 + \|a\|_2^2 \log \|a\|_2, \tag{3.11}$$

which is the same as

$$\tau(a^2 \log a) \leqslant \sum_{(m,n)\in\mathbb{Z}^2} (|m| + |n|)|a_{m,n}|^2 + \tau(a^2) \log(\tau(a^2))^{1/2}.$$

Our main goal was of course to prove the conjecture in general using Weissler's method [3.17], however, because of the noncommutativity, in the last step we encountered a technical problem. So we decided to restrict ourselves to a certain class of elements. Now we will prove the conjecture for the case $m = sn$ for some $s \in \mathbb{Z} \setminus \{0\}$ and later on in Section 4 we will give more details of what we have set up for the general case and explain what the problem is in this setting.

Theorem 3.3.2. *Let $a = \sum_{n\in\mathbb{Z}} a_n U^n V^{sn}$ be in A_θ^∞ where $s \in \mathbb{Z} \setminus \{0\}$, and such that $a > 0$. Then*

$$\tau(a^2 \log a) \leqslant \sum_{n\in\mathbb{Z}} (1+|s|)|n||a_n|^2 + \|a\|_2^2 \log \|a\|_2, \tag{3.12}$$

which is the same as

$$\tau(a^2 \log a) \leqslant \sum_{n\in\mathbb{Z}} (1+|s|)|n||a_n|^2 + \tau(a^2) \log(\tau(a^2))^{\frac{1}{2}}.$$

Proof. First suppose $\tau(a) = 1$, i.e. $a_0 = 1$, and suppose that at most finitely many number of a_n's are nonzero. Put $x = a - 1 = \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} a_n U^n V^{sn}$. Using the fact that $\|a\|_2^2 = 1 + \|x\|_2^2$, it can be shown that

$$\|a\|_2^2 \log \|a\|_2 \geqslant \frac{1}{2}\|x\|_2^2.$$

Following Weissler, we shall prove the theorem by proving the stronger inequality

$$0 \leqslant \sum_{n\in\mathbb{Z}} (1+|s|)|n||a_n|^2 + \frac{1}{2}\|x\|_2^2 - \tau(a^2 \log a). \tag{3.13}$$

For a complex number r we define

$$x_r = \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} a_n r^{(1+|s|)|n|} U^n V^{sn}$$

and $P_r(a) = 1 + x_r$. By Proposition 3.2.2, for r in some complex neighborhood of the interval $[0,1]$, we can define $\log P_r(a)$.

Let

$$G(r) = \sum_{n\in\mathbb{Z}} (1+|s|) r^{2(1+|s|)|n|} |n||a_n|^2 + \frac{1}{2}\|x_r\|_2^2 - \tau((P_r(a))^2 \log P_r(a)).$$

Therefore, to prove (3.13) it suffices to show $G(1) \geqslant 0$. The function $G(r)$ is analytic in a complex neighborhood of $[0,1]$. So to prove $G(1) \geqslant 0$, using Lemma 3.2.1, we shall show that all the coefficients of the expansion of $G(r)$ around $r = 0$ are nonnegative.

First note that for r with small enough $|r|$ we have $\|x_r\|_2 < 1$ (note that the sum is a finite sum). So

$$(P_r(a))^2 \log P_r(a) = (1 + x_r)^2 \log(1 + x_r)$$

$$= (1+2x_r+x_r^2)(1-\frac{1}{2}x_r^2+\frac{1}{3}x_r^3-\frac{1}{4}x_r^4+\cdots)$$

$$= x_r+\frac{3}{2}x_r^2+2\sum_{k=3}^{\infty}(-1)^{k-1}x_r^k\frac{(k-3)!}{k!}.$$

So

$$G(r) = \sum_{n\in\mathbb{Z}}(1+|s|)r^{2(1+|s|)|n|}|n||a_n|^2+\frac{1}{2}\|x_r\|_2^2 \tag{3.14}$$

$$-\tau(x_r)-\frac{3}{2}\tau(x_r^2)+2\sum_{k=3}^{\infty}(-1)^k\frac{(k-3)!}{k!}\tau(x_r^k).$$

Using the facts that $\tau(x_r)=0$ and

$$\tau(x_r^2)=\|x_r\|_2^2=\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}}r^{2(1+|s|)|n|}|a_n|^2,$$

combined with (3.14), we get

$$G(r) = 2\sum_{\substack{n\in\mathbb{Z}\\ n\geqslant 0}}((1+|s|)n-1)r^{2(1+|s|)n}|a_n|^2+2\sum_{k=3}^{\infty}g_k(r), \tag{3.15}$$

where $g_k(r)=(-1)^k\frac{(k-3)!}{k!}\tau(x_r^k)$. Now we try to find the Taylor expansion of $\tau(x_r^k)$.

First we need to fix some notation. Let

$$M=\{n\in\mathbb{Z}:\, n\neq 0, a_n\neq 0\},$$

$$M_1=\{n\in M:\, n>0\}.$$

For a function $P: M\longrightarrow\mathbb{Z}_0^+$, we put

$$M_P=\{n\in M:\, P(n)\neq 0\}.$$

So (M_P,P) is a multiset. Indeed, the multiplicity of n is $P(n)$. Moreover, let $\mathcal{S}(M_P)$ be the set of all permutations of the multiset (M_P,P). Let I_k be the set of all functions $P: M\longrightarrow\mathbb{Z}_0^+$ such that

$$\sum_{n\in M}P(n)=k,$$

and $I_{k,0}$ be the set of all functions in I_k such that

$$\sum_{n\in M}P(n)n=0.$$

For $P: M\longrightarrow\mathbb{Z}_0^+$, we also define

$$Q_P: M_1\longrightarrow\mathbb{Z}_0^+$$

by $Q_P(n) = P(-n)$.

Using the multinomial expansion of x_r we have

$$x_r^k = \sum_{P\in I_k} \left(\prod_{n\in M} \left(a_n r^{(1+|s|)|n|} \right)^{P(n)} \right) \left(\sum_{\sigma\in\mathcal{S}(M_P)} \prod_{i=1}^{k} U^{\sigma(n_i^P)} V^{s\sigma(n_i^P)} \right)$$

where n_i^P, for $i = 1, 2, \ldots k$, is a labeling of elements of M_P when $P \in I_k$. Then

$$\tau(x_r^k) = \sum_{P\in I_{k,0}} \left(\prod_{n\in M} \left(a_n r^{(1+|s|)|n|} \right)^{P(n)} \right) \tau \left(\sum_{\sigma\in\mathcal{S}(M_P)} \prod_{i=1}^{k} U^{\sigma(n_i^P)} V^{s\sigma(n_i^P)} \right).$$

So we have

$$\tau(x_r^k) = \sum_{P\in I_{k,0}} \prod_{n\in M_1} \left(a_n r^{(1+|s|)n} \right)^{P(n)} \prod_{-n\in M_1} \left(a_n r^{-(1+|s|)n} \right)^{P(n)}$$

$$\times \tau \left(\sum_{\sigma\in\mathcal{S}(M_P)} \prod_{i=1}^{k} U^{\sigma(n_i^P)} V^{s\sigma(n_i^P)} \right).$$

Hence

$$\tau(x_r^k) = \sum_{P\in I_{k,0}} \prod_{n\in M_1} \left(a_n r^{(1+|s|)n} \right)^{P(n)} \prod_{n\in M_1} \left(a_{-n} r^{(1+|s|)n} \right)^{Q_P(n)}$$

$$\times \tau \left(\sum_{\sigma\in\mathcal{S}(M_P)} \prod_{i=1}^{k} U^{\sigma(n_i^P)} V^{s\sigma(n_i^P)} \right).$$

Then since a is self-adjoint, using (3.4) we have

$$\tau(x_r^k) = \sum_{(P,Q)\in H_k} \prod_{n\in M_1} \left(a_n r^{(1+|s|)n} \right)^{P(n)} \prod_{n\in M_1} \left(\overline{a_n} r^{(1+|s|)n} \right)^{Q(n)} \tag{3.16}$$

$$\times e^{\left(-2\pi i s\theta \sum_{n\in M_1} Q(n)n^2\right)} \tau \left(\sum_{\sigma\in\mathcal{S}(M_{P,Q})} \prod_{i=1}^{k} U^{\sigma(n_i^{P,Q})} V^{s\sigma(n_i^{P,Q})} \right),$$

where H_k is the set of all pairs (P, Q) such that

$$P : M_1 \longrightarrow \mathbb{Z}_0^+,$$

$$Q : M_1 \longrightarrow \mathbb{Z}_0^+,$$

$$\sum_{n\in M_1} P(n)n = \sum_{n\in M_1} Q(n)n, \tag{3.17}$$

$$\sum_{n\in M_1} P(n) + \sum_{n\in M_1} Q(n) = k. \tag{3.18}$$

Also $(M_{P,Q},[P,Q])$ is a multiset defined by

$$M_{P,Q}=M_P^+\cup M_Q^-,$$

where

$$M_P^+=\{n\in M_1:\ P(n)\neq 0\},$$

$$M_Q^-=\{n\in M:\ -n\in M_1, Q(-n)\neq 0\},$$

and

$$[P,Q]:M_{P,Q}\longrightarrow \mathbb{Z}_0^+,$$

is defined by

$$[P,Q](n)=\begin{cases}P(n) & n\in M_P^+\\ Q(-n) & n\in M_Q^-\end{cases}.$$

Also, $n_i^{P,Q}$ for $i=1,2,\ldots,k$ is a labeling for elements of $M_{P,Q}$.

So regarding (3.16), we see that

$$\tau(x_r^k)=\sum_{(P,Q)\in H_k}\prod_{n\in M_1}\left(a_n r^{(1+|s|)n}\right)^{P(n)}\prod_{n\in M_1}\left(\overline{a_n} r^{(1+|s|)n}\right)^{Q(n)}$$

$$\times e^{(-2\pi i s\theta\sum\limits_{n\in M_1}Q(n)n^2)}\sum_{\sigma\in\mathcal{S}(M_{P,Q})}\tau\left(\prod_{i=1}^k U^{\sigma(n_i^{P,Q})}V^{s\sigma(n_i^{P,Q})}\right).$$

Now we will calculate $\tau\left(\prod_{i=1}^k U^{\sigma(n_i^{P,Q})}V^{s\sigma(n_i^{P,Q})}\right)$, for $\sigma\in\mathcal{S}(M_{P,Q})$, the set of permutations of the multiset $M_{P,Q}$. For simplicity we drop the superscripts P,Q. Using (3.17), we have

$$\sum_{i=1}^k\sigma(n_i)=0. \tag{3.19}$$

Hence

$$\tau\left(\prod_{i=1}^k U^{\sigma(n_i)}V^{s\sigma(n_i)}\right)=\tau\left(e^{2\pi i\theta B_\sigma}U^{\sum\limits_{i=1}^k\sigma(n_i)}V^{s\sum\limits_{i=1}^k\sigma(n_i)}\right)=e^{2\pi i\theta B_\sigma},$$

where for $\sigma\in\mathcal{S}(M_{P,Q})$,

$$B_\sigma=\frac{s}{2}\sum_{n\in M_1}(P(n)+Q(n))n^2.$$

In fact, we know that

$$\begin{gathered}B_\sigma=-s\sigma(n_2)\sigma(n_1)\\ -s\sigma(n_3)\left[\sigma(n_1)+\sigma(n_2)\right]\\ -s\sigma(n_4)\left[\sigma(n_1)+\sigma(n_2)+\sigma(n_3)\right]-\cdots\\ -s\sigma(n_{k-1})\left[\sigma(n_1)+\sigma(n_2)+\cdots+\sigma(n_{k-2})\right]\end{gathered}$$

$$-s\sigma(n_k)\left[\sigma(n_1)+\sigma(n_2)+\cdots+\sigma(n_{k-1})\right].$$

We also define

$$\begin{gathered}A_\sigma = s\sigma(n_1)\left[\sigma(n_1)+\sigma(n_2)+\cdots+\sigma(n_k)\right]\\ +s\sigma(n_2)\left[\sigma(n_2)+\sigma(n_3)+\cdots+\sigma(n_k)\right]\\ +s\sigma(n_3)\left[\sigma(n_3)+\sigma(n_4)+\cdots+\sigma(n_k)\right]\\ +s\sigma(n_4)\left[\sigma(n_4)+\sigma(n_5)+\cdots+\sigma(n_k)\right]+\cdots\\ +s\sigma(n_{k-1})\left[\sigma(n_{k-1})+\sigma(n_k)\right]\\ +s\sigma(n_k)\sigma(n_k).\end{gathered}$$

Using (3.19), we get $B_\sigma - A_\sigma = 0$. So $B_\sigma = \frac{1}{2}(B_\sigma + A_\sigma)$. On the other hand,

$$B_\sigma + A_\sigma = \sum_{i=1}^{k} s(\sigma(n_i))^2 = \sum_{i=1}^{k} s(n_i)^2$$

$$= \sum_{n\in M_P^+} P(n)sn^2 + \sum_{n\in M_Q^-} Q(-n)sn^2 = \sum_{n\in M_1} P(n)sn^2 + \sum_{-n\in M_1} Q(-n)sn^2$$

$$= \sum_{n\in M_1} P(n)sn^2 + \sum_{n\in M_1} Q(n)sn^2.$$

So we have proved

$$B_\sigma = \frac{s}{2}\sum_{n\in M_1}(P(n)+Q(n))n^2.$$

Therefore,

$$\begin{gathered}\tau(x_r^k) = \sum_{(P,Q)\in H_k}\prod_{n\in M_1}\left(a_n r^{(1+|s|)n}\right)^{P(n)}\prod_{n\in M_1}\left(\overline{a_n} r^{(1+|s|)n}\right)^{Q(n)}\\ \times e^{(-2\pi i s\theta\sum\limits_{n\in M_1}Q(n)n^2)}\sum_{\sigma\in\mathcal{S}(M_{P,Q})} e^{2\pi i\theta B_\sigma}.\end{gathered}$$

Since

$$|\mathcal{S}(M_{P,Q})| = \frac{k!}{\prod\limits_{n\in M_1} P(n)!Q(n)!},$$

we see that

$$\tau(x_r^k) = k!\sum_{(P,Q)\in H_k}\prod_{n\in M_1}\left(a_n r^{(1+|s|)n}\right)^{P(n)}\prod_{n\in M_1}\left(\overline{a_n} r^{(1+|s|)n}\right)^{Q(n)}$$

$$\times\frac{1}{\prod\limits_{n\in M_1}P(n)!Q(n)!}e^{-2\pi i s\theta\sum\limits_{(m,n)\in M_1}Q(n)n^2}e^{\pi i s\theta\left(\sum\limits_{n\in M_1}(P(n)+Q(n))n^2\right)}$$

$$= k! \sum_{(P,Q)\in H_k} \prod_{n\in M_1} \left(a_n r^{(1+|s|)n}\right)^{P(n)} \prod_{n\in M_1} \left(\overline{a_n} r^{(1+|s|)n}\right)^{Q(n)}$$

$$\times \frac{1}{\prod\limits_{n\in M_1} P(n)!Q(n)!} e^{\pi i s\theta \sum\limits_{n\in M_1} (P(n)-Q(n))n^2}$$

$$= k! \sum_{(P,Q)\in H_k} r^{\left(\sum\limits_{n\in M_1} (1+|s|)n(P(n)+Q(n))\right)}$$

$$\times e^{\pi i s\theta \sum\limits_{n\in M_1} P(n)n^2} \prod_{n\in M_1} \frac{(a_n)^{P(n)}}{P(n)!} e^{-\pi i s\theta \sum\limits_{n\in M_1} Q(n)n^2} \prod_{n\in M_1} \frac{(\overline{a_n})^{Q(n)}}{Q(n)!}.$$

Now for a function $P : M \longrightarrow \mathbb{Z}_0^+$, define

$$D(P) = e^{\pi i s\theta \sum\limits_{n\in M_1} P(n)n^2} \prod_{n\in M_1} \frac{(-a_n)^{P(n)}}{P(n)!}. \tag{3.20}$$

Then we have

$$\tau(x_r^k) = k! \sum_{(P,Q)\in H_k} (-1)^{\left(\sum\limits_{n\in M_1} P(n)+Q(n)\right)} r^{\left(\sum\limits_{n\in M_1} (1+|s|)n(P(n)+Q(n))\right)} D(P)\overline{D(Q)}.$$

So

$$\tau(x_r^k) = (-1)^k k! \sum_{l=2}^{\infty} r^{2(1+|s|)l} \left(\sum_{(P,Q)\in G_l} D(P)\overline{D(Q)} \right), \tag{3.21}$$

where G_l the set of all pairs (P,Q) such that

$$P : M_1 \longrightarrow \mathbb{Z}_0^+,$$

$$Q : M_1 \longrightarrow \mathbb{Z}_0^+,$$

and

$$\sum_{n\in M_1} P(n) + \sum_{n\in M_1} Q(n) = k,$$

$$\sum_{n\in M_1} P(n)n = \sum_{n\in M_1} Q(n)n = l,$$

One should note that in (3.21), l starts from 2. Here we shall show why that is the case:

$$\sum_{n\in M_1} P(n) \leqslant \sum_{n\in M_1} P(n)n. \tag{3.22}$$

Similarly we have

$$\sum_{n\in M_1} Q(n) \leqslant \sum_{n\in M_1} Q(n)n, \tag{3.23}$$

So

$$k=\sum_{n\in M_1}P(n)+\sum_{n\in M_1}Q(n)\leqslant\sum_{n\in M_1}P(n)n+\sum_{n\in M_1}Q(n)n=2l$$

So for a fixed k, $\frac{k}{2}\leqslant l$ and since k is at least 3, $l\geqslant 2$.

Now for l and $t\geqslant 1$ define

$$C(t,l):=\sum_{P\in H_{t,l}}D(P),$$

where $H_{t,l}$ is the set of all functions $P:M_1\longrightarrow\mathbb{Z}_0^+$ such that

$$\sum_{n\in M_1}P(n)=t, \tag{3.24}$$

and

$$\sum_{n\in M_1}P(n)n=l. \tag{3.25}$$

When there is no such P then the sum is taken to be 0. For instance if $t>l$ then there is no such P, for

$$t=\sum_{n\in M_1}P(n)\leqslant\sum_{n\in M_1}P(n)n=l.$$

Then we have

$$\sum_{(P,Q)\in G_l}D(P)\overline{D(Q)}=\sum_{t=1}^{k-1}\sum_{\substack{\mathcal{Q}\in H_{k-t,l}\\ \mathcal{P}\in H_{t,l}}}D(\mathcal{P})\overline{D(\mathcal{Q})}$$

$$=\sum_{t=1}^{k-1}\left(\sum_{\mathcal{P}\in H_{t,l}}D(\mathcal{P})\right)\left(\sum_{\mathcal{Q}\in H_{k-t,l}}\overline{D(\mathcal{Q})}\right)=\sum_{t=1}^{k-1}C(t,l)\overline{C(k-t,l)}.$$

Now using this in (3.21), we get

$$\tau(x_r^k)=(-1)^k k!\sum_{l=2}^{\infty}r^{2(1+|s|)l}\sum_{t=1}^{k-1}C(t,l)\overline{C(k-t,l)},$$

and this implies

$$\sum_{k=3}^{N}g_k(r)=\sum_{k=3}^{N}(k-3)!\sum_{l=2}^{\infty}r^{2(1+|s|)l}\sum_{t=1}^{k-1}C(t,l)\overline{C(k-t,l)}$$

$$=\sum_{l=2}^{\infty}r^{2(1+|s|)l}\sum_{k=3}^{N}(k-3)!\sum_{t=1}^{k-1}C(t,l)\overline{C(k-t,l)}$$

$$=\sum_{l=2}^{\infty}r^{2(1+|s|)l}\sum_{\substack{i=1\\ 3\leq i+j\leq N}}^{l}\sum_{j=1}^{l}(i+j-3)!C(i,l)\overline{C(j,l)}.$$

Therefore, for $N \geqslant 2l$ the coefficient of $r^{2(1+|s|)l}$ in $\sum_{k=3}^{N} g_k(r)$ is

$$\sum_{\substack{i=1 \\ i+j\geq 3}}^{l} \sum_{j=1}^{l} (i+j-3)! C(i,l)\overline{C(j,l)}, \tag{3.26}$$

and this is also the coefficient of of $r^{2(1+|s|)l}$ in $\sum_{k=3}^{\infty} g_k(r)$.

Now we show that for $l \geq 0$,

$$C(1,l) = -a_l e^{\pi i s l^2 \theta}.$$

In fact, this is true, for if $l \notin M$, then $a_l = 0$ and also $H_{1,l} = \varnothing$ which implies $C(1,l) = 0$. If $l \in M$, then

$$P(n) = \begin{cases} 1 & n = l \\ 0 & \text{otherwise} \end{cases}$$

is the only element in $H_{1,l}$. So using (3.20), we have

$$C(1,l) = \sum_{P \in H_{1,l}} D(P) = -a_l e^{\pi i s l^2 \theta}. \tag{3.27}$$

Recall that

$$G(r) = 2\sum_{\substack{n\in\mathbb{Z} \\ n\geqslant 0}} ((1+|s|)n - 1) r^{2(1+|s|)n} |a_n|^2 + 2\sum_{k=3}^{\infty} g_k(r).$$

Therefore the coefficient of $r^{2(1+|s|)l}$, $(l \geqslant 2)$ in $G(r)$ is

$$2((1+|s|)l - 1)|a_l|^2 + 2\sum_{\substack{i=1 \\ i+j\geq 3}}^{l} \sum_{j=1}^{l} (i+j-3)! C(i,l)\overline{C(j,l)}.$$

Using (3.27), this is equal to

$$2((1+|s|)l - 1) C(1,l)\overline{C(1,l)} + 2\sum_{\substack{i=1 \\ i+j\geq 3}}^{l} \sum_{j=1}^{l} (i+j-3)! C(i,l)\overline{C(j,l)},$$

which can be written as

$$2\sum_{i=1}^{l} \sum_{j=1}^{l} A_l(i,j) C(i,l)\overline{C(j,l)}, \tag{3.28}$$

where the matrix A_l defined by

$$A_l(i,j) = \begin{cases} (1+|s|)l - 1 & i = j = 1 \\ (i+j-3)! & i+j \geqslant 3 \end{cases}.$$

In [3.17] it has been shown that for $l \geqslant 2$, A_l is a positive semi-definite matrix. Hence the coefficient of $r^{2(1+|s|)l}$, ($l \geqslant 2$) in $G(r)$ is positive. So we have proved (3.12) for a positive element a with $a_0 = 1$ in which only finitely many coefficients are non-zero.

Homogeneity of (3.12) implies that it should hold for a positive element a (with only finitely many non-zero coefficients), even if $a_0 \neq 1$.

Finally, we shall prove (3.12) for an arbitrary strictly positive element of the form $\sum\limits_{n\in\mathbb{Z}} a_n U^n V^{sn}$. For $a = \sum\limits_{n\in\mathbb{Z}} a_n U^n V^{sn}$ and $b = \sum\limits_{n\in\mathbb{Z}} b_n U^n V^{sn}$ in A_θ^∞, we define

$$a * b = \sum_{p\in\mathbb{Z}} (a*b)_p U^p V^{sp}$$

where $(a*b)_p = a_p b_p$. We also define d_j in A_θ^∞ for $j \geqslant 0$ by

$$d_j = \sum_{n\in\mathbb{Z}} d_n^j U^n V^{sn},$$

where

$$d_n^j = \begin{cases} 1 & |n| \leq j \\ 0 & \text{otherwise} \end{cases}.$$

Then we have

$$\|(d_j * a) - a\|_2^2 = \sum_{n\in\mathbb{Z}} |d_n^j a_n - a_n|^2 = \sum_{|n|>j} |a_n|^2.$$

So

$$\lim_{j\to\infty} d_j * a = a, \tag{3.29}$$

in the $\|.\|_2$ topology.

Moreover,

$$\begin{aligned} \|(d_j * a) - a\| &= \|\sum_{n\in\mathbb{Z}} (d_n^j a_n - a_n) U^n V^{sn}\| \\ &\leq \sum_{|n|>j} |a_n|, \end{aligned}$$

which implies

$$\lim_{j\to\infty} d_j * a = a,$$

in the C*-norm topology. Now let

$$a = \sum_{n\in\mathbb{Z}} a_n U^n V^{sn}$$

be a strictly positive element in A_θ^∞ and F be a complex open neighborhood of $\sigma(a)$ in the right half plane away from the y-axis. Since a is strictly positive, we can choose such a set. Therefore, using Proposition 3.2.1, we see that for large enough j, $\sigma(d_j * a)$ is inside F. On the other hand, since a is self-adjoint, (3.4) implies that for $n \in \mathbb{Z}$,

$$a_n = \bar{a}_{-n} e^{-2\pi i n^2 \theta}.$$

So for $n \in \mathbb{Z}$, and $j \in \mathbb{N}$ we have

$$d_n^j a_n = d_n^j \bar{a}_{-n} e^{-2\pi i n^2 \theta},$$

which guarantees that $d_j * a$ is self-adjoint. Therefore, $\sigma(d_j * a) \subset \mathbb{R}$ Hence for large enough j, $d_j * a$ is strictly positive and we can define $\log(d_j * a)$.

Since for $j \geqslant 0$, $d_j * a$ is an element of A_θ^∞ which has at most finitely many non-zero coefficients, we shall apply (3.12) to $d_j * a$ for large enough j and we will get

$$\tau((d_j * a)^2 \log(d_j * a)) \leqslant \sum_{n \in \mathbb{Z}} (1 + |s|)|n||d_n^j|^2 |a_n|^2 + \|d_j * a\|_2^2 \log \|d_j * a\|_2. \tag{3.30}$$

Let

$$h = \sum_{n \in \mathbb{Z}} (1 + |s|)^{\frac{1}{2}} |n|^{\frac{1}{2}} a_n U^n V^{sn}.$$

Then

$$\|d_j * h\|_2^2 = \sum_{n \in \mathbb{Z}} (1 + |s|)|n||d_n^j|^2 |a_n|^2,$$

and

$$\lim_{j \to \infty} d_j * h = h \tag{3.31}$$

in the $\|.\|_2$ topology.

Thus

$$\lim_{j \to \infty} \sum_{n \in \mathbb{Z}} (1 + |s|)|n||d_n^j|^2 |a_n|^2 = \lim_{j \to \infty} \|d_j * h\|_2^2 = \lim_{j \to \infty} \|h\|_2^2 = \sum_{n \in \mathbb{Z}} (1 + |s|)|n||a_n|^2. \tag{3.32}$$

To prove (3.12), taking the limit of (3.30) as $j \longrightarrow \infty$, we use (3.29), (3.32) and also the continuity of τ with respect to $\|.\|_2$. In fact, τ is continuous with respect to $\|.\|_2$, for one can show (See [3.10] Theorem 3.3.2.) for $a \in A_\theta$,

$$|\tau(a)|^2 \leq \|\tau\| \tau(a^* a).$$

$\square$

3.4 Towards Proving the Conjecture

In this section, as promised in Section 3, we will give the details of what we have done towards proving Conjecture 3.3.1 and we will explain what the remaining technical problem is. It seems to us that a part of the solution should involve the noncommutative binomial theorem of Choi, Elliott, and Yui in [3.2], and its extension in [3.5]. Another possible approach would be to first try to prove this inequality for rational values of θ and then extend it to irrational θ's. In what follows we will use the assumptions of Conjecture 3.3.1.

First suppose $\tau(a) = 1$, i.e. $a_{0,0} = 1$ and suppose that at most finitely many of the $a_{m,n}$'s are nonzero. Put $x = a - 1 = \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} a_{m,n}U^mV^n$. Using the fact that

$$\|a\|_2^2 = 1 + \|x\|_2^2,$$

it can be shown that

$$\|a\|_2^2 \log \|a\|_2 \geqslant \frac{1}{2}\|x\|_2^2.$$

We are going to try to prove the conjecture by proving an stronger inequality:

$$0 \leqslant \sum_{(m,n)\in\mathbb{Z}^2} (|m| + |n|)|a_{m,n}|^2 + \frac{1}{2}\|x\|_2^2 - \tau(a^2 \log a). \tag{3.33}$$

For a complex number r we define

$$x_r = \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} a_{m,n} r^{(|m|+|n|)} U^m V^n$$

and $P_r(a) = 1 + x_r$. By Proposition 3.2.2, for r in some complex neighborhood of the interval $[0, 1]$, we can define $\log P_r(a)$.

Let

$$G(r) = \sum_{(m,n)\in\mathbb{Z}^2} r^{2(|m|+|n|)}(|m| + |n|)|a_{m,n}|^2 + \frac{1}{2}\|x_r\|_2^2 - \tau((P_r(a))^2 \log P_r(a)).$$

Therefore, to prove (3.33) it suffices to show that $G(1) \geqslant 0$. The function $G(r)$ is analytic in a complex neighborhood of $[0, 1]$. So to prove $G(1) \geqslant 0$, using Lemma 3.2.1, we need to show that all the coefficients of the expansion of $G(r)$ around $r = 0$ are nonnegative.

First note that for r with small enough $|r|$ we have $\|x_r\|_2 < 1$ (note that the sum is a finite sum). So

$$\begin{aligned}(P_r(a))^2 \log P_r(a) &= (1 + x_r)^2 \log(1 + x_r) \\ &= (1 + 2x_r + x_r^2)(1 - \frac{1}{2}x_r^2 + \frac{1}{3}x_r^3 - \frac{1}{4}x_r^4 + \cdots) \\ &= x_r + \frac{3}{2}x_r^2 + 2\sum_{k=3}^{\infty}(-1)^{k-1}x_r^k \frac{(k-3)!}{k!}.\end{aligned}$$

So

$$\begin{aligned}G(r) = &\sum_{(m,n)\in\mathbb{Z}^2} r^{2(|m|+|n|)}(|m| + |n|)|a_{m,n}|^2 + \frac{1}{2}\|x_r\|_2^2 \\ &-\tau(x_r) - \frac{3}{2}\tau(x_r^2) + 2\sum_{k=3}^{\infty}(-1)^k \frac{(k-3)!}{k!}\tau(x_r^k).\end{aligned} \tag{3.34}$$

Using the facts that $\tau(x_r) = 0$ and

$$\tau(x_r^2) = \|x_r\|_2^2 = \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} r^{2(|m|+|n|)}|a_{m,n}|^2$$

in (3.34), we get

$$G(r) = 2 \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ m\geqslant 0, n\geqslant 0}} (m+n-1) r^{2(m+n)} |a_{m,n}|^2 + 2\sum_{k=3}^{\infty} g_k(r) \tag{3.35}$$

where $g_k(r) = (-1)^k \dfrac{(k-3)!}{k!} \tau(x_r^k)$. Now let us try to find the Taylor expansion of $\tau(x_r^k)$.

First we need to fix some notation. Let

$$M = \{(m,n) \in \mathbb{Z}^2 : (m,n) \neq 0, a_{m,n} \neq 0\},$$

$$M_1 = \{(m,n) \in M : m \geqslant 0, n \geqslant 0\},$$
$$M_2 = \{(m,n) \in M : m > 0, n < 0\}.$$

For a function $P : M \longrightarrow \mathbb{Z}_0^+$, we put

$$M_P = \{(m,n) \in M : P(m,n) \neq 0\}.$$

So (M_P, P) is a multiset. Indeed, the multiplicity of (m,n) is $P(m,n)$. Moreover, let $\mathcal{S}(M_P)$ be the set of all permutations of the multiset (M_P, P). For $\sigma \in \mathcal{S}(M_P)$, by $\sigma_1(m,n)$ and $\sigma_2(m,n)$ we mean the first and the second components of $\sigma(m,n)$, respectively. Let I_k be the set of all functions $P : M \longrightarrow \mathbb{Z}_0^+$ such that

$$\sum_{(m,n)\in M} P(m,n) = k,$$

and $I_{k,0}$ be the set of all functions in I_k such that

$$\sum_{(m,n)\in M} P(m,n)m = \sum_{(m,n)\in M} P(m,n)n = 0.$$

For $P : M \longrightarrow \mathbb{Z}_0^+$, we also define

$$Q_P : M_1 \cup M_2 \longrightarrow \mathbb{Z}_0^+$$

by $Q_P(m,n) = P(-m,-n)$.

Using the multinomial expansion of x_r we have

$$x_r^k = \sum_{P\in I_k} \left(\prod_{(m,n)\in M} \left(a_{m,n} r^{|m|+|n|}\right)^{P(m,n)} \right) \left(\sum_{\sigma\in\mathcal{S}(M_P)} \prod_{i=1}^{k} U^{\sigma_1(m_i^P, n_i^P)} V^{\sigma_2(m_i^P, n_i^P)} \right)$$

where (m_i^P, n_i^P), for $i = 1, 2, \cdots k$, is a labeling of elements of M_P when $P \in I_k$. Then

$$\tau(x_r^k) = \sum_{P\in I_{k,0}} \left(\prod_{(m,n)\in M} \left(a_{m,n} r^{|m|+|n|}\right)^{P(m,n)} \right) \tau \left(\sum_{\sigma\in\mathcal{S}(M_P)} \prod_{i=1}^{k} U^{\sigma_1(m_i^P, n_i^P)} V^{\sigma_2(m_i^P, n_i^P)} \right).$$

So we have

$$\tau(x_r^k) = \sum_{P\in I_{k,0}} \prod_{(m,n)\in M_1} \left(a_{m,n} r^{m+n}\right)^{P(m,n)} \prod_{(-m,-n)\in M_1} \left(a_{m,n} r^{-m-n}\right)^{P(m,n)}$$

$$\times \prod_{(m,n)\in M_2} \left(a_{m,n} r^{m-n}\right)^{P(m,n)} \prod_{(-m,-n)\in M_2} \left(a_{m,n} r^{n-m}\right)^{P(m,n)}$$

$$\times \tau\left(\sum_{\sigma\in\mathcal{S}(M_P)} \prod_{i=1}^{k} U^{\sigma_1(m_i^P,n_i^P)} V^{\sigma_2(m_i^P,n_i^P)}\right).$$

Then

$$\tau(x_r^k) = \sum_{P\in I_{k,0}} \prod_{(m,n)\in M_1} \left(a_{m,n} r^{m+n}\right)^{P(m,n)} \prod_{(m,n)\in M_1} \left(a_{-m,-n} r^{m+n}\right)^{Q_P(m,n)}$$

$$\times \prod_{(m,n)\in M_2} \left(a_{m,n} r^{m-n}\right)^{P(m,n)} \prod_{(m,n)\in M_2} \left(a_{-m,-n} r^{m-n}\right)^{Q_P(m,n)}$$

$$\times \tau\left(\sum_{\sigma\in\mathcal{S}(M_P)} \prod_{i=1}^{k} U^{\sigma_1(m_i^P,n_i^P)} V^{\sigma_2(m_i^P,n_i^P)}\right).$$

Then since a is self-adjoint, using (3.4) we have

$$\tau(x_r^k) = \sum_{(P,Q)\in H_k} \prod_{(m,n)\in M_1} \left(a_{m,n} r^{m+n}\right)^{P(m,n)} \prod_{(m,n)\in M_1} \left(\overline{a_{m,n}} r^{m+n}\right)^{Q(m,n)} \tag{3.36}$$

$$\times \prod_{(m,n)\in M_2} \left(a_{m,n} r^{m-n}\right)^{P(m,n)} \prod_{(m,n)\in M_2} \left(\overline{a_{m,n}} r^{m-n}\right)^{Q(m,n)}$$

$$\times e^{\left(-2\pi i\theta \sum\limits_{(m,n)\in M_1} Q(m,n)mn\right)} e^{\left(-2\pi i\theta \sum\limits_{(m,n)\in M_2} Q(m,n)mn\right)}$$

$$\times \tau\left(\sum_{\sigma\in\mathcal{S}(M_{P,Q})} \prod_{i=1}^{k} U^{\sigma_1(m_i^{P,Q},n_i^{P,Q})} V^{\sigma_2(m_i^{P,Q},n_i^{P,Q})}\right),$$

where H_k is the set of all pairs (P,Q) such that

$$P : M_1 \cup M_2 \longrightarrow \mathbb{Z}_0^+,$$

$$Q : M_1 \cup M_2 \longrightarrow \mathbb{Z}_0^+,$$

$$\sum_{(m,n)\in M_1\cup M_2} P(m,n)n = \sum_{(m,n)\in M_1\cup M_2} Q(m,n)n, \tag{3.37}$$

$$\sum_{(m,n)\in M_1\cup M_2} P(m,n)m = \sum_{(m,n)\in M_1\cup M_2} Q(m,n)m, \tag{3.38}$$

$$\sum_{(m,n)\in M_1\cup M_2} P(m,n) + \sum_{(m,n)\in M_1\cup M_2} Q(m,n) = k \tag{3.39}$$

and $(M_{P,Q}, [P,Q])$ is a multiset defined by

$$M_{P,Q} = M_P^{1,2} \cup M_Q^{-1,-2}$$

where

$$M_P^{1,2} = \{(m,n) \in M_1 \cup M_2 :\ P(m,n) \neq 0\},$$

$$M_Q^{-1,-2} = \{(m,n) \in M :\ (m,n) \notin M_1 \cup M_2, Q(-m,-n) \neq 0\},$$

and

$$[P,Q] : M_{P,Q} \longrightarrow \mathbb{Z}_0^+,$$

is defined by

$$[P,Q](m,n) = \begin{cases} P(m,n) & (m,n) \in M_P^{1,2} \\ Q(-m,-n) & (m,n) \in M_Q^{-1,-2} \end{cases}.$$

Also, $(m_i^{P,Q}, n_i^{P,Q})$ for $i = 1, 2, \ldots, k$ is a labeling for elements of $M_{P,Q}$.

Now we show that for $(P,Q) \in H_k$,

$$\tau\left(\sum_{\sigma \in \mathcal{S}(M_{P,Q})} \prod_{i=1}^{k} U^{\sigma_1(m_i^{P,Q}, n_i^{P,Q})} V^{\sigma_2(m_i^{P,Q}, n_i^{P,Q})}\right) \tag{3.40}$$

$$= e^{\pi i\theta \sum_{(m,n)\in M_1 \cup M_2} (P(m,n)+Q(m,n))mn} B_{P,Q},$$

where $B_{P,Q}$ is a real number which depends on P and Q. Indeed, for $(P,Q) \in H_k$ and $\sigma \in \mathcal{S}(M_{P,Q})$ we have (for simplicity we drop the superscript P,Q)

$$\prod_{i=1}^{k} U^{\sigma_1(m_i,n_i)} V^{\sigma_2(m_i,n_i)} = e^{2\pi i\theta B_\sigma} U^{(\sum_{i=1}^{k} \sigma_1(m_i,n_i))} V^{(\sum_{i=1}^{k} \sigma_2(m_i,n_i))}, \tag{3.41}$$

where

$$\begin{gathered} B_\sigma = -\sigma_1(m_2,n_2)\sigma_2(m_1,n_1) \\ -\sigma_1(m_3,n_3)\left[\sigma_2(m_1,n_1) + \sigma_2(m_2,n_2)\right] \\ -\sigma_1(m_4,n_4)\left[\sigma_2(m_1,n_1) + \sigma_2(m_2,n_2) + \sigma_2(m_3,n_3)\right] - \cdots \\ -\sigma_1(m_k,n_k)\left[\sigma_2(m_1,n_1) + \sigma_2(m_2,n_2) + \sigma_2(m_3,n_3) + \cdots + \sigma_2(m_{k-1},n_{k-1})\right]. \end{gathered}$$

Since $(P,Q) \in H_k$, (3.37) and (3.38) implies

$$\sum_{i=1}^{k} \sigma_1(m_i,n_i) = 0, \tag{3.42}$$

and

$$\sum_{i=1}^{k} \sigma_2(m_i,n_i) = 0. \tag{3.43}$$

So using (3.41),

$$\tau(\prod_{i=1}^{k} U^{\sigma_1(m_i,n_i)} V^{\sigma_2(m_i,n_i)}) = e^{2\pi i\theta B_\sigma}. \tag{3.44}$$

Let

$$\begin{aligned}
A_\sigma = \sigma_1(m_1,n_1)\left[\sigma_2(m_1,n_1) + \sigma_2(m_2,n_2) + \cdots \sigma_2(m_k,n_k)\right] \\
+\sigma_1(m_2,n_2)\left[\sigma_2(m_2,n_2) + \sigma_2(m_3,n_3) + \cdots \sigma_2(m_k,n_k)\right] \\
+\sigma_1(m_3,n_3)\left[\sigma_2(m_3,n_3) + \sigma_2(m_4,n_4) + \cdots \sigma_2(m_k,n_k)\right] + \cdots \\
+\sigma_1(m_{k-1},n_{k-1})\left[\sigma_2(m_{k-1},n_{k-1}) + \sigma_2(m_k,n_k)\right] \\
+\sigma_1(m_k,n_k)\sigma_2(m_k,n_k).
\end{aligned}$$

Using (3.42) and (3.43), one can check that $B_\sigma - A_\sigma = 0$. So

$$B_\sigma = \frac{1}{2}(B_\sigma + A_\sigma). \tag{3.45}$$

We also set

$$D_\sigma = \sum_{j=2}^{k}\sum_{i=1}^{j-1}\left[\sigma_1(m_i,n_i)\sigma_2(m_j,n_j) - \sigma_1(m_j,n_j)\sigma_2(m_i,n_i)\right].$$

One can see that

$$D_\sigma = \sum_{j=1}^{k-1}\sum_{i=j+1}^{k}\left[\sigma_1(m_j,n_j)\sigma_2(m_i,n_i) - \sigma_1(m_i,n_i)\sigma_2(m_j,n_j)\right]. \tag{3.46}$$

Then we see that

$$\begin{aligned}
B_\sigma + A_\sigma &= D_\sigma + \sum_{i=1}^{k}\sigma_1(m_i,n_i)\sigma_2(m_i,n_i) \\
&= D_\sigma + \sum_{(m,n)\in M_{P,Q}} [P,Q](m,n)mn \\
&= D_\sigma + \sum_{(m,n)\in M_{P,Q}^{1,2}} P(m,n)mn + \sum_{(m,n)\in M_{P,Q}^{-1,-2}} Q(-m,-n)mn \\
&= D_\sigma + \sum_{(m,n)\in M_1\cup M_2} P(m,n)mn + \sum_{(-m,-n)\in M_1\cup M_2} Q(-m,-n)mn \\
&= D_\sigma + \sum_{(m,n)\in M_1\cup M_2} P(m,n)mn + \sum_{(m,n)\in M_1\cup M_2} Q(m,n)mn.
\end{aligned}$$

Therefore, by (3.45) we have

$$B_\sigma = \frac{1}{2}\left[D_\sigma + \sum_{(m,n)\in M_1\cup M_2}(P(m,n)+Q(m,n))mn\right].$$

Then regarding (3.44) we have

$$\tau(\prod_{i=1}^{k} U^{\sigma_1(m_i,n_i)} V^{\sigma_2(m_i,n_i)}) = e^{2\pi i\theta B_\sigma}$$

$$= e^{\pi i\theta \sum\limits_{(m,n)\in M_1\cup M_2} (P(m,n)+Q(m,n))mn} e^{\pi i\theta D_\sigma}.$$

Now if we define

$$B_{P,Q} = \sum_{\sigma\in\mathcal{S}(M_{P,Q})} e^{\pi i\theta D_\sigma},$$

we see that

$$\tau\left(\sum_{\sigma\in\mathcal{S}(M_{P,Q})} \prod_{i=1}^{k} U^{\sigma_1(m_i^{P,Q},n_i^{P,Q})} V^{\sigma_2(m_i^{P,Q},n_i^{P,Q})}\right) = e^{\pi i\theta \sum\limits_{(m,n)\in M_1\cup M_2} (P(m,n)+Q(m,n))mn} B_{P,Q}.$$

So we have proved (3.40). Now we will show that $B_{P,Q}$ is a real number. Indeed, for $\sigma \in \mathcal{S}(M_{P,Q})$, we define $\beta_\sigma \in \mathcal{S}(M_{P,Q})$ by

$$\beta_\sigma(m_i,n_i) = \sigma(m_{k-i+1},n_{k-i+1}), \quad i = 1,2,\ldots,k.$$

Then we have

$$D_{\beta_\sigma} = \sum_{j=2}^{k}\sum_{i=1}^{j-1} [\beta_{\sigma 1}(m_i,n_i)\beta_{\sigma 2}(m_j,n_j) - \beta_{\sigma 1}(m_j,n_j)\beta_{\sigma 2}(m_i,n_i)]$$

$$= \sum_{j=2}^{k}\sum_{i=1}^{j-1} \sigma_1(m_{k-i+1},n_{k-i+1})\sigma_2(m_{k-j+1},n_{k-j+1})$$

$$-\sum_{j=2}^{k}\sum_{i=1}^{j-1} \sigma_1(m_{k-j+1},n_{k-j+1})\sigma_2(m_{k-i+1},n_{k-i+1})$$

$$= \sum_{t=k-1}^{1}\sum_{s=k}^{t+1} [\sigma_1(m_s,n_s)\sigma_2(m_t,n_t) - \sigma_1(m_t,n_t)\sigma_2(m_s,n_s)]$$

$$= \sum_{t=1}^{k-1}\sum_{s=t+1}^{k} [\sigma_1(m_s,n_s)\sigma_2(m_t,n_t) - \sigma_1(m_t,n_t)\sigma_2(m_s,n_s)]$$

$$= -\sum_{t=1}^{k-1}\sum_{s=t+1}^{k} [\sigma_1(m_t,n_t)\sigma_2(m_s,n_s) - \sigma_1(m_s,n_s)\sigma_2(m_t,n_t)] = -D_\sigma,$$

where in the last equality we have used (3.46). Now we have

$$B_{P,Q} = \sum_{\sigma\in\mathcal{S}(M_{P,Q})} e^{\pi i\theta D_\sigma} = \frac{1}{2}(\sum_{\sigma\in\mathcal{S}(M_{P,Q})} e^{\pi i\theta D_\sigma} + \sum_{\sigma\in\mathcal{S}(M_{P,Q})} e^{\pi i\theta D_\sigma})$$

$$= \frac{1}{2}\Big(\sum_{\sigma\in\mathcal{S}(M_{P,Q})} e^{\pi i\theta D_\sigma} + \sum_{\sigma\in\mathcal{S}(M_{P,Q})} e^{\pi i\theta D_{\beta_\sigma}} \Big)$$

$$= \frac{1}{2}\Big(\sum_{\sigma\in\mathcal{S}(M_{P,Q})} e^{\pi i\theta D_\sigma} + \sum_{\sigma\in\mathcal{S}(M_{P,Q})} e^{-\pi i\theta D_\sigma} \Big) = \frac{1}{2} \sum_{\sigma\in\mathcal{S}(M_{P,Q})} (e^{\pi i\theta D_\sigma} + e^{-\pi i\theta D_\sigma})$$

So $B_{P,Q}$ is real.

Now using (3.36), we see that

$$\tau(x_r^k) = \sum_{(P,Q)\in H_k} \prod_{(m,n)\in M_1} \big(a_{m,n} r^{m+n}\big)^{P(m,n)} \prod_{(m,n)\in M_1} \big(\overline{a_{m,n}} r^{m+n}\big)^{Q(m,n)}$$
$$\times \prod_{(m,n)\in M_2} \big(a_{m,n} r^{m-n}\big)^{P(m,n)} \prod_{(m,n)\in M_2} \big(\overline{a_{m,n}} r^{m-n}\big)^{Q(m,n)}$$
$$\times e^{-2\pi i\theta \sum\limits_{(m,n)\in M_1\cup M_2} Q(m,n)mn} e^{\pi i\theta \sum\limits_{(m,n)\in M_1\cup M_2} (P(m,n)+Q(m,n))mn} B_{P,Q}.$$

Then we have

$$\tau(x_r^k) = \sum_{(P,Q)\in H_k} r^{\left(\sum\limits_{(m,n)\in M_1} (P(m,n)+Q(m,n))(m+n) + \sum\limits_{(m,n)\in M_2} (P(m,n)+Q(m,n))(m-n)\right)}$$
$$\times e^{\pi i\theta \sum\limits_{(m,n)\in M_1\cup M_2} P(m,n)mn} e^{-\pi i\theta \sum\limits_{(m,n)\in M_1\cup M_2} Q(m,n)mn}$$
$$\times \prod_{(m,n)\in M_1\cup M_2} a_{m,n}^{P(m,n)} \prod_{(m,n)\in M_1\cup M_2} \overline{a_{m,n}}^{Q(m,n)} B_{P,Q}$$
$$= \sum_{(P,Q)\in H_k} r^{\left(\sum\limits_{(m,n)\in M_1\bigcup M_2} P(m,n)m + \sum\limits_{(m,n)\in M_1\bigcup M_2} Q(m,n)m\right)}$$
$$\times r^{\left(\sum\limits_{(m,n)\in M_1} P(m,n)n - \sum\limits_{(m,n)\in M_2} Q(m,n)n + \sum\limits_{(m,n)\in M_1} Q(m,n)n - \sum\limits_{(m,n)\in M_2} P(m,n)n\right)}$$
$$\times e^{\pi i\theta \sum\limits_{(m,n)\in M_1\cup M_2} P(m,n)mn} e^{-\pi i\theta \sum\limits_{(m,n)\in M_1\cup M_2} Q(m,n)mn}$$
$$\times \prod_{(m,n)\in M_1\cup M_2} a_{m,n}^{P(m,n)} \prod_{(m,n)\in M_1\cup M_2} \overline{a_{m,n}}^{Q(m,n)} B_{P,Q}.$$

So

$$\tau(x_r^k) = \sum_{\substack{l+s=2 \\ l\geq 0, s\geq 0}}^{\infty} r^{2(l+s)} \sum_{(P,Q)\in G_{l,s}} e^{\pi i\theta \sum\limits_{(m,n)\in M_1\cup M_2} P(m,n)mn} \tag{3.47}$$
$$\times e^{-\pi i\theta \sum\limits_{(m,n)\in M_1\cup M_2} Q(m,n)mn} \prod_{(m,n)\in M_1\cup M_2} a_{m,n}^{P(m,n)} \prod_{(m,n)\in M_1\cup M_2} \overline{a_{m,n}}^{Q(m,n)} B_{P,Q},$$

where $G_{l,s}$ is the set of all pairs (P,Q) such that

$$P : M_1 \cup M_2 \longrightarrow \mathbb{Z}_0^+,$$

$$Q : M_1 \cup M_2 \longrightarrow \mathbb{Z}_0^+,$$

and

$$\sum_{(m,n)\in M_1\cup M_2} P(m,n) + \sum_{(m,n)\in M_1\cup M_2} Q(m,n) = k,$$

$$\sum_{(m,n)\in M_1\cup M_2} P(m,n)m = \sum_{(m,n)\in M_1\cup M_2} Q(m,n)m = l,$$

$$\sum_{(m,n)\in M_1} P(m,n)n - \sum_{(m,n)\in M_2} Q(m,n)n = \sum_{(m,n)\in M_1} Q(m,n)n - \sum_{(m,n)\in M_2} P(m,n)n = s.$$

One should note that in (3.47) $(l+s)$ starts from 2. Here we shall show why that is the case: for $(m,n) \in M_1$ if $m = 0$, then $n \neq 0$. So we have

$$\sum_{(m,n)\in M_1} P(m,n) \leqslant \sum_{(m,n)\in M_1} P(m,n)m + \sum_{(m,n)\in M_1} P(m,n)n. \tag{3.48}$$

Similarly we have

$$\sum_{(m,n)\in M_2} P(m,n) \leqslant \sum_{(m,n)\in M_2} P(m,n)m - \sum_{(m,n)\in M_2} P(m,n)n, \tag{3.49}$$

$$\sum_{(m,n)\in M_1} Q(m,n) \leqslant \sum_{(m,n)\in M_1} Q(m,n)m + \sum_{(m,n)\in M_1} Q(m,n)n, \tag{3.50}$$

$$\sum_{(m,n)\in M_2} Q(m,n) \leqslant \sum_{(m,n)\in M_2} Q(m,n)m - \sum_{(m,n)\in M_2} Q(m,n)n. \tag{3.51}$$

So

$$\begin{aligned} k = \sum_{(m,n)\in M_1\cup M_2} P(m,n) + \sum_{(m,n)\in M_1\cup M_2} Q(m,n) \leqslant \\ \sum_{(m,n)\in M_1\cup M_2} P(m,n)m + \sum_{(m,n)\in M_1\cup M_2} Q(m,n)m \\ + \sum_{(m,n)\in M_1} P(m,n)n - \sum_{(m,n)\in M_2} Q(m,n)n \\ + \sum_{(m,n)\in M_1} Q(m,n)n - \sum_{(m,n)\in M_2} P(m,n)n \\ = 2(s+l). \end{aligned}$$

So for a fixed k, $\dfrac{k}{2} \leqslant l+s$ and since k is at least 3, $l+s \geqslant 2$.

Let $\tilde{G}_{l,s}$ be the set of all pairs $(\tilde{P}, \tilde{Q})$ such that

$$\tilde{P} : M \longrightarrow \mathbb{Z}_0^+,$$

$$\tilde{Q} : M \longrightarrow \mathbb{Z}_0^+,$$

and

$$\sum_{(m,n)\in M_1\cup M_2}\tilde{P}(m,n)+\sum_{(m,n)\in M_1\cup M_2}\tilde{Q}(m,n)=k,$$

$$\sum_{(m,n)\in M_1\cup M_2}\tilde{P}(m,n)m=\sum_{(m,n)\in M_1\cup M_2}\tilde{Q}(m,n)m=l,$$

$$\sum_{(m,n)\in M_1}\tilde{P}(m,n)n-\sum_{(m,n)\in M_2}\tilde{Q}(m,n)n=\sum_{(m,n)\in M_1}\tilde{Q}(m,n)n-\sum_{(m,n)\in M_2}\tilde{P}(m,n)n=s.$$

There exists a one-to-one correspondence between $\tilde{G}_{l,s}$ and $G_{l,s}$. In fact, for $(P,Q)\in G_{l,s}$, we can define

$$\tilde{P}(m,n)=\begin{cases} P(m,n) & (m,n)\in M_1\cup M_2\\ Q(-m,-n) & (m,n)\notin M_1\cup M_2\end{cases},$$

and

$$\tilde{Q}(m,n)=\begin{cases} Q(m,n) & (m,n)\in M_1\cup M_2\\ P(-m,-n) & (m,n)\notin M_1\cup M_2\end{cases}.$$

Using this correspondence and the fact that $B_{P,Q}=B_{\tilde{P}\restriction_{M_1\cup M_2},\tilde{Q}\restriction_{M_1\cup M_2}}$ in (3.47), we have

$$\tau(x_r^k)=\sum_{\substack{l+s=2\\ l\geq 0,s\geq 0}}^{\infty} r^{2(l+s)}\sum_{(\tilde{P},\tilde{Q})\in\tilde{G}_{l,s}} e^{\pi i\theta\sum\limits_{(m,n)\in M_1\cup M_2}\tilde{P}(m,n)mn} \tag{3.52}$$

$$\times e^{-\pi i\theta\sum\limits_{(m,n)\in M_1\cup M_2}\tilde{Q}(m,n)mn}\prod_{(m,n)\in M_1\cup M_2} a_{m,n}^{\tilde{P}(m,n)}\prod_{(m,n)\in M_1\cup M_2}\overline{a_{m,n}}^{\tilde{Q}(m,n)}B_{\tilde{P},\tilde{Q}}.$$

Now if we can decompose $B_{\tilde{P},\tilde{Q}}$ into two terms $B_{\tilde{P}}$ and $B_{\tilde{Q}}$, i.e.,

$$B_{\tilde{P},\tilde{Q}}=B_{\tilde{P}}B_{\tilde{Q}}, \tag{3.53}$$

such that $B_{\tilde{P}}$ and $B_{\tilde{Q}}$ depend only on $\tilde{P}$ and $\tilde{Q}$, respectively, then we can continue the proof of Theorem 3.3.2. Indeed, if (3.53) holds, for a function

$$P: M\longrightarrow \mathbb{Z}_0^+,$$

we can define

$$D(P)=e^{\pi i\theta\sum\limits_{(m,n)\in M_1\cup M_2}P(m,n)mn}\prod_{(m,n)\in M_1\cup M_2}\left(-a_{m,n}\right)^{P(m,n)}B_P, \tag{3.54}$$

and the rest would be similar to the proof of Theorem 3.3.2.

Bibliography

[3.1] R. A. Adams, F. H. Clarke, *Gross' logarithmic Sobolev inequality: a simple proof*, Amer. J. Math. 101 (1979), 1265–1270.

[3.2] M.-D. Choi, G. A. Elliott and N. Yui, *Gauss p olynomials and the rotation algebras*, Inv. Math. 99 (1990), 225-246.

[3.3] A. Connes, *C^*-algèbres et géométrie différentielle*. C.R. Acad. Sc. Paris, t. 290, Série A, 599-604, 1980.

[3.4] A. Connes, *Noncommutative geometry*. Academic Press (1994).

[3.5] C. Farsi, *A noncommutative n-nomial formula*, Rocky Mountain J. Math. 37, no. 5, (2007), 1527-1540

[3.6] L. Gross, *Logarithmic Sobolev Inequalities*, Amer. J. Math. 97 (1975), 1061-1083.

[3.7] L. Gross, *Logarithmic Sobolev Inequalities and Contractivity of Semigroups*, LNM, Vol. 1563, Springer, New York, 1993.

[3.8] E. Hsu, *Logarithmic Sobolev Inequalities on Path Spaces Over Riemannian Manifolds*. Comm. Math. Phys. 189 (1997), no. 1, 9-16.

[3.9] M. Khalkhali, *Basic Noncommutative Geometry*. Second edition. EMS Series of Lectures in Mathematics. European Mathematical Society (EMS), Zürich, 2013.

[3.10] G. J. Murphy, *C*-algebras and operator theory*, Academic Press, London, 1990.

[3.11] E. Nelson, *The free Markoff field*, J. Funct. Anal. 12 (1973), 211-227.

[3.12] L. Nirenberg, *On elliptic partial differential equations*, Ann. Scuola Normale Superiore di Pisa, Ser III, 13 (1959), 1-47.

[3.13] M. A. Rieffel, *C*-algebras associated with irrational rotations*, Pacific J. Math. 93 (1981), no. 2, 415–429. MR623572 (83b:46087)

[3.14] J. Rosenberg, *Noncommutative variations on Laplace's equation*, Anal. PDE 1 (2008), no. 1, 95–114, arXiv:0802.4033.

[3.15] O. S. Rothaus, *Lower bounds for eigenvalues of regular Sturm-Liouville operators and the logarithmic Sobolev inequality*, Duke Math. Journal 45 (1978), 351-362.

[3.16] W. Rudin. *Functional Analysis*, McGraw-Hill, Inc. Singapore, 1991.

[3.17] F. Weissler, *Logarithmic Sobolev inequalities and hypercontractive estimates on the circle*, J. Funct. Anal. 37 (1980), 218-234.

Chapter 4

A Scalar Curvature Formula for the Noncommutative 3-Torus

4.1 Introduction

Since the beginning of noncommutative geometry in [4.2], noncommutative tori have proved to be an invaluable model to understand and test many aspects of noncommutative geometry. Curvature, one of the most important geometric invariants, is among those aspects. Defining a suitable curvature concept in the noncommutative setting is an important problem at the heart of noncommutative geometry. More precisely, we are interested in curvature invariants of noncommutative Riemannian manifolds. In contrast it should be noted that curvature of connections and the corresponding Chern-Weil theory in the noncommutative setting has already been defined in [4.2].

In their pioneering work [4.5], Connes and Tretkoff (cf. also [4.1] for a preliminary version) took a first step in this direction and proved a Gauss-Bonnet theorem for a curved noncommutative two torus equipped with a conformally deformed metric. They gave a spectral definition of curvature and computed its trace. This result was extended in [4.6] to noncommutative tori equipped with an arbitrary translation invariant complex structure and conformal perurbation of its metric. The full computation of curvature in these examples was done independely and simultaneously in [4.4] and [4.7]. This line of work has been continued and extended in different directions in many papers [4.8].

The approach used in the aforementioned papers is based on the heat kernel techniques and Connes' pseudodifferential calculus. In this paper using a similar technique we will give a formula for the scalar curvature of a curved noncommutative 3-torus. This would be the first odd dimensional case that has been studied among the noncommutative tori. In [4.12] a general pattern for the scalar curvature of even dimensional noncommutatuve tori is found. While a similar question in the odd dimensional case needs a close study of the first.

This paper is organized as follows. In Section 2, we recall some facts about the heat kernel expansion in the commutative case. In Section 3, we recall basic facts about higher dimensional noncommutative tori and their flat geometry. Then we perturb the standard volume form on this space conformally and analyse the corresponding perturbed

Laplacian. In Section 4, we recall the pseudodifferential calculus of [4.3] for $\mathbb{T}^3_\theta$. In Section 5, we review the derivation of the small time heat kernel expansion for the perturbed Laplacian, using the pseudodifferential calculus. Then we perform the computation of the scalar curvature for $\mathbb{T}^3_\theta$, and find explicit formulas for the local functions that describe the curvature in terms of the modular automorphism of the conformally perturbed volume form and derivatives of the Weyl factor.

4.2 Heat Kernel Expansion and Scalar Curvature

To motivate the definition of scalar curvature in our noncommutative setting, let us first recall Gilkey's theorem on asymptotic expansion of heat kernels. Let (M, g) be a closed, oriented Riemannian manifold of dimension n, endowed with the metric g and $\triangle$ be the Laplace operator acting on $C^\infty(M)$, the algebra of smooth functions on M. If C is a contour going counterclockwise around the nonnegative part of the x-axis without touching it, then using the Cauchy integral formula

$$e^{-t\triangle} = \frac{1}{2\pi i}\int_C e^{-t\lambda}(\triangle - \lambda)^{-1} d\lambda$$

and approximating the operator $(\triangle - \lambda)^{-1}$ by a pseudodifferential operator $R(\lambda)$ one can find an asymptotic expansion for the smooth kernel $K(t, x, y)$ of $e^{-t\triangle}$ on the diagonal [4.9].

More precisely, using the formula for the symbol of the product of two pseudo differential operators one can inductively find an asymptotic expansion $\sum_{j=0}^{\infty} r_j(x, \xi, \lambda)$ for the symbol of $R(\lambda)$ such that $r_j(x, \xi, \lambda)$ is a symbol of order $-2-j$ depending on the complex parameter λ, where $j \in \mathbb{N} \cup \{0\}$, $x \in M$ and $\xi \in \mathbb{R}^n$. Then one can see that for $t > 0$, the operator $e^{-t\triangle}$ has a smooth kernel $K(t, x, y)$ and as $t \longrightarrow 0^+$ there exist asymptotic expansion

$$K(t, x, x) \sim t^{-n/2} \sum_{m=0}^{\infty} a_{2m}(x) t^m,$$

where

$$a_{2m}(x) = \frac{1}{2\pi i} \iint_C e^{-\lambda} r_{2m}(x, \xi, \lambda) d\lambda d\xi. \tag{4.1}$$

It follows that

$$\mathrm{Tr}_{L^2} e^{-t\triangle} \sim t^{-n/2} \sum_{m=0}^{\infty} a_{2m} t^m$$

where

$$a_{2m} = \int_M a_{2m}(x) \,\mathrm{dvol}(x).$$

Moreover $a_2(x)$ is a constant multiple of the scalar curvature of M at the point x, so that a_2 is the total scalar curvature [4.9]. In what follows we will explain how we exploit these facts to define the scalar curvature of the curved noncommutative 3-torus by analogy.

4.3 A Curved Noncommutative 3-Torus

Let $\theta = (\theta_{k\ell}) \in M_3(\mathbb{R})$ be a skew symmetric matrix. The universal C*-algebra generated by three unitary elements u_1, u_2, u_3 subject to the relations

$$u_k u_\ell = e^{2\pi i \theta_{k\ell}} u_\ell u_k, \qquad k, \ell = 1, 2, 3$$

is called the noncommutative 3-torus and is denoted by A_θ^3. It has a positive faithful normalized trace denoted by τ. This C*-algebra is indeed a noncommutative deformation of $C(\mathbb{T}^3)$, the algebra of continuous functions on the 3-torus.

For $r = (r_1, r_2, r_3) \in \mathbb{Z}^3$ we set

$$u^r = \exp(\pi i(r_1\theta_{12}r_2 + r_1\theta_{13}r_3 + r_2\theta_{23}r_3))u_1^{r_1}u_2^{r_2}u_3^{r_3}.$$

There is an action α of the 3-torus $\mathbb{T}^3$ on A_θ^3 which is defined by

$$\alpha_z(u^r) = z^r u^r$$

where $z = (z_1, z_2, z_3) \in \mathbb{T}^3$ and $z^r = z_1^{r_1} z_2^{r_2} z_3^{r_3}$. Let $\mathbb{T}_\theta^3$ be the set of all elements $a \in A_\theta^3$ for which the map

$$\alpha(a) : \mathbb{T}^3 \longrightarrow A_\theta^3, \quad z \mapsto \alpha_z(a),$$

is a smooth map. This set is a unital dense subalgebra of A_θ^3 and it is called the algebra of smooth elements of A_θ^3. In fact, it is the analogue of $C^\infty(\mathbb{T}^3)$, the algebra of smooth functions on the 3-torus. It is known that

$$\mathbb{T}_\theta^3 = \left\{ \sum_{r\in\mathbb{Z}^3} a_r u^r : (a_r) \text{ is a rapidly decreasing sequence indexed by } \mathbb{Z}^3 \right\}.$$

By rapidly decreasing we mean for all $k \in \mathbb{N}$,

$$\mathrm{Sup}(1 + |r|^2)^k |a_r|^2 < \infty.$$

The trace on A_θ^3, plays the role of integration in the noncommutative setting and extracts the constant term of the elements of $\mathbb{T}_\theta^3$, i.e.

$$\tau(\sum_{r\in\mathbb{Z}^3} a_r u^r) = a_0.$$

The algebra $\mathbb{T}_\theta^3$ also possesses three derivations, uniquely defined by the relations

$$\delta_j(\sum_{r\in\mathbb{Z}^3} a_r u^r) = \sum_{r\in\mathbb{Z}^3} r_j a_r u^r, \qquad j = 1, 2, 3.$$

These derivations are noncommutative counterparts of the partial derivatives on $C^\infty(\mathbb{T}^3)$ and they satisfy the integration by parts relation i.e.

$$\tau(a\delta_j(b)) = -\tau(\delta_j(a)b), \quad a, b \in \mathbb{T}_\theta^3.$$

More details can be seen in [4.10].

Our next goal is to introduce a spectral triple (A_θ^3, H, D) which encodes the geometry of A_θ^3 with a flat metric. Then we will define the Laplace operator $\triangle$ by $\triangle = D^2$, and perturbing the metric in a conformal class we will define a Laplace type operator $\triangle_\varphi$ by $\triangle_\varphi = k \triangle k$, where $k \in A_\theta^3$ is a positive element representing the conformal class of the metric on A_θ^3. Finally, we shall use $\triangle_\varphi = k \triangle k$ to study the geometry of A_θ^3 with a conformally perturbed metric. Then by analogy with (4.1), we define the scalar curvature of A_θ^3 with the perturbed metric to be

$$\frac{1}{2\pi i}\int_{\mathbb{R}^3}\int_C e^{-\lambda} b_2(\xi,\lambda)\, d\lambda\, d\xi, \tag{4.2}$$

where C is a contour going counterclockwise around the nonnegative part of the x-axis and $b_2(\xi,\lambda)$ is the third term in the asymptotic expansion of the symbol of $(k \triangle k - \lambda)^{-1}$. We will find the first three terms of this asymptotic expansion by Connes' pseudodifferential calculus [4.2] and finally we will compute (4.2).

Let $\langle .,. \rangle_\tau$ be the inner product on A_θ^3 defined by

$$\langle a, b\rangle_\tau = \tau(b^* a), \quad a, b \in A_\theta^3.$$

We denote the completion of A_θ^3 with respect to this inner product by H_τ. It is indeed the representation space in the GNS construction associated to τ. Let $H = H_\tau \otimes \mathbb{C}^2$ and $\pi : A_\theta^3 \longrightarrow B(H)$ be the representation defined by

$$\pi(a) = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \quad a \in A_\theta^3.$$

We define the Dirac operator D on H by

$$D := -i\sum_{j=1}^{3} \sigma_j \delta_j,$$

where σ_j for $j = 1, 2, 3$ are Pauli spin matrices i.e.

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

It can be shown that (A_θ^3, H, D) is a spectral triple. Moreover, we define the Laplace operator by $\triangle = D^2$. It can be seen that

$$\triangle = \sum_{j=1}^{3} \delta_j^2 \otimes I.$$

Next we define a conformal perturbation of this spectral triple. Let $h \in \mathbb{T}_\theta^3$ be a smooth self adjoint element. We associate to h a positive linear functional $\varphi = \varphi_h$ on A_θ^3. This positive linear functional is defined by

$$\varphi(a) = \tau(ae^{-h}), \quad a \in A_\theta^3.$$

Let Δ be the modular operator for φ, i.e.

$$\Delta(a) = e^{-h} a e^{h}, \quad a \in A_\theta^3,$$

and $\{\eta_t\}$, $t \in \mathbb{R}$ be a 1-parameter group of automorphisms of A_θ^3 defined by

$$\eta_t(a) = \Delta^{-it}(a), \quad a \in A_\theta^3.$$

Unlike τ, φ is not a trace. But it satisfies the KMS condition at $\beta = 1$ for $\{\eta_t\}$. In other words,

$$\varphi(ab) = \varphi(b\eta_i(a)), \quad a, b \in A_\theta^3.$$

In analogy with the 2-dimensional case we define a Laplace type operator by $\triangle_\varphi = k \triangle k$, where $k = e^{h/2}$ should be thought as the left multiplication operator by k.

4.4 Connes' Pseudodifferential Calculus

In this section, we will recall Connes' pseudodifferential calculus that was introduced in [4.2].

For $n \in \mathbb{N} \cup \{0\}$, a *differential operator* on $\mathbb{T}_\theta^3$ of order n is a polynomial in $\delta_1, \delta_2, \delta_3$ of the form

$$P(\delta_1, \delta_2, \delta_3) = \sum_{|j| \leqslant n} a_j \delta_1^{j_1} \delta_2^{j_2} \delta_3^{j_3}$$

where $j = (j_1, j_2, j_3) \in \mathbb{Z}_{\geqslant 0}^3$, $|j| = j_1 + j_2 + j_3$ and $a_j \in \mathbb{T}_\theta^3$. Now we extend this definition to pseudodifferential operators.

Definition 4.4.1. A smooth function $\rho : \mathbb{R}^3 \to \mathbb{T}_\theta^3$ is called a *symbol of order* $n \geq 0$ if for all nonnegative integers $i_1, i_2, i_3, j_1, j_2, j_3$ there exists a constant C, such that

$$\|\delta_1^{i_1} \delta_2^{i_2} \delta_3^{i_3} (\partial_1^{j_1} \partial_2^{j_2} \partial_3^{j_3} \rho(\xi))\| \leq C(1 + |\xi|)^{n - |j|},$$

and if there exists a smooth function $k : \mathbb{R}^3 \setminus \{(0,0,0)\} \to \mathbb{T}_\theta^3$ such that

$$\lim_{\lambda \to \infty} \lambda^{-n} \rho(\lambda\xi_1, \lambda\xi_2, \lambda\xi_3) = k(\xi_1, \xi_2, \xi_3).$$

In the last definition by $\partial_1, \partial_2, \partial_3$ we mean partial derivatives, i.e.

$$\partial_1 = \partial/\partial\xi_1, \quad \partial_2 = \partial/\partial\xi_2, \quad \partial_3 = \partial/\partial\xi_3.$$

The space of symbols of order n is denoted by S_n. To any symbol $\rho \in S_n$, an operator P_ρ on $\mathbb{T}_\theta^3$ is associated which is given by

$$P_\rho(a) = (2\pi)^{-3} \iint e^{-iz.\xi} \rho(\xi) \alpha_z(a) dz d\xi, \qquad a \in \mathbb{T}_\theta^3,$$

and is called a *pseudodifferential operator*.

Definition 4.4.2. Let ρ and ρ' be symbols of order k. They are called *equivalent* if and only if $\rho - \rho' \in S_n$ for all $n \in \mathbb{Z}$. This equivalence relation is denoted by $\rho \sim \rho'$.

The next proposition which plays a key role in our computations in this paper, shows that the space of pseudodifferential operators is an algebra. Given the differential operators P and Q, by the next proposition we can find the symbols of PQ and P^* up to the equivalence relation $\sim$, where P^* is the adjoint of P with respect to the inner product $\langle \cdot, \cdot \rangle_\tau$ on H_τ (See [4.5]).

Proposition 4.4.1. Let ρ and ρ' be the symbols of the pseudodifferential operators P and Q. Then PQ and P^* are pseudodifferential operators, and $\sigma(PQ)$ and $\sigma(P^*)$, symbols of PQ and P^* respectively, can be obtained by the following formulas

$$\sigma(PQ) \sim \sum_{(\ell_1,\ell_2,\ell_3)\in(\mathbb{Z}\geqslant 0)^3} \frac{1}{\ell_1!\ell_2!\ell_3!} \partial_1^{\ell_1}\partial_2^{\ell_2}\partial_3^{\ell_3}(\rho(\xi))\delta_1^{\ell_1}\delta_2^{\ell_2}\delta_3^{\ell_3}(\rho'(\xi)),$$

$$\sigma(P^*) \sim \sum_{(\ell_1,\ell_2,\ell_3)\in(\mathbb{Z}\geqslant 0)^3} \frac{1}{\ell_1!\ell_2!\ell_3!} \partial_1^{\ell_1}\partial_2^{\ell_2}\partial_3^{\ell_3}\delta_1^{\ell_1}\delta_2^{\ell_2}\delta_3^{\ell_3}(\rho(\xi))^*.$$

4.5 The Main Result

In this section, using Connes' pseudodifferential calculus, we will define the scalar curvature of the curved noncommutative 3-torus and we will compute it.

To define the scalar curvature of the noncommutative 3-torus we analyze $\triangle_\varphi = k \triangle k$ on H_τ. Exploiting the formula in Proposition 4.4.1 and considering k as a pseudodifferential operator of order 0 with the symbol $\sigma(k) = k$, plus the fact that the symbol of $\triangle$ is

$$\sigma(\triangle) = \sum_{i=1}^{3} \xi_i^2,$$

we can find the symbol of $k \triangle k$. Indeed, we can show that

$$\sigma(k \triangle k) = a_0(\xi) + a_1(\xi) + a_2(\xi),$$

where $\xi = (\xi_1, \xi_2, \xi_3)$ and

$$a_0(\xi) = \sum_{i=1}^{3} k\delta_i\delta_i(k),$$

$$a_1(\xi) = 2\sum_{i=1}^{3} \xi_i k\delta_i(k),$$

$$a_2(\xi) = \sum_{i=1}^{3} k^2\xi_i^2.$$

Let $\lambda \in \mathbb{C}$. We need to find an asymptotic expansion of the symbol of $(k \triangle k - \lambda)^{-1}$. Indeed, we have to find an operator R_λ such that

$$\sigma(R_\lambda \cdot (k \triangle k - \lambda)) \sim \sigma(I)$$

where I is the identity operator. Using the formula in Proposition 4.4.1 and following the steps in page 52 of [4.9], we can find a recursive formula for the terms of an asymptotic expansion of $(k \triangle k - \lambda)^{-1}$. In fact, one can show that

$$\sigma(k \triangle k - \lambda)^{-1} \sim \sum_{n=0}^{\infty} b_n(\xi, \lambda),$$

where $b_n(\xi, \lambda)$ is a symbol of order $-2-n$ given by the following recursive formula:

$$b_0(\xi, \lambda) = (k^2 \sum_{i=1}^{3} \xi_i^2 - \lambda)^{-1},$$

$$b_n(\xi, \lambda) = - \sum_{\substack{2+j+\ell_1+\ell_2+\ell_3-m=n \\ 0 \le j < n,\ 0 \le m \le 2}} \frac{1}{\ell_1! \ell_2! \ell_3!} \partial_1^{\ell_1} \partial_2^{\ell_2} \partial_3^{\ell_3} (b_j) \delta_1^{\ell_1} \delta_2^{\ell_2} \delta_3^{\ell_3} (a_m) b_0, \tag{4.3}$$

for $n \geqslant 1$.

Now we are able to define the scalar curvature of A_θ^3 with the perturbed metric. Indeed, (4.1) motivates us to define the scalar curvature of A_θ^3 with the perturbed metric as follows:

Definition 4.5.1. Let C be a contour going counterclockwise around the nonnegative part of the x-axis and $b_2(\xi, \lambda)$ for $\lambda \in \mathbb{C}$ be the third term in the asymptotic expansion of the symbol of $(k \triangle k - \lambda)^{-1}$. Then the scalar curvature of A_θ^3 with the perturbed metric is defined to be the element $S \in A_\theta^3$ given by

$$S = \frac{1}{2\pi i} \int_{\mathbb{R}^3} \int_C e^{-\lambda} b_2(\xi, \lambda) d\lambda d\xi.$$

Let

$$\alpha(\lambda) = \int_{\mathbb{R}^3} b_2(\xi, \lambda) d\xi.$$

The function α is homogeneous of degree $-1/2$ with respect to λ. We also define

$$\beta(\lambda) = \lambda^{-1/2} \alpha(\lambda).$$

The function β is homogeneous of degree -1 with respect to λ. For the square root function we consider the nonnegative part of the real axis as the branch cut. Then we have

$$S = \frac{1}{2\pi i} \beta(-1) \int_C \frac{e^{-\lambda}}{-\lambda^{1/2}} d\lambda.$$

To compute the latter, we consider the contour $C = C_1 + C_2 + C_3$, where $C_1 = re^{i\pi/4}$ for $r \in (\infty, 1)$, $C_2 = e^{i\theta}$ for $\theta \in (\pi/4, 7\pi/4)$ and $C_3 = re^{7i\pi/4}$ for $r \in (1, \infty)$. One can see that

$$\int_{C1} \frac{e^{-\lambda}}{-\lambda^{1/2}} d\lambda = (-1)^{7/8} e^{\frac{i\pi}{8}} \sqrt{\pi} \text{Erfc} \left[(-1)^{1/8} \right],$$

$$\int_{C_2} \frac{e^{-\lambda}}{-\lambda^{1/2}} d\lambda = \sqrt{\pi} \left(-\mathrm{Erf}\left[(-1)^{1/8}\right] + \mathrm{Erf}\left[(-1)^{7/8}\right] \right),$$

and

$$\int_{C_3} \frac{e^{-\lambda}}{-\lambda^{1/2}} d\lambda = (-1)^{1/8} e^{\frac{7i\pi}{8}} \sqrt{\pi} \left(1 + \mathrm{Erf}\left[(-1)^{7/8}\right] \right).$$

Therefore,

$$\int_C \frac{e^{-\lambda}}{-\lambda^{1/2}} d\lambda = -2\sqrt{\pi}$$

and this implies that

$$S = \frac{1}{2\pi i} \int_{\mathbb{R}^3} \int_C e^{-\lambda} b_2(\xi, \lambda) d\xi d\lambda = \frac{-1}{\sqrt{\pi}} \alpha(-1).$$

By this argument, to find S, it suffices to work with $\lambda = -1$ and compute

$$\alpha(-1) = \int_{\mathbb{R}^3} b_2(\xi, -1) d\xi.$$

We devote the rest of the paper to the calculation of $\alpha(-1)$.

4.6 The Computation of $b_2(\xi, -1)$

In this section we will use the recursive formula (4.3) to find $b_2(\xi, -1)$. In what follows, we set $b_n = b_n(\xi, -1)$ for $n \in \mathbb{N}$.

We know that

$$b_0 = (k^2 \sum_{i=1}^{3} \xi_i^2 + 1)^{-1}.$$

Now using (4.3), we have

$$b_1 = -b_0 a_1 b_0 - (\sum_{i=1}^{3} \partial_i(b_0) \delta_i(a_2)) b_0.$$

Simplifying the above formula we obtain

$$\begin{aligned} b_1 = &\, 2\xi_1^3 b_0^2 k^3 \delta_1(k) b_0 + 2\xi_2 \xi_1^2 b_0^2 k^3 \delta_2(k) b_0 + 2\xi_3 \xi_1^2 b_0^2 k^3 \delta_3(k) b_0 + 2\xi_2^2 \xi_1 b_0^2 k^3 \delta_1(k) b_0 \\ &+ 2\xi_3^2 \xi_1 b_0^2 k^3 \delta_1(k) b_0 + 2\xi_2^3 b_0^2 k^3 \delta_2(k) b_0 + 2\xi_3^3 b_0^2 k^3 \delta_3(k) b_0 + 2\xi_2 \xi_3^2 b_0^2 k^3 \delta_2(k) b_0 \\ &+ 2\xi_2^2 \xi_3 b_0^2 k^3 \delta_3(k) b_0 + 2\xi_1^3 b_0^2 k^2 \delta_1(k) b_0 k + 2\xi_2 \xi_1^2 b_0^2 k^2 \delta_2(k) b_0 k \\ &+ 2\xi_3 \xi_1^2 b_0^2 k^2 \delta_3(k) b_0 k + 2\xi_2^2 \xi_1 b_0^2 k^2 \delta_1(k) b_0 k + 2\xi_3^2 \xi_1 b_0^2 k^2 \delta_1(k) b_0 k \\ &+ 2\xi_2^3 b_0^2 k^2 \delta_2(k) b_0 k + 2\xi_3^3 b_0^2 k^2 \delta_3(k) b_0 k + 2\xi_2 \xi_3^2 b_0^2 k^2 \delta_2(k) b_0 k \\ &+ 2\xi_2^2 \xi_3 b_0^2 k^2 \delta_3(k) b_0 k - 2\xi_1 b_0 k \delta_1(k) b_0 - 2\xi_2 b_0 k \delta_2(k) b_0 - 2\xi_3 b_0 k \delta_3(k) b_0. \end{aligned}$$

Now we use b_1 in (4.3) to obtain b_2. We have

$$b_2 = -b_0 a_0 - b_1 a_1$$
$$-\partial_1(b_0)\delta_1(a_1) - \partial_2(b_0)\delta_2(a_1) - \partial_3(b_0)\delta_3(a_1)$$
$$-\partial_1(b_1)\delta_1(a_2) - \partial_2(b_1)\delta_2(a_2) - \partial_3(b_1)\delta_3(a_2)$$
$$-\partial_{12}(b_0)\delta_1(\delta_2(a_2)) - \partial_{13}(b_0)\delta_1(\delta_3(a_2)) - \partial_{23}(b_0)\delta_2(\delta_3(a_2))$$
$$-(1/2)\partial_{11}(b_0)\delta_1^2(a_2) - (1/2)\partial_{22}(b_0)\delta_2^2(a_2) - (1/2)\partial_{33}(b_0)\delta_3^2(a_2).$$

After doing computations we get a simplified formula for b_2. Here we exhibit the first 10 terms of b_2. The entire formula can be seen in Appendix A.

$$b_2 = -b_0 k\delta_1(\delta_1(k))\, b_0 - b_0 k\delta_2(\delta_2(k))\, b_0 - b_0 k\delta_3(\delta_3(k))\, b_0 + 6\xi_1^2 b_0^2 k^2 \delta_1(k)^2 b_0 + 2\xi_2^2 b_0^2 k^2 \delta_1(k)^2 b_0$$
$$+2\xi_3^2 b_0^2 k^2 \delta_1(k)^2 b_0 + 2\xi_1^2 b_0^2 k^2 \delta_2(k)^2 b_0 + 6\xi_2^2 b_0^2 k^2 \delta_2(k)^2 b_0 + 2\xi_3^2 b_0^2 k^2 \delta_2(k)^2 b_0 + 2\xi_1^2 b_0^2 k^2 \delta_3(k)^2 b_0 + \cdots$$

4.7 Integrating $b_2(\xi, -1)$ over $\mathbb{R}^3$

In this section, first we will change the variables and then we will use a rearrangement lemma to integrate $b_2(\xi, -1)$ over $\mathbb{R}^3$.

To integrate $b_2(\xi, -1)$ with respect to $\xi = (\xi_1, \xi_2, \xi_3)$, we apply the spherical change of coordinates

$$\xi_1 = r\sin\Phi\cos\theta, \quad \xi_2 = r\sin\Phi\sin\theta, \quad \xi_3 = r\cos\Phi,$$

where $0 \le \theta \le 2\pi, 0 \le \Phi \le \pi$ and $0 \le r \le \infty$. Considering this change of coordinates and integrating the formula for $b_2(\xi, -1)$ in Appendix A, with respect to θ and Φ, one finds that

$$\int_{\mathbb{R}^3} b_2(\xi, -1) d\xi,$$

up to an overall factor of $4\pi/3$, is

$$\int_0^\infty B(r) dr,$$

where

$$B(r) = 4r^8 b_0^2 k^2 \delta_i(k) b_0^2 k^4 \delta_i(k) b_0 + 4r^8 b_0^2 k^3 \delta_i(k) b_0^2 k^3 \delta_i(k) b_0$$
$$+8r^8 b_0^3 k^4 \delta_i(k) b_0 k^2 \delta_i(k) b_0 + 8r^8 b_0^3 k^5 \delta_i(k) b_0 k \delta_i(k) b_0$$
$$+8r^8 b_0^3 k^5 \delta_i(k) b_0 \delta_i(k) b_0 k + 4r^8 b_0^2 k^2 \delta_i(k) b_0^2 k^3 \delta_i(k) b_0 k$$
$$+4r^8 b_0^2 k^3 \delta_i(k) b_0^2 k^2 \delta_i(k) b_0 k + 8r^8 b_0^3 k^4 \delta_i(k) b_0 k \delta_i(k) b_0 k - 8r^6 b_0^3 k^4 \delta_i(k)^2 b_0$$
$$-4r^6 b_0^3 k^5 \delta_i(\delta_i(k))\, b_0 - 4r^6 b_0^3 k^4 \delta_i(\delta_i(k))\, b_0 k - r^6 b_0 k \delta_i(k) b_0^2 k^3 \delta_i(k) b_0$$
$$-14r^6 b_0^2 k^2 \delta_i(k) b_0 k^2 \delta_i(k) b_0 - 18r^6 b_0^2 k^3 \delta_i(k) b_0 k \delta_i(k) b_0$$
$$-14r^6 b_0^2 k^3 \delta_i(k) b_0 \delta_i(k) b_0 k - 4r^6 b_0 k \delta_i(k) b_0^2 k^2 \delta_i(k) b_0 k$$

$$-10r^6b_0^2k^2\delta_i(k)b_0k\delta_i(k)b_0k+10r^4b_0^2k^2\delta_i(k)^2b_0$$
$$+7r^4b_0^2k^3\delta_i\left(\delta_i(k)\right)b_0+r^4b_0^2k^2\delta_i\left(\delta_i(k)\right)b_0k+10r^4b_0k\delta_i(k)b_0k\delta_i(k)b_0$$
$$+6r^4b_0k\delta_i(k)b_0\delta_i(k)b_0k-3r^2b_0k\delta_i\left(\delta_i(k)\right)b_0.$$

In the above sum $b_0=(r^2k^2+1)^{-1}$ and summation over i is understood.

Moreover, one can see that for $x\in\mathbb{T}^3_\theta$, $xk^n=k^n\Delta^{n/2}(x)$. Using this relation plus the fact that $b_0k=kb_0$, we can see that

$$B(r)=4r^8k^6b_0^2\Delta^2(\delta_i(k))b_0^2\delta_i(k)b_0+4r^8k^6b_0^2\Delta^{3/2}(\delta_i(k))b_0^2\delta_i(k)b_0$$
$$+8r^8k^6b_0^3\Delta(\delta_i(k))b_0\delta_i(k)b_0+8r^8k^6b_0^3\Delta^{1/2}(\delta_i(k))b_0\delta_i(k)b_0$$
$$+8r^8k^6b_0^3\Delta^{1/2}(\delta_i(k))b_0\Delta^{1/2}(\delta_i(k))b_0+4r^8k^6b_0^2\Delta^2(\delta_i(k))b_0^2\Delta^{1/2}(\delta_i(k))b_0$$
$$+4r^8k^6b_0^2\Delta^{3/2}(\delta_i(k))b_0^2\Delta^{1/2}(\delta_i(k))b_0+8r^8k^6b_0^3\Delta(\delta_i(k))b_0\Delta^{1/2}(\delta_i(k))b_0$$
$$-8r^6k^4b_0^3\delta_i(k)^2b_0-4r^6k^5b_0^3\delta_i\left(\delta_i(k)\right)b_0-4r^6k^5b_0^3\Delta^{1/2}(\delta_i\left(\delta_i(k)\right))b_0$$
$$-4r^6k^4b_0\Delta^{3/2}(\delta_i(k))b_0^2\delta_i(k)b_0-14r^6k^4b_0^2\Delta(\delta_i(k))b_0\delta_i(k)b_0$$
$$-18r^6k^4b_0^2\Delta^{1/2}(\delta_i(k))b_0\delta_i(k)b_0-14r^6k^4b_0^2\Delta^{1/2}(\delta_i(k))b_0\Delta^{1/2}(\delta_i(k))b_0$$
$$-4r^6k^4b_0\Delta^{3/2}(\delta_i(k))b_0^2\Delta^{1/2}(\delta_i(k))b_0-10r^6k^4b_0^2\Delta(\delta_i(k))b_0\Delta^{1/2}(\delta_i(k))b_0$$
$$+10r^4k^2b_0^2\delta_i(k)^2b_0+7r^4k^3b_0^2\delta_i\left(\delta_i(k)\right)b_0+3r^4k^3b_0^2\Delta^{1/2}(\delta_i\left(\delta_i(k)\right))b_0$$
$$+10r^4k^2b_0\Delta^{1/2}(\delta_i(k))b_0\delta_i(k)b_0+6r^4k^2b_0\Delta^{1/2}(\delta_i(k))b_0\Delta^{1/2}(\delta_i(k))b_0$$
$$-3r^2kb_0\delta_i\left(\delta_i(k)\right)b_0.$$

To integrate the terms of $B(r)$, we need a lemma similar to the rearrangement lemma in [4.4]. In the following lemma we will use exactly the same method as in the proof of the rearrangement lemma in [4.4], to prove a slightly different statement.

Lemma 4.7.1. *Let $\rho_j\in\mathbb{T}^3_\theta$ and $m_j\in\mathbb{Z}$, for $j=0,1,2,\dots,l$. Then*

$$\int_0^\infty(k^2u+1)^{-m_0}\rho_1(k^2u+1)^{-m_1}\cdots\rho_l(k^2u+1)^{-m_l}u^{(\sum_{j=0}^{j=l}m_j-3/2)}du$$

$$=k^{(-2\sum_{j=0}^{j=l}m_j+1)}F_{m_0,m_1,m_1,\dots,m_l}(\Delta_{(1)},\Delta_{(2)},\dots,\Delta_{(l)})(\rho_1\rho_2\cdots\rho_l),$$

where

$$F_{m_0,m_1,m_1,\dots,m_l}(u_1,u_2,\dots,u_l)=\int_0^\infty(u+1)^{-m_0}\prod_{j=1}^{j=l}\left(u\prod_{h=1}^{h=j}u_h+1\right)^{-m_j}u^{(\sum_{j=0}^{j=l}m_j-3/2)}du,$$

and $\Delta_{(j)}$ means that Δ acts on the jth factor, for $j=0,1,2,\dots,l$.

Proof. Let G_n and $G_{n,\alpha}$ be the inverse Fourier transforms of the functions defined respectively by

$$g_n(t) = (e^{t/2} + e^{-t/2})^{-n}$$

and

$$H_{n,\alpha}(t) = e^{(n-\alpha)t}(e^t + 1)^{-n},$$

where $n \in \mathbb{N}$ and $\alpha \in (0, n)$. Then $G_{n,\alpha}(s) = G_n(s - i(n/2 - \alpha))$. So we have

$$H_{n,\alpha}(t) = \int_{-\infty}^{\infty} G_n(s - i(n/2 - \alpha))e^{-ist}ds. \tag{4.4}$$

Let J be the integral in the left hand side of the equation in the lemma. Now we use the substitutions $u = e^s$ and $k = e^{f/2}$ to compute J. Therefore, we have

$$J = \int_{-\infty}^{\infty} (e^{(s+f)} + 1)^{-m_0}\rho_1(e^{(s+f)} + 1)^{-m_1} \cdots \rho_l(e^{(s+f)} + 1)^{-m_l}e^{(\sum\limits_{j=0}^{j=l} m_j - 1)s}e^{s/2}ds.$$

Then for $j = 0, 1, 2, \ldots, l$, we pick a positive real number α_j such that $\sum\limits_{j=0}^{j=l}\alpha_j = 1$. We also set $\beta_j = -\sum\limits_{i=j}^{i=l}(m_i - \alpha_i)$. Replacing $(e^{(s+f)} + 1)^{-m_j}$ by $e^{(m_j-\alpha_j)(f+s)}(e^{(s+f)} + 1)^{-m_j}$ in J, we get

$$J = e^{-(\sum\limits_{j=0}^{j=l} m_j - 1)f}\int_{-\infty}^{\infty} H_{m_0,\alpha_0}(s + f)\Delta^{\beta_1}(\rho_1)H_{m_1,\alpha_1}(s + f) \cdots \Delta^{\beta_l}(\rho_l)H_{m_l,\alpha_l}(s + f)e^{s/2}ds.$$

Let $\rho'_j = \Delta^{\beta_j}(\rho_j)$. Using (4.4), J can be written as an integral of the form

$$e^{-(\sum\limits_{j=0}^{j=l} m_j - 1)f}H_{m_0,\alpha_0}(s + f)\rho'_1e^{-i(s+f)t_1}\rho'_2 \cdots e^{-i(s+f)t_{l-1}}\rho'_le^{-i(s+f)t_l}e^{s/2} \tag{4.5}$$

with respect to the measure

$$\prod_{j=1}^{j=l} G_{m_j,\alpha_j}(t_j)dt_jds.$$

Now we can write the term (4.5) as

$$e^{-(\sum\limits_{j=0}^{j=l} m_j - 1)f}H_{m_0,\alpha_0}(s + f)e^{-i(\sum\limits_{j=1}^{j=l} t_j)(s+f)}\prod_{h=1}^{h=l}\Delta^{-i\sum\limits_{j=h}^{j=l} t_j}(\rho'_h)e^{s/2}.$$

We also have

$$\int_{-\infty}^{\infty} H_{m_0,\alpha_0}(s + f)e^{-i(\sum\limits_{j=1}^{j=l} t_j)(s+f)}e^{s/2}ds =$$

$$e^{-f/2}\int_{-\infty}^{\infty} e^{(s+f)/2}H_{m_0,\alpha_0}(s+f)e^{-i(\sum\limits_{j=1}^{j=l}t_j)(s+f)}ds = 2\pi e^{-f/2}P_{m_0,\alpha_0}(-\sum_{j=1}^{j=l}t_j),$$

where P_{m_0,α_0} is the inverse Fourier transform of the function $e^{s/2}H_{m_0,\alpha_0}(s)$.

So we have

$$J = 2\pi e^{-f/2}e^{-(\sum\limits_{j=0}^{j=l}m_j-1)f}\int\prod_{h=1}^{h=l}\Delta^{-i\sum\limits_{j=h}^{j=l}t_j}(\rho'_h)P_{m_0,\alpha_0}(-\sum_{j=1}^{j=l}t_j)\prod_{j=1}^{j=l}G_{m_j,\alpha_j}(t_j)dt_j.$$

Replacing ρ'_j by $\Delta^{\beta_j}(\rho_j)$ we have

$$\Delta^{-i\sum\limits_{j=h}^{j=l}t_j}(\rho'_h) = \Delta^{-i\sum\limits_{j=h}^{j=l}t_j+\beta_h}(\rho_h).$$

Now we replace the last term by

$$u_h^{-i\sum\limits_{j=h}^{j=l}t_j+\beta_h}.$$

We define

$$F_{m_0,m_1,m_1,\ldots,m_l}(u_1,u_2,\ldots,u_l) = 2\pi e^{-f/2}\int\prod_{h=1}^{h=l}u_h^{-i\sum\limits_{j=h}^{j=l}t_j+\beta_h}P_{m_0,\alpha_0}(-\sum_{j=1}^{j=l}t_j)\prod_{j=1}^{j=l}G_{m_j,\alpha_j}(t_j)dt_j.$$

Moreover, we can write

$$2\pi P_{m_0,\alpha_0}(-\sum_{j=1}^{j=l}t_j) = \int_{-\infty}^{\infty} e^{s/2}H_{m_0,\alpha_0}(s)e^{-i(\sum\limits_{j=1}^{j=l}t_j)(s)}ds.$$

Using this and assuming that $u_h = e^{s_h}$, we can do the integration. Then, the coefficient of t_j in the exponent is

$$-is - i\sum_{h=1}^{h=j}s_h.$$

So integrating in t_j gives the Fourier transform of G_{m_j,α_j} at $s+\sum\limits_{h=1}^{h=j}s_h$. On the other hand we have

$$e^{(m_j-\alpha_j)(s+\sum\limits_{h=1}^{h=j}s_h)}(e^{(s+\sum\limits_{h=1}^{h=j}s_h)}+1)^{-m_j} = e^{(m_j-\alpha_j)s}\left(\prod_{h=1}^{h=j}u_h\right)^{(m_j-\alpha_j)}\left(e^s\prod_{h=1}^{h=j}u_h+1\right)^{-m_j}.$$

When we multiply these terms from $j=1$ to $j=l$, the exponent of u_h is $\sum\limits_{j=h}^{j=l}(m_j-\alpha_j)$. So $u_h^{\beta_h}$ disappears and we get

$$F_{m_0,m_1,m_2,\ldots,m_l}(u_1,u_2,\ldots,u_l) =$$

$$e^{-f/2}\int_{-\infty}^{\infty}(e^s+1)^{-m_0}\prod_{h=1}^{h=l}\left(e^s\prod_{h=1}^{h=j}u_h+1\right)^{-m_j}e^{(\sum_{j=0}^{j=l}m_j-1)s}e^{s/2}ds.$$

□

In Lemma 4.7.1, it is clear that

$$F_{m_0,m_1,m_1,\dots,m_l}(u_1,u_2,\dots,u_l)=H_{m_0,m_1,m_1,\dots,m_l}(u_1,u_1u_2,\dots,u_1\cdots u_l),$$

where

$$H_{m_0,m_1,m_1,\dots,m_l}(u_1,u_2,\dots,u_l)=\int_0^{\infty}(u+1)^{-m_0}\prod_{j=1}^{j=l}(uu_h+1)^{-m_j}u^{(\sum_{j=0}^{j=l}m_j-3/2)}du.$$

We only need some of these functions:

$$H_{1,1}(x)=\int_0^{\infty}(u+1)^{-1}(ux+1)^{-1}u^{1/2}du=\frac{\pi}{x+\sqrt{x}},$$

$$H_{1,1,1}(x,y)=\int_0^{\infty}(u+1)^{-1}(ux+1)^{-1}(uy+1)^{-1}u^{3/2}du=$$
$$\frac{\pi\left(\sqrt{x}+\sqrt{y}+1\right)}{\left(\sqrt{x}+1\right)\sqrt{x}\left(y+\sqrt{y}\right)\left(\sqrt{x}+\sqrt{y}\right)},$$

$$H_{2,1}(x)=\int_0^{\infty}(u+1)^{-2}(ux+1)^{-1}u^{3/2}du=\frac{\frac{2\pi}{\sqrt{x}}+\pi}{2\left(\sqrt{x}+1\right)^2},$$

$$H_{2,1,1}(x,y)=\int_0^{\infty}(u+1)^{-2}(ux+1)^{-1}(uy+1)^{-1}u^{5/2}du=$$
$$\frac{\pi\left(\sqrt{x}\left(\sqrt{y}+2\right)^2+x\left(\sqrt{y}+2\right)+2\left(\sqrt{y}+1\right)^2\right)}{2\left(\sqrt{x}+1\right)^2\sqrt{x}\left(\sqrt{y}+1\right)^2\sqrt{y}\left(\sqrt{x}+\sqrt{y}\right)},$$

$$H_{1,2,1}(x,y)=\int_0^{\infty}(u+1)^{-1}(ux+1)^{-2}(uy+1)^{-1}u^{5/2}du=$$
$$\frac{\pi\left(2x^{3/2}+4x\left(\sqrt{y}+1\right)+2\sqrt{x}\left(\sqrt{y}+1\right)^2+y+\sqrt{y}\right)}{2\left(\sqrt{x}+1\right)^2x^{3/2}\left(\sqrt{y}+1\right)\sqrt{y}\left(\sqrt{x}+\sqrt{y}\right)^2},$$

$$H_{2,2,1}(x,y)=\int_0^{\infty}(u+1)^{-2}(ux+1)^{-2}(uy+1)^{-1}u^{7/2}du=$$
$$\frac{\pi\left(2\left(x^{3/2}+4x+4\sqrt{x}+1\right)y+\left(7x^{3/2}+x^2+13x+7\sqrt{x}+1\right)\sqrt{y}\right)}{2\left(\sqrt{x}+1\right)^3x^{3/2}\left(\sqrt{y}+1\right)^2\sqrt{y}\left(\sqrt{x}+\sqrt{y}\right)^2}+$$
$$\frac{\pi\left(\left(x+3\sqrt{x}+1\right)y^{3/2}+2\left(\sqrt{x}+1\right)^3\sqrt{x}\right)}{2\left(\sqrt{x}+1\right)^3x^{3/2}\left(\sqrt{y}+1\right)^2\sqrt{y}\left(\sqrt{x}+\sqrt{y}\right)^2},$$

$$H_{3,1}(x) = \int_0^\infty (u+1)^{-3}(ux+1)^{-1}u^{5/2}du = \frac{\pi\left(3x+9\sqrt{x}+8\right)}{8\left(\sqrt{x}+1\right)^3\sqrt{x}},$$

and

$$H_{3,1,1}(x,y) = \int_0^\infty (u+1)^{-3}(ux+1)^{-1}(uy+1)^{-1}u^{7/2}du =$$

$$\frac{\pi\left(-\frac{3x+9\sqrt{x}+8}{\left(\sqrt{x}+1\right)^3\sqrt{x}} - \frac{8}{\sqrt{y}+1} - \frac{5}{\left(\sqrt{y}+1\right)^2} - \frac{2}{\left(\sqrt{y}+1\right)^3} + \frac{8}{\sqrt{y}}\right)}{8(x-y)}.$$

Now with the notations that we have set up, we can state and prove the main result of this paper:

Theorem 4.7.2. *The scalar curvature of A_θ^3 with the perturbed metric, is the element $S \in A_\theta^3$ given by the following formula multiplied by $-4\sqrt{\pi}/3$.*

$$S = k^{-2}(-\frac{3}{2}H_{1,1} + \frac{7}{2}H_{2,1} - 2H_{3,1})(\Delta_{(1)})(\delta_i\left(\delta_i(k)\right))$$

$$+k^{-3}(3H_{1,1,1} - 7H_{2,1,1} + 4H_{3,1,1})(\Delta_{(1)}, \Delta_{(1)}\Delta_{(2)})(\Delta^{1/2}(\delta_i(k))\Delta^{1/2}(\delta_i(k))))$$

$$+k^{-3}(5H_{1,1,1} - 9H_{2,1,1} + 4H_{3,1,1})(\Delta_{(1)}, \Delta_{(1)}\Delta_{(2)})(\Delta^{1/2}(\delta_i(k))\delta_i(k))$$

$$+k^{-2}(\frac{3}{2}H_{2,1} - 2H_{3,1})(\Delta_{(1)})(\Delta^{1/2}(\delta_i\left(\delta_i(k)\right))) + k^{-3}(5H_{2,1} - 4H_{3,1})(\Delta_{(1)})(\delta_i(k)^2)$$

$$+k^{-3}(-5H_{2,1,1} + 2H_{3,1,1})(\Delta_{(1)}, \Delta_{(1)}\Delta_{(2)})(\Delta(\delta_i(k))\Delta^{1/2}(\delta_i(k)))$$

$$k^{-3}(-2H_{1,2,1} + 2H_{2,2,1})(\Delta_{(1)}, \Delta_{(1)}\Delta_{(2)})(\Delta^{3/2}(\delta_i(k))\Delta^{1/2}(\delta_i(k)))$$

$$k^{-3}(-7H_{2,1,1} + 4H_{3,1,1})(\Delta_{(1)}, \Delta_{(1)}\Delta_{(2)})(\Delta(\delta_i(k))\delta_i(k))$$

$$k^{-3}(-2H_{1,2,1} + 2H_{2,2,1})(\Delta_{(1)}, \Delta_{(1)}\Delta_{(2)})(\Delta^{3/2}(\delta_i(k))\delta_i(k))$$

$$+2k^{-3}H_{2,2,1}(\Delta_{(1)}, \Delta_{(1)}\Delta_{(2)})(\Delta^2(\delta_i(k))\Delta^{1/2}(\delta_i(k)))$$

$$+2k^{-3}H_{2,2,1}(\Delta_{(1)}, \Delta_{(1)}\Delta_{(2)})(\Delta^2(\delta_i(k))\delta_i(k)).$$

Proof. It suffices to find

$$\int_0^\infty B(r)dr.$$

For that we only need to use the substitution $r^2 = u$, and then apply Lemma 4.7.1. $\square$

Bibliography

[4.1] P. B. Cohen, A. Connes, *Conformal geometry of the irrational rotation algebra*. Preprint MPI / 92-93.

[4.2] A. Connes, *C^*-algèbres et géométrie différentielle*. C.R. Acad. Sc. Paris, t. 290, Série A, 599-604, 1980.

[4.3] A. Connes, *Noncommutative geometry*. Academic Press (1994).

[4.4] A. Connes and H. Moscovici, *Modular curvature for noncommutative two-tori*. J. Amer. Math. Soc., 27(3):639-684, 2014.

[4.5] A. Connes, P. Tretkoff, *The Gauss-Bonnet theorem for the noncommutative two torus*, *Noncommutative geometry, arithmetic, and related topics*, 141-158, Johns Hopkins Univ. Press, Baltimore, MD, 2011.

[4.6] F. Fathizadeh, M. Khalkhali, *The Gauss-Bonnet theorem for noncommutative two tori with a general conformal structure*, J. Noncommut. Geom., 6 (2012), no. 3, 457-480.

[4.7] F. Fathizadeh and M. Khalkhali, *Scalar curvature for the noncommutative two torus. J. Noncommut.* Geom., 7(4):1145-1183, 2013.

[4.8] F. Fathizadeh and M. Khalkhali. Scalar curvature for noncommutative four-tori, J. Noncommut. Geom., 9(2), 473-503, 2015.

[4.9] P. Gilkey, *Invariance theory, the heat equation, and the Atiyah-Singer index theorem*, Mathematics Lecture Series, 11. Publish or Perish, Inc., Wilmington, DE, 1984.

[4.10] J.M. Gracia-Bondía, J. Várilly, H. Figueroa. *Elements of noncommutative geometry*, Birkhäuser Advanced Texts: Basler Lehrbücher, 2001.

[4.11] M. Khalkhali, *Basic Noncommutative Geometry*, Second edition. EMS Series of Lectures in Mathematics. European Mathematical Society (EMS), Zürich, 2013.

[4.12] Y. Liu, *Modular curvature for toric noncommutative manifolds*, arXiv:1510.04668.

Appendix A

The Formula for $b_2(\xi, -1)$

In Section 4.5 we gave the first 10 terms of $b_2(\xi, -1)$. Here we have the entire formula:

$$b_2 = -b_0 k \delta_1 \left(\delta_1(k)\right) b_0 - b_0 k \delta_2 \left(\delta_2(k)\right) b_0 - b_0 k \delta_3 \left(\delta_3(k)\right) b_0$$

$$+6\xi_1^2 b_0^2 k^2 \delta_1(k)^2 b_0 + 2\xi_2^2 b_0^2 k^2 \delta_1(k)^2 b_0 + 2\xi_3^2 b_0^2 k^2 \delta_1(k)^2 b_0 + 2\xi_1^2 b_0^2 k^2 \delta_2(k)^2 b_0$$

$$+6\xi_2^2 b_0^2 k^2 \delta_2(k)^2 b_0 + 2\xi_3^2 b_0^2 k^2 \delta_2(k)^2 b_0 + 2\xi_1^2 b_0^2 k^2 \delta_3(k)^2 b_0 + 2\xi_2^2 b_0^2 k^2 \delta_3(k)^2 b_0$$

$$+6\xi_3^2 b_0^2 k^2 \delta_3(k)^2 b_0 + 5\xi_1^2 b_0^2 k^3 \delta_1 \left(\delta_1(k)\right) b_0 + \xi_2^2 b_0^2 k^3 \delta_1 \left(\delta_1(k)\right) b_0$$

$$+\xi_3^2 b_0^2 k^3 \delta_1 \left(\delta_1(k)\right) b_0 + 4\xi_1\xi_2 b_0^2 k^3 \delta_1 \left(\delta_2(k)\right) b_0 + 4\xi_1\xi_3 b_0^2 k^3 \delta_1 \left(\delta_3(k)\right) b_0$$

$$+4\xi_1\xi_2 b_0^2 k^3 \delta_2 \left(\delta_1(k)\right) b_0 + \xi_1^2 b_0^2 k^3 \delta_2 \left(\delta_2(k)\right) b_0 + 5\xi_2^2 b_0^2 k^3 \delta_2 \left(\delta_2(k)\right) b_0$$

$$+\xi_3^2 b_0^2 k^3 \delta_2 \left(\delta_2(k)\right) b_0 + 4\xi_2\xi_3 b_0^2 k^3 \delta_2 \left(\delta_3(k)\right) b_0 + 4\xi_1\xi_3 b_0^2 k^3 \delta_3 \left(\delta_1(k)\right) b_0$$

$$+4\xi_2\xi_3 b_0^2 k^3 \delta_3 \left(\delta_2(k)\right) b_0 + \xi_1^2 b_0^2 k^3 \delta_3 \left(\delta_3(k)\right) b_0 + \xi_2^2 b_0^2 k^3 \delta_3 \left(\delta_3(k)\right) b_0$$

$$+5\xi_3^2 b_0^2 k^3 \delta_3 \left(\delta_3(k)\right) b_0 - 8\xi_1^4 b_0^3 k^4 \delta_1(k)^2 b_0 - 8\xi_1^2\xi_2^2 b_0^3 k^4 \delta_1(k)^2 b_0$$

$$-8\xi_1^2\xi_3^2 b_0^3 k^4 \delta_1(k)^2 b_0 - 8\xi_1^2\xi_2^2 b_0^3 k^4 \delta_2(k)^2 b_0 - 8\xi_2^4 b_0^3 k^4 \delta_2(k)^2 b_0$$

$$-8\xi_2^2\xi_3^2 b_0^3 k^4 \delta_2(k)^2 b_0 - 8\xi_1^2\xi_3^2 b_0^3 k^4 \delta_3(k)^2 b_0 - 8\xi_2^2\xi_3^2 b_0^3 k^4 \delta_3(k)^2 b_0$$

$$-8\xi_3^4 b_0^3 k^4 \delta_3(k)^2 b_0 - 4\xi_1^4 b_0^3 k^5 \delta_1 \left(\delta_1(k)\right) b_0 - 4\xi_1^2\xi_2^2 b_0^3 k^5 \delta_1 \left(\delta_1(k)\right) b_0$$

$$-4\xi_1^2\xi_3^2 b_0^3 k^5 \delta_1 \left(\delta_1(k)\right) b_0 - 8\xi_1^3\xi_2 b_0^3 k^5 \delta_1 \left(\delta_2(k)\right) b_0 - 8\xi_1\xi_2^3 b_0^3 k^5 \delta_1 \left(\delta_2(k)\right) b_0$$

$$-8\xi_1\xi_2\xi_3^2 b_0^3 k^5 \delta_1 \left(\delta_2(k)\right) b_0 - 8\xi_1^3\xi_3 b_0^3 k^5 \delta_1 \left(\delta_3(k)\right) b_0$$

$$-8\xi_1\xi_2^2\xi_3 b_0^3 k^5 \delta_1 \left(\delta_3(k)\right) b_0 - 8\xi_1\xi_3^3 b_0^3 k^5 \delta_1 \left(\delta_3(k)\right) b_0 - 4\xi_1^2\xi_2^2 b_0^3 k^5 \delta_2 \left(\delta_2(k)\right) b_0$$

$$-4\xi_2^4 b_0^3 k^5 \delta_2 \left(\delta_2(k)\right) b_0 - 4\xi_2^2\xi_3^2 b_0^3 k^5 \delta_2 \left(\delta_2(k)\right) b_0 - 8\xi_1^2\xi_2\xi_3 b_0^3 k^5 \delta_2 \left(\delta_3(k)\right) b_0$$

$$-8\xi_2^3\xi_3 b_0^3 k^5 \delta_2 \left(\delta_3(k)\right) b_0 - 8\xi_2\xi_3^3 b_0^3 k^5 \delta_2 \left(\delta_3(k)\right) b_0 - 4\xi_1^2\xi_3^2 b_0^3 k^5 \delta_3 \left(\delta_3(k)\right) b_0$$

$$-4\xi_2^2\xi_3^2 b_0^3 k^5 \delta_3 \left(\delta_3(k)\right) b_0 - 4\xi_3^4 b_0^3 k^5 \delta_3 \left(\delta_3(k)\right) b_0$$

$$+4\xi_1\xi_2 b_0^2 k^2 \delta_1(k)\delta_2(k) b_0 + 4\xi_1\xi_3 b_0^2 k^2 \delta_1(k)\delta_3(k) b_0$$

$$+\xi_1^2 b_0^2 k^2 \delta_1 \left(\delta_1(k)\right) b_0 k + \xi_2^2 b_0^2 k^2 \delta_1 \left(\delta_1(k)\right) b_0 k + \xi_3^2 b_0^2 k^2 \delta_1 \left(\delta_1(k)\right) b_0 k$$

$$+4\xi_1\xi_2 b_0^2 k^2 \delta_2(k)\delta_1(k) b_0 + 4\xi_2\xi_3 b_0^2 k^2 \delta_2(k)\delta_3(k) b_0 + \xi_1^2 b_0^2 k^2 \delta_2 \left(\delta_2(k)\right) b_0 k$$

$$+\xi_2^2 b_0^2 k^2 \delta_2\left(\delta_2(k)\right) b_0 k + \xi_3^2 b_0^2 k^2 \delta_2\left(\delta_2(k)\right) b_0 k + 4\xi_1\xi_3 b_0^2 k^2 \delta_3(k)\delta_1(k) b_0$$
$$+4\xi_2\xi_3 b_0^2 k^2 \delta_3(k)\delta_2(k) b_0 + \xi_1^2 b_0^2 k^2 \delta_3\left(\delta_3(k)\right) b_0 k + \xi_2^2 b_0^2 k^2 \delta_3\left(\delta_3(k)\right) b_0 k$$
$$+\xi_3^2 b_0^2 k^2 \delta_3\left(\delta_3(k)\right) b_0 k - 8\xi_1^3\xi_2 b_0^3 k^4 \delta_1(k)\delta_2(k) b_0 - 8\xi_1\xi_2^3 b_0^3 k^4 \delta_1(k)\delta_2(k) b_0$$
$$-8\xi_1\xi_2\xi_3^2 b_0^3 k^4 \delta_1(k)\delta_2(k) b_0 - 8\xi_1^3\xi_3 b_0^3 k^4 \delta_1(k)\delta_3(k) b_0$$
$$-8\xi_1\xi_2^2\xi_3 b_0^3 k^4 \delta_1(k)\delta_3(k) b_0 - 8\xi_1\xi_3^3 b_0^3 k^4 \delta_1(k)\delta_3(k) b_0 - 4\xi_1^4 b_0^3 k^4 \delta_1\left(\delta_1(k)\right) b_0 k$$
$$-4\xi_1^2\xi_2^2 b_0^3 k^4 \delta_1\left(\delta_1(k)\right) b_0 k - 4\xi_1^2\xi_3^2 b_0^3 k^4 \delta_1\left(\delta_1(k)\right) b_0 k$$
$$-8\xi_1^3\xi_2 b_0^3 k^4 \delta_1\left(\delta_2(k)\right) b_0 k - 8\xi_1\xi_2^3 b_0^3 k^4 \delta_1\left(\delta_2(k)\right) b_0 k$$
$$-8\xi_1\xi_2\xi_3^2 b_0^3 k^4 \delta_1\left(\delta_2(k)\right) b_0 k - 8\xi_1^3\xi_3 b_0^3 k^4 \delta_1\left(\delta_3(k)\right) b_0 k$$
$$-8\xi_1\xi_2^2\xi_3 b_0^3 k^4 \delta_1\left(\delta_3(k)\right) b_0 k - 8\xi_1\xi_3^3 b_0^3 k^4 \delta_1\left(\delta_3(k)\right) b_0 k$$
$$-8\xi_1^3\xi_2 b_0^3 k^4 \delta_2(k)\delta_1(k) b_0 - 8\xi_1\xi_2^3 b_0^3 k^4 \delta_2(k)\delta_1(k) b_0$$
$$-8\xi_1\xi_2\xi_3^2 b_0^3 k^4 \delta_2(k)\delta_1(k) b_0 - 8\xi_1^2\xi_2\xi_3 b_0^3 k^4 \delta_2(k)\delta_3(k) b_0$$
$$-8\xi_2^3\xi_3 b_0^3 k^4 \delta_2(k)\delta_3(k) b_0 - 8\xi_2\xi_3^3 b_0^3 k^4 \delta_2(k)\delta_3(k) b_0$$
$$-4\xi_1^2\xi_2^2 b_0^3 k^4 \delta_2\left(\delta_2(k)\right) b_0 k - 4\xi_2^4 b_0^3 k^4 \delta_2\left(\delta_2(k)\right) b_0 k$$
$$-4\xi_2^2\xi_3^2 b_0^3 k^4 \delta_2\left(\delta_2(k)\right) b_0 k - 8\xi_1^2\xi_2\xi_3 b_0^3 k^4 \delta_2\left(\delta_3(k)\right) b_0 k$$
$$-8\xi_2^3\xi_3 b_0^3 k^4 \delta_2\left(\delta_3(k)\right) b_0 k - 8\xi_2\xi_3^3 b_0^3 k^4 \delta_2\left(\delta_3(k)\right) b_0 k$$
$$-8\xi_1^3\xi_3 b_0^3 k^4 \delta_3(k)\delta_1(k) b_0 - 8\xi_1\xi_2^2\xi_3 b_0^3 k^4 \delta_3(k)\delta_1(k) b_0$$
$$-8\xi_1\xi_3^3 b_0^3 k^4 \delta_3(k)\delta_1(k) b_0 - 8\xi_1^2\xi_2\xi_3 b_0^3 k^4 \delta_3(k)\delta_2(k) b_0$$
$$-8\xi_2^3\xi_3 b_0^3 k^4 \delta_3(k)\delta_2(k) b_0 - 8\xi_2\xi_3^3 b_0^3 k^4 \delta_3(k)\delta_2(k) b_0$$
$$-4\xi_1^2\xi_3^2 b_0^3 k^4 \delta_3\left(\delta_3(k)\right) b_0 k - 4\xi_2^2\xi_3^2 b_0^3 k^4 \delta_3\left(\delta_3(k)\right) b_0 k$$
$$-4\xi_3^4 b_0^3 k^4 \delta_3\left(\delta_3(k)\right) b_0 k + 6\xi_1^2 b_0 k \delta_1(k) b_0 k \delta_1(k) b_0$$
$$+2\xi_2^2 b_0 k \delta_1(k) b_0 k \delta_1(k) b_0 + 2\xi_3^2 b_0 k \delta_1(k) b_0 k \delta_1(k) b_0$$
$$+4\xi_1\xi_2 b_0 k \delta_1(k) b_0 k \delta_2(k) b_0 + 4\xi_1\xi_3 b_0 k \delta_1(k) b_0 k \delta_3(k) b_0$$
$$+2\xi_1^2 b_0 k \delta_1(k) b_0 \delta_1(k) b_0 k + 2\xi_2^2 b_0 k \delta_1(k) b_0 \delta_1(k) b_0 k$$
$$+2\xi_3^2 b_0 k \delta_1(k) b_0 \delta_1(k) b_0 k - 4\xi_1^4 b_0 k \delta_1(k) b_0^2 k^3 \delta_1(k) b_0$$
$$-4\xi_1^2\xi_2^2 b_0 k \delta_1(k) b_0^2 k^3 \delta_1(k) b_0 - 4\xi_1^2\xi_3^2 b_0 k \delta_1(k) b_0^2 k^3 \delta_1(k) b_0$$
$$-4\xi_1^3\xi_2 b_0 k \delta_1(k) b_0^2 k^3 \delta_2(k) b_0 - 4\xi_1\xi_2^3 b_0 k \delta_1(k) b_0^2 k^3 \delta_2(k) b_0$$
$$-4\xi_1\xi_2\xi_3^2 b_0 k \delta_1(k) b_0^2 k^3 \delta_2(k) b_0 - 4\xi_1^3\xi_3 b_0 k \delta_1(k) b_0^2 k^3 \delta_3(k) b_0$$
$$-4\xi_1\xi_2^2\xi_3 b_0 k \delta_1(k) b_0^2 k^3 \delta_3(k) b_0 - 4\xi_1\xi_3^3 b_0 k \delta_1(k) b_0^2 k^3 \delta_3(k) b_0$$
$$+4\xi_1\xi_2 b_0 k \delta_2(k) b_0 k \delta_1(k) b_0 + 2\xi_1^2 b_0 k \delta_2(k) b_0 k \delta_2(k) b_0$$
$$+6\xi_2^2 b_0 k \delta_2(k) b_0 k \delta_2(k) b_0 + 2\xi_3^2 b_0 k \delta_2(k) b_0 k \delta_2(k) b_0$$
$$+4\xi_2\xi_3 b_0 k \delta_2(k) b_0 k \delta_3(k) b_0 + 2\xi_1^2 b_0 k \delta_2(k) b_0 \delta_2(k) b_0 k$$
$$+2\xi_2^2 b_0 k \delta_2(k) b_0 \delta_2(k) b_0 k + 2\xi_3^2 b_0 k \delta_2(k) b_0 \delta_2(k) b_0 k$$

$$-4\xi_1^3\xi_2 b_0 k\delta_2(k) b_0^2 k^3\delta_1(k) b_0 - 4\xi_1\xi_2^3 b_0 k\delta_2(k) b_0^2 k^3\delta_1(k) b_0$$
$$-4\xi_1\xi_2\xi_3^2 b_0 k\delta_2(k) b_0^2 k^3\delta_1(k) b_0 - 4\xi_1^2\xi_2^2 b_0 k\delta_2(k) b_0^2 k^3\delta_2(k) b_0$$
$$-4\xi_2^4 b_0 k\delta_2(k) b_0^2 k^3\delta_2(k) b_0 - 4\xi_2^2\xi_3^2 b_0 k\delta_2(k) b_0^2 k^3\delta_2(k) b_0$$
$$-4\xi_1^2\xi_2\xi_3 b_0 k\delta_2(k) b_0^2 k^3\delta_3(k) b_0 - 4\xi_2^3\xi_3 b_0 k\delta_2(k) b_0^2 k^3\delta_3(k) b_0$$
$$-4\xi_2\xi_3^3 b_0 k\delta_2(k) b_0^2 k^3\delta_3(k) b_0 + 4\xi_1\xi_3 b_0 k\delta_3(k) b_0 k\delta_1(k) b_0$$
$$+4\xi_2\xi_3 b_0 k\delta_3(k) b_0 k\delta_2(k) b_0 + 2\xi_1^2 b_0 k\delta_3(k) b_0 k\delta_3(k) b_0$$
$$+2\xi_2^2 b_0 k\delta_3(k) b_0 k\delta_3(k) b_0 + 6\xi_3^2 b_0 k\delta_3(k) b_0 k\delta_3(k) b_0$$
$$+2\xi_1^2 b_0 k\delta_3(k) b_0\delta_3(k) b_0 k + 2\xi_2^2 b_0 k\delta_3(k) b_0\delta_3(k) b_0 k$$
$$+2\xi_3^2 b_0 k\delta_3(k) b_0\delta_3(k) b_0 k - 4\xi_1^3\xi_3 b_0 k\delta_3(k) b_0^2 k^3\delta_1(k) b_0$$
$$-4\xi_1\xi_2^2\xi_3 b_0 k\delta_3(k) b_0^2 k^3\delta_1(k) b_0 - 4\xi_1\xi_3^3 b_0 k\delta_3(k) b_0^2 k^3\delta_1(k) b_0$$
$$-4\xi_1^2\xi_2\xi_3 b_0 k\delta_3(k) b_0^2 k^3\delta_2(k) b_0 - 4\xi_2^3\xi_3 b_0 k\delta_3(k) b_0^2 k^3\delta_2(k) b_0$$
$$-4\xi_2\xi_3^3 b_0 k\delta_3(k) b_0^2 k^3\delta_2(k) b_0 - 4\xi_1^2\xi_3^2 b_0 k\delta_3(k) b_0^2 k^3\delta_3(k) b_0$$
$$-4\xi_2^2\xi_3^2 b_0 k\delta_3(k) b_0^2 k^3\delta_3(k) b_0 - 4\xi_3^4 b_0 k\delta_3(k) b_0^2 k^3\delta_3(k) b_0$$
$$-10\xi_1^4 b_0^2 k^2\delta_1(k) b_0 k^2\delta_1(k) b_0 - 12\xi_1^2\xi_2^2 b_0^2 k^2\delta_1(k) b_0 k^2\delta_1(k) b_0$$
$$-2\xi_2^4 b_0^2 k^2\delta_1(k) b_0 k^2\delta_1(k) b_0 - 12\xi_1^2\xi_3^2 b_0^2 k^2\delta_1(k) b_0 k^2\delta_1(k) b_0$$
$$-4\xi_2^2\xi_3^2 b_0^2 k^2\delta_1(k) b_0 k^2\delta_1(k) b_0 - 2\xi_3^4 b_0^2 k^2\delta_1(k) b_0 k^2\delta_1(k) b_0$$
$$-8\xi_1^3\xi_2 b_0^2 k^2\delta_1(k) b_0 k^2\delta_2(k) b_0 - 8\xi_1\xi_2^3 b_0^2 k^2\delta_1(k) b_0 k^2\delta_2(k) b_0$$
$$-8\xi_1\xi_2\xi_3^2 b_0^2 k^2\delta_1(k) b_0 k^2\delta_2(k) b_0 - 8\xi_1^3\xi_3 b_0^2 k^2\delta_1(k) b_0 k^2\delta_3(k) b_0$$
$$-8\xi_1\xi_2^2\xi_3 b_0^2 k^2\delta_1(k) b_0 k^2\delta_3(k) b_0 - 8\xi_1\xi_3^3 b_0^2 k^2\delta_1(k) b_0 k^2\delta_3(k) b_0$$
$$+4\xi_1^6 b_0^2 k^2\delta_1(k) b_0^2 k^4\delta_1(k) b_0 + 8\xi_1^4\xi_2^2 b_0^2 k^2\delta_1(k) b_0^2 k^4\delta_1(k) b_0$$
$$+4\xi_1^2\xi_2^4 b_0^2 k^2\delta_1(k) b_0^2 k^4\delta_1(k) b_0 + 8\xi_1^4\xi_3^2 b_0^2 k^2\delta_1(k) b_0^2 k^4\delta_1(k) b_0$$
$$+8\xi_1^2\xi_2^2\xi_3^2 b_0^2 k^2\delta_1(k) b_0^2 k^4\delta_1(k) b_0 + 4\xi_1^2\xi_3^4 b_0^2 k^2\delta_1(k) b_0^2 k^4\delta_1(k) b_0$$
$$+4\xi_1^5\xi_2 b_0^2 k^2\delta_1(k) b_0^2 k^4\delta_2(k) b_0 + 8\xi_1^3\xi_2^3 b_0^2 k^2\delta_1(k) b_0^2 k^4\delta_2(k) b_0$$
$$+4\xi_1\xi_2^5 b_0^2 k^2\delta_1(k) b_0^2 k^4\delta_2(k) b_0 + 8\xi_1^3\xi_2\xi_3^2 b_0^2 k^2\delta_1(k) b_0^2 k^4\delta_2(k) b_0$$
$$+8\xi_1\xi_2^3\xi_3^2 b_0^2 k^2\delta_1(k) b_0^2 k^4\delta_2(k) b_0 + 4\xi_1\xi_2\xi_3^4 b_0^2 k^2\delta_1(k) b_0^2 k^4\delta_2(k) b_0$$
$$+4\xi_1^5\xi_3 b_0^2 k^2\delta_1(k) b_0^2 k^4\delta_3(k) b_0 + 8\xi_1^3\xi_2^2\xi_3 b_0^2 k^2\delta_1(k) b_0^2 k^4\delta_3(k) b_0$$
$$+4\xi_1\xi_2^4\xi_3 b_0^2 k^2\delta_1(k) b_0^2 k^4\delta_3(k) b_0 + 8\xi_1^3\xi_3^3 b_0^2 k^2\delta_1(k) b_0^2 k^4\delta_3(k) b_0$$
$$+8\xi_1\xi_2^2\xi_3^3 b_0^2 k^2\delta_1(k) b_0^2 k^4\delta_3(k) b_0 + 4\xi_1\xi_3^5 b_0^2 k^2\delta_1(k) b_0^2 k^4\delta_3(k) b_0$$
$$-8\xi_1^3\xi_2 b_0^2 k^2\delta_2(k) b_0 k^2\delta_1(k) b_0 - 8\xi_1\xi_2^3 b_0^2 k^2\delta_2(k) b_0 k^2\delta_1(k) b_0$$
$$-8\xi_1\xi_2\xi_3^2 b_0^2 k^2\delta_2(k) b_0 k^2\delta_1(k) b_0 - 2\xi_1^4 b_0^2 k^2\delta_2(k) b_0 k^2\delta_2(k) b_0$$
$$-12\xi_1^2\xi_2^2 b_0^2 k^2\delta_2(k) b_0 k^2\delta_2(k) b_0 - 10\xi_2^4 b_0^2 k^2\delta_2(k) b_0 k^2\delta_2(k) b_0$$
$$-4\xi_1^2\xi_3^2 b_0^2 k^2\delta_2(k) b_0 k^2\delta_2(k) b_0 - 12\xi_2^2\xi_3^2 b_0^2 k^2\delta_2(k) b_0 k^2\delta_2(k) b_0$$

$$-2\xi_3^4b_0^2k^2\delta_2(k)b_0k^2\delta_2(k)b_0-8\xi_1^2\xi_2\xi_3b_0^2k^2\delta_2(k)b_0k^2\delta_3(k)b_0$$
$$-8\xi_2^3\xi_3b_0^2k^2\delta_2(k)b_0k^2\delta_3(k)b_0-8\xi_2\xi_3^3b_0^2k^2\delta_2(k)b_0k^2\delta_3(k)b_0$$
$$+4\xi_1^5\xi_2b_0^2k^2\delta_2(k)b_0^2k^4\delta_1(k)b_0+8\xi_1^3\xi_2^3b_0^2k^2\delta_2(k)b_0^2k^4\delta_1(k)b_0$$
$$+4\xi_1\xi_2^5b_0^2k^2\delta_2(k)b_0^2k^4\delta_1(k)b_0+8\xi_1^3\xi_2\xi_3^2b_0^2k^2\delta_2(k)b_0^2k^4\delta_1(k)b_0$$
$$+8\xi_1\xi_2^3\xi_3^2b_0^2k^2\delta_2(k)b_0^2k^4\delta_1(k)b_0+4\xi_1\xi_2\xi_3^4b_0^2k^2\delta_2(k)b_0^2k^4\delta_1(k)b_0$$
$$+4\xi_1^4\xi_2^2b_0^2k^2\delta_2(k)b_0^2k^4\delta_2(k)b_0+8\xi_1^2\xi_2^4b_0^2k^2\delta_2(k)b_0^2k^4\delta_2(k)b_0$$
$$+4\xi_2^6b_0^2k^2\delta_2(k)b_0^2k^4\delta_2(k)b_0+8\xi_1^2\xi_2^2\xi_3^2b_0^2k^2\delta_2(k)b_0^2k^4\delta_2(k)b_0$$
$$+8\xi_2^4\xi_3^2b_0^2k^2\delta_2(k)b_0^2k^4\delta_2(k)b_0+4\xi_2^2\xi_3^4b_0^2k^2\delta_2(k)b_0^2k^4\delta_2(k)b_0$$
$$+4\xi_1^4\xi_2\xi_3b_0^2k^2\delta_2(k)b_0^2k^4\delta_3(k)b_0+8\xi_1^2\xi_2^3\xi_3b_0^2k^2\delta_2(k)b_0^2k^4\delta_3(k)b_0$$
$$+4\xi_2^5\xi_3b_0^2k^2\delta_2(k)b_0^2k^4\delta_3(k)b_0+8\xi_1^2\xi_2\xi_3^3b_0^2k^2\delta_2(k)b_0^2k^4\delta_3(k)b_0$$
$$+8\xi_2^3\xi_3^3b_0^2k^2\delta_2(k)b_0^2k^4\delta_3(k)b_0+4\xi_2\xi_3^5b_0^2k^2\delta_2(k)b_0^2k^4\delta_3(k)b_0$$
$$-8\xi_1^3\xi_3b_0^2k^2\delta_3(k)b_0k^2\delta_1(k)b_0-8\xi_1\xi_2^2\xi_3b_0^2k^2\delta_3(k)b_0k^2\delta_1(k)b_0$$
$$-8\xi_1\xi_3^3b_0^2k^2\delta_3(k)b_0k^2\delta_1(k)b_0-8\xi_1^2\xi_2\xi_3b_0^2k^2\delta_3(k)b_0k^2\delta_2(k)b_0$$
$$-8\xi_2^3\xi_3b_0^2k^2\delta_3(k)b_0k^2\delta_2(k)b_0-8\xi_2\xi_3^3b_0^2k^2\delta_3(k)b_0k^2\delta_2(k)b_0$$
$$-2\xi_1^4b_0^2k^2\delta_3(k)b_0k^2\delta_3(k)b_0-4\xi_1^2\xi_2^2b_0^2k^2\delta_3(k)b_0k^2\delta_3(k)b_0$$
$$-2\xi_2^4b_0^2k^2\delta_3(k)b_0k^2\delta_3(k)b_0-12\xi_1^2\xi_3^2b_0^2k^2\delta_3(k)b_0k^2\delta_3(k)b_0$$
$$-12\xi_2^2\xi_3^2b_0^2k^2\delta_3(k)b_0k^2\delta_3(k)b_0-10\xi_3^4b_0^2k^2\delta_3(k)b_0k^2\delta_3(k)b_0$$
$$+4\xi_1^5\xi_3b_0^2k^2\delta_3(k)b_0^2k^4\delta_1(k)b_0+8\xi_1^3\xi_2^2\xi_3b_0^2k^2\delta_3(k)b_0^2k^4\delta_1(k)b_0$$
$$+4\xi_1\xi_2^4\xi_3b_0^2k^2\delta_3(k)b_0^2k^4\delta_1(k)b_0+8\xi_1^3\xi_3^3b_0^2k^2\delta_3(k)b_0^2k^4\delta_1(k)b_0$$
$$+8\xi_1\xi_2^2\xi_3^3b_0^2k^2\delta_3(k)b_0^2k^4\delta_1(k)b_0+4\xi_1\xi_3^5b_0^2k^2\delta_3(k)b_0^2k^4\delta_1(k)b_0$$
$$+4\xi_1^4\xi_2\xi_3b_0^2k^2\delta_3(k)b_0^2k^4\delta_2(k)b_0+8\xi_1^2\xi_2^3\xi_3b_0^2k^2\delta_3(k)b_0^2k^4\delta_2(k)b_0$$
$$+4\xi_2^5\xi_3b_0^2k^2\delta_3(k)b_0^2k^4\delta_2(k)b_0+8\xi_1^2\xi_2\xi_3^3b_0^2k^2\delta_3(k)b_0^2k^4\delta_2(k)b_0$$
$$+8\xi_2^3\xi_3^3b_0^2k^2\delta_3(k)b_0^2k^4\delta_2(k)b_0+4\xi_2\xi_3^5b_0^2k^2\delta_3(k)b_0^2k^4\delta_2(k)b_0$$
$$+4\xi_1^4\xi_3^2b_0^2k^2\delta_3(k)b_0^2k^4\delta_3(k)b_0+8\xi_1^2\xi_2^2\xi_3^2b_0^2k^2\delta_3(k)b_0^2k^4\delta_3(k)b_0$$
$$+4\xi_2^4\xi_3^2b_0^2k^2\delta_3(k)b_0^2k^4\delta_3(k)b_0+8\xi_1^2\xi_3^4b_0^2k^2\delta_3(k)b_0^2k^4\delta_3(k)b_0$$
$$+8\xi_2^2\xi_3^4b_0^2k^2\delta_3(k)b_0^2k^4\delta_3(k)b_0+4\xi_3^6b_0^2k^2\delta_3(k)b_0^2k^4\delta_3(k)b_0$$
$$-14\xi_1^4b_0^2k^3\delta_1(k)b_0k\delta_1(k)b_0-16\xi_1^2\xi_2^2b_0^2k^3\delta_1(k)b_0k\delta_1(k)b_0$$
$$-2\xi_2^4b_0^2k^3\delta_1(k)b_0k\delta_1(k)b_0-16\xi_1^2\xi_3^2b_0^2k^3\delta_1(k)b_0k\delta_1(k)b_0$$
$$-4\xi_2^2\xi_3^2b_0^2k^3\delta_1(k)b_0k\delta_1(k)b_0-2\xi_3^4b_0^2k^3\delta_1(k)b_0k\delta_1(k)b_0$$
$$-12\xi_1^3\xi_2b_0^2k^3\delta_1(k)b_0k\delta_2(k)b_0-12\xi_1\xi_2^3b_0^2k^3\delta_1(k)b_0k\delta_2(k)b_0$$
$$-12\xi_1\xi_2\xi_3^2b_0^2k^3\delta_1(k)b_0k\delta_2(k)b_0-12\xi_1^3\xi_3b_0^2k^3\delta_1(k)b_0k\delta_3(k)b_0$$
$$-12\xi_1\xi_2^2\xi_3b_0^2k^3\delta_1(k)b_0k\delta_3(k)b_0-12\xi_1\xi_3^3b_0^2k^3\delta_1(k)b_0k\delta_3(k)b_0$$

$$-10\xi_1^4 b_0^2 k^3 \delta_1(k) b_0 \delta_1(k) b_0 k - 12\xi_1^2\xi_2^2 b_0^2 k^3 \delta_1(k) b_0 \delta_1(k) b_0 k$$
$$-2\xi_2^4 b_0^2 k^3 \delta_1(k) b_0 \delta_1(k) b_0 k - 12\xi_1^2\xi_3^2 b_0^2 k^3 \delta_1(k) b_0 \delta_1(k) b_0 k$$
$$-4\xi_2^2\xi_3^2 b_0^2 k^3 \delta_1(k) b_0 \delta_1(k) b_0 k - 2\xi_3^4 b_0^2 k^3 \delta_1(k) b_0 \delta_1(k) b_0 k$$
$$-8\xi_1^3\xi_2 b_0^2 k^3 \delta_1(k) b_0 \delta_2(k) b_0 k - 8\xi_1\xi_2^3 b_0^2 k^3 \delta_1(k) b_0 \delta_2(k) b_0 k$$
$$-8\xi_1\xi_2\xi_3^2 b_0^2 k^3 \delta_1(k) b_0 \delta_2(k) b_0 k - 8\xi_1^3\xi_3 b_0^2 k^3 \delta_1(k) b_0 \delta_3(k) b_0 k$$
$$-8\xi_1\xi_2^2\xi_3 b_0^2 k^3 \delta_1(k) b_0 \delta_3(k) b_0 k - 8\xi_1\xi_3^3 b_0^2 k^3 \delta_1(k) b_0 \delta_3(k) b_0 k$$
$$+4\xi_1^6 b_0^2 k^3 \delta_1(k) b_0^2 k^3 \delta_1(k) b_0 + 8\xi_1^4\xi_2^2 b_0^2 k^3 \delta_1(k) b_0^2 k^3 \delta_1(k) b_0$$
$$+4\xi_1^2\xi_2^4 b_0^2 k^3 \delta_1(k) b_0^2 k^3 \delta_1(k) b_0 + 8\xi_1^4\xi_3^2 b_0^2 k^3 \delta_1(k) b_0^2 k^3 \delta_1(k) b_0$$
$$+8\xi_1^2\xi_2^2\xi_3^2 b_0^2 k^3 \delta_1(k) b_0^2 k^3 \delta_1(k) b_0 + 4\xi_1^2\xi_3^4 b_0^2 k^3 \delta_1(k) b_0^2 k^3 \delta_1(k) b_0$$
$$+4\xi_1^5\xi_2 b_0^2 k^3 \delta_1(k) b_0^2 k^3 \delta_2(k) b_0 + 8\xi_1^3\xi_2^3 b_0^2 k^3 \delta_1(k) b_0^2 k^3 \delta_2(k) b_0$$
$$+4\xi_1\xi_2^5 b_0^2 k^3 \delta_1(k) b_0^2 k^3 \delta_2(k) b_0 + 8\xi_1^3\xi_2\xi_3^2 b_0^2 k^3 \delta_1(k) b_0^2 k^3 \delta_2(k) b_0$$
$$+8\xi_1\xi_2^3\xi_3^2 b_0^2 k^3 \delta_1(k) b_0^2 k^3 \delta_2(k) b_0 + 4\xi_1\xi_2\xi_3^4 b_0^2 k^3 \delta_1(k) b_0^2 k^3 \delta_2(k) b_0$$
$$+4\xi_1^5\xi_3 b_0^2 k^3 \delta_1(k) b_0^2 k^3 \delta_3(k) b_0 + 8\xi_1^3\xi_2^2\xi_3 b_0^2 k^3 \delta_1(k) b_0^2 k^3 \delta_3(k) b_0$$
$$+4\xi_1\xi_2^4\xi_3 b_0^2 k^3 \delta_1(k) b_0^2 k^3 \delta_3(k) b_0 + 8\xi_1^3\xi_3^3 b_0^2 k^3 \delta_1(k) b_0^2 k^3 \delta_3(k) b_0$$
$$+8\xi_1\xi_2^2\xi_3^3 b_0^2 k^3 \delta_1(k) b_0^2 k^3 \delta_3(k) b_0 + 4\xi_1\xi_3^5 b_0^2 k^3 \delta_1(k) b_0^2 k^3 \delta_3(k) b_0$$
$$-12\xi_1^3\xi_2 b_0^2 k^3 \delta_2(k) b_0 k \delta_1(k) b_0 - 12\xi_1\xi_2^3 b_0^2 k^3 \delta_2(k) b_0 k \delta_1(k) b_0$$
$$-12\xi_1\xi_2\xi_3^2 b_0^2 k^3 \delta_2(k) b_0 k \delta_1(k) b_0 - 2\xi_1^4 b_0^2 k^3 \delta_2(k) b_0 k \delta_2(k) b_0$$
$$-16\xi_1^2\xi_2^2 b_0^2 k^3 \delta_2(k) b_0 k \delta_2(k) b_0 - 14\xi_2^4 b_0^2 k^3 \delta_2(k) b_0 k \delta_2(k) b_0$$
$$-4\xi_1^2\xi_3^2 b_0^2 k^3 \delta_2(k) b_0 k \delta_2(k) b_0 - 16\xi_2^2\xi_3^2 b_0^2 k^3 \delta_2(k) b_0 k \delta_2(k) b_0$$
$$-2\xi_3^4 b_0^2 k^3 \delta_2(k) b_0 k \delta_2(k) b_0 - 12\xi_1^2\xi_2\xi_3 b_0^2 k^3 \delta_2(k) b_0 k \delta_3(k) b_0$$
$$-12\xi_2^3\xi_3 b_0^2 k^3 \delta_2(k) b_0 k \delta_3(k) b_0 - 12\xi_2\xi_3^3 b_0^2 k^3 \delta_2(k) b_0 k \delta_3(k) b_0$$
$$-8\xi_1^3\xi_2 b_0^2 k^3 \delta_2(k) b_0 \delta_1(k) b_0 k - 8\xi_1\xi_2^3 b_0^2 k^3 \delta_2(k) b_0 \delta_1(k) b_0 k$$
$$-8\xi_1\xi_2\xi_3^2 b_0^2 k^3 \delta_2(k) b_0 \delta_1(k) b_0 k - 2\xi_1^4 b_0^2 k^3 \delta_2(k) b_0 \delta_2(k) b_0 k$$
$$-12\xi_1^2\xi_2^2 b_0^2 k^3 \delta_2(k) b_0 \delta_2(k) b_0 k - 10\xi_2^4 b_0^2 k^3 \delta_2(k) b_0 \delta_2(k) b_0 k$$
$$-4\xi_1^2\xi_3^2 b_0^2 k^3 \delta_2(k) b_0 \delta_2(k) b_0 k - 12\xi_2^2\xi_3^2 b_0^2 k^3 \delta_2(k) b_0 \delta_2(k) b_0 k$$
$$-2\xi_3^4 b_0^2 k^3 \delta_2(k) b_0 \delta_2(k) b_0 k - 8\xi_1^2\xi_2\xi_3 b_0^2 k^3 \delta_2(k) b_0 \delta_3(k) b_0 k$$
$$-8\xi_2^3\xi_3 b_0^2 k^3 \delta_2(k) b_0 \delta_3(k) b_0 k - 8\xi_2\xi_3^3 b_0^2 k^3 \delta_2(k) b_0 \delta_3(k) b_0 k$$
$$+4\xi_1^5\xi_2 b_0^2 k^3 \delta_2(k) b_0^2 k^3 \delta_1(k) b_0 + 8\xi_1^3\xi_2^3 b_0^2 k^3 \delta_2(k) b_0^2 k^3 \delta_1(k) b_0$$
$$+4\xi_1\xi_2^5 b_0^2 k^3 \delta_2(k) b_0^2 k^3 \delta_1(k) b_0 + 8\xi_1^3\xi_2\xi_3^2 b_0^2 k^3 \delta_2(k) b_0^2 k^3 \delta_1(k) b_0$$
$$+8\xi_1\xi_2^3\xi_3^2 b_0^2 k^3 \delta_2(k) b_0^2 k^3 \delta_1(k) b_0 + 4\xi_1\xi_2\xi_3^4 b_0^2 k^3 \delta_2(k) b_0^2 k^3 \delta_1(k) b_0$$
$$+4\xi_1^4\xi_2^2 b_0^2 k^3 \delta_2(k) b_0^2 k^3 \delta_2(k) b_0 + 8\xi_1^2\xi_2^4 b_0^2 k^3 \delta_2(k) b_0^2 k^3 \delta_2(k) b_0$$
$$+4\xi_2^6 b_0^2 k^3 \delta_2(k) b_0^2 k^3 \delta_2(k) b_0 + 8\xi_1^2\xi_2^2\xi_3^2 b_0^2 k^3 \delta_2(k) b_0^2 k^3 \delta_2(k) b_0$$

$$+8\xi_2^4\xi_3^2b_0^2k^3\delta_2(k)b_0^2k^3\delta_2(k)b_0 + 4\xi_2^2\xi_3^4b_0^2k^3\delta_2(k)b_0^2k^3\delta_2(k)b_0$$
$$+4\xi_1^4\xi_2\xi_3b_0^2k^3\delta_2(k)b_0^2k^3\delta_3(k)b_0 + 8\xi_1^2\xi_2^3\xi_3b_0^2k^3\delta_2(k)b_0^2k^3\delta_3(k)b_0$$
$$+4\xi_2^5\xi_3b_0^2k^3\delta_2(k)b_0^2k^3\delta_3(k)b_0 + 8\xi_1^2\xi_2\xi_3^3b_0^2k^3\delta_2(k)b_0^2k^3\delta_3(k)b_0$$
$$+8\xi_2^3\xi_3^3b_0^2k^3\delta_2(k)b_0^2k^3\delta_3(k)b_0 + 4\xi_2\xi_3^5b_0^2k^3\delta_2(k)b_0^2k^3\delta_3(k)b_0$$
$$-12\xi_1^3\xi_3b_0^2k^3\delta_3(k)b_0k\delta_1(k)b_0 - 12\xi_1\xi_2^2\xi_3b_0^2k^3\delta_3(k)b_0k\delta_1(k)b_0$$
$$-12\xi_1\xi_3^3b_0^2k^3\delta_3(k)b_0k\delta_1(k)b_0 - 12\xi_1^2\xi_2\xi_3b_0^2k^3\delta_3(k)b_0k\delta_2(k)b_0$$
$$-12\xi_2^3\xi_3b_0^2k^3\delta_3(k)b_0k\delta_2(k)b_0 - 12\xi_2\xi_3^3b_0^2k^3\delta_3(k)b_0k\delta_2(k)b_0$$
$$-2\xi_1^4b_0^2k^3\delta_3(k)b_0k\delta_3(k)b_0 - 4\xi_1^2\xi_2^2b_0^2k^3\delta_3(k)b_0k\delta_3(k)b_0$$
$$-2\xi_2^4b_0^2k^3\delta_3(k)b_0k\delta_3(k)b_0 - 16\xi_1^2\xi_3^2b_0^2k^3\delta_3(k)b_0k\delta_3(k)b_0$$
$$-16\xi_2^2\xi_3^2b_0^2k^3\delta_3(k)b_0k\delta_3(k)b_0 - 14\xi_3^4b_0^2k^3\delta_3(k)b_0k\delta_3(k)b_0$$
$$-8\xi_1^3\xi_3b_0^2k^3\delta_3(k)b_0\delta_1(k)b_0k - 8\xi_1\xi_2^2\xi_3b_0^2k^3\delta_3(k)b_0\delta_1(k)b_0k$$
$$-8\xi_1\xi_3^3b_0^2k^3\delta_3(k)b_0\delta_1(k)b_0k - 8\xi_1^2\xi_2\xi_3b_0^2k^3\delta_3(k)b_0\delta_2(k)b_0k$$
$$-8\xi_2^3\xi_3b_0^2k^3\delta_3(k)b_0\delta_2(k)b_0k - 8\xi_2\xi_3^3b_0^2k^3\delta_3(k)b_0\delta_2(k)b_0k$$
$$-2\xi_1^4b_0^2k^3\delta_3(k)b_0\delta_3(k)b_0k - 4\xi_1^2\xi_2^2b_0^2k^3\delta_3(k)b_0\delta_3(k)b_0k$$
$$-2\xi_2^4b_0^2k^3\delta_3(k)b_0\delta_3(k)b_0k - 12\xi_1^2\xi_3^2b_0^2k^3\delta_3(k)b_0\delta_3(k)b_0k$$
$$-12\xi_2^2\xi_3^2b_0^2k^3\delta_3(k)b_0\delta_3(k)b_0k - 10\xi_3^4b_0^2k^3\delta_3(k)b_0\delta_3(k)b_0k$$
$$+4\xi_1^5\xi_3b_0^2k^3\delta_3(k)b_0^2k^3\delta_1(k)b_0 + 8\xi_1^3\xi_2^2\xi_3b_0^2k^3\delta_3(k)b_0^2k^3\delta_1(k)b_0$$
$$+4\xi_1\xi_2^4\xi_3b_0^2k^3\delta_3(k)b_0^2k^3\delta_1(k)b_0 + 8\xi_1^3\xi_3^3b_0^2k^3\delta_3(k)b_0^2k^3\delta_1(k)b_0$$
$$+8\xi_1\xi_2^2\xi_3^3b_0^2k^3\delta_3(k)b_0^2k^3\delta_1(k)b_0 + 4\xi_1\xi_3^5b_0^2k^3\delta_3(k)b_0^2k^3\delta_1(k)b_0$$
$$+4\xi_1^4\xi_2\xi_3b_0^2k^3\delta_3(k)b_0^2k^3\delta_2(k)b_0 + 8\xi_1^2\xi_2^3\xi_3b_0^2k^3\delta_3(k)b_0^2k^3\delta_2(k)b_0$$
$$+4\xi_2^5\xi_3b_0^2k^3\delta_3(k)b_0^2k^3\delta_2(k)b_0 + 8\xi_1^2\xi_2\xi_3^3b_0^2k^3\delta_3(k)b_0^2k^3\delta_2(k)b_0$$
$$+8\xi_2^3\xi_3^3b_0^2k^3\delta_3(k)b_0^2k^3\delta_2(k)b_0 + 4\xi_2\xi_3^5b_0^2k^3\delta_3(k)b_0^2k^3\delta_2(k)b_0$$
$$+4\xi_1^4\xi_3^2b_0^2k^3\delta_3(k)b_0^2k^3\delta_3(k)b_0 + 8\xi_1^2\xi_2^2\xi_3^2b_0^2k^3\delta_3(k)b_0^2k^3\delta_3(k)b_0$$
$$+4\xi_2^4\xi_3^2b_0^2k^3\delta_3(k)b_0^2k^3\delta_3(k)b_0 + 8\xi_1^2\xi_3^4b_0^2k^3\delta_3(k)b_0^2k^3\delta_3(k)b_0$$
$$+8\xi_2^2\xi_3^4b_0^2k^3\delta_3(k)b_0^2k^3\delta_3(k)b_0 + 4\xi_3^6b_0^2k^3\delta_3(k)b_0^2k^3\delta_3(k)b_0$$
$$+8\xi_1^6b_0^3k^4\delta_1(k)b_0k^2\delta_1(k)b_0 + 16\xi_1^4\xi_2^2b_0^3k^4\delta_1(k)b_0k^2\delta_1(k)b_0$$
$$+8\xi_1^2\xi_2^4b_0^3k^4\delta_1(k)b_0k^2\delta_1(k)b_0 + 16\xi_1^4\xi_3^2b_0^3k^4\delta_1(k)b_0k^2\delta_1(k)b_0$$
$$+16\xi_1^2\xi_2^2\xi_3^2b_0^3k^4\delta_1(k)b_0k^2\delta_1(k)b_0 + 8\xi_1^2\xi_3^4b_0^3k^4\delta_1(k)b_0k^2\delta_1(k)b_0$$
$$+8\xi_1^5\xi_2b_0^3k^4\delta_1(k)b_0k^2\delta_2(k)b_0 + 16\xi_1^3\xi_2^3b_0^3k^4\delta_1(k)b_0k^2\delta_2(k)b_0$$
$$+8\xi_1\xi_2^5b_0^3k^4\delta_1(k)b_0k^2\delta_2(k)b_0 + 16\xi_1^3\xi_2\xi_3^2b_0^3k^4\delta_1(k)b_0k^2\delta_2(k)b_0$$
$$+16\xi_1\xi_2^3\xi_3^2b_0^3k^4\delta_1(k)b_0k^2\delta_2(k)b_0 + 8\xi_1\xi_2\xi_3^4b_0^3k^4\delta_1(k)b_0k^2\delta_2(k)b_0$$
$$+8\xi_1^5\xi_3b_0^3k^4\delta_1(k)b_0k^2\delta_3(k)b_0 + 16\xi_1^3\xi_2^2\xi_3b_0^3k^4\delta_1(k)b_0k^2\delta_3(k)b_0$$

$$+8\xi_1\xi_2^4\xi_3b_0^3k^4\delta_1(k)b_0k^2\delta_3(k)b_0+16\xi_1^3\xi_3^3b_0^3k^4\delta_1(k)b_0k^2\delta_3(k)b_0$$
$$+16\xi_1\xi_2^2\xi_3^3b_0^3k^4\delta_1(k)b_0k^2\delta_3(k)b_0+8\xi_1\xi_3^5b_0^3k^4\delta_1(k)b_0k^2\delta_3(k)b_0$$
$$+8\xi_1^5\xi_2b_0^3k^4\delta_2(k)b_0k^2\delta_1(k)b_0+16\xi_1^3\xi_2^3b_0^3k^4\delta_2(k)b_0k^2\delta_1(k)b_0$$
$$+8\xi_1\xi_2^5b_0^3k^4\delta_2(k)b_0k^2\delta_1(k)b_0+16\xi_1^3\xi_2\xi_3^2b_0^3k^4\delta_2(k)b_0k^2\delta_1(k)b_0$$
$$+16\xi_1\xi_2^3\xi_3^2b_0^3k^4\delta_2(k)b_0k^2\delta_1(k)b_0+8\xi_1\xi_2\xi_3^4b_0^3k^4\delta_2(k)b_0k^2\delta_1(k)b_0$$
$$+8\xi_1^4\xi_2^2b_0^3k^4\delta_2(k)b_0k^2\delta_2(k)b_0+16\xi_1^2\xi_2^4b_0^3k^4\delta_2(k)b_0k^2\delta_2(k)b_0$$
$$+8\xi_2^6b_0^3k^4\delta_2(k)b_0k^2\delta_2(k)b_0+16\xi_1^2\xi_2^2\xi_3^2b_0^3k^4\delta_2(k)b_0k^2\delta_2(k)b_0$$
$$+16\xi_2^4\xi_3^2b_0^3k^4\delta_2(k)b_0k^2\delta_2(k)b_0+8\xi_2^2\xi_3^4b_0^3k^4\delta_2(k)b_0k^2\delta_2(k)b_0$$
$$+8\xi_1^4\xi_2\xi_3b_0^3k^4\delta_2(k)b_0k^2\delta_3(k)b_0+16\xi_1^2\xi_2^3\xi_3b_0^3k^4\delta_2(k)b_0k^2\delta_3(k)b_0$$
$$+8\xi_2^5\xi_3b_0^3k^4\delta_2(k)b_0k^2\delta_3(k)b_0+16\xi_1^2\xi_2\xi_3^3b_0^3k^4\delta_2(k)b_0k^2\delta_3(k)b_0$$
$$+16\xi_2^3\xi_3^3b_0^3k^4\delta_2(k)b_0k^2\delta_3(k)b_0+8\xi_2\xi_3^5b_0^3k^4\delta_2(k)b_0k^2\delta_3(k)b_0$$
$$+8\xi_1^5\xi_3b_0^3k^4\delta_3(k)b_0k^2\delta_1(k)b_0+16\xi_1^3\xi_2^2\xi_3b_0^3k^4\delta_3(k)b_0k^2\delta_1(k)b_0$$
$$+8\xi_1\xi_2^4\xi_3b_0^3k^4\delta_3(k)b_0k^2\delta_1(k)b_0+16\xi_1^3\xi_3^3b_0^3k^4\delta_3(k)b_0k^2\delta_1(k)b_0$$
$$+16\xi_1\xi_2^2\xi_3^3b_0^3k^4\delta_3(k)b_0k^2\delta_1(k)b_0+8\xi_1\xi_3^5b_0^3k^4\delta_3(k)b_0k^2\delta_1(k)b_0$$
$$+8\xi_1^4\xi_2\xi_3b_0^3k^4\delta_3(k)b_0k^2\delta_2(k)b_0+16\xi_1^2\xi_2^3\xi_3b_0^3k^4\delta_3(k)b_0k^2\delta_2(k)b_0$$
$$+8\xi_2^5\xi_3b_0^3k^4\delta_3(k)b_0k^2\delta_2(k)b_0+16\xi_1^2\xi_2\xi_3^3b_0^3k^4\delta_3(k)b_0k^2\delta_2(k)b_0$$
$$+16\xi_2^3\xi_3^3b_0^3k^4\delta_3(k)b_0k^2\delta_2(k)b_0+8\xi_2\xi_3^5b_0^3k^4\delta_3(k)b_0k^2\delta_2(k)b_0$$
$$+8\xi_1^4\xi_3^2b_0^3k^4\delta_3(k)b_0k^2\delta_3(k)b_0+16\xi_1^2\xi_2^2\xi_3^2b_0^3k^4\delta_3(k)b_0k^2\delta_3(k)b_0$$
$$+8\xi_2^4\xi_3^2b_0^3k^4\delta_3(k)b_0k^2\delta_3(k)b_0+16\xi_1^2\xi_3^4b_0^3k^4\delta_3(k)b_0k^2\delta_3(k)b_0$$
$$+16\xi_2^2\xi_3^4b_0^3k^4\delta_3(k)b_0k^2\delta_3(k)b_0+8\xi_3^6b_0^3k^4\delta_3(k)b_0k^2\delta_3(k)b_0$$
$$+8\xi_1^6b_0^3k^5\delta_1(k)b_0k\delta_1(k)b_0+16\xi_1^4\xi_2^2b_0^3k^5\delta_1(k)b_0k\delta_1(k)b_0$$
$$+8\xi_1^2\xi_2^4b_0^3k^5\delta_1(k)b_0k\delta_1(k)b_0+16\xi_1^4\xi_3^2b_0^3k^5\delta_1(k)b_0k\delta_1(k)b_0$$
$$+16\xi_1^2\xi_2^2\xi_3^2b_0^3k^5\delta_1(k)b_0k\delta_1(k)b_0+8\xi_1^2\xi_3^4b_0^3k^5\delta_1(k)b_0k\delta_1(k)b_0$$
$$+8\xi_1^5\xi_2b_0^3k^5\delta_1(k)b_0k\delta_2(k)b_0+16\xi_1^3\xi_2^3b_0^3k^5\delta_1(k)b_0k\delta_2(k)b_0$$
$$+8\xi_1\xi_2^5b_0^3k^5\delta_1(k)b_0k\delta_2(k)b_0+16\xi_1^3\xi_2\xi_3^2b_0^3k^5\delta_1(k)b_0k\delta_2(k)b_0$$
$$+16\xi_1\xi_2^3\xi_3^2b_0^3k^5\delta_1(k)b_0k\delta_2(k)b_0+8\xi_1\xi_2\xi_3^4b_0^3k^5\delta_1(k)b_0k\delta_2(k)b_0$$
$$+8\xi_1^5\xi_3b_0^3k^5\delta_1(k)b_0k\delta_3(k)b_0+16\xi_1^3\xi_2^2\xi_3b_0^3k^5\delta_1(k)b_0k\delta_3(k)b_0$$
$$+8\xi_1\xi_2^4\xi_3b_0^3k^5\delta_1(k)b_0k\delta_3(k)b_0+16\xi_1^3\xi_3^3b_0^3k^5\delta_1(k)b_0k\delta_3(k)b_0$$
$$+16\xi_1\xi_2^2\xi_3^3b_0^3k^5\delta_1(k)b_0k\delta_3(k)b_0+8\xi_1\xi_3^5b_0^3k^5\delta_1(k)b_0k\delta_3(k)b_0$$
$$+8\xi_1^6b_0^3k^5\delta_1(k)b_0\delta_1(k)b_0k+16\xi_1^4\xi_2^2b_0^3k^5\delta_1(k)b_0\delta_1(k)b_0k$$
$$+8\xi_1^2\xi_2^4b_0^3k^5\delta_1(k)b_0\delta_1(k)b_0k+16\xi_1^4\xi_3^2b_0^3k^5\delta_1(k)b_0\delta_1(k)b_0k$$
$$+16\xi_1^2\xi_2^2\xi_3^2b_0^3k^5\delta_1(k)b_0\delta_1(k)b_0k+8\xi_1^2\xi_3^4b_0^3k^5\delta_1(k)b_0\delta_1(k)b_0k$$

$$+8\xi_1^5\xi_2 b_0^3 k^5\delta_1(k)b_0\delta_2(k)b_0k + 16\xi_1^3\xi_2^3 b_0^3 k^5\delta_1(k)b_0\delta_2(k)b_0k$$
$$+8\xi_1\xi_2^5 b_0^3 k^5\delta_1(k)b_0\delta_2(k)b_0k + 16\xi_1^3\xi_2\xi_3^2 b_0^3 k^5\delta_1(k)b_0\delta_2(k)b_0k$$
$$+16\xi_1\xi_2^3\xi_3^2 b_0^3 k^5\delta_1(k)b_0\delta_2(k)b_0k + 8\xi_1\xi_2\xi_3^4 b_0^3 k^5\delta_1(k)b_0\delta_2(k)b_0k$$
$$+8\xi_1^5\xi_3 b_0^3 k^5\delta_1(k)b_0\delta_3(k)b_0k + 16\xi_1^3\xi_2^2\xi_3 b_0^3 k^5\delta_1(k)b_0\delta_3(k)b_0k$$
$$+8\xi_1\xi_2^4\xi_3 b_0^3 k^5\delta_1(k)b_0\delta_3(k)b_0k + 16\xi_1^3\xi_3^3 b_0^3 k^5\delta_1(k)b_0\delta_3(k)b_0k$$
$$+16\xi_1\xi_2^2\xi_3^3 b_0^3 k^5\delta_1(k)b_0\delta_3(k)b_0k + 8\xi_1\xi_3^5 b_0^3 k^5\delta_1(k)b_0\delta_3(k)b_0k$$
$$+8\xi_1^5\xi_2 b_0^3 k^5\delta_2(k)b_0k\delta_1(k)b_0 + 16\xi_1^3\xi_2^3 b_0^3 k^5\delta_2(k)b_0k\delta_1(k)b_0$$
$$+8\xi_1\xi_2^5 b_0^3 k^5\delta_2(k)b_0k\delta_1(k)b_0 + 16\xi_1^3\xi_2\xi_3^2 b_0^3 k^5\delta_2(k)b_0k\delta_1(k)b_0$$
$$+16\xi_1\xi_2^3\xi_3^2 b_0^3 k^5\delta_2(k)b_0k\delta_1(k)b_0 + 8\xi_1\xi_2\xi_3^4 b_0^3 k^5\delta_2(k)b_0k\delta_1(k)b_0$$
$$+8\xi_1^4\xi_2^2 b_0^3 k^5\delta_2(k)b_0k\delta_2(k)b_0 + 16\xi_1^2\xi_2^4 b_0^3 k^5\delta_2(k)b_0k\delta_2(k)b_0$$
$$+8\xi_2^6 b_0^3 k^5\delta_2(k)b_0k\delta_2(k)b_0 + 16\xi_1^2\xi_2^2\xi_3^2 b_0^3 k^5\delta_2(k)b_0k\delta_2(k)b_0$$
$$+16\xi_2^4\xi_3^2 b_0^3 k^5\delta_2(k)b_0k\delta_2(k)b_0 + 8\xi_2^2\xi_3^4 b_0^3 k^5\delta_2(k)b_0k\delta_2(k)b_0$$
$$+8\xi_1^4\xi_2\xi_3 b_0^3 k^5\delta_2(k)b_0k\delta_3(k)b_0 + 16\xi_1^2\xi_2^3\xi_3 b_0^3 k^5\delta_2(k)b_0k\delta_3(k)b_0$$
$$+8\xi_2^5\xi_3 b_0^3 k^5\delta_2(k)b_0k\delta_3(k)b_0 + 16\xi_1^2\xi_2\xi_3^3 b_0^3 k^5\delta_2(k)b_0k\delta_3(k)b_0$$
$$+16\xi_2^3\xi_3^3 b_0^3 k^5\delta_2(k)b_0k\delta_3(k)b_0 + 8\xi_2\xi_3^5 b_0^3 k^5\delta_2(k)b_0k\delta_3(k)b_0$$
$$+8\xi_1^5\xi_2 b_0^3 k^5\delta_2(k)b_0\delta_1(k)b_0k + 16\xi_1^3\xi_2^3 b_0^3 k^5\delta_2(k)b_0\delta_1(k)b_0k$$
$$+8\xi_1\xi_2^5 b_0^3 k^5\delta_2(k)b_0\delta_1(k)b_0k + 16\xi_1^3\xi_2\xi_3^2 b_0^3 k^5\delta_2(k)b_0\delta_1(k)b_0k$$
$$+16\xi_1\xi_2^3\xi_3^2 b_0^3 k^5\delta_2(k)b_0\delta_1(k)b_0k + 8\xi_1\xi_2\xi_3^4 b_0^3 k^5\delta_2(k)b_0\delta_1(k)b_0k$$
$$+8\xi_1^4\xi_2^2 b_0^3 k^5\delta_2(k)b_0\delta_2(k)b_0k + 16\xi_1^2\xi_2^4 b_0^3 k^5\delta_2(k)b_0\delta_2(k)b_0k$$
$$+8\xi_2^6 b_0^3 k^5\delta_2(k)b_0\delta_2(k)b_0k + 16\xi_1^2\xi_2^2\xi_3^2 b_0^3 k^5\delta_2(k)b_0\delta_2(k)b_0k$$
$$+16\xi_2^4\xi_3^2 b_0^3 k^5\delta_2(k)b_0\delta_2(k)b_0k + 8\xi_2^2\xi_3^4 b_0^3 k^5\delta_2(k)b_0\delta_2(k)b_0k$$
$$+8\xi_1^4\xi_2\xi_3 b_0^3 k^5\delta_2(k)b_0\delta_3(k)b_0k + 16\xi_1^2\xi_2^3\xi_3 b_0^3 k^5\delta_2(k)b_0\delta_3(k)b_0k$$
$$+8\xi_2^5\xi_3 b_0^3 k^5\delta_2(k)b_0\delta_3(k)b_0k + 16\xi_1^2\xi_2\xi_3^3 b_0^3 k^5\delta_2(k)b_0\delta_3(k)b_0k$$
$$+16\xi_2^3\xi_3^3 b_0^3 k^5\delta_2(k)b_0\delta_3(k)b_0k + 8\xi_2\xi_3^5 b_0^3 k^5\delta_2(k)b_0\delta_3(k)b_0k$$
$$+8\xi_1^5\xi_3 b_0^3 k^5\delta_3(k)b_0k\delta_1(k)b_0 + 16\xi_1^3\xi_2^2\xi_3 b_0^3 k^5\delta_3(k)b_0k\delta_1(k)b_0$$
$$+8\xi_1\xi_2^4\xi_3 b_0^3 k^5\delta_3(k)b_0k\delta_1(k)b_0 + 16\xi_1^3\xi_3^3 b_0^3 k^5\delta_3(k)b_0k\delta_1(k)b_0$$
$$+16\xi_1\xi_2^2\xi_3^3 b_0^3 k^5\delta_3(k)b_0k\delta_1(k)b_0 + 8\xi_1\xi_3^5 b_0^3 k^5\delta_3(k)b_0k\delta_1(k)b_0$$
$$+8\xi_1^4\xi_2\xi_3 b_0^3 k^5\delta_3(k)b_0k\delta_2(k)b_0 + 16\xi_1^2\xi_2^3\xi_3 b_0^3 k^5\delta_3(k)b_0k\delta_2(k)b_0$$
$$+8\xi_2^5\xi_3 b_0^3 k^5\delta_3(k)b_0k\delta_2(k)b_0 + 16\xi_1^2\xi_2\xi_3^3 b_0^3 k^5\delta_3(k)b_0k\delta_2(k)b_0$$
$$+16\xi_2^3\xi_3^3 b_0^3 k^5\delta_3(k)b_0k\delta_2(k)b_0 + 8\xi_2\xi_3^5 b_0^3 k^5\delta_3(k)b_0k\delta_2(k)b_0$$
$$+8\xi_1^4\xi_3^2 b_0^3 k^5\delta_3(k)b_0k\delta_3(k)b_0 + 16\xi_1^2\xi_2^2\xi_3^2 b_0^3 k^5\delta_3(k)b_0k\delta_3(k)b_0$$
$$+8\xi_2^4\xi_3^2 b_0^3 k^5\delta_3(k)b_0k\delta_3(k)b_0 + 16\xi_1^2\xi_3^4 b_0^3 k^5\delta_3(k)b_0k\delta_3(k)b_0$$

$$+16\xi_2^2\xi_3^4b_0^3k^5\delta_3(k)b_0k\delta_3(k)b_0 + 8\xi_3^6b_0^3k^5\delta_3(k)b_0k\delta_3(k)b_0$$
$$+8\xi_1^5\xi_3b_0^3k^5\delta_3(k)b_0\delta_1(k)b_0k + 16\xi_1^3\xi_2^2\xi_3b_0^3k^5\delta_3(k)b_0\delta_1(k)b_0k$$
$$+8\xi_1\xi_2^4\xi_3b_0^3k^5\delta_3(k)b_0\delta_1(k)b_0k + 16\xi_1^3\xi_3^3b_0^3k^5\delta_3(k)b_0\delta_1(k)b_0k$$
$$+16\xi_1\xi_2^2\xi_3^3b_0^3k^5\delta_3(k)b_0\delta_1(k)b_0k + 8\xi_1\xi_3^5b_0^3k^5\delta_3(k)b_0\delta_1(k)b_0k$$
$$+8\xi_1^4\xi_2\xi_3b_0^3k^5\delta_3(k)b_0\delta_2(k)b_0k + 16\xi_1^2\xi_2^3\xi_3b_0^3k^5\delta_3(k)b_0\delta_2(k)b_0k$$
$$+8\xi_2^5\xi_3b_0^3k^5\delta_3(k)b_0\delta_2(k)b_0k + 16\xi_1^2\xi_2\xi_3^3b_0^3k^5\delta_3(k)b_0\delta_2(k)b_0k$$
$$+16\xi_2^3\xi_3^3b_0^3k^5\delta_3(k)b_0\delta_2(k)b_0k + 8\xi_2\xi_3^5b_0^3k^5\delta_3(k)b_0\delta_2(k)b_0k$$
$$+8\xi_1^4\xi_3^2b_0^3k^5\delta_3(k)b_0\delta_3(k)b_0k + 16\xi_1^2\xi_2^2\xi_3^2b_0^3k^5\delta_3(k)b_0\delta_3(k)b_0k$$
$$+8\xi_2^4\xi_3^2b_0^3k^5\delta_3(k)b_0\delta_3(k)b_0k + 16\xi_1^2\xi_3^4b_0^3k^5\delta_3(k)b_0\delta_3(k)b_0k$$
$$+16\xi_2^2\xi_3^4b_0^3k^5\delta_3(k)b_0\delta_3(k)b_0k + 8\xi_3^6b_0^3k^5\delta_3(k)b_0\delta_3(k)b_0k$$
$$-4\xi_1^4b_0k\delta_1(k)b_0^2k^2\delta_1(k)b_0k - 4\xi_1^2\xi_2^2b_0k\delta_1(k)b_0^2k^2\delta_1(k)b_0k$$
$$-4\xi_1^2\xi_3^2b_0k\delta_1(k)b_0^2k^2\delta_1(k)b_0k - 4\xi_1^3\xi_2b_0k\delta_1(k)b_0^2k^2\delta_2(k)b_0k$$
$$-4\xi_1\xi_2^3b_0k\delta_1(k)b_0^2k^2\delta_2(k)b_0k - 4\xi_1\xi_2\xi_3^2b_0k\delta_1(k)b_0^2k^2\delta_2(k)b_0k$$
$$-4\xi_1^3\xi_3b_0k\delta_1(k)b_0^2k^2\delta_3(k)b_0k - 4\xi_1\xi_2^2\xi_3b_0k\delta_1(k)b_0^2k^2\delta_3(k)b_0k$$
$$-4\xi_1\xi_3^3b_0k\delta_1(k)b_0^2k^2\delta_3(k)b_0k - 4\xi_1^3\xi_2b_0k\delta_2(k)b_0^2k^2\delta_1(k)b_0k$$
$$-4\xi_1\xi_2^3b_0k\delta_2(k)b_0^2k^2\delta_1(k)b_0k - 4\xi_1\xi_2\xi_3^2b_0k\delta_2(k)b_0^2k^2\delta_1(k)b_0k$$
$$-4\xi_1^2\xi_2^2b_0k\delta_2(k)b_0^2k^2\delta_2(k)b_0k - 4\xi_2^4b_0k\delta_2(k)b_0^2k^2\delta_2(k)b_0k$$
$$-4\xi_2^2\xi_3^2b_0k\delta_2(k)b_0^2k^2\delta_2(k)b_0k - 4\xi_1^2\xi_2\xi_3b_0k\delta_2(k)b_0^2k^2\delta_3(k)b_0k$$
$$-4\xi_2^3\xi_3b_0k\delta_2(k)b_0^2k^2\delta_3(k)b_0k - 4\xi_2\xi_3^3b_0k\delta_2(k)b_0^2k^2\delta_3(k)b_0k$$
$$-4\xi_1^3\xi_3b_0k\delta_3(k)b_0^2k^2\delta_1(k)b_0k - 4\xi_1\xi_2^2\xi_3b_0k\delta_3(k)b_0^2k^2\delta_1(k)b_0k$$
$$-4\xi_1\xi_3^3b_0k\delta_3(k)b_0^2k^2\delta_1(k)b_0k - 4\xi_1^2\xi_2\xi_3b_0k\delta_3(k)b_0^2k^2\delta_2(k)b_0k$$
$$-4\xi_2^3\xi_3b_0k\delta_3(k)b_0^2k^2\delta_2(k)b_0k - 4\xi_2\xi_3^3b_0k\delta_3(k)b_0^2k^2\delta_2(k)b_0k$$
$$-4\xi_1^2\xi_3^2b_0k\delta_3(k)b_0^2k^2\delta_3(k)b_0k - 4\xi_2^2\xi_3^2b_0k\delta_3(k)b_0^2k^2\delta_3(k)b_0k$$
$$-4\xi_3^4b_0k\delta_3(k)b_0^2k^2\delta_3(k)b_0k - 6\xi_1^4b_0^2k^2\delta_1(k)b_0k\delta_1(k)b_0k$$
$$-8\xi_1^2\xi_2^2b_0^2k^2\delta_1(k)b_0k\delta_1(k)b_0k - 2\xi_2^4b_0^2k^2\delta_1(k)b_0k\delta_1(k)b_0k$$
$$-8\xi_1^2\xi_3^2b_0^2k^2\delta_1(k)b_0k\delta_1(k)b_0k - 4\xi_2^2\xi_3^2b_0^2k^2\delta_1(k)b_0k\delta_1(k)b_0k$$
$$-2\xi_3^4b_0^2k^2\delta_1(k)b_0k\delta_1(k)b_0k - 4\xi_1^3\xi_2b_0^2k^2\delta_1(k)b_0k\delta_2(k)b_0k$$
$$-4\xi_1\xi_2^3b_0^2k^2\delta_1(k)b_0k\delta_2(k)b_0k - 4\xi_1\xi_2\xi_3^2b_0^2k^2\delta_1(k)b_0k\delta_2(k)b_0k$$
$$-4\xi_1^3\xi_3b_0^2k^2\delta_1(k)b_0k\delta_3(k)b_0k - 4\xi_1\xi_2^2\xi_3b_0^2k^2\delta_1(k)b_0k\delta_3(k)b_0k$$
$$-4\xi_1\xi_3^3b_0^2k^2\delta_1(k)b_0k\delta_3(k)b_0k + 4\xi_1^6b_0^2k^2\delta_1(k)b_0^2k^3\delta_1(k)b_0k$$
$$+8\xi_1^4\xi_2^2b_0^2k^2\delta_1(k)b_0^2k^3\delta_1(k)b_0k + 4\xi_1^2\xi_2^4b_0^2k^2\delta_1(k)b_0^2k^3\delta_1(k)b_0k$$
$$+8\xi_1^4\xi_3^2b_0^2k^2\delta_1(k)b_0^2k^3\delta_1(k)b_0k + 8\xi_1^2\xi_2^2\xi_3^2b_0^2k^2\delta_1(k)b_0^2k^3\delta_1(k)b_0k$$

$$+4\xi_1^2\xi_3^4b_0^2k^2\delta_1(k)b_0^2k^3\delta_1(k)b_0k + 4\xi_1^5\xi_2b_0^2k^2\delta_1(k)b_0^2k^3\delta_2(k)b_0k$$
$$+8\xi_1^3\xi_2^3b_0^2k^2\delta_1(k)b_0^2k^3\delta_2(k)b_0k + 4\xi_1\xi_2^5b_0^2k^2\delta_1(k)b_0^2k^3\delta_2(k)b_0k$$
$$+8\xi_1^3\xi_2\xi_3^2b_0^2k^2\delta_1(k)b_0^2k^3\delta_2(k)b_0k + 8\xi_1\xi_2^3\xi_3^2b_0^2k^2\delta_1(k)b_0^2k^3\delta_2(k)b_0k$$
$$+4\xi_1\xi_2\xi_3^4b_0^2k^2\delta_1(k)b_0^2k^3\delta_2(k)b_0k + 4\xi_1^5\xi_3b_0^2k^2\delta_1(k)b_0^2k^3\delta_3(k)b_0k$$
$$+8\xi_1^3\xi_2^2\xi_3b_0^2k^2\delta_1(k)b_0^2k^3\delta_3(k)b_0k + 4\xi_1\xi_2^4\xi_3b_0^2k^2\delta_1(k)b_0^2k^3\delta_3(k)b_0k$$
$$+8\xi_1^3\xi_3^3b_0^2k^2\delta_1(k)b_0^2k^3\delta_3(k)b_0k + 8\xi_1\xi_2^2\xi_3^3b_0^2k^2\delta_1(k)b_0^2k^3\delta_3(k)b_0k$$
$$+4\xi_1\xi_3^5b_0^2k^2\delta_1(k)b_0^2k^3\delta_3(k)b_0k - 4\xi_1^3\xi_2b_0^2k^2\delta_2(k)b_0k\delta_1(k)b_0k$$
$$-4\xi_1\xi_2^3b_0^2k^2\delta_2(k)b_0k\delta_1(k)b_0k - 4\xi_1\xi_2\xi_3^2b_0^2k^2\delta_2(k)b_0k\delta_1(k)b_0k$$
$$-2\xi_1^4b_0^2k^2\delta_2(k)b_0k\delta_2(k)b_0k - 8\xi_1^2\xi_2^2b_0^2k^2\delta_2(k)b_0k\delta_2(k)b_0k$$
$$-6\xi_2^4b_0^2k^2\delta_2(k)b_0k\delta_2(k)b_0k - 4\xi_1^2\xi_3^2b_0^2k^2\delta_2(k)b_0k\delta_2(k)b_0k$$
$$-8\xi_2^2\xi_3^2b_0^2k^2\delta_2(k)b_0k\delta_2(k)b_0k - 2\xi_3^4b_0^2k^2\delta_2(k)b_0k\delta_2(k)b_0k$$
$$-4\xi_1^2\xi_2\xi_3b_0^2k^2\delta_2(k)b_0k\delta_3(k)b_0k - 4\xi_2^3\xi_3b_0^2k^2\delta_2(k)b_0k\delta_3(k)b_0k$$
$$-4\xi_2\xi_3^3b_0^2k^2\delta_2(k)b_0k\delta_3(k)b_0k + 4\xi_1^5\xi_2b_0^2k^2\delta_2(k)b_0^2k^3\delta_1(k)b_0k$$
$$+8\xi_1^3\xi_2^3b_0^2k^2\delta_2(k)b_0^2k^3\delta_1(k)b_0k + 4\xi_1\xi_2^5b_0^2k^2\delta_2(k)b_0^2k^3\delta_1(k)b_0k$$
$$+8\xi_1^3\xi_2\xi_3^2b_0^2k^2\delta_2(k)b_0^2k^3\delta_1(k)b_0k + 8\xi_1\xi_2^3\xi_3^2b_0^2k^2\delta_2(k)b_0^2k^3\delta_1(k)b_0k$$
$$+4\xi_1\xi_2\xi_3^4b_0^2k^2\delta_2(k)b_0^2k^3\delta_1(k)b_0k + 4\xi_1^4\xi_2^2b_0^2k^2\delta_2(k)b_0^2k^3\delta_2(k)b_0k$$
$$+8\xi_1^2\xi_2^4b_0^2k^2\delta_2(k)b_0^2k^3\delta_2(k)b_0k + 4\xi_2^6b_0^2k^2\delta_2(k)b_0^2k^3\delta_2(k)b_0k$$
$$+8\xi_1^2\xi_2^2\xi_3^2b_0^2k^2\delta_2(k)b_0^2k^3\delta_2(k)b_0k + 8\xi_2^4\xi_3^2b_0^2k^2\delta_2(k)b_0^2k^3\delta_2(k)b_0k$$
$$+4\xi_2^2\xi_3^4b_0^2k^2\delta_2(k)b_0^2k^3\delta_2(k)b_0k + 4\xi_1^4\xi_2\xi_3b_0^2k^2\delta_2(k)b_0^2k^3\delta_3(k)b_0k$$
$$+8\xi_1^2\xi_2^3\xi_3b_0^2k^2\delta_2(k)b_0^2k^3\delta_3(k)b_0k + 4\xi_2^5\xi_3b_0^2k^2\delta_2(k)b_0^2k^3\delta_3(k)b_0k$$
$$+8\xi_1^2\xi_2\xi_3^3b_0^2k^2\delta_2(k)b_0^2k^3\delta_3(k)b_0k + 8\xi_2^3\xi_3^3b_0^2k^2\delta_2(k)b_0^2k^3\delta_3(k)b_0k$$
$$+4\xi_2\xi_3^5b_0^2k^2\delta_2(k)b_0^2k^3\delta_3(k)b_0k - 4\xi_1^3\xi_3b_0^2k^2\delta_3(k)b_0k\delta_1(k)b_0k$$
$$-4\xi_1\xi_2^2\xi_3b_0^2k^2\delta_3(k)b_0k\delta_1(k)b_0k - 4\xi_1\xi_3^3b_0^2k^2\delta_3(k)b_0k\delta_1(k)b_0k$$
$$-4\xi_1^2\xi_2\xi_3b_0^2k^2\delta_3(k)b_0k\delta_2(k)b_0k - 4\xi_2^3\xi_3b_0^2k^2\delta_3(k)b_0k\delta_2(k)b_0k$$
$$-4\xi_2\xi_3^3b_0^2k^2\delta_3(k)b_0k\delta_2(k)b_0k - 2\xi_1^4b_0^2k^2\delta_3(k)b_0k\delta_3(k)b_0k$$
$$-4\xi_1^2\xi_2^2b_0^2k^2\delta_3(k)b_0k\delta_3(k)b_0k - 2\xi_2^4b_0^2k^2\delta_3(k)b_0k\delta_3(k)b_0k$$
$$-8\xi_1^2\xi_3^2b_0^2k^2\delta_3(k)b_0k\delta_3(k)b_0k - 8\xi_2^2\xi_3^2b_0^2k^2\delta_3(k)b_0k\delta_3(k)b_0k$$
$$-6\xi_3^4b_0^2k^2\delta_3(k)b_0k\delta_3(k)b_0k + 4\xi_1^5\xi_3b_0^2k^2\delta_3(k)b_0^2k^3\delta_1(k)b_0k$$
$$+8\xi_1^3\xi_2^2\xi_3b_0^2k^2\delta_3(k)b_0^2k^3\delta_1(k)b_0k + 4\xi_1\xi_2^4\xi_3b_0^2k^2\delta_3(k)b_0^2k^3\delta_1(k)b_0k$$
$$+8\xi_1^3\xi_3^3b_0^2k^2\delta_3(k)b_0^2k^3\delta_1(k)b_0k + 8\xi_1\xi_2^2\xi_3^3b_0^2k^2\delta_3(k)b_0^2k^3\delta_1(k)b_0k$$
$$+4\xi_1\xi_3^5b_0^2k^2\delta_3(k)b_0^2k^3\delta_1(k)b_0k + 4\xi_1^4\xi_2\xi_3b_0^2k^2\delta_3(k)b_0^2k^3\delta_2(k)b_0k$$
$$+8\xi_1^2\xi_2^3\xi_3b_0^2k^2\delta_3(k)b_0^2k^3\delta_2(k)b_0k + 4\xi_2^5\xi_3b_0^2k^2\delta_3(k)b_0^2k^3\delta_2(k)b_0k$$

$$+8\xi_1^2\xi_2\xi_3^3b_0^2k^2\delta_3(k)b_0^2k^3\delta_2(k)b_0k+8\xi_2^3\xi_3^3b_0^2k^2\delta_3(k)b_0^2k^3\delta_2(k)b_0k$$
$$+4\xi_2\xi_3^5b_0^2k^2\delta_3(k)b_0^2k^3\delta_2(k)b_0k+4\xi_1^4\xi_3^2b_0^2k^2\delta_3(k)b_0^2k^3\delta_3(k)b_0k$$
$$+8\xi_1^2\xi_2^2\xi_3^2b_0^2k^2\delta_3(k)b_0^2k^3\delta_3(k)b_0k+4\xi_2^4\xi_3^2b_0^2k^2\delta_3(k)b_0^2k^3\delta_3(k)b_0k$$
$$+8\xi_1^2\xi_3^4b_0^2k^2\delta_3(k)b_0^2k^3\delta_3(k)b_0k+8\xi_2^2\xi_3^4b_0^2k^2\delta_3(k)b_0^2k^3\delta_3(k)b_0k$$
$$+4\xi_3^6b_0^2k^2\delta_3(k)b_0^2k^3\delta_3(k)b_0k+4\xi_1^6b_0^2k^3\delta_1(k)b_0^2k^2\delta_1(k)b_0k$$
$$+8\xi_1^4\xi_2^2b_0^2k^3\delta_1(k)b_0^2k^2\delta_1(k)b_0k+4\xi_1^2\xi_2^4b_0^2k^3\delta_1(k)b_0^2k^2\delta_1(k)b_0k$$
$$+8\xi_1^4\xi_3^2b_0^2k^3\delta_1(k)b_0^2k^2\delta_1(k)b_0k+8\xi_1^2\xi_2^2\xi_3^2b_0^2k^3\delta_1(k)b_0^2k^2\delta_1(k)b_0k$$
$$+4\xi_1^2\xi_3^4b_0^2k^3\delta_1(k)b_0^2k^2\delta_1(k)b_0k+4\xi_1^5\xi_2b_0^2k^3\delta_1(k)b_0^2k^2\delta_2(k)b_0k$$
$$+8\xi_1^3\xi_2^3b_0^2k^3\delta_1(k)b_0^2k^2\delta_2(k)b_0k+4\xi_1\xi_2^5b_0^2k^3\delta_1(k)b_0^2k^2\delta_2(k)b_0k$$
$$+8\xi_1^3\xi_2\xi_3^2b_0^2k^3\delta_1(k)b_0^2k^2\delta_2(k)b_0k+8\xi_1\xi_2^3\xi_3^2b_0^2k^3\delta_1(k)b_0^2k^2\delta_2(k)b_0k$$
$$+4\xi_1\xi_2\xi_3^4b_0^2k^3\delta_1(k)b_0^2k^2\delta_2(k)b_0k+4\xi_1^5\xi_3b_0^2k^3\delta_1(k)b_0^2k^2\delta_3(k)b_0k$$
$$+8\xi_1^3\xi_2^2\xi_3b_0^2k^3\delta_1(k)b_0^2k^2\delta_3(k)b_0k+4\xi_1\xi_2^4\xi_3b_0^2k^3\delta_1(k)b_0^2k^2\delta_3(k)b_0k$$
$$+8\xi_1^3\xi_3^3b_0^2k^3\delta_1(k)b_0^2k^2\delta_3(k)b_0k+8\xi_1\xi_2^2\xi_3^3b_0^2k^3\delta_1(k)b_0^2k^2\delta_3(k)b_0k$$
$$+4\xi_1\xi_3^5b_0^2k^3\delta_1(k)b_0^2k^2\delta_3(k)b_0k+4\xi_1^5\xi_2b_0^2k^3\delta_2(k)b_0^2k^2\delta_1(k)b_0k$$
$$+8\xi_1^3\xi_2^3b_0^2k^3\delta_2(k)b_0^2k^2\delta_1(k)b_0k+4\xi_1\xi_2^5b_0^2k^3\delta_2(k)b_0^2k^2\delta_1(k)b_0k$$
$$+8\xi_1^3\xi_2\xi_3^2b_0^2k^3\delta_2(k)b_0^2k^2\delta_1(k)b_0k+8\xi_1\xi_2^3\xi_3^2b_0^2k^3\delta_2(k)b_0^2k^2\delta_1(k)b_0$$
$$k+4\xi_1\xi_2\xi_3^4b_0^2k^3\delta_2(k)b_0^2k^2\delta_1(k)b_0k+4\xi_1^4\xi_2^2b_0^2k^3\delta_2(k)b_0^2k^2\delta_2(k)b_0k$$
$$+8\xi_1^2\xi_2^4b_0^2k^3\delta_2(k)b_0^2k^2\delta_2(k)b_0k+4\xi_2^6b_0^2k^3\delta_2(k)b_0^2k^2\delta_2(k)b_0k$$
$$+8\xi_1^2\xi_2^2\xi_3^2b_0^2k^3\delta_2(k)b_0^2k^2\delta_2(k)b_0k+8\xi_2^4\xi_3^2b_0^2k^3\delta_2(k)b_0^2k^2\delta_2(k)b_0k$$
$$+4\xi_2^2\xi_3^4b_0^2k^3\delta_2(k)b_0^2k^2\delta_2(k)b_0k+4\xi_1^4\xi_2\xi_3b_0^2k^3\delta_2(k)b_0^2k^2\delta_3(k)b_0k$$
$$+8\xi_1^2\xi_2^3\xi_3b_0^2k^3\delta_2(k)b_0^2k^2\delta_3(k)b_0k+4\xi_2^5\xi_3b_0^2k^3\delta_2(k)b_0^2k^2\delta_3(k)b_0k$$
$$+8\xi_1^2\xi_2\xi_3^3b_0^2k^3\delta_2(k)b_0^2k^2\delta_3(k)b_0k+8\xi_2^3\xi_3^3b_0^2k^3\delta_2(k)b_0^2k^2\delta_3(k)b_0k$$
$$+4\xi_2\xi_3^5b_0^2k^3\delta_2(k)b_0^2k^2\delta_3(k)b_0k+4\xi_1^5\xi_3b_0^2k^3\delta_3(k)b_0^2k^2\delta_1(k)b_0k$$
$$+8\xi_1^3\xi_2^2\xi_3b_0^2k^3\delta_3(k)b_0^2k^2\delta_1(k)b_0k+4\xi_1\xi_2^4\xi_3b_0^2k^3\delta_3(k)b_0^2k^2\delta_1(k)b_0k$$
$$+8\xi_1^3\xi_3^3b_0^2k^3\delta_3(k)b_0^2k^2\delta_1(k)b_0k+8\xi_1\xi_2^2\xi_3^3b_0^2k^3\delta_3(k)b_0^2k^2\delta_1(k)b_0k$$
$$+4\xi_1\xi_3^5b_0^2k^3\delta_3(k)b_0^2k^2\delta_1(k)b_0k+4\xi_1^4\xi_2\xi_3b_0^2k^3\delta_3(k)b_0^2k^2\delta_2(k)b_0k$$
$$+8\xi_1^2\xi_2^3\xi_3b_0^2k^3\delta_3(k)b_0^2k^2\delta_2(k)b_0k+4\xi_2^5\xi_3b_0^2k^3\delta_3(k)b_0^2k^2\delta_2(k)b_0k$$
$$+8\xi_1^2\xi_2\xi_3^3b_0^2k^3\delta_3(k)b_0^2k^2\delta_2(k)b_0k+8\xi_2^3\xi_3^3b_0^2k^3\delta_3(k)b_0^2k^2\delta_2(k)b_0k$$
$$+4\xi_2\xi_3^5b_0^2k^3\delta_3(k)b_0^2k^2\delta_2(k)b_0k+4\xi_1^4\xi_3^2b_0^2k^3\delta_3(k)b_0^2k^2\delta_3(k)b_0k$$
$$+8\xi_1^2\xi_2^2\xi_3^2b_0^2k^3\delta_3(k)b_0^2k^2\delta_3(k)b_0k+4\xi_2^4\xi_3^2b_0^2k^3\delta_3(k)b_0^2k^2\delta_3(k)b_0k$$
$$+8\xi_1^2\xi_3^4b_0^2k^3\delta_3(k)b_0^2k^2\delta_3(k)b_0k+8\xi_2^2\xi_3^4b_0^2k^3\delta_3(k)b_0^2k^2\delta_3(k)b_0k$$
$$+4\xi_3^6b_0^2k^3\delta_3(k)b_0^2k^2\delta_3(k)b_0k+8\xi_1^6b_0^3k^4\delta_1(k)b_0k\delta_1(k)b_0k$$

$$+16\xi_1^4\xi_2^2b_0^3k^4\delta_1(k)b_0k\delta_1(k)b_0k + 8\xi_1^2\xi_2^4b_0^3k^4\delta_1(k)b_0k\delta_1(k)b_0k$$
$$+16\xi_1^4\xi_3^2b_0^3k^4\delta_1(k)b_0k\delta_1(k)b_0k + 16\xi_1^2\xi_2^2\xi_3^2b_0^3k^4\delta_1(k)b_0k\delta_1(k)b_0k$$
$$+8\xi_1^2\xi_3^4b_0^3k^4\delta_1(k)b_0k\delta_1(k)b_0k + 8\xi_1^5\xi_2b_0^3k^4\delta_1(k)b_0k\delta_2(k)b_0k$$
$$+16\xi_1^3\xi_2^3b_0^3k^4\delta_1(k)b_0k\delta_2(k)b_0k + 8\xi_1\xi_2^5b_0^3k^4\delta_1(k)b_0k\delta_2(k)b_0k$$
$$+16\xi_1^3\xi_2\xi_3^2b_0^3k^4\delta_1(k)b_0k\delta_2(k)b_0k + 16\xi_1\xi_2^3\xi_3^2b_0^3k^4\delta_1(k)b_0k\delta_2(k)b_0k$$
$$+8\xi_1\xi_2\xi_3^4b_0^3k^4\delta_1(k)b_0k\delta_2(k)b_0k + 8\xi_1^5\xi_3b_0^3k^4\delta_1(k)b_0k\delta_3(k)b_0k$$
$$+16\xi_1^3\xi_2^2\xi_3b_0^3k^4\delta_1(k)b_0k\delta_3(k)b_0k + 8\xi_1\xi_2^4\xi_3b_0^3k^4\delta_1(k)b_0k\delta_3(k)b_0k$$
$$+16\xi_1^3\xi_3^3b_0^3k^4\delta_1(k)b_0k\delta_3(k)b_0k + 16\xi_1\xi_2^2\xi_3^3b_0^3k^4\delta_1(k)b_0k\delta_3(k)b_0k$$
$$+8\xi_1\xi_3^5b_0^3k^4\delta_1(k)b_0k\delta_3(k)b_0k + 8\xi_1^5\xi_2b_0^3k^4\delta_2(k)b_0k\delta_1(k)b_0k$$
$$+16\xi_1^3\xi_2^3b_0^3k^4\delta_2(k)b_0k\delta_1(k)b_0k + 8\xi_1\xi_2^5b_0^3k^4\delta_2(k)b_0k\delta_1(k)b_0k$$
$$+16\xi_1^3\xi_2\xi_3^2b_0^3k^4\delta_2(k)b_0k\delta_1(k)b_0k + 16\xi_1\xi_2^3\xi_3^2b_0^3k^4\delta_2(k)b_0k\delta_1(k)b_0k$$
$$+8\xi_1\xi_2\xi_3^4b_0^3k^4\delta_2(k)b_0k\delta_1(k)b_0k + 8\xi_1^4\xi_2^2b_0^3k^4\delta_2(k)b_0k\delta_2(k)b_0k$$
$$+16\xi_1^2\xi_2^4b_0^3k^4\delta_2(k)b_0k\delta_2(k)b_0k + 8\xi_2^6b_0^3k^4\delta_2(k)b_0k\delta_2(k)b_0k$$
$$+8b_0^3k^4\delta_3(k)b_0k\delta_3(k)b_0k\xi_3^6 + 8b_0^3k^4\delta_3(k)b_0k\delta_1(k)b_0k\xi_1\xi_3^5$$
$$+8b_0^3k^4\delta_2(k)b_0k\delta_3(k)b_0k\xi_2\xi_3^5 + 8b_0^3k^4\delta_3(k)b_0k\delta_2(k)b_0k\xi_2\xi_3^5$$
$$+16b_0^3k^4\delta_3(k)b_0k\delta_3(k)b_0k\xi_1^2\xi_3^4 + 8b_0^3k^4\delta_2(k)b_0k\delta_2(k)b_0k\xi_2^2\xi_3^4$$
$$+16b_0^3k^4\delta_3(k)b_0k\delta_3(k)b_0k\xi_2^2\xi_3^4 + 16b_0^3k^4\delta_3(k)b_0k\delta_1(k)b_0k\xi_1^3\xi_3^3$$
$$+16b_0^3k^4\delta_2(k)b_0k\delta_3(k)b_0k\xi_2^3\xi_3^3 + 16b_0^3k^4\delta_3(k)b_0k\delta_2(k)b_0k\xi_2^3\xi_3^3$$
$$+16b_0^3k^4\delta_3(k)b_0k\delta_1(k)b_0k\xi_1\xi_2^2\xi_3^3 + 16b_0^3k^4\delta_2(k)b_0k\delta_3(k)b_0k\xi_1^2\xi_2\xi_3^3$$
$$+16b_0^3k^4\delta_3(k)b_0k\delta_2(k)b_0k\xi_1^2\xi_2\xi_3^3 + 8b_0^3k^4\delta_3(k)b_0k\delta_3(k)b_0k\xi_1^4\xi_3^2$$
$$+16b_0^3k^4\delta_2(k)b_0k\delta_2(k)b_0k\xi_2^4\xi_3^2 + 8b_0^3k^4\delta_3(k)b_0k\delta_3(k)b_0k\xi_2^4\xi_3^2$$
$$+16b_0^3k^4\delta_2(k)b_0k\delta_2(k)b_0k\xi_1^2\xi_2^2\xi_3^2 + 16b_0^3k^4\delta_3(k)b_0k\delta_3(k)b_0k\xi_1^2\xi_2^2\xi_3^2$$
$$+8b_0^3k^4\delta_3(k)b_0k\delta_1(k)b_0k\xi_1^5\xi_3 + 8b_0^3k^4\delta_2(k)b_0k\delta_3(k)b_0k\xi_2^5\xi_3$$
$$+8b_0^3k^4\delta_3(k)b_0k\delta_2(k)b_0k\xi_2^5\xi_3 + 8b_0^3k^4\delta_3(k)b_0k\delta_1(k)b_0k\xi_1\xi_2^4\xi_3$$
$$+16b_0^3k^4\delta_2(k)b_0k\delta_3(k)b_0k\xi_1^2\xi_2^3\xi_3 + 16b_0^3k^4\delta_3(k)b_0k\delta_2(k)b_0k\xi_1^2\xi_2^3\xi_3$$
$$+16b_0^3k^4\delta_3(k)b_0k\delta_1(k)b_0k\xi_1^3\xi_2^2\xi_3 + 8b_0^3k^4\delta_2(k)b_0k\delta_3(k)b_0k\xi_1^4\xi_2\xi_3$$
$$+8b_0^3k^4\delta_3(k)b_0k\delta_2(k)b_0k\xi_1^4\xi_2\xi_3.$$

Curriculum Vitae

Sajad Sadeghi

Education

- **Ph.D. in Mathematics (2012-2016)**
  The University of Western Ontario, London, ON, Canada.
- **M.Sc. in Mathematics (2007-2010)**
  Tarbiat Modares University, Tehran, Iran.
- **B.Sc. in Mathematics (2002-2007)**
  Shahid Beheshti University, Tehran, Iran.

Research Interests

Noncommutative geometry, Operator algebras and C*-algebras, Spectral geometry, Geometric analysis, Harmonic analysis

Publications

- **A Scalar Curvature Formula for the Noncommutative Three Torus**, joint with M. Khalkhali and A. Moatadelro, to be submitted.
- **On Logarithmic Sobolev Inequality for the Noncommutative Two Torus**, joint with M. Khalkhali, arXiv:1601.04242, January 2016 (accepted for publication in Journal of Pseudo-Differential Operators and Applications).

Talks and Presentations

- Spin Manifolds and Dirac Operators, NCG Seminar, UWO, January 2015.
- An Operator Algebraic Proof of Uniformization Theorem, NCG Seminar, UWO, April 2014.
- Dirac Operators and Geodesic Metric on the Sierpinski Gasket, NCG Seminar, UWO, March 2013.

Conferences and Programs Attended

- Noncommutative Geometry and Spectral Invariants, Université du Québec à Montréal, Montréal, Canada, June 29- July 3, 2015.
- Workshop on the Geometry of Noncommutative Manifolds, Fields Institute, Toronto, Canada, March 16-18, 2015.
- Noncommutative Geometry and its Applications, Hausdorff Research Institute for Mathematics, Bonn, Germany, September 1-30, 2014.
- Program on Noncommutative Geometry and Quantum Groups, Fields Institute, Toronto, Canada, June 17-28, 2013.
- Winter School on Operator Algebras, RIMS, Kyoto University, Kyoto, Japan, December 7-16, 2011.
- Conference on C*-Algebras and Related Topics, RIMS, Kyoto University, Kyoto, Japan, September 5-9, 2011.

Scholarships

- Western Graduate Research Scholarship: The University of Western Ontario, 2012-2016.