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Optimal Search

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OPTIMAL SEARCH[1]

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ABSTRACT

This paper presents a number of fundamental results on the existence and properties of expected value maximizing search rules for problems in which searchers may choose both the number of samples taken and the size of each sample. These rules include fixed-sample-size rules and sequential rules as special cases. Also presented are conditions sufficient for the optimal rules to reduce to sequential or fixed-sample-size rules.

1. INTRODUCTION

Since the seminal papers of Stigler [12], [13], the literature on search problems has concentrated upon two types of search strategies; fixed-sample-size (fss) and sequential strategies. Two ideas which appeared in this literature were that, ceteris paribus, a best (in the sense of maximizing expected utility) sequential strategy dominated any fss strategy and that an optimal search strategy was necessarily sequential. Lately, several authors have shown these ideas are incorrect. Benhabib and Bull [1], Gal, Landsberger and Levykson [5], Manning and Morgan [9] and Morgan [10], [11] have all noted that there is a class of search strategies which dominate both fss and sequential strategies. The purpose of this paper is to present a collection of fundamental results on the existence and properties of these new search strategies.

The usual formulation of search problems views the searcher as periodically drawing observations from a population and eventually taking some terminal action which depends upon what he has observed. If he uses a sequential strategy, then the searcher is required to draw exactly one observation at a time and to wait until that observation is received before deciding if he should draw another (see DeGroot [4, p. 280]). The total number of observations drawn is unknown until the moment the searcher receives an observation which induces a halt to his search. The usual view of a fss strategy is that it is a sequential strategy constrained in a particular way. The fss searcher is thought of as choosing the total number of observations ex ante and then doggedly drawing this number of observations sequentially, irrespective of what he actually observes.

Naturally, such a fss strategy is dominated by a sequential strategy.

There are, however, other fss strategies. The authors of [1], [5], [9], [10] and [11] all consider a fss strategy which allows the searcher to draw only one sample but allows all the observations in the sample to be drawn simultaneously. This is a common practice in reality. For example, a firm seeking another to undertake a project will often choose to ask other firms to submit tenders together. A Ph.D. graduate seeking an academic post will usually apply to several universities at once, rather than wait for a reply to one application before sending out another. The advantage of such a fss strategy is, of course, that it allows information to be gathered quickly; in particular, more quickly than a sequential strategy. Against this is the disadvantage that using a fss strategy may result in overinvestment in information, a possibility avoided by a sequential strategy. Which of these strategies is to be preferred, therefore, depends upon the relative values of the advantage and disadvantage. These values will vary from one type of search problem to another, making fss strategies preferred to sequential strategies in some cases but not in others.

The new class of optimal search strategies mentioned in the first paragraph combines the flexibility of fss strategies to allow simultaneous receipt of a number of observations with the flexibility of sequential strategies to choose whether or not to sample again. In some cases these optimal strategies will be fss strategies. In other cases they will be sequential strategies. For a broad class of problems, however, the optimal strategies will be neither fss nor sequential strategies.

Section 2 presents an example illustrating the remarks made in this introduction. Section 3 presents a formal description of the class of search problems considered. Section 4 explains why the optimal search strategies dominate the best fss and best sequential search strategies. The existence and form of a set of optimal search rules is established in Section 5. Section 6 describes circumstances in which the best fss or best sequential search strategies are optimal. Some concluding comments are offered in Section 7.

2. AN EXAMPLE

Deborah, Dolores and Dorothy Decisionmaker are three sisters. Each needs her own house built before the winter snows arrive. Builders need one month to prepare quotes and charge a standard fee of $1000 for each quote when it is requested. Winter is only three months away and construction requires one month so the sisters must select a quote in two months or less from now. To avoid family rivalries the sisters have all decided to build the same type of house, a dwelling for which builders will quote either $50,000 or $60,000, with respective probabilities of 0.2 and 0.8. Total payment is made to the builder when construction is begun and a completed house yields the occupier a stream of services with a present value of $80,000. Each sister has a monthly rate of time preference $\delta = 0.01$.

Deborah figures out that her best sequential strategy is to now ask one builder for a quote. She will receive this quote in one month's time. She should accept a quote for $50,000 but reject a quote of $60,000 in favour of purchasing a second quote. The lower of the two quotes must be accepted at the end of the second month so that the house can be built before the winter snows arrive. Using the notation that $p_n^{min} = \min\{p_1,\ldots,p_n\}$ is the smallest of n received quotes, $n \geq 1$, Deborah calculates that the expected present value of this best sequential strategy is

$$(2.1) \qquad W_{seq} = -1000 + \frac{1}{1.01} E[\max\{\frac{1}{1.01} 80000 - p_1^{min}, \\ -1000 + \frac{1}{1.01} E[\frac{1}{1.01} 80000 - p_2^{min} | p_1^{min}]\}] \\ = \$20623.70$$

Dolores was always the most spontaneous of the three sisters. She decides to get all her quotations at once and asks four builders for quotes simultaneously. To Deborah's surprise, Dolores' fss strategy has an expected present value of

$$(2.2) \qquad W_{fss} = -4000 + \frac{1}{1.01} E[\frac{1}{1.01} 80000 - p_4^{min}] \\ = \$20863.29$$

Dorothy simply does what she expects to be best. She decides to ask three builders for quotes and to accept any offer for \$50000 in one month's time. If all quotes are for \$60000 then she will ask a further four builders for quotes and, in one more month's time, will accept the lowest of the seven quotes received. Although she does not know it, the expected present value of her (optimal) strategy is

$$(2.3) \qquad W^* = -3000 + \frac{1}{1.01} E[\max\{\frac{1}{1.01} 80000 - p_3^{min}, \\ -4000 + \frac{1}{1.01} E[\frac{1}{1.01} 80000 - p_7^{min} | p_3^{min}]\}] \\ = \$21688.58$$

Later, on a winter's evening when all three sisters are snug inside their houses, they realise that Dorothy's strategy was best of all because it combined the freedom to choose her sample size at any decision point with the freedom to stop or continue her search past the decision point. The sequential strategy was suboptimal because it constrained Deborah to purchasing only one quote at each of the two decision points. The fss strategy was suboptimal because Delores was constrained to searching over only one of the two months available.

3. THE SEARCH PROBLEM

Search consists of drawing observations from a population $X \subset R$. At time t, where $t = 1,\ldots,T$ and $T \geq 1$, the searcher must decide whether or not to continue his search. If he chooses to stop at t then he selects the terminal action offering the highest utility from those currently available to him. If he chooses to continue to search then he must decide the size n^t of the sample of observations to make on X at t. It is assumed that these observations become available to the searcher one period later, at t+1.[2] Search is costly and the searcher's problem is to determine the search rule ρ^* which will maximize his expected utility.

At time t=1 no observations on X have been received. Accordingly define the singleton set

$$(3.1) \qquad Y^1 := \{y^1\}$$

where y^1 is the vector conditioning F^1, the searcher's initial estimate of the c.d.f. over X. At times t the searcher can request n^t observations on X. If $n^t \geq 1$, then at t+1 he will receive the vector of observations

$$(3.2) \qquad x^t := (x_1^t,\ldots,x_{n^t}^t) \in X^t.$$

x_j^t is the j^{th} observation requested at t and X^t is the n^t-fold Cartesian product of X. The vector of all observations received by t is

$$(3.3)\quad y^t := \begin{cases} y^1, & \text{if } t = 1 \\ y^{t-1}, & \text{if } n^{t-1} = 0 \text{ for } 2 \le t \le T \\ (y^{t-1}, x^{t-1}), & \text{if } n^{t-1} \ge 1 \text{ for } 2 \le t \le T \end{cases}$$

where

$$(3.4)\quad y^t \in Y^t := \begin{cases} Y^{t-1}, & \text{if } n^{t-1} = 0 \\ Y^{t-1} \times X^{t-1}, & \text{if } n^{t-1} \ge 1 \end{cases} \text{; for } 2 \le t \le T.$$

At time t the searcher will use y^t to choose $n^t \in I^+$, the set of non-negative integers, and to decide

$$(3.5)\quad \xi^t(y^t) = \begin{cases} 0, & \text{if search is to continue at } t \\ 1, & \text{if search is to stop at } t. \end{cases}$$

Definition 1: A sequence of functions $\nu := \{\nu^t\}_{t=1}^{T}$ where $\nu^t : Y^t \to I^+$ is called a sample size rule.

Definition 2: A sequence of functions $\xi := \{\xi^t\}_{t=1}^{T}$ where $\xi^t : Y^t \to \{0,1\}$ is called a stopping rule.

Definition 3: A sequence of functions $\rho := \{\rho^t\}_{t=1}^{T}$ where $\rho^t := (\xi^t, \nu^t) : Y^t \to \{0,1\} \times I^+$ is called a search rule.[3]

$c(n^t)$ denotes the financial cost to the searcher of drawing n^t observations on X at any t. These accumulate across periods so that at t the searcher's net wealth is

$$(3.6)\quad w^t := w^1 - \sum_{j=1}^{t-1} c(n^j), \text{ for } t \ge 2.$$

The searcher's preferences are represented by his (indirect) utility function $I(x, w^t)$.[4] Non-financial costs of search are represented at any t by $K(n^t)$, the psychic cost of drawing n^t observations on X at t (e.g., sampling X may be physically fatiguing). Both the financial and psychic costs of sampling at t are incurred at t. Unlike financial costs, psychic search costs need not accumulate across periods.[5]

An expected utility maximizing search rule requires the searcher to compare, at each t, the highest currently available utility from stopping at t with the highest utility expected from continuing to search at t. This comparison requires that these utilities be both well defined for each $y^t \in Y^t$ and each $t=1,\ldots,T$.

Let $\mathcal{G}^t$ denote the set of observations still available to the searcher at time t. The composition of $\mathcal{G}^t$ will depend upon the searcher's ability to recall observations received prior to time t. For example, if the searcher has no recall then

$$(3.7)\quad \mathcal{G}^t = \tilde{\mathcal{G}}^t := \begin{cases} \{x_1^{t-1},\ldots,x_{n^{t-1}}^{t-1}\}, & \text{if } n^{t-1} > 0 \text{ and } t \geq 2 \\ \phi & \text{, if } n^{t-1} = 0 \text{ or } t = 1. \end{cases}$$

Alternatively, if the searcher has full recall then

$$(3.8)\qquad \mathcal{G}^t := \bigcup_{j=1}^{t} \tilde{\mathcal{G}}^t .$$

Clearly many forms of partial recall can be encompassed in this framework.[6] Define u^{*t} as the utility of the searcher's best terminal action at t; that is,

$$(3.9)\qquad u^{*t} := \max\{I^*(w^t),\ \max_{x\in\mathcal{G}^t} I(x,w^t)\};\quad t=1,\ldots,T.$$

$I(x,w^t)$ is the utility of choosing observation x with wealth w^t remaining and $I^*(w^t)$ is the utility at t of the searcher's best non-search generated alternative.[7] $u^{*1} := I^*(w^1)$ since $\mathcal{G}^1 = \phi$.

At each t the searcher must compare u^{*t} to the expected present valued utility of continuing to search past t. In order to form such expectations the searcher must be able to construct a sequence $\{F^t(x)\}_{t=1}^{T}$ of marginal subjective probability measures $F^t: X \to [0,1]$ which, for any observed y^t, are his current estimates of the c.d.f. over X; t=1,...,T. Using F^t the searcher evaluates the expected present valued utility of continuing to search past t using the search rule $\rho = (\xi,\nu)$. Using (3.5) this can be written as

$$(3.10)\quad W_t^T(y^t,\rho) := -K(n^t) + \beta E_{F^t}[\xi^{t+1}(y^{t+1})u^{*t+1} + (1-\xi^{t+1}(y^{t+1}))W_{t+1}^T(y^{t+1},\rho)|y^t]$$

for t=1,...,T-1 with

$$(3.11)\quad W_T^T(y^T,\rho) := \beta I^*(w^T).$$

$\beta := 1/(1+\delta)$ where $\delta \geq 0$ is the searcher's rate of time preference. (3.10) states that the expected value of continuing to search at t is the discounted expected value of the best terminal action available at t+1 given search stops at t+1 (i.e., $\xi^{t+1}(y^{t+1}) = 1$), plus the discounted expected value of continuing search at t+1 given search continues (i.e., $\xi^{t+1}(y^{t+1}) = 0$), with each possibility weighted by its respective probability of occurring. (3.11) states that no search is possible past T and that any offers available at T are unavailable past T, leaving the searcher after T with only the action of allocating his remaining wealth w^T to non-search generated alternatives. The expected present valued utility of any search rule ρ at time t is therefore

$$(3.12)\quad V_t^T(y^t,\rho) = \max\{u^{*t}, W_t^T(y^t,\rho)\};\quad t=1,\ldots,T.$$

Let $\mathfrak{N}$ denote the set of all sample size rules and let S denote the set of all stopping rules. Then $P := S \times \mathfrak{N}$ is the set of all search rules.

Definition 4: A search rule ρ^* is called optimal if and only if, for any $y^t \in Y^t$ and any $t=1,\ldots,T$

$$(3.13) \qquad V_t^T(y^t,\rho^*) \geq V_t^T(y^t,\rho), \ \forall \rho \in P.$$

This completes the description of the model. The following section demonstrates that, when ρ^* exists, it dominates both the best fss search rule and the best sequential search rule. Section 5 shows ρ^* exists in a very wide class of problems.

4. DOMINATION OF THE FSS AND SEQUENTIAL RULES

By definition, any sequential search strategy constrains the searcher to taking exactly one extra observation on X whenever he continues his search. The set of all sequential search rules is therefore

$$(4.1) \qquad P_{seq} := \{(\xi,\nu_{seq}) | \nu_{seq}^t(\cdot) \equiv 1 \ \forall \ t=1,\ldots,T\} \subset P.$$

For any given y^t the expected value of the best sequential search rule ρ_{seq} is

$$(4.2) \qquad V_t^T(y^t,\rho_{seq}) = \max_{\rho \in P_{seq}} V_t^T(y^t,\rho) = \max_{\xi \in S} V_t^T(y^t,(\xi,\nu_{seq})).$$

By definition, any fss search strategy constrains the searcher to taking no more than one sample of observations from X by requiring search to cease at or before t=2. The set of all fss search rules is therefore

$$(4.3) \qquad P_{fss} := \{(\xi_{fss},\nu) | \xi_{fss}^1(\cdot) \equiv 1\} \subset P.$$

For any given y^t the expected value of the best fss search rule ρ_{fss} is

$$(4.4) \qquad V_t^T(y^t,\rho_{fss}) = \max_{\rho \in P_{fss}} V_t^T(y^t,\rho) = \max_{\nu \in \mathfrak{N}} V_t^T(y^t,(\xi_{fss},\nu)).$$

For any given y^t the expected value of the optimal search rule ρ^* is

$$(4.5)\qquad V_t^T(y^t,\rho^*) = \max_{\rho\in P} V_t^T(y^t,\rho) = \max_{\xi\in S,\nu\in\mathfrak{N}} V_t^T(y^t,(\xi,\nu)).$$

Comparing (4.2), (4.4) and (4.5) shows

$$(4.6)\qquad V_t^T(y^t,\rho^*) \geq \max\{V_t^T(y^t,\rho_{seq}),V_t^T(y^t,\rho_{fss})\}.$$

This establishes the following statement.

<u>Proposition 1</u>

ρ^* dominates both ρ_{seq} and ρ_{fss}.

In Section 6 some particular search models are discussed for which ρ^* is either ρ_{seq} or ρ_{fss}. The next section addresses the more immediate tasks of establishing the existence and forms of ρ^* in a wide class of problems.

5. THE OPTIMAL SEARCH RULES

This section begins by imposing a weak condition which ensures the searcher is able to form expectations about the value of continued search. The remainder of the section establishes that this weak condition is sufficient for the existence of an optimal search rule. The supporting argument explains the notion of "regular" search rules. Then each member of a non-empty subset of regular search rules is proved to be optimal by showing it has an expected value equal to the supremum of the expected values of all the search rules in P.

Consider any vector $y = (x^1,x^2,\ldots,x^T)$ of observations on X. If the searcher has full ex ante knowledge of y then the present valued utility of searching out y cannot exceed

$$(5.1)\ Z(y):= \max\{I^*(w^1),\ \sup_{2\leq t\leq T}(-\sum_{j=1}^{t-1}\beta^{j-1}K(n^j)+\beta^j\max\{I^*(w^t),\ \max_{x\in\mathcal{G}^t} I(x,w^t)\})\}.$$

It is assumed that

(5.2) $E[|Z(y)|] < \infty$.

This condition ensures the expected present valued utility of continued search is always finite and, in particular, ensures

(5.3) $\sup_{\rho \in P} V_t^T(y^t,\rho) < \infty,\ \forall\, y^t \in Y^t,\ \forall t=1,\dots,T.$

Later (5.2) is shown to be sufficient for the existence of ρ^*.

The idea of "regularity" is well established in the literature on sequential decision making (see, for example, [4, pp. 289-290]). In essence, any search rule in P_{seq} is called regular if it requires a further single observation to be taken on X only when this is expected to increase the searcher's utility. It is straightforward to extend the idea of regularity from P_{seq} to P ; that is, to search rules which permit samples with sizes different from unity to be drawn from X.

Definition 5: A search rule $\rho \in P$ is said to be regular if

(5.3a) $V_1^T(y^1,\rho) \geq I^*(w^1)$

and if, for any $y^t \in Y^t$,

(5.3b) $\xi^t(y^t) = 0$ only if $u^{*t} < W_t^T(y^t,\rho),\ \forall t=1,\dots,T.$

That is, a search rule is regular only if its expected value is at least as great as the utility of not searching and if it commands the searcher to continue only when continuing is expected to improve his utility. Definition 5 does not restrict $n^t = 1$ for any $t=1,\dots,T-1$.

Let $\mathcal{R} \subset P$ denote the set of regular search rules. A moment's reflection upon definition 5 will provide the reader with the intuition that, if an optimal search rule ρ^* exists, then there must exist a regular search rule which is "as good as" ρ^* and is, therefore, optimal also. Lemmas 1, 2 and 3 prove this intuition correct and, thereby, allow the question of the existence of optimal search rules to be confined to $\mathfrak{R}$ since, if $\mathfrak{R}$ contains no optimal search rules then, necessarily, neither does $\mathcal{F}$.

Lemma 1:

If there exists a non-regular search rule ρ' then there exists a regular search rule ρ'' such that

$$V_t^T(y^t,\rho') \le V_t^T(y^t,\rho'') \text{ for any } y^t \in Y^t;\ t=1,\dots,T.$$

Proof: The argument given here closely follows an argument provided by DeGroot [4, p. 289] for sequential decision procedures. There are two possibilities.

EITHER $\rho' := (\xi',\nu')$ is such that, for all $y^t \in Y^t$ and any $t=1,\dots,T$

(5.4) $\xi'^t(y^t) = 0$ and $u^{*t} \ge W_t^T(y^t,\rho')$.

Together, (5.4) and (3.5) imply

(5.5) $V_t^T(y^t,\rho') = W_t^T(y^t,\rho')$.

Let $\rho'' := (\xi'',\nu'')$ where $\xi''^t(\cdot) \equiv 1$ and $\nu''^t(\cdot) \equiv 0$ $\forall t=1,\dots,T$.

ρ'' exists, is regular and, from (3.5), (5.4) and (5.5),

(5.6) $V_t^T(y^t,\rho'') = u^{*t} \ge W_t^T(y^t,\rho') = V_t^T(y^t,\rho')$.

OR $\rho' := (\xi',\nu')$ is such that, for some t and some $y^t \in Y^t$,

(5.7) $\xi'^t(y^t) = 0$ and $u^{*t} < W_t^T(y^t,\rho')$.

Together, (5.7) and (3.5) imply

(5.8) $V_t^T(y^t,\rho') = W_t^T(y^t,\rho')$.

Let $\rho'' := \rho'$. Then ρ'' exists, is regular and

(5.9) $V_t^T(y^t,\rho') = V_t^T(y^t,\rho'')$.

(5.6) and (5.9) together establish the result.

Q.E.D.

Lemma 2

$\mathcal{R}$, the set of regular search rules, is not empty.

Proof: Let $\rho := (\xi,\nu)$ where $\xi^t(\cdot) \equiv 1$ and $\nu^t(\cdot) \equiv 0$ for all $t=1,\ldots,T$. ρ exists. If $\rho \in \mathcal{R}$, then $\mathcal{R} \neq \phi$. If $\rho \notin \mathcal{R}$, then $\mathcal{R} \neq \phi$ by Lemma 1.

Q.E.D.

Lemma 3

If an optimal search rule exists, then there exists an optimal search rule which is regular.

Proof: Suppose ρ' is an optimal search rule. The proof is trivial if $\rho' \in \mathcal{R}$. Suppose $\rho' \notin \mathcal{R}$. Then, by Lemma 1, there exists $\rho'' \in \mathcal{R}$ such that

(5.10) $$V_t^T(y^t,\rho') \leq V_t^T(y^t,\rho'').$$

(5.10) contradicts the optimality of ρ' unless (5.10) is an equality. But then ρ'' is also an optimal search rule.

Q.E.D.

Corollary 1

An optimal search rule exists if and only if a regular optimal search rule exists.

Proof: If $\rho' \in P$ is optimal then, by Lemma 3, there exists the optimal rule $\rho'' \in \mathcal{R}$. If $\rho'' \in \mathcal{R}$ is optimal then $\rho'' \in P$ also since $\mathcal{R} \subset P$.

Q.E.D.

This result justifies devoting the remainder of the argument presented in this section to exploring only $\mathcal{R}$ for optimal search rules. A consequence of Lemma 3 which is useful in this exploration is the following remark.

Remark 1: For any t=1,...,T and for any $y^t \in Y^t$,

(5.11) $$\sup_{\rho \in P} V_t^T(y^t, \rho) = \sup_{\rho \in \mathcal{R}} V_t^T(y^t, \rho).$$

Two existence results are now established. First of all, the existence of an optimal sample size rule ν^* is proved for any stopping rule $\xi \in S$. Then a particular subset of stopping rules is combined with ν^* to give a particular subset of regular search rules. Each of these search rules is proved to be optimal.

Definition 6

A sample size rule ν^* is optimal for $\xi \in S$ if and only if, for any $y^t \in Y^t$ and any t=1,...,T,

(5.12) $$W_t^T(y^t, (\xi, \nu^*)) \geq W_t^T(y^t, (\xi, \nu)) \text{ for all } \nu \in \mathfrak{N}.$$

Theorem 1

If $E[|Z(y)|] < \infty$, then there exists a sample size rule ν^* which is optimal for any stopping rule $\xi \in S$.

Proof: For any t=1,...,T, for any $y^t \in Y^t$ and for any $\xi \in S$, let

(5.13) $$\overline{W}_t^T(y^t, \xi) := \sup_{\nu \in \mathfrak{N}} W_t^T(y^t, (\xi, \nu)).$$

$\overline{W}_t^T(y^t, \xi)$ exists since $\mathfrak{N} \neq \phi$ and since

(5.14) $$\overline{W}_t^T(y^t, \xi) \leq E[|Z(y)|] < \infty.$$

Let $\{\nu_i\}_{i=1}^{\infty}$ be a sequence of sample size rules such that

(5.15) $$\lim_{i \to \infty} W_t^T(y^t, (\xi, \nu_i)) = \overline{W}_t^T(y^t, \xi).$$

For any $i \geq 1$, define $\hat{\nu}_i \in \{\nu_1, \ldots, \nu_i\}$ as the sample size rule which provides the highest expected present valued utility of continued search; that is, $\hat{\nu}_i$ is such that, for any $i \geq 1$, (5.16) $W_t^T(y^t,(\xi,\hat{\nu}_i)) = \max\{W_t^T(y^t,(\xi,\nu_1)),\ldots,W_t^T(y^t,(\xi,\nu_i))\}$.

From (5.16) and (5.15),

(5.17) $\lim_{i\to\infty} W_t^T(y^t,(\xi,\hat{\nu}_i)) = \overline{W}_t^T(y^t,\xi)$.

Let

(5.18) $\nu^* := \lim_{i\to\infty} \hat{\nu}_i$

and notice that, together, (5.13) and (5.14) imply

(5.19) $W_t^T(y^t,(\xi,\hat{\nu}_i)) < \infty$ for all $i \geq 1$.

(5.14), (5.18) and (5.19) show $\{\hat{\nu}_i\}_{i=1}^{\infty}$ satisfies the conditions of Lesbesgue's Dominated Convergence Theorem. It follows from that theorem, (5.17) and (5.18) that

(5.20) $W_t^T(y^t,(\xi,\nu^*)) = W_t^T(y^t,(\xi,\lim_{i\to\infty}\hat{\nu}_i)) = \lim_{i\to\infty} W_t^T(y^t,(\xi,\hat{\nu}_i)) = \overline{W}_t^T(y^t,\xi)$.

That is, ν^* satisfies (5.12).

Q.E.D.

Since ν^* exists, subject to (5.2), it remains to combine ν^* with the following stopping rules and to prove the optimality of the resulting search rules.

<u>Definition 7</u>

$$(5.21)\quad S^* := \{\xi\epsilon S \mid \xi^t(y^t) = \begin{cases} 0; & \text{if } u^{*t} < W_t^T(y^t,(\xi,\nu)) \\ \begin{array}{l} 0 \text{ with probability } \rho^t(y^t) \text{ and} \\ 1 \text{ with probability } 1-\rho^t(y^t) \end{array} & ;\text{ if } u^{*t} = W_t^T(y^t,(\xi,\nu)) \\ 1; & \text{if } u^{*t} > W_t^T(y^t,(\xi,\nu)). \end{cases}$$

$$\forall t=1,\ldots,T \text{ and } \forall y^t \in Y^t \text{ and } \forall \nu \in \mathfrak{N}\}.$$

Denote the general element of $S*$ by ξ^* so that $\rho^* := (\xi^*, \nu^*)$ is the general element of the set $P^* := S^* \times \{\nu^*\}$; that is, P^* is the set of all the search rules formed by pairing ν^* with each of the stopping rules in S^*. It is clear from a comparison of (5.21), (5.3a) and (5.3b) that each ρ^* is regular. Hence $P^* \subset \mathcal{R}$. A consequence of this and Remark 1 is the following remark.

Remark 2

(5.22) For any t=1,...,T and for any $y^t \in Y^t$,

$$\sup_{\rho \in P^*} V_t^T(y^t, \rho) \leq \sup_{\rho \in \mathcal{R}} V_t^T(y^t, \rho) = \sup_{\rho \in P} V_t^T(y^t, \rho).$$

The last part remaining of the argument of this section is to show that any $\rho^* \in P^*$ is an optimal search rule.

Theorem 2

If $E[|Z(y)|] < \infty$ then P^* is a non-empty set of optimal search rules.

Proof: $P^* := S^* \times \{\nu^*\} \neq \phi$ since $S^* \neq \phi$ and $\{\nu^*\} \neq \phi$.

The rest of the proof must establish two points. These are that, for all t=1,...,T and for all $y^t \in Y^t$,

(5.23) $$V_t^T(y^t, \rho') = V_t^T(y^t, \rho''), \ \forall \rho', \rho'' \in P^*,$$

and that, for any $\rho^* \in P^*$,

(5.24) $$V_t^T(y^t, \rho^*) \geq \sup_{\rho \in P} V_t^T(y^t, \rho)$$

since (5.24) and (5.22) will together establish (3.13) for ρ^*. It is convenient to begin by establishing (5.24) first.

Take any $\rho^* \in P^*$. Using (3.12) and Theorem 1 shows that, for any t=1,...,T and for any $y^t \in Y^t$,

$$(5.25)\qquad V_t^T(y^t,\rho^*) \equiv V_t^T(y^t,(\xi^*,\nu^*)) \equiv \max\{u^{*t},W_t^T(y^t,(\xi^*,\nu^*))\}$$

$$\geq \max\{u^{*t},W_t^T(y^t,(\xi^*,\nu))\} \equiv V_t^T(y^t,(\xi^*,\nu)) \text{ for all } \nu \in \mathfrak{N}.$$

Using (3.10), (3.11) and (3.12) $V_t^T(y^t,(\xi^*,\nu))$ can be expanded, for any $\nu \in \mathfrak{N}$, as

$$(5.26)\qquad V_t^T(y^t,(\xi^*,\nu)) = \xi^{*t}(y^t)u^{*t} + (1-\xi^*(y^t))W_t^T(y^t,(\xi^*,\nu))$$

$$= \xi^{*t}(y^t)u^{*t} + (1-\xi^{*t}(y^t))(-K(n^t)+\beta E_{F^t}[\xi^{*t+1}(y^{t+1})u^{*t+1} +$$
$$(1-\xi^{*t+1}(y^{t+1}))(-K(n^{t+1})+\beta E_{F^{t+1}}[\xi^{*t+2}(y^{t+2})u^{*t+2}+\ldots+$$
$$(1-\xi^{*T-1}(y^{T-1}))(-K(n^{T-1})+\beta E_{F^{T-1}}[u^{*T}|y^{T-1}]\ldots|y^{t+1}]|y^t]\ .$$

Comparing each of the terms nested in (5.26) to (5.21) shows (5.26) is

$$(5.27)\qquad V_t^T(y^t,(\xi^*,\nu)) = \max\{u^{*t}, -K(n^t)+\beta E_{F^t}[\max\{u^{*t+1}, -K(n^{t+1})+\beta E_{F^{t+1}}[\max\{u^{*t+2},$$
$$\ldots,\ -K(n^{T-1})+\beta E_{F^{T-1}}[u^{*T}|y^{T-1}]\ldots|y^{t+1}]|y^t]\}$$

$$\geq \xi^t(y^t)u^{*t} + (1-\xi^t(y^t))(-K(n^t)+\beta E_{F^t}[\xi^{t+1}(y^{t+1})u^{*t+1} +$$
$$(1-\xi^{t+1}(y^{t+1}))(-K(n^{t+1})+\beta E_{F^{t+1}}[\xi^{t+2}(y^{t+2})u^{*t+2}+\ldots+$$
$$(1-\xi^{T-1}(y^{T-1}))(-K(n^{T-1})+\beta E_{F^{T-1}}[u^{*T}|y^{T-1}]\ldots|y^{t+1}]|y^t]$$

$$\equiv V_t^T(y^t,(\xi,\nu)) \text{ for all } \xi \in S\ .$$

Combining (5.25) and (5.27) shows

$$(5.28)\qquad V_t^T(y^t,\rho^*) \geq V_t^T(y^t,\rho),\ \forall \rho \in P$$

that is, ρ^* is optimal. Finally, (5.23) is easily established by noting that, since both ρ' and ρ'' are optimal,

$$V_t^T(y^t,\rho') \geq V_t^T(y^t,\rho'') \quad \text{and} \quad V_t^T(y^t,\rho') \leq V_t^T(y^t,\rho'').$$

Q.E.D.

This establishes the existence and form of each of the optimal search rules $\rho^* \epsilon P^*$. At each t the searcher deduces the size of the sample he should draw from X at t in order to maximize the expected present value of continued search. Having deduced this expected value, he continues his search by drawing this sample if the expectation exceeds the best terminal utility currently available, stops if the expectation is smaller than the best terminal utility and is indifferent between stopping and continuing when the expectation and the best terminal utility are equal. This procedure is well defined if (5.2) is satisfied.

The above existence proof is conceptually different from the proofs of existence of best sequential search rules given by Chow and Robbins [3], De Groot [4], Kohn and Shavell [7] and Yahav [14]. Their proofs utilize the fact that $\nu^t_{seq}(\cdot) \equiv 1$ for all t to establish that a best sequential search rule is the "most regular" sequential rule in that, for any given vector of observations, the best sequential rule extends search over at least as many periods as does any other regular sequential rule. The optimal search rule described above does not generally extend search over at least as many periods as any other regular search rule and, consequently, the above existence proof differs in its structure from those cited above. The reason that optimal search may last for fewer periods than best sequential search is that optimal search permits the searcher to gather observations more rapidly than does sequential search. This makes it more likely that an acceptable offer will be discovered sooner.

6. WHEN ARE THE BEST FSS AND SEQUENTIAL STRATEGIES OPTIMAL?

Since the fss and sequential search strategies have each received considerable attention in the literature, it is useful to discover sets of conditions which constrain ρ^* to either the fss strategy or the sequential strategy. In at least two instances, all three strategies are indistinguishable.

The optimal search rules contained in P^* are distinguished from each other only by the probability, $p^t(y^t)$, with which they order a continuation of search at t in the event of a tie between u^{*t} and $W_t^T(y^t,\rho^*)$ (see (5.21)). The previous literature has concentrated upon the particular best sequential search rule for which $p^t(\cdot) \equiv 0 \; \forall t$ i.e. always stop in the event of a tie. For comparability to the previous literature, this section confines its discussion to the optimal search rule $\rho^{**} \in P^*$ for which $p^t(\cdot) \equiv 0 \; \forall t$. This is merely a convenience since all the optimal rules in P^* have the same expected value.

The conditions which are necessary and sufficient for $\rho^{**} = \rho_{fss}$ are the conditions under which search will never continue past $t=2$, i.e.,

$$(6.1) \quad \max\{I^*(w^2), \max_{x \in G^2} I(x,w^2)\} \geq W_2^T(y^2,\rho^{**}), \; \forall \, y^2 \in Y^2.$$

$I^*(w^1)$ is the opportunity cost of searching. If it is low enough (the value of living through the winter in a cave for the three sisters of section 2) then search will occur at t=1 even in the face of high search costs or high rates of time preference, provided the searched for commodity (e.g. shelter) is sufficiently valuable. However, once any observation taken on X is received, at t=2, then the searcher gains the opportunity of obtaining the commodity. If marginal search costs or discounting are large enough to make the value of continuing to search no greater than the value of obtaining the commodity, then search stops at t=2. This is the meaning of (6.1). For example, with $\delta = 0.01$ the sisters will always choose to stop searching at t=2 if the cost of each quotation exceeds \$1790 since, under these conditions, the value of accepting even a quotation of \$60000 is $\frac{\$80000}{1.01} - \$60000 = \$19207.9$ while the expected value of continuing to search $W_2^3(\cdot,\rho^{**}) \leq \$19207.9$. A little arithmetic shows that under these conditions the optimal sample size $\nu^*(y^1)$ at t=1 is unity, so that optimal search consists of taking just one observation at t=1 and then stopping at t=2. Under these conditions, therefore, all three (optimal, fss and sequential) search strategies are indistinguishable. The same indistinguishability arises in the (trivial) case where it is sub-optimal to search at all since, if $I^*(w^1) \geq W_1^T(y^1,\rho^*)$, then $I^*(w^1) \geq W_1^T(y^1,\rho_{seq})$ and $I^*(w^1) \geq W_1^T(y^1,\rho_{fss})$ also.

Condition (6.1) may be forced upon a searcher in ways other than the above. For example, firms often search by asking for tenders because preparing tenders for commercial projects is usually a time consuming exercise. If the searching firm's contractural opportunities will vanish before a second sample of firms could supply further tenders, then the optimal search strategy is directly constrained to being the best fss strategy. Condition (6.1) is also more likely to arise when the searcher does not enjoy any recall privileges since, while $W_2^T(y^2,\rho^{**})$ decreases as recall privileges are withdrawn, $\max\{I^*(w^2), \max_{x \in \mathcal{C}^2} I(x,w^2)\}$ is unaffected.

Sequential search is the most time intensive of the three search strategies discussed in this paper. Consequently ρ_{seq} will usually be sub-optimal when T is small or δ is large. For example, when $T = 2$ and quotations cost \$1000 each, Deborah's sequential strategy has an expected value strictly below that of Dolores' fss strategy for any $\delta \geq 0$. Conversely, large T, small δ and large marginal intraperiod sampling costs for observations additional to the first will assist ρ_{seq} towards optimality. The following proposition confirms this intuition.

Proposition 2

If (i) $\delta = 0$, $T = \infty$,

(ii) the searcher has full recall,

(iii) $c(n), K(n) \geq 0$, convex and non-decreasing, and

(iv) $I(\cdot, w)$ is strictly increasing w.r.t. w,

then the optimal search rule is sequential.

Proof:

See the Appendix.

The set of sufficient conditions described in the above proposition is, of course, only a subset of the set of conditions sufficient for the best sequential strategy to be optimal. Nevertheless, the result suggests that sequential strategies may be sub-optimal in a wider class of search problems than seems to be commonly supposed. Simulation studies conducted by the authors suggest that ρ_{seq} is particularly likely to be sub-optimal when the future is discounted, even at low rates. On the other hand, the removal of full recall seems to have only a small effect on the expected value of ρ_{seq} when T is not small. Establishing conditions sufficient for ρ_{seq} to be optimal seems to be an elusive task unless particular functional forms are assumed.

7. CONCLUDING REMARKS

This paper has presented a number of fundamental results on the existence and properties of optimal search strategies for a wide class of problems in which the searcher may choose both the number of samples taken and the size of each sample. These results are of considerable generality but do not, of course, extend to problems outside the set of search problems considered here. While the more important aspects of search are captured by the problems addressed, some search phenomena have been excluded e.g. uncertain recall, uncertain decision horizons and consumption of rewards as search proceeds. Finally, the degree of suboptimality of sequential and fixed-sample-size search rules is not described here for every problem considered. Results concerning the degree of suboptimality will be specific to the functional forms and parameter values of each problem. Simulation studies conducted by the authors show the expected value of the optimal variable sample size strategies can substantially exceed the expected values of the best sequential and best fixed-sample-size strategies in important cases.

FOOTNOTES

[1]This article was written while Morgan was on sabbatical leave from the Flinders University of South Australia at the University of Western Ontario. The authors wish to acknowledge helpful comments received from Noel Cressie, John Darroch, Alan Harrison, John McDonald, John McMillan and Victoria Zinde-Walsh. Any errors remain our responsibility.

[2]Other assumptions are possible. For example, it may take more than one period for observations to be received. In such cases the decision to stop or to continue search will be made in partial ignorance of the outcomes of previously taken search decisions. The model presented here extends naturally to such cases.

[3]A point of potential confusion should be avoided here. If, for some y^t, $\nu^t(y^t) = n^t > 0$ then the searcher is committed to taking n^t further observations on X only if $\xi^t(y^t) = 0$ also. After all, if $\xi^t(y^t) = 1$ (i.e. search stops at t) it is clearly not welfare improving to obtain another n^t (valueless) observations on X.

[4]This utility function should be interpreted as the direct or indirect utility function appropriate to whichever more specific formulation of the problem is of interest to the reader. For an example of one such formulation see [9].

[5]It is straightforward, but of little economic interest, to allow c(•) and K(•) to differ across periods.

[6]Uncertain recall is not admitted in this model. For sequential search models with uncertain recall see [6] and [8].

[7]With one exception (see [1]), previous search models assume the "fall-back reward" $I^*(w^t)$ is independent of the search activity. In many cases this is not so. For instance, if search is wealth reducing then the search activity reduces the wealth available for purchases of other than searched for commodities--in which case $I^*(w^t)$ is decreasing w.r.t. t.

APPENDIX

This appendix contains the proofs of Proposition 2 and Lemma A-1, a result used in the proof of Proposition 2.

Lemma A-1

Let a, b, c, be random variables defined over R^1 with c.d.f.'s F, G, H respectively. Then

$$E_{FGH}[\max\{\max(a,b),c\}] \le E_F[\max\{a, E_{GH}[\max\{\max(a,b),c\}|a]\}].$$

Proof

$$E_{FGH}[\max\{\max(a,b),c\}] = \int_{-\infty}^{\infty} E_{GH}[\max\{\max(a,b),c\}|a]dF(a)$$

$$\le \int_{-\infty}^{\infty} \max\{a, E_{GH}[\max\{\max(a,b),c\}|a]\}dF(a).$$

Q.E.D.

Proof of Proposition 2

The result is trivial if $c(n) \equiv K(n) \equiv \delta \equiv 0$ since then any regular search strategy will continue search until a terminal utility level of $\max\{I^*(w^1), \max_{x \in X} I(x,w^1)\}$ is achieved, i.e., for any $y^t \in Y^t$ and any $t \ge 1$,

(A.1) $V_t^T(y^t,\rho) = \max\{I^*(w^1), \max_{x \in X} I(x,w^1)\}, \forall \rho \in \mathcal{R}.$

Since $\rho_{seq} \in \mathcal{R}$ and $P^* \subset \mathcal{R}$, (A.1) implies ρ_{seq} is optimal.

Now suppose either $c(n) \ge 0$ or $K(n) \ge 0$ with $>$ for $n \ge 1$. The following argument establishes that if optimal search is continued at t then $n^{*t} = 1$. The argument begins by establishing that if optimal search is continued at t then $n^{*t} \ge 1$. Suppose instead that $n^{*t} = 0$. Then, since he has full recall, the searcher's problem at t+1 is exactly the same as at t, so

$n^{*t+1} = 0$ also. Continuing this argument ad infinitum shows $n^{*t} = 0$ $\Rightarrow n^{*t+1} = n^{*t+2} = \ldots = 0$. Hence

$$V_t^T(y^t, \rho^{**}) = u^{*t} \Rightarrow \xi^{**t}(y^t) = 1$$

i.e. if $n^{*t} = 0$, optimal search stops at t. This establishes that optimal search continues at t only if

$$\text{(A.2)} \quad n^{*t} \geq 1.$$

The residue of the proof shows that if optimal search continues at t then $n^{*t} \leq 1$, from which, with (A.2), $n^{*t} = 1$ follows immediately.

Writing $V_t^T(y^t, \rho^{**}) \equiv V_t^T(y^t, (\xi^{**}, \nu^{**}))$ in the form of (5.27) with $\delta = 0$ and $T = \infty$ shows, for any $y^t \in Y^t$ and any $t \geq 1$,

$$\text{(A.3)} \quad V_t^T(y^t, \rho^{**}) = \max\{u^{*t}, -K(n^{*t}) + E_{F^t}[\max\{u^{*t+1}, -K(n^{*t+1}) + E_{F^{t+1}}[\max\{u^{*t+2}, \ldots\} | y^{t+1}]\} | y^t]\}$$

$$= \max\{u^{*t}, E_{F^t}[\max\{u^{*t+1} - K(n^{*t}), E_{F^{t+1}}[\max\{u^{*t+2} - K(n^{*t}) - K(n^{*t+1}), \ldots\} | y^{t+1}]\} | y^t]\}.$$

It is notationally convenient to normalize the searcher's utility function so that $K(0) = 0$. $K(n)$ is non-negative, convex and non-decreasing so

$$K(1) + K(n-1) \leq K(n), \text{ for } n \geq 1.$$

Using this in (A.3) gives

$$\text{(A.4)}\quad V_t^T(y^t,\rho^{**}) \le \max\{u^{*t}, E_{F^t}[\max\{u^{*t+1} - K(1) - K(n^{*t}-1), E_{F^{t+1}}[\max\{u^{*t+2} - K(1) - K(n^{*t}-1) - K(n^{*t+1}),\ldots\}|y^{t+1}]\}|y^t]\}.$$

At time $t \geq 2$ the searcher's wealth is $w^t = w^1 - \sum_{j=1}^{t-1} c(n^{*j})$. $c(n)$ is non-negative, convex and non-decreasing so

$$\text{(A.5)}\quad c(1) + c(n^{*t}-1) \le c(n^{*t}).$$

Substituting from (3.9) and (A.5) into (A.4) gives

$$\text{(A.6)}\quad V_t^T(y^t,\rho^{**})$$

$$\le \max\{u^{*t}, E_{F^t}[\max\{\max(I^*(w^t-c(1)-c(n^{*t}-1)), \max_{x\in G^{t+1}} I(x,w^t-c(1)-c(n^{*t}-1))) - K(1) - K(n^{*t}-1),$$

$$E_{F^{t+1}}[\max\{\max(I^*(w^t-c(1)-c(n^{*t}-1)-c(n^{*t+1})), \max_{x\in G^{t+2}} I(x,w^t-c(1)-c(n^{*t}-1)-c(n^{*t+1})))$$

$$-K(1) - K(n^{*t}-1) - K(n^{*t+1}),\ldots\}|y^{t+1}]\}|y^t]\}.$$

Note that $G^{t+1} = G^t \cup \{x_1^t,\ldots,x_{n^{*t}}^t\}$. Rewriting (A.6) in the form of the expectation described in Lemma A-1 gives

$$\text{(A.7)}\quad V_t^T(y^t,\rho^{**}) \le \max\{u^{*t}, E_{F^t}[\max\{\underbrace{\max(I^*(w^t-c(1)-c(n^{*t}-1)), \max_{x\in G^t\cup\{x_1^t\}} I(x,w^t-c(1)-c(n^{*t}-1)))}_{a}$$

$$\underbrace{-K(1) - K(n^{*t}-1)}, \underbrace{\max(I^*(w^t-c(1)-c(n^{*t}-1)), \max_{x\in G^t\cup\{x_1^t,\ldots,x_{n^{*t}}^t\}} I(x,w^t-c(1)-c(n^{*t}-1)))}_{b}$$

$$\underbrace{-K(1) - K(n^{*t}-1)}, \underbrace{E_{F^{t+1}}[\cdot]}_{c}\}|y^t]\}.$$

Apply Lemma A-1 to $E_{F^t}[\ldots]$ in (A.7) as indicated to show

$$(A.8)\quad V_t^T(y^t,\rho^{**}) \le \max\{u^{*t}, E_{F^t}[\max\{\max(I^*(w^t-c(1)-c(n^{*t}-1)), \max_{x\in G^t\cup\{x_1^t\}} I(x,w^t-c(1)-c(n^{*t}-1)))$$

$$-K(1)-K(n^{*t}-1), E_{F^{t+1}}[\max\{\max(I^*(w^t-c(1)-c(n^{*t}-1)),$$

$$\max_{x\in G^t\cup\{x_1^t,\ldots,x^t_{n^{*t}}\}} I(x,w^t-c(1)-c(n^{*t}-1))) - K(1)-K(n^{*t}-1), E_{F^{t+2}}[\cdot]\}|(y^t,x_1^t)]\}|y^t]\}$$

$$\le \max\{u^{*t}, E_{F^t}[\max\{\max(I^*(w^t-c(1)), \max_{x\in G^t\cup\{x_1^t\}} I(x,w^t-c(1))) - K(1),$$

$$E_{F^{t+1}}[\max\{\max(I^*(w^t-c(1)-c(n^{*t}-1)), \max_{x\in G^t\cup\{x_1^t,\ldots,x^t_{n^*_t}\}} I(x,w^t-c(1)-c(n^{*t}-1)))$$

$$-K(1)-K(n^{*t}-1), E_{F^{t+2}}[\cdot]\}|(y^t,x_1^t)]\}|y^t]\}.$$

Suppose $n^{*t} \ge 2$. Then the second inequality in (A.8) is strict since then $n^{*t}-1 \ge 1$ and so either $c(n^{*t}-1) > 0$ or $K(n^{*t}-1) > 0$. But this means the (assumed optimal) sampling strategy of taking $n^{*t} \ge 2$ observations on X at t has an expected value strictly less than the sampling strategy of taking exactly one observation on X with the option of taking the remaining $n^{*t}-1$ observations at t+1. This contradicts the optimality of ρ^{**} (and of all other rules $\rho^* \in P^*$ since all the rules in P^* have the same expected value). Hence

$$(A.9)\quad n^{*t} \le 1.$$

(A.2) and (A.9) together establish

$$(A.10)\quad n^{*t} = 1$$

i.e. optimal search continues at t if and only if it is optimal to draw <u>exactly one</u> additional observation on X at t: the optimal strategy is sequential.

Q.E.D.

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