1982

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Citation of this paper:

Davies, James B.. "Intergenerational Transfers, Redistribution, and Inequality." Department of Economics Research Reports, 8206. London, ON: Department of Economics, University of Western Ontario (1982).
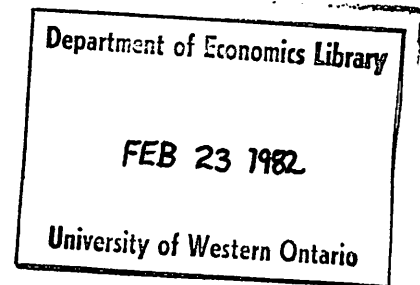
RESEARCH REPORT 8206

INTERGENERATIONAL TRANSFERS,
REDISTRIBUTION, AND INEQUALITY

by

James B. Davies

February, 1982

I. Introduction

It has long been thought that inheritance exerts an unambiguously disequalizing force. Although there has not been similar unanimity concerning the long-term impact of redistribution, at least a sizeable group of economists have believed it favorable. Recently both positions have come under attack in steady-state models of income distribution embodying altruistically-motivated intergenerational transfers.

Both Laitner (1979a and b) and Becker and Tomes (1979) show that inheritance is unambiguously <u>equalizing</u>. Further, Becker and Tomes show by examples that in at least some cases a linear redistributive tax-transfer scheme can have a zero or positive effect on steady-state income inequality.[1] In fact they conjecture (p. 1178) that the failure of attempts at redistribution to reduce measured inequality in income very much in the U.S. over the last half century might well be due to the operation of the intergenerational mechanisms that can produce this perverse effect.

Predictions of an equalizing impact of inheritance and a perverse effect of redistribution may both appear provocative. However, it is not surprising that with altruistic transfers intergenerational consumption-smoothing reduces <u>intra</u>generational inequality. What is not clear is whether the perverse effect of redistribution is also a "natural" outcome. This paper demonstrates this is not the case.

The paper shows that a perverse effect of redistribution depends on a severe decline in the scale of inheritances and (therefore) mean lifetime wealth. A reduction in the scale of inheritance reduces the scope for its (unambiguously) equalizing effect. The drop in mean lifetime wealth ensures that the amount available for redistribution will rise only slightly, or

actually decline. (It is neither necessary nor sufficient for a perverse impact, however, to be beyond the peak of the "Laffer curve".) It is argued here that with the most "realistic" parameters the fall in inheritances and lifetime wealth induced by redistribution is not sufficiently severe to produce a disequalizing result.

There are two principal factors that determine the sharpness of the drop in mean inheritance brought on by increased redistribution. One is the elasticity of substitution in parents' utility functions between their own consumption and their children's lifetime wealth. The higher this elasticity, the greater is the drop in equilibrium mean inheritance. The other factor is the level of the interest rate relative to the rate of growth of the economy. The higher the interest rate, the greater is the impact of a given absolute change in parents' consumption on children's inheritances. In consequence, the more sensitive is mean inheritance to changes in redistribution.

Both independent empirical evidence and comparisons of alternative equilibria of the model suggest that an intergenerational elasticity of substitution below unity is the "leading case." With an elasticity below unity a disequalizing impact of redistribution can only be obtained if the interest rate exceeds the growth rate by some critical amount. Illustrative computations show that the critical interest rates range from the moderately, to the implausibly high. Hence, with "real-world" parameters in the present model redistribution is equalizing. The explanation for the failure of redistribution in the U.S. over the last 50 years must apparently be sought elsewhere.

The paper proceeds as follows. First we set out the determinants of family income, in Section II. Lifetime earnings are determined by wealth-maximizing investment in human capital. In addition to earnings people receive inheritances in non-human form, and either pay taxes or receive transfers according to a linear tax-transfer scheme based on pre-tax lifetime wealth. Rental rates on both human and non-human capital are exogenous; that is, we examine a "small open economy".

In Section III the consumption choice problem is outlined and the impact of taxation on desired transfers is investigated. We are then able, in Section IV, to study the determinants of equilibrium growth, making use of the aggregate consumption function implied by Section III. Section V solves differential equations for income and consumption. The latter are written as functions of past shocks to earnings ability and the various parameters of the model. It is a short step to derive the equilibrium means, variances, and coefficients of variation.

Section VI studies the impact of changes in the attempted scale of redistribution on equilibrium mean income and consumption, and inequality. The magnitude of these effects is studied in Section VII, which presents examples of alternative equilibria.

II. Determinants of Family Income

We consider a model in which reproduction is asexual. A single parent has a single child. Thus the roles of differential fertility, estate division, and mating are not considered. In the absence of government, a member of generation t would have lifetime wealth R_t, composed of lifetime earnings and inheritances, E_t and I_t respectively. The former is the product of the fraction of time devoted to working, $1 - s_t$, the human capital stock, H_t^*, and the rental rate on human capital, w_t:

$$(1) \qquad E_t = (1 - s_t)w_t H_t^*$$

where s_t = the fraction of time spent in human capital formation. The inheritance, on the other hand is

$$(2) \qquad I_t = rB_{t-1} = r(R_{t-1} - C_{t-1})$$

where B_{t-1} is the bequest left by the previous generation and r is one plus the interest rate. The rental on non-human capital, r, is assumed constant for all time, while that on human capital grows at the constant percentage rate $q - 1$. Factor prices are given from outside the model. In other words we examine a "small open economy".

In this model lifetime earnings are the result of optimal investment in human capital in a one period setting.[2] Although, since there is only one period, one does not need to borrow to finance schooling, the absence of borrowing difficulties means that the model has a perfect capital market spirit. With a fixed amount of time available for learning or earning the individual will maximize (1) by choosing an optimal s_t. The first-order condition is

(3) $\quad \dfrac{\partial E_t}{\partial s_t} = (1 - s_t)w_t \dfrac{\partial H^*}{\partial s_t} - w_t H^* = 0.$

In other words one selects s_t to equalize the addition to post-school earnings and the (foregone-earnings) cost of education at the margin.[3]

A crucial element in our model is the relationship between the E's of successive generations. If one wants a stationary distribution of E_t (except for scale) and imperfect heritability, the only tractable formulation appears to be linear regression to the mean. Pursuing the discussion in terms of _net_ human capital, $H_t = (1 - s_t)H_t^*$, we require

(4) $\quad H_t = (1 - v)\bar{H} + vH_{t-1} + \epsilon_t$

where ϵ_t is a random term with mean zero and finite variance and $\bar{H}$ is the constant mean net human capital.[4] For stationary variance in H_t, $V(H_t)$, we require

(5) $\quad V(H_t) = \dfrac{V(\epsilon_t)}{1 - v^2}$

To obtain (4) as the result of maximizing behavior we may assume that "ability", A_t, is subject to the same kind of linear regression:

(6) $\quad A_t = (1 - v_A)\bar{A} + v_A A_{t-1} + \epsilon_t^A$

and that for given s_t, human capital production is proportional to ability. For example, suppose

(7) $\quad H_t^* = \alpha s_t^{\xi} A_t, \qquad 0 \leq \xi \leq 1$

Then optimal s_t is uniform for all at

$$s_t = \dfrac{\xi}{\xi + 1}$$

and we obtain (4) directly, where

$$v = \alpha \left(\frac{\xi}{\xi+1}\right)^{\xi} v_A$$

and

$$\epsilon_t = \alpha \left(\frac{\xi}{\xi+1}\right)^{\xi} \epsilon_t^A.$$

The degree of heritability of earnings, given by v, plays an important role in the subsequent analysis. Note that v^2 is the proportion of the variance of H_t explained by variation in H_{t-1}. Empirical estimates suggest a value of v^2 in the range $0.25 - 0.5$, that is $0.5 \leq v \leq .707$.[5]

Finally, how does the government redistribute income? It is assumed to impose a linear redistributive tax-transfer scheme based on lifetime wealth. (Such a scheme, of course, has no impact on optimal human capital investment.) Lifetime wealth after-tax is therefore:

$$(8) \qquad R_t = (1 - u)(E_t + I_t + G_t)$$

where u is the constant marginal tax rate and $(1 - u)G_t$ is guaranteed lifetime wealth (provided $E_t + I_t \geq 0$). Denoting mean values of E_t and I_t, $\bar{E}_t$ and $\bar{I}_t$ respectively, the self-financing level of G_t is clearly

$$G_t = u(\bar{E}_t + \bar{I}_t + G_t)$$

or

$$(9) \qquad G_t = \left(\frac{u}{1-u}\right)(\bar{E}_t + \bar{I}_t).$$

Substituting (9) into (8), and taking expectations, we find

$$(10) \qquad \bar{R}_t = \bar{E}_t + \bar{I}_t$$

so that (9) may also be written:

$$(9') \qquad G_t = \left(\frac{u}{1-u}\right)\bar{R}_t.$$

III. Determinants of Transfers

We assume that the utility of generation t is an additively separable iso-elastic function of the consumption of the current generation, C_t, and the lifetime wealth of the succeeding generation, R_{t+1}:[6]

$$(11) \qquad U_t = \begin{cases} \dfrac{C_t^{1-\gamma}}{1-\gamma} + \beta \dfrac{R_{t-1}^{1-\gamma}}{1-\gamma} \quad , & \gamma \neq 1 \\[2em] \ell n C_t + \beta \ell n R_{t+1} \quad , & \gamma = 1 \end{cases}$$

where β reflects the strength of altrusim, and γ is the inverse of the elasticity of substitution between parental consumption and children's income, σ. We find below that whether this elasticity is above, below, or equal to unity is one principal determinant of the impact of the redistributive scheme. There is a certain amount of empirical evidence which suggests $\sigma < 1$ is most "realistic".

The parent maximizes (11) subject to a constraint relating expenditure on C_t and R_{t+1} to total resources. Let the pre-tax lifetime wealth of generation t before transfers be denoted

$$(12) \qquad \underline{R}_t^m = E_t + G_t.$$

Substituting (2) and (12) into (8)[7]

$$(13) \qquad R_{t+1} = (1 - u)[\underline{R}_{t+1}^m + r(R_t - C_t)].$$

Pre-tax lifetime wealth of generation t+1 is

$$(14) \qquad R_{t+1}^m = \frac{R_{t+1}}{(1-u)}$$

Using (14) we may write the constraint (13) in more familiar form:

(13′) $c_t + \dfrac{R^m_{t+1}}{r} = R_t + \dfrac{R^m_{t+1}}{r}$.

Maximizing (11) subject to (13′), from the first order conditions we obtain

(15) $c_t = \theta [R_t + \dfrac{R^m_{t+1}}{r}] = \theta Z_t$

where Z_t may be referred to as "family resources", and

(16) $\theta = \{1 + \beta^{1/\gamma}[r(1-u)]^{\frac{1-\gamma}{\gamma} - 1}\}$.

The consumption plan is thus one in which a fraction θ of family resources is appropriated for the consumption of the parent. This fraction falls with increases in the degree of altruism, β. It is related in a more complex way to changes in r and u and differences in γ. What is most important in this paper is how it reacts to a change in u:

(17) $\dfrac{\partial \theta}{\partial u} = \theta \left(\dfrac{1-\theta}{1-u}\right)\left(\dfrac{1-\gamma}{\gamma}\right) \begin{cases} > 0 , & \gamma < 1 \\ = 0 , & \gamma = 1 \\ < 0 , & \gamma > 1 \end{cases}$

It is demonstrated later that it is the $\gamma < 1$ case which is most conducive to a perverse effect of redistribution. Intuitively, one can see at this stage that this makes sense. If θ actually _rises_ when u increases, inter-generational transfers must surely decline in aggregate. This will be dis-equalizing for the reasons already outlined in the introduction.

Finally, we should comment on the fact that under (15) $c_t > R_t$, and therefore $I_{t+1} < 0$, is possible. This does not offend on the grounds of realism. Negative bequests--transfers from child to parent--are often observed (as pointed out, e.g., by Shorrocks, 1979, p. 419). If one extends the definition to include in-kind transfers--meals, visits, etc.--it seems

clear that the balance of transfers over a lifetime may frequently be from child to parent. The reason that negative transfers are upsetting is that they depend on the child's volition, yet we have not assumed any altruism on the part of the child toward his parent. This anomaly can only be removed by entering the consumption of all generations in the utility function as in Laitner (1979a and b) and Lowry (1981). The justification for using the simpler approach adopted here is discussed in footnote 6.

IV. Equilibrium Growth

We assume the rental rate on human capital grows at the constant percentage rate q - 1. For stationarity we require mean resources and consumption, $\bar{R}_t$ and $\bar{C}_t$ to grow at the same rate. This implies constant growth for $\bar{I}_t$ and G_t as well. The question we first address is what relationship between aggregate consumption and life wealth must hold for equilibrium growth to proceed.

Mean life wealth, $\bar{R}_t$, equals the sum of $\bar{E}_t$ and $\bar{I}_t$, or

$$\bar{R}_t = \bar{E}_t + r(\bar{R}_{t-1} - \bar{C}_{t-1}).$$

If both $\bar{R}_t$ and $\bar{C}_t$ grow at the rate q - 1 this can be written:

$$\bar{R}_t = \bar{E}_t + \frac{r}{q}(\bar{R}_t - \bar{C}_t).$$

The required relationship between $\bar{C}_t$ and $\bar{R}_t$ simplifies to

$$(18) \qquad \bar{C}_t = (\frac{q}{r})\bar{E}_t + (1 - \frac{q}{r})\bar{R}_t$$

This relationship is illustrated in Figure 1 for the r > q case. Since $\bar{R}_t = \bar{E}_t + \bar{I}_t$, the vertical distance between the 45° line and $\bar{E}_t$ gives equilibrium $\bar{I}_t$. As shown, with $\bar{I}_t = 0$ the only possible equilibrium is one where $\bar{C}_t = \bar{E}_t$. (This is also true with r ≤ q.) In other words, in order to preserve $\bar{I} = 0$, in each generation all resources must be consumed. Thus when r/q changes, the stationary condition pivots around the $(\bar{E}_t, \bar{E}_t)$ point.

Note that when $r = q$ the stationary condition requires simply that $\bar{C}_t$ must equal $\bar{E}_t$, irrespective of $\bar{R}_t$. The reason is that any $\bar{I}_t$ will grow at the rate q, since $r = q$, as long as there are no fresh additions to the stock of inherited wealth in any generation. Hence with $r = q$, consumption must exhaust earnings, but leave inherited wealth untouched in each generation.

The relationship between $\bar{C}_t$, $\bar{I}_t$ and $\bar{R}_t$ when $\bar{C}_t \neq \bar{E}_t$ can be studied with the help of Figure 2. Here we plot $\bar{C}_t - \bar{E}_t$, that is what might be referred to as "consumption out of inheritances", against $\bar{I}_t$. In all cases $\bar{C}_t - \bar{E}_t$ is simply a multiple, $1 - \frac{q}{r}$, of $\bar{I}_t$. In other words, with given r/q, there is a particular fraction of this generation's inheritance that must be consumed in order that $\bar{I}_{t+1} = q\bar{I}_t$, and this fraction is __independent__ of the level of $\bar{I}_t$. A higher r/q naturally raises this required fraction of consumption.

In order to characterize equilibrium growth fully we must examine the behavioral relation between $\bar{R}_t$ and $\bar{C}_t$. Using (12), (15) gives the "consumption function"

$$\bar{C}_t = \theta(\bar{R}_t + \frac{\bar{E}_{t+1} + G_{t+1}}{r})$$

Since $\bar{E}_t$ and $\bar{R}_t$ grow at the rate q, using (9') this may be written

$$\bar{C}_t = \theta\left[\bar{R}_t + \frac{\bar{E}_t + q(\frac{u}{1-u})\bar{R}_t}{r}\right]$$

which simplifies to

(19) $\qquad \bar{C}_t = \theta \{ [1 + \frac{qu}{r(1-u)}]\bar{R}_t + (\frac{q}{r})\bar{E}_t \}.$

This expression has a ready interpretation. Parents consume part of their own resources, on average $\theta\bar{R}_t$. In addition they project the resources of the next generation, believing that

$$\bar{E}_{t+1} = q\, \bar{E}_t$$

and $\qquad G_{t+1} = q\, G_t = q(\frac{u}{1-u})\bar{R}_t$

These amounts appear in (19) discounted at the rate r.

The unique equilibrium relationship between $\bar{C}_t$ and $\bar{R}_t$ is given by the point where the line representing the consumption function (19) intersects that based on (18). Figures 3 and 4 illustrate this for the $r = q$ and $r > q$ cases, respectively.

Figures 3 and 4 make clear just how θ, $\frac{r}{q}$, and u affect $\bar{R}_t$ and $\bar{C}_t$. An increase in the "propensity to consume", θ, out of family resources shifts the consumption function upward and increases its slope. Thus an increased propensity to consume lowers equilibrium $\bar{R}_t$. Note that a positive effect on $\bar{C}_t$ can, however, be obtained. When $r < q$ (the case not shown in the diagrams here) the stationary condition is downward-sloping. The decline in $\bar{C}_t$ as $\bar{R}_t$ falls to the new equilibrium in this case halts before $\bar{C}_t$ returns to its original level, since the stationary condition requires <u>higher</u> $\bar{C}_t$ at lower $\bar{R}_t$.

An increase in $\frac{r}{q}$ shifts both behavioral and stationary conditions down. While it <u>lowers</u> the slope of the consumption function, it <u>increases</u> that of the stationary condition. Equilibrium $\bar{R}_t$ therefore rises. (Figures 3 and 4, as drawn, differ only in that the latter uses a higher value of r/q. The impact on equilibrium $\bar{R}_t$ is evident.)

Figures 3 and 4 also allow us to anticipate the analytic results of Section VI on the impact of a change in u on aggregates. When $\gamma = 1$, θ is unaffected by changes in u. It is therefore clear from (19) that a rise in u rotates the consumption function upward. In all cases this reduces equilibrium $\bar{R}_t$. Hence redistribution leads to an unambiguous decline in wealth in this model when $\gamma = 1$. However, the impact on $\bar{C}_t$ is in the same direction in only one case. When $r > q$, $\bar{C}_t$ declines but with $r = q$ there is no change. (With $r < q$, $\bar{C}_t$ actually increases.) In Section VI we find that all these results hold for any value of γ.

We find, in Section VI, that the levels of γ and r/q are the crucial determinants of whether increased redistribution is actually equalizing. It is demonstrated that what is required for a disequalizing impact is a <u>large</u> effect of u on equilibrium $\bar{R}_t$. Figures 3 and 4 allow us to see why lower values of γ (i.e., high σ's), and high values of r/q produce the largest values of $\left|\frac{\partial \bar{R}_t}{\partial u}\right|$, and are therefore least conducive to an equalizing effect.

We showed above that increases in both u and θ lower $\bar{R}_t$. It is therefore clear that $\left|\frac{\partial \bar{R}_t}{\partial u}\right|$ will be greatest when $\frac{\partial \theta}{\partial u} > 0$, or (from 17) $\gamma < 1$. The impact of $\frac{r}{q}$ is a little more complicated.

Figures 3 and 4 compare the effects of raising u in the $r = q$ and $r > q$ cases. In fact the diagrams were drawn on the assumptions that $\bar{E}_t = 1.0$, $\theta = \frac{1}{2}$ (and $\gamma = 1$ so that $\frac{\partial \theta}{\partial u} = 0$), and u rises from 0 to 0.5. In Figure 3 the no-tax equilibrium $\bar{R}_t = 1.0$, while the 50% tax rate lowers steady-state $\bar{R}_t$ to 0.5. On the other hand, in Figure 4, with $\frac{r}{q} = 1.5$, the increase in u takes $\bar{R}_t$ from 2.0 to 2/3, a greater change in proportional (and absolute) terms.

Why is the proportional impact of a rise in u on equilibrium $\bar{R}_t$ greater with higher r/q? The diagrams show that raising r/q has two effects. First, the slope of the behavioral condition actually becomes less sensitive to a change in u. By itself this would make $\left|\frac{\partial \bar{R}_t}{\partial u}\right|$ lower with higher r/q. Second, however, the stationary condition becomes steeper. The effect this has is particularly evident in the comparison of Figures 3 and 4.

The inclined stationary condition of Figure 4 means that when u rises it is not sufficient for $\bar{R}_t(\bar{I}_t)$ to drop until $\bar{C}_t$ returns to its original level. In Figure 3 this would have returned $\bar{C}_t - \bar{E}_t$ to zero and equilibrium would obtain once more. With the inclined stationary condition of Figure 4, however, when $\bar{C}_t$ has fallen to its original level $\bar{C}_t-\bar{E}_t/\bar{I}_t$ will, of course, exceed to unique equilibrium values. A further decline in $\bar{R}_t(\bar{I}_t)$ must take place so that $\bar{C}_t$ can fall sufficiently to restore the required value of $\bar{C}_t-\bar{E}_t/\bar{I}_t$.

More intuitively, the higher is $\frac{r}{q}$ the greater is the change in equilibrium $\bar{C}_t$ associated with a change in $\bar{I}_t$. This is just a consequence of the greater fecundity of saving. Think for a moment in terms of disequilibrium dynamics. With $\bar{C}_t$ out of equilibrium--say too high due to a rise in u-- the first step in adjustment would give $\bar{I}_{t+1} < q\bar{I}_t$, and likely $\bar{I}_{t+1} < \bar{I}_t$. A process of declining mean inheritance is set in motion. With each decline in $\bar{I}$, however, high r/q gives a large change in the equilibrium value of $\bar{C}$. With a stable consumption function this means that the change in $\bar{I}$ and $\bar{R}$ required to bring desired $\bar{C}$ down to equilibrium $\bar{C}$ will be large. At bottom this is due to the fact that a dollar's worth of saving buys a greater

increase in child's inheritance with a higher interest rate, which is what is behind the steeper stationary condition with higher r/q.

Finally in this section we may solve (18) and (19) for the equilibrium values of $\bar{R}_t$, $\bar{C}_t$ and $\bar{I}_t$. First we have

(20) $\qquad \bar{R}_t = [\dfrac{(1-\theta)(1-u)}{(1-\delta) - (1-\theta)u}]\bar{E}_t = [\dfrac{(1-\theta)(1-u)}{\phi}]\bar{E}_t$

where $\delta = \dfrac{r}{q}(1-\theta)(1-u)$. It is easy to confirm that (20) implies $\dfrac{\partial \bar{R}_t}{\partial \theta} < 0$, $\dfrac{\partial \bar{R}_t}{\partial(r/q)} > 0$, and $\dfrac{\partial \bar{R}_t}{\partial u} < 0$ (in the θ fixed case), agreeing with the diagrammatic analysis. Mean consumption, on the other hand, is

(21) $\qquad \bar{C}_t = \dfrac{q\theta}{r\phi}\,\bar{E}_t$

Rewriting ϕ as

$$\phi = 1 + (1-\theta)[u(\dfrac{r}{q}-1) - \dfrac{r}{q}]$$

it is clear that (21) agrees with the diagrammatic result that θ fixed,

$$\dfrac{\partial \bar{C}_t}{\partial u} \gtrless 0 \quad \text{as} \quad \dfrac{r}{q} \lessgtr 1$$

Since $\bar{I}_t = \dfrac{r}{q}(\bar{R}_t - \bar{C}_t)$ we may use (20) and (21) to derive

(22) $\qquad \bar{I}_t = (\dfrac{\delta-\theta}{\phi})\,\bar{E}_t$

It is readily confirmed that $\dfrac{\partial \bar{I}_t}{\partial u} < 0$ (holding θ constant), $\dfrac{\partial \bar{I}_t}{\partial \theta} < 0$, and $\dfrac{\partial \bar{I}_t}{\partial(\frac{r}{q})} > 0$ as the foregoing diagrammatic analysis implied. Note also that

$$\bar{I}_t \gtrless 0 \quad \text{as} \quad \delta = \dfrac{r}{q}(1-\theta)(1-u) \gtrless \theta$$

or

$$\bar{I}_t \gtrless 0 \quad \text{as} \quad \theta \lessgtr \dfrac{r(1-u)}{q + r(1-u)}$$

Since $\dfrac{r(1-u)}{q + r(1-u)} \leq \dfrac{r}{q+r}$, and the latter fraction may itself not be very high, it is clear that negative mean inheritance is more than a curiosity in this model. This is especially true with $u > 0$.

V. Equilibrium Inequality

In this section I show that under certain parameter restrictions the model gives distributions of lifetime earnings, wealth, and consumption which all have finite stationary coefficients of variation.[8] The next section performs comparative static exercises with the results.

Lifetime Wealth:

"Money" lifetime wealth, R_t^m, consists of E_t, G_t, and I_t. Noting that $E_t = w_t H_t$, and recalling (15) this may be written

$$(23) \qquad R_t^m = w_t H_t + G_t + (1-u)(1-\theta)rR_{t-1}^m - \theta(w_t H_t + G_t)$$

$$= (1-\theta)[w_t H_t + G_t + (1-u)rR_{t-1}^m].$$

Solving this first-order difference equation we find

$$R_t^m = (1-\theta) \sum_{i=o}^{\infty} [r(1-\theta)(1-u)]^i (w_{t-i}H_{t-i} + G_{t-i}), \quad r(1-\theta)(1-u) < 1.$$

We can express this relationship in terms of current parameters and past random shocks by substituting in

$$(24) \qquad H_{t-i} = \bar{H} + \sum_{j=o}^{\infty} v^j \epsilon_{t-i-j}$$

and noting from (9) and (14) that

$$(25) \qquad G_{t-i} = \frac{u}{q^i} \bar{R}_t^m .$$

We now have

$$(26) \qquad R_t^m = (1-\theta) \sum_{i=o}^{\infty} \delta^i \{ (w_t \bar{H} + u\bar{R}_t^m) + w_t \sum_{j=o}^{\infty} v^j \epsilon_{t-i-j} \} .$$

The expectation of $\epsilon_{t-i-j} = 0$, all i,j. Thus, as long as $0 < \delta < 1$ (a mild restriction),

$$(27) \qquad \bar{R}^m_t = \frac{(1 - \theta)w_t \bar{H}}{\emptyset}$$

as expected from (20). Note that $\emptyset = (1 - \delta) - (1 - \theta)u > 0$ is required for $\bar{R}^m_t$, $\bar{R}_t > 0$. This condition is satisfied whenever $r \leq q$, and also with values of $r > q$ in the apparently "realistic" range.[9]

Returning to (26) we see it can be rewritten:

$$(28) \qquad R^m_t = \bar{R}^m_t + (1 - \theta)w_t \sum_{i=o}^{\infty} \delta^i \sum_{j=o}^{\infty} v^j \epsilon_{t-i-j}$$

or

$$(29) \qquad R^m_t \begin{cases} = \bar{R}^m_t + (1 - \theta)w_t \sum_{i=o}^{\infty} (\dfrac{\delta^{i+1} - v^{i+1}}{\delta - v}) \epsilon_{t-i} \quad ; \quad (\delta \neq v) \\[2em] = \bar{R}^m_t + (1 - \theta)w_t \sum_{i=o}^{\infty} \delta^i (i+1) \epsilon_{t-i} \qquad ; \quad (\delta = v) \end{cases}$$

Finally, for $\delta \neq v$, the variance of R^m_t is:

$$(30) \qquad V(R^m_t) = (1 - \theta)^2 w_t^2 [\frac{1 + \delta v}{(1 - \delta v)(1 - \delta^2)}] \quad V(H) = \frac{V(R_t)}{(1 - u)^2}$$

Note that $0 < \delta^2 < 1$ and $0 < \delta v < 1$ are required for positive values of $V(R^m_t)$. Both conditions are satisfied, however, given the previously noted restrictions on δ and v.

In order to study inequality, as reflected in the coefficient of variation, we divide (30) by the square of (27) to get:

$$(31) \qquad CV^2(R^m) = CV^2(R) = [\frac{(1 + \delta v)\emptyset^2}{(1 - \delta v)(1 - \delta^2)}] \quad CV^2(H).$$

This relationship immediately draws attention to a fool-proof method of redistributing lifetime income. The term in brackets does not depend at all on the distribution of H. One may therefore redistribute R_t without any perverse results simply by redistributing H_t. In other words a linear

redistributive tax-transfer scheme based on net lifetime labor income would be unambiguously equalizing, for lifetime wealth. This adds some weight to the standard arguments against taxing capital or investment income (i.e., favorable impacts on efficiency, and growth; and distribution via an increase in labor's share brought about by faster capital accumulation).

A second point which is clear is that a decrease in intergenerational earnings mobility (a rise in v) must increase equilibrium inequality. This in turn implies that inequality in R is _always_ less than in H for positive values of u. $CV^2(R)$ takes on a maximum with $v=1$. Since $\phi = (1-\delta) - (1-\theta)u < (1-\delta)$

$$(32) \qquad CV^2(R)\big|_{v=1} = \frac{\phi^2}{(1-\delta)^2} \, CV^2(H) < CV^2(H). \qquad 0 < u < 1$$

That is, inequality in R is always less than in H. This puts an upper bound on any damage that might be caused by redistribution in a case where it has a perverse effect.

Consumption:

In equilibrium the distributions of C and R must be identical except for scale. From (13) and (15)

$$(33) \qquad R_{t+1} = \frac{r(1-\theta)(1-u)}{\theta} \, C_t.$$

This proportionality means that $CV^2(C_t) = CV^2(R_{t+1})$, and since in equilibrium CV's are constant we may simply say $CV^2(C) = CV^2(R)$. This removes one element of ambiguity from the measurement of "inequality" in this model. It also makes it a little easier to understand the unambiguously equalizing effect of inheritance. In the absence of perfect heritability of earnings $(v=1)$, transfers must be used to prevent extremes in C_t matching those of E_t within a family dynasty. While this seems to have strong intuition, some have suggested in

informal treatments that a disequalizing effect is possible in this type

of model. (See, e.g., Tomes, 1981a, pp. 928-930.)

The reason that an ambiguous impact of inheritance in the type

of model considered here might be expected is that there are two competing

"effects" of inheritance whose relative importance does vary. On the one

hand, for parents of equal R_t, transfers to children are inversely related

to E_{t+1}. This is highly equalizing. On the other hand, allowing R_t to vary,

with $v > 0$ expected inheritance and expected earnings ability are positively

related. The overall correlation of E_{t+1} and I_{t+1} can therefore not be

signed.[10] What we show here is that although the relative importance of these

"effects" can vary with $v < 1$, their net result is always equalizing.

Inheritances:

The degree of "inequality" in inheritances is not studied here.[11]

Analysis of the reaction of $CV^2(I)$ to changes in u is unilluminating. Suppose

that at $u = 0$, $\delta > \theta$, and we therefore have $CV^2(I) > \theta$. Raising u we may

reach a point where $\delta = \theta$ and $\overline{I}_t = 0$. As we do so $CV^2(I)$ goes to infinity.

Further increases in u may give $\delta < \theta$ and $\overline{I}_t < 0$. Correspondingly $CV^2(I)$ will come

down from infinity and start to decrease. It is not clear, however, that

inequality in I_t in any fundamental sense would initially rise with u and then

fall. In addition the dispersion of I_t bears no simple relationship to "economic

inequality" in this model and is therefore of limited interest.

VI. Effects of Redistribution

Looking first at the effects of redistribution on aggregates, we find

that

(34) $$\frac{\partial \overline{R}_t}{\partial u} = \left(\frac{-\theta(1-\theta)}{\gamma\phi^2}\right)w_t\overline{H} < 0.$$

Mean lifetime wealth <u>always</u> declines when u rises. Note that since

$\bar{R}_t = w_t\bar{H} + \bar{I}_t$, $\frac{\partial\bar{R}}{\partial u} = \frac{\partial\bar{I}}{\partial u}$, and the mean inheritance must therefore fall.

Since $\bar{I}_t = r(\bar{R}_{t-1} - \bar{C}_{t-1})$ this also means that the absolute difference between

$\bar{C}_t$ and $\bar{R}_t$ must decline. From our discussion in Section IV we might anticipate

that the same would hold true for the percentage difference. In fact

(35) $\qquad \dfrac{\partial(\frac{\bar{C}_t}{\bar{R}_t})}{\partial u} = \dfrac{q\theta}{r\gamma(1-\theta)(1-u)^2} > 0$

so that the "propensity to save" out of R_t unambiguously falls as u rises.

Turning to the impact on the <u>level</u> of consumption

(36) $\qquad \dfrac{\partial\bar{C}_t}{\partial u} = \dfrac{\theta[(1-\theta)(1-u)-\delta]}{\delta\gamma\theta}\,\bar{R}_t$.

When r = q, $\delta = (1-u)(1-\theta)$, giving $\frac{\partial\bar{C}}{\partial u} = 0$, as we knew from the fact that $\bar{C}_t = w_t\bar{H}$

under these circumstances. When r > q, $\frac{\partial\bar{C}}{\partial u} < 0$ because $\delta > (1-u)(1-\theta)$. Finally,

with r < q, $\frac{\partial\bar{C}}{\partial u} > 0$. Hence, although redistribution always reduces equilibrium

$\bar{R}$, as shown diagrammatically earlier, it is possible for the standard

of consumption to rise. The leading case is, however, clearly r > q.

Redistribution "normally" will reduce mean inheritances, lifetime resources, and

consumption.

What is the impact on inequality? Consider first the r = q case:

(37) $\qquad CV^2(R)\big|_{r=q} = [\dfrac{\theta^2(1+\delta v)}{(1-\delta v)(1-\delta^2)}]CV^2(H)$.

Now

(38) $\qquad \dfrac{\partial\delta}{\partial u} = \dfrac{-\delta}{1-u}[1+\theta(\dfrac{1-\gamma}{\gamma})]$.

For finite γ this is always negative. Thus we can see from (37) that with

$\gamma \geq 1$ and, consequently, $\frac{\partial \theta}{\partial u} \leq 0$, $\left. \frac{\partial CV^2(R)}{\partial u} \right|_{r=q}$ must be negative. On the other

hand, as the examples computed in the next section confirm, with $\gamma < 1$

the $\frac{\partial \theta}{\partial u} > 0$ influence may be strong enough to make $\frac{\partial CV^2(R)}{\partial u}$ rise, depending

on parameter values.

Unfortunately, when we allow $r \neq q$ the situation becomes less clear-

cut. When $r \neq q$, θ in (37) is replaced by $\phi = (1-\delta) - (1-\theta)u$ or

$$(39) \qquad \phi = (1 - \frac{r}{q})(1-u) + \theta[\frac{r}{q} + (1 - \frac{r}{q})u].$$

We have

$$\frac{\partial \phi}{\partial u} = (\frac{r}{q} - 1)(1-\theta) + [\frac{r}{q} + (1 - \frac{r}{q})u]\frac{\partial \theta}{\partial u}.$$

Since the coefficient on $\frac{\partial \theta}{\partial u}$ here is positive, we still get clear-cut results

with $r < q$. When $r < q$ the first term is negative, and with $\gamma \geq 1$, $\frac{\partial \phi}{\partial u}$ will

consequently be negative, giving $\frac{\partial CV^2(R)}{\partial u} < 0$. However, when $r > q$ the first

term is _positive_, and $\frac{\partial \phi}{\partial u} > 0$ is possible even with $\gamma \geq 1$. One can see that

if r is only slightly greater than q the $\frac{\partial \phi}{\partial u} > 0$ result will only be given

with quite low values of γ. (See (17).) As r/q rises, however, $\frac{\partial \phi}{\partial u} > 0$ can

be obtained with increasingly high values of γ.

To what extent does the $\frac{\partial \phi}{\partial u} > 0$ possibility with $\gamma \geq 1$ modify the clear-

cut $r = q$ situation? This is a question which can best be answered by

choosing plausible values for other parameters and seeing how high r/q

must rise before $\frac{\partial CV^2(R)}{\partial u} > 0$ is obtained with $\gamma \geq 1$. This is done in the

next section. The conclusion is that some disequalizing effect is obtained

with $\gamma = 1$ at an (annual) interest rate of 4.5%, while with $\gamma = 2$ an interest

rate of 7% is required. These rates are higher than the usual estimates of

mean rates of return on household wealth in the U.S., as pointed out in the next section. Hence we may perhaps think of the $\gamma \geq 1$ case as one where redistribution works, without doing too much violence to the facts. Caution is certainly warranted, however.

As mentioned earlier there is _some_ empirical evidence suggesting that $\gamma \geq 1$ should be considered the "leading case".[12] Taken separately the additive components of the utility function $\dfrac{c_t^{1-\gamma}}{1-\gamma}$ and $\beta \dfrac{R_{t+1}^{1-\gamma}}{1-\gamma}$ each display the constant relative risk aversion form, with γ the index of risk aversion. What evidence there is suggests $\gamma > 1$ in the risk aversion context. In studies of consumer demand γ has been estimated as the elasticity of the marginal utility of income and values greater than unity are generally obtained. (See Stern, 1977.) Another small body of literature estimates the elasticity of substitution between consumption at different points in the life cycle of a single generation. This provides estimates uniformly below unity. Finally, in a model of intergenerational transfers similar to that studied here, but with more than one child per family, Tomes (1981b) estimates an elasticity of substitution between the incomes of siblings of 0.86. While none of this evidence bears _directly_ on the intergenerational elasticity of substitution considered here, it does at least suggest that $\gamma \geq 1$ may be the "leading case". This presumption is strengthened by the fact that the equilibria computed in the next section are most "true-to-life" with $\gamma \geq 1$.

It may seem hard to understand how a perverse distributional result can be obtained in this model. The key is to realize that both C_t and R_{t+1} are proportional to Z_t. Letting π_1 and π_2 be the shares of R_t and $\dfrac{R^m_{t+1}}{r}$ in Z_t respectively, and noting that $CV^2(\dfrac{R^m_{t+1}}{r}) = CV^2(\underline{R}^m_{t+1})$ and does not vary with t, we may write

$$(40) \qquad CV^2(C) = CV^2(Z) = \pi_1^2 CV^2(R) + \pi_2^2 CV^2(\underline{R}^m) + 2\rho CV(R) CV(\underline{R}^m)$$

where ρ is the correlation coefficient for R_t and $\underline{R}^m_{t+1}$. Since ρ is typically small,[13] and $CV^2(C) = CV^2(R)$, substituting in for $\underline{R}^m$ we have:

$$(41) \qquad CV^2(C) \approx (\dfrac{\pi_2^2}{1 - \pi_1^2}) CV^2(E + G).$$

There are two main effects of a rise in u on which (41) allows us to focus. First, unless $\overline{R}$ is more than (negatively) unit elastic with respect to u, $\dfrac{\partial G}{\partial u} > 0$ and $CV^2(E + G)$ falls as u increases. On the other hand, the shares π_1 and π_2 fall and rise respectively. The first influence lowers inequality. What is the effect of the second?

As we have noted above, $CV^2(R)$ is at a maximum with $v = 1$. It is readily confirmed that with this limiting value of v, $CV^2(R) = CV^2(E + G)$.[14] Hence in all cases with $0 \leq v < 1$, $CV^2(R) < CV^2(E + G)$. Thus the shift in weight from $CV^2(R)$ to $CV^2(E + G)$ in (40) when u rises tends to <u>increase</u> equilibrium inequality.

The result of the change in weights can be explained more intuitively. We have a model where inheritance is unambiguously equalizing. An increase in the scale of redistribution results in a decrease in the scale of inheritance, which is reflected in the drop in $\overline{R}$. The decreased scale of inheritance gives it less opportunity to equalize the underlying distribution of $E + G$.

We can now see why the conclusion that lower values of γ are more likely to give a perverse distributional effect is obtained. First, the lower γ the

more severe is the effect of a rise in u on $\bar{R}$. The greater shift in weights
in (40) is therefore obtained when γ is lower. However, in addition, the
greater drop in $\bar{R}$ with lower γ leads to a smaller increase in G. Hence
with lower γ, $CV^2(E + G)$ drops less. In fact if γ is sufficiently small G
may actually _fall_ leading to an _increase_ in $CV^2(E + G)$.[15]

The complete intuition therefore is as follows. The lower γ the
greater is the reduction in the scale of inheritances as u rises. The less
therefore is their equalizing impact on the underlying distributions of
E + G. But in addition to this, the severe drop in $\bar{I}$ and $\bar{R}$ with lower γ
reduces the amount available for redistribution.

The reduced efficacy of redistribution at higher levels of r/q can
also be readily explained with the help of (40). From (20) and (34), the
% change in $\bar{R}_t$ with respect to a change in u is

$$\frac{1}{R_t} \frac{\partial \bar{R}_t}{\partial u} = \frac{-\theta}{\gamma \emptyset (1-u)} < 0.$$

Higher values of r/q reduce $\emptyset$ and therefore make $\bar{R}_t$ more responsive,
proportionally, to a change in u. This means that the higher is r/q the
greater is the weight-shifting influence of a rise in u in (40).

What is the influence of other parameters on the efficacy of
redistribution? Those of greatest interest are the heritability of earnings,
v, the degree of altruism, β, and the tax rate u itself. First we have

(42) $\qquad \dfrac{\partial CV^2(R)}{\partial v} = \dfrac{2\delta \emptyset^2}{(1-\delta^2)(1-\delta v)^2} CV^2(H) > 0.$

By the same argument used to sign $\dfrac{\partial CV^2(R)}{\partial u}$ above, it is clear that an increase in
u will reduce $\dfrac{\partial CV^2(R)}{\partial v}$ for $r \geq q$ and $\gamma \geq 1$. Hence

In terms of (40) the explanation for greater efficacy of redistribution with higher v is that as v increases, $CV^2(R)$ rises toward $CV^2(E + G)$. Hence with high v the shift in weights in (40), which tends to raise inequality, has less scope to operate than the drop in $CV^2(E + G)$, which works in the opposite direction.

Turning to B, (16) makes clear that the higher is β the more sensitive will be θ to a change in u. Hence in the γ > 1 case redistribution will be more effective with higher β since $\frac{\partial\theta}{\partial u}$ (which is positive) will be higher. On the other hand, with γ < 1, $\frac{\partial\theta}{\partial u}$ (which is negative) will be lower with higher β and redistribution will have a greater tendency towards a perverse result. The greater the subjective importance of intergenerational transfers the greater will be the effect of a change in their price.

The impact of the level of u itself is a little more difficult to analyze. In the r = q, γ = 1 case it is possible to show that $\frac{\partial CV^2(R)}{\partial u}$ is likely to decline in absolute value as u rises, giving a diminishing marginal impact of redistribution.[16] However, with r = q and γ ≠ 1 it is not possible to sign this effect. Thus the computations of the next section show that with γ > 1, as u rises $\frac{dCV^2(R)}{\partial u}$ initially drops, but then rises. Redistribution then becomes more effective as u increases. On the other hand, with γ < 1 in the examples computed, at u = 0 $\frac{\partial CV^2(r)}{\partial u} < 0$, while as u rises so does $\frac{\partial CV^2(R)}{\partial u}$, eventually becoming positive. Loosely, the reason for this pattern in the γ < 1 case is that when u = 0 $\frac{\partial G}{\partial u}$ must be positive. Equal absolute increases in u give decreasing percentage increases in $\frac{u}{1-u}$ as u rises. Hence the percentage rate of increase of

$$G_t = (\frac{u}{1-u})\bar{R}_t$$

will decline unless the percentage rate of decrease of $\bar{R}_t$ is falling

sufficiently fast.

Finally, what is the relationship between the possibility of a

perverse effect of redistribution in this model and the shape of the "Laffer

curve"? In terms of (40) the equalizing impact of higher u is reduced for

lower values of γ partly because the increase in G resulting from a given

rise in u is less. Another way of putting this is to say that the "Laffer

curve" is less steep with lower γ. Also, note that the claim, in the previous

paragraph, that G_t is likely to rise at a decreasing rate as u increases

(needed to explain the declining efficacy of redistribution as u rises with

$\gamma < 1$) is an assertion that the elasticity of the "Laffer curve" is less than

unity.

In this model the "Laffer curve" is given by the average tax revenue

(= average transfers)

$$(43) \qquad u\bar{R}_t = G'_t = \frac{u(1-\theta)(1-u)}{(1-\delta)-(1-\theta)u} \; E_t = \frac{u(1-\theta)(1-u)}{\phi} \; \bar{E}_t.$$

Since $\phi > 0$ at $u = 0$ and $u = 1$, we see that $G'_t = 0$ at both $u = 0$ and $u = 1$.

We also have

$$(44) \qquad \frac{\partial G'_t}{\partial u} = \bar{R}_t [1 - \frac{u\theta}{(1-u)\gamma\phi}].$$

In the special case where $r = q$ this gives the easily-interpreted expression:

$$(45) \qquad \frac{\partial G'_t}{\partial u} \bigg|_{r=q} = \bar{R}_t [1 - \frac{u}{(1-u)\gamma}].$$

Thus G' has a slope of unity at $u = 0$, and $-\infty$ at $u = 1$. Since

$$(46) \qquad \frac{\partial^2 G'_t}{\partial u^2} \bigg|_{r=q} = -\bar{R}_t [\frac{1+\gamma}{(1-u)\gamma^2}] < 0$$

at least with $r = q$ the Laffer curve is concave, and has a single peak at

$u = (1+\gamma)^{-1}$. Finally, note that the elasticity of the curve is just:

$$(47) \qquad \eta_u^{G'} \bigg|_{r=q} = 1 - \frac{u}{(1-u)\gamma} .$$

We can see from (47) that we were correct in attributing the lower equalizing effect of redistribution with smaller γ's to a lower responsiveness of G to u. Also, the explanation of the declining efficacy of redistribution as u rises in the $\gamma < 1$ case given above is confirmed. At u = 0 a small increase in u produces the same % rise in G as it would with $\gamma \geq 1$. It is therefore not surprising that increased redistribution is initially equalizing even with low γ.

While the shape of the Laffer curve clearly helps to explain our results it is only part of the story. It should be emphasized that being past the peak of the curve is neither necessary nor sufficient for a perverse effect of redistribution. It is not necessary since the shift of weights in (40) can give CV(C) increasing even when G is still rising. It is not sufficient because $G = \frac{G'}{1-u}$ and G continues to rise after the peak in the Laffer curve has been reached. In fact

$$(48) \qquad \frac{\partial G_t}{\partial u} = (1 - \frac{u\theta}{\gamma\phi}) \frac{\bar{R}_t}{(1-u)^2}$$

so that $\frac{\partial G_t}{\partial u}$ is only negative with $u > \frac{\gamma\phi}{\theta}$. With $r > q$ this condition can only be satisfied with $\gamma < 1$. In terms of our discussion of (40), although the Laffer curve "exists" for $\gamma \geq 1$, $\frac{\partial G_t}{\partial u}$ is always positive and $\frac{\partial CV^2 (E + G)}{\partial u}$ is always negative. Hence there is always a competition between the weight-shifting, and $CV^2 (E + G)$ falling, influences with $\gamma \geq 1$.

VII. Examples

In this section the magnitudes of the effects discussed above are explored by computing alternative equilibria. In addition the analysis of the impact of increased redistribution given in the preceding section is illustrated in two cases: one with a perverse effect of redistribution, and one with a favorable effect.

Table I shows equilibrium values of CV(C) under a range of assumptions on r, γ, v, β, and u. Throughout, q is maintained at the annual rate of 1.015.[17] Popular estimates of the mean household rate of return on net worth would place r at about 1.03 on an annual basis. (See, e.g., Boskin, 1978, p. 19; and Feldstein, Green, and Sheshinski, 1978, p. 64.) Alternative values of 1.015 and 1.045 are used for sensitivity-testing. A generation is assumed to equal 25 years, so that annual r's of 1.015, 1.03 and 1.045 correspond to r's, per generation, of 1.451, 2.094, and 3.005 respectively. CV(H) is set at 0.55.[18] Alternative values of $\gamma = \frac{1}{2}$, 1 and 2 are used. As argued in Section II, the plausible range for v is approximately (0.5, .7071). Table I uses the limits of this range alternatively. Finally, there is no empirical evidence on β. The values used, 1 and $\frac{1}{2}$, represent perfect, and somewhat less than perfect, altruism respectively.

Looking first at the r = q = 1.015 case shown in the first four columns of Table I, we see that with $\gamma \geq 1$ there is a strong equalizing impact of redistribution. With $\gamma = 1$ a tax rate of 50% reduces CV(C) from 13 to 26%, while with $\gamma = 2$ the same tax rate lowers the coefficient of variation from 23 to 31%. With $\gamma = \frac{1}{2}$, on the other hand, redistribution only works when u is raised from 0 to .25, except in one case where u = 0.5 gives a further improvement.

Raising r to the "best guess" value of r = 1.030 alters the quantitative,

but not the qualitative picture. With $\gamma \geq 1$ redistribution still works in

all the cases shown, although the proportional impact of a 50% tax rate, for

example, falls to the range 9 to 23% with $\gamma = 1$ and 20 to 31% with $\gamma = 2$.

With $\gamma = \frac{1}{2}$, in three of the four cases shown, even u = 0.25 is disequalizing,

Even a little redistribution may be a bad thing with a high intergenerational

elasticity of substitution.

The r = 1.045 results indicate that some caution in assessing the

impact of redistribution is in order. In three of four cases, with $\gamma = 1$,

there is a point beyond which increased redistribution is disequalizing.

In two of the four cases this point comes <u>before</u> u = 0.5, so that the result

cannot be dismissed as an artifact of an extreme tax system. With $\gamma = 2$

we still have an equalizing effect in all cases considered. Computations

not shown in the table indicate that r = 1.07 is required before perverse

results show up in this case.

We may thus conclude that if one has a strong prior in favor of $\gamma \approx 2$

(or higher), the present model argues for an equalizing effect of redistribution

with plausible interest rates. On the other hand if one is not certain that

$\gamma > 1$ and prefers $\gamma \approx 1$, a more cautious evaluation is in order. An interest

rate of 4.5% may seem "high", but is clearly not outside the plausible range.

Unitary intergenerational elasticity of substitution and moderately high

interest rates apparently add up to a disequalizing effect of redistribution.

Table I also shows that the strength of tax effects is quite sensitive

to the values of v and β. With r = 1.03, $\gamma = 1$, and $\beta = 1$, for example, raising

v from 0.5 to 0.7071, increases the favorable distributional effect of $u = \frac{1}{2}$

by 56%. Similarly, an increase of β from 0.5 to 1.0 improves the distributional impact by 67% when $r = 1.03$, $\gamma = \frac{1}{2}$, and $v = 0.5$.

Note that the impact of β on the efficacy of redistribution differs between the $\gamma \gtrless 1$ cases, as shown in the previous section. With $\gamma = \frac{1}{2}$, higher β leads to a significantly less favorable impact of redistribution.

Table II, which shows the values of $\overline{I}/\overline{R}$ found in the alternative equilibria, can help us to judge which parameter combinations give the most "realistic" results. There are two criteria one may reasonably apply. Casual empiricism suggests that the ratio if $\overline{I}/\overline{R}$ is likely a low positive number, or, if negative, not too far below zero. The second prior one might have is that although changes in u may alter $\overline{I}/\overline{R}$ they should not do so too violently.

Table II shows, first, that the parameter values $\beta = 0.5$ and $r = 1.015$ can both be rejected on the grounds that they each only produce positive $\overline{I}/\overline{R}$ in one case, and for both parameters that is a no-tax case. Second, if one supposes that $0.25 \leq u \leq 0.5$ best approximates actual tax systems, with $r = 1.030$ either $\gamma = 0.5$ or 1.0 might be considered realistic since $\overline{I}/\overline{R}$ is a low positive fraction at some u in this range in both cases. On the same criterion the $r = 1.045$, $\gamma = 1.0$ combination does not appear unrealistic.

On the other hand, from the point of view of sensitivity of $\overline{I}/\overline{R}$ to u, the $\gamma = 0.5$ results in Table II look quite unrealistic. With $\gamma = 0.5$, and $r = 1.03$, for example, $\overline{I}/\overline{R}$ falls from .965 to -.468 as u rises from 0 to 0.5. It is difficult to believe that a change of this magnitude would occur in the real world, especially as we have gone from a regime of very low tax rates to moderately heavy taxation over the last 50 or 60 years without

any apparent change in the relative importance of inherited wealth on this scale. On the sensitivity count the $\gamma = 1$ or 2 results clearly are most "reasonable".[19]

Table III illustrates the analysis of distributional effects provided in the previous section. As u rises, the weight $\pi_1 (\overline{R}/\overline{Z})$ declines while $\pi_2 (\overline{R}^m_{t+1}/\overline{Z})$ increases. As the table shows, in all cases CV(R) < CV(E+G), so that this shift in weights is disequalizing. The table also shows that CV(E+G) declines whenever G increases with u. Note that the _increase_ in CV(E+G) with $\gamma = 0.5$ when u rises above 0.25 occurs because G actually declines.

The table confirms the importance of the factors studied in the previous section. In all cases $CV^2(R)$ moves in the same direction as its $\pi_1^2 \, CV^2(R) + \pi_2^2 \, CV^2(\underline{R}^m)$ component. The correlation of R_t and $\underline{R}^m_{t+1}$ is so low that changes in the $2\rho CV(R) \, CV(\underline{R}^m)$ term are swamped by those in the terms on which we have focused.

Finally, Table III shows the relationship between perverse results and the shape of the "Laffer curve". In the $\gamma = 0.5$ case this curve peaks between u = 0 and 0.25, while with $\gamma = 1.0$ the peak occurs between u = 0.5 and 0.75. As explained in the previous section, being beyond this peak is neither necessary nor sufficient for a perverse distributional impact. The table illustrates the lack of sufficiency. With $\gamma = 1.0$, CV(R) continues to fall beyond the peak of the Laffer curve. The table shows that the drop in mean tax revenue G', is insufficient to prevent G from increasing and CV(E+G) declining, with a strong equalizing effect. The fact that with $\gamma = 0.5$ in the cases shown CV(R) always moves in the opposite direction to G' is fortuitous. It is not difficult to construct cases where CV(R) and G' move in the same direction.

VIII. Conclusion

This paper has analyzed the effect of a linear redistributive tax-transfer scheme under exogenous factor prices, where generations are linked by altruistically-motivated transfers. Under certain conditions an equilibrium in which lifetime earnings and wealth, consumption, and inheritance all grow at the same percentage rate and have stationary coefficients of variation is obtained. An important feature of the equilibrium is that coefficients of variation of lifetime wealth and consumption are the same. The analysis of "inequality" in this model is therefore less ambiguous than might be expected.

Strikingly, lifetime wealth and consumption are always more equal than lifetime earnings in the present model. This holds true irrespective of the marginal tax rate. Since the distributions of lifetime wealth, consumption, and earnings would coincide in the absence of inheritance, this means that, from the point of view of the coefficient of variation, inheritance is unambiguously equalizing.

This extends the similar result obtained by Laitner, 1979a and b, in the absence of a tax-transfer scheme. Although there is thus an upper bound on the inequality that can be obtained when redistribution is in force, the question of whether redistribution can have a perverse distributional effect remains open. The circumstances under which a perverse effect may be obtained have been investigated both analytically and by means of examples.

The analysis shows that the type of redistribution examined always leads to lower equilibrium mean inheritance and lifetime wealth. In the most plausible case, where the rate of interest exceeds that of economic growth,

there is also an unambiguous negative impact on mean consumption. There are two crucial determinants of the impact on inequality. The greater is the elasticity of substitution in the parents' utility function between their own consumption and children's lifetime wealth, σ; and the higher is the interest rate, r, relative to the rate of growth of the economy, q, the less effective will be redistribution in achieving its equalizing aim.

We have argued, from independent empirical evidence, and on the basis of the realism of alternative equilibria of the model under differing parameterizations, that $\sigma < 1$ is the "leading case". With $\sigma < 1$, redistribution is always equalizing if the interest rate does not exceed the growth rate. Further, computations of alternative equilibria show that the critical interest rate above which redistribution has a perverse result ranges from moderately high ($\sigma = 1$) to implausibly high ($\sigma \geq 2$). Hence with "real world" parameters the model suggests redistribution would in fact be equalizing. This casts some doubt on the conjecture of Becker and Tomes (1979) that the failure of redistributive efforts in the U.S. to equalize the distribution of income very much over about the last 50 years (particularly marked in the post-war period), might represent a real-world example of the type of redistributive failure possible in models of the type studied here.

The explanation for the critical importance of σ and r/q has been carefully considered. High values of either promote a sharp drop in mean inheritance and lifetime wealth when there is an attempted increase in the scale of redistribution. (Increased redistribution requires a higher marginal tax rate on inherited wealth. This causes a substitution away from intergenerational saving that is greater the higher is σ, and has a greater impact on what is actually inherited the higher is r/q.) Since inheritance

is unambiguously equalizing, it is not surprising that a drop in its overall scale is disequalizing. A sharp fall in lifetime wealth can make the amount available for redistribution rise only slightly, or actually decline. (Being past the peak of the "Laffer curve" is, however, neither necessary nor sufficient for a perverse effect of redistribution.)

Other influences on the efficacy of redistribution include the marginal tax rate, u; earnings heritability, v, and the degree of altruism, β. In the $\sigma > 1$ case, where redistribution may be disequalizing, a higher level of u is more conducive to a disequalizing effect. In the arguably more realistic $\sigma \leq 1$ case, however, higher u may actually lead to a more favorable effect of a given absolute increase in u.

Higher values of v always make redistribution more equalizing. On the other hand, higher β tends to magnify whatever effect of redistribution obtains. With $\sigma \leq 1$ as the leading case, the most propitious world for redistribution is one with low intergenerational mobility in the earnings distribution and a high degree of parental altruism towards children.

Notes

[1] Becker and Tomes did not call attention to the equalizing effect of inheritance. Their equation (20), however, shows that both inequality in "endowment luck" (ex ante earnings ability) and in "market luck" (ex post additions to or subtraction from earnings ability) have impact coefficients on intragenerational income inequality which are less than unity.

Bevan (1979) shows, by simulation, that it is possible to construct a model with altruistic intergenerational transfers in which inheritance is disequalizing. (See his Table 7.) The contrast with Laitner's results appears to occur because Bevan uses a positive correlation of the earnings abilities of successive generations. Although Becker and Tomes also have a positive correlation, their model differs from Bevan's in allowing negative transfers. This may explain why inheritance cannot be disequalizing in their model.

Note that the models investigated in Bevan and Stiglitz (1979) and Stiglitz (1978,b) use simple intergenerational transfer rules (e.g. proportionality to lifetime income) which set them aside from the type of model considered here.

[2] The simple model used here is an example of the class first analyzed by BenPorath (1967).

[3] One result of using a one period model is that no interest rate affects the human capital investment decision. Nonetheless if contemplating comparative statics with r in this model, one ought to bear in mind that in multiperiod (i.e., more "realistic") models changes in r will alter the distribution of lifetime earnings. Such a comparative statics exercise is not performed here.

[4]Note that to avoid having any $H_t \leq 0$ we must restrict ϵ_t so that $\epsilon_t > -(1-v)\bar{H}$. That is, we cannot allow negative shocks larger than the constant term $(1-v)\bar{H}$ in absolute value since for low H_{t-1} these could give $H_t < 0$. While $H_t < 0$ is not necessarily "unrealistic", the existence of non-positive lifetime earnings leads to severe analytical problems. While a utility function defined for negative values of consumption could be introduced (e.g., the constant absolute risk aversion function), allowing negative consumption is peculiar, to say the least. Note that the requirement that $\epsilon_t > -(1-v)\bar{H}$ rules out a normal distribution for H_t.

[5]The proportion of the variance of earnings explained by "family background" may be viewed as an estimate of v^2. Blinder [1976, p. 621] reports average R^2 of .248 in four regressions, which suggests $v^2 = 0.25$. In a survey of studies using sibling data Griliches [1979, p. 559] concludes that about 30% of the variance in log earnings may be explained by family background. Finally, Taubman [1976, p. 867] obtains upper and lower bounds of 0.3 and 0.55 for the combined influence of genetics and family environment using identical twins data. All these studies use earnings for a single year, that is their estimates are downward-biased due to transitory earnings. Griliches suggests that correcting this bias could raise R^2 as high as 0.5. Hence the range of plausible estimates is something like $0.25 \leq v^2 \leq 0.5$.

[6]An alternative way of viewing this setup is as a limiting case of the class of utility functions which depend on the consumption of a finite number of future generations. (In such a framework the current generation assumes, erroneously, that the last generation considered will consume its entire lifetime wealth.) It might be thought preferable to consider

the other limit, where the number of future generations considered goes to infinity. This problem is, however, far less tractable. In particular it is difficult to deal with imperfect correlation of the earnings abilities of successive generations in such a framework. (Note that both Laitner, 1979a and b; and Lowry, 1981, who use the infinite horizon, assume a zero correlation.) This would not be the case, of course, if one assumed perfect foresight with respect to the earnings capacities of all future generations--a rather artificial approach.

[7]The price of R_{t+1} from the parent's point of view is $[r(1-u)]^{-1}$. While it will be appreciated that a rise in u makes it more costly for parents to raise R_{t+1}, it is also important to realize that it makes it more attractive to raise C_t by making negative transfers. For each dollar of negative transfer the tax liability of the child is reduced by u dollars. One consequence is that as u goes to unity, intergenerational transfers do not go to zero. With u=1 it is true that positive transfers are taxed at a rate of 100%. Negative transfers, on the other hand, are subsidized at a rate of 100%. The result is that as u goes to unity, $C_t = Z_t$ in all families, and all I_t's are substantially negative.

[8]The only inequality index used in this paper is the coefficient of variation, since it is the only index for which analytical results are available. This is unfortunate since alternative indexes, based on different social welfare functions, differ in both ordinal and cardinal rankings of degrees of inequality. The coefficient of variation is particularly sensitive to the extremes of a distribution. In the income distribution context this normally means it is most sensitive to the upper tail of the distribution. See Sen (1973) for a lucid discussion of the properties of alternative inequality indexes.

[9]Referring to Figure 1, part (a), we can see that as q/r falls the intersection between the behavioral and stationary conditions for equilibrium growth occurs at higher and higher values of $\bar{R}_t$, eventually going to infinite $\bar{R}_t$ at a critical q/r. If q/r falls any more equilibrium $\bar{R}_t$ then switches to an extremely low negative value and begins to rise. Equilibria of the latter ("unrealistic") type are not obtained in the examples of the next section where q and r are set at 1.45 and 2.09 respectively for generations of 25 years. (These figures correspond to annual q and r of 1.015 and 1.03.)

[10]Note that a positive correlation of inheritances and earnings is not _sufficient_ for a disequalizing effect. To see this, consider an example where two identically distributed variates X_1 and X_2 form the sum Y. Then

$$CV^2(Y) = (\frac{1}{2} + 2\rho)CV^2(X)$$

where $CV^2(X) = CV^2(X_1) = CV^2(X_2)$, and ρ is the correlation coefficient of X_1 and X_2. Clearly for a disequalizing effect in this example one requires $\rho > \frac{1}{4}$.

[11]The expression for $CV^2(I)$ is extremely messy, and comparative statics are difficult for reasons made clear in this subsection.

[12]The evidence referred to in this paragraph, except for Stern (1977) and the Tomes (1981) piece, is carefully summarized in Davies (1981, pp. 573-574).

[13]One might suppose that with high v's the model would give fairly high ρ's. In fact the computations of the next section show that this is not the case, and that we are warranted in ignoring changes in the third

term of (40). The reason ρ is typically low is that R_t includes I_t as well as E_t. We know that inheritances are always equalizing, so that the correlation of I_t and E_t (and therefore E_{t+1}) is low or even negative. As shown in the computations of the next section the result is that ρ is typically in the range $0 \le \rho \le 0.1$.

[14] $CV^2(R)|_{v=1} = (\frac{\phi}{1-\delta})^2 CV^2(E)$. Since $V(E+G) = V(E)$, $CV^2(E+G) = (\frac{\bar{E}}{E+G})^2 CV^2(E)$. Recalling that $G = (\frac{u}{1-u})\bar{R}$ we have $CV^2(E+G) = [\frac{\bar{E}}{\bar{E} + \frac{u(1-\theta)}{\phi}\bar{E}}] CV^2(E)$. Substituting in for ϕ we find that $CV^2(E+G)|_{v=1} = (\frac{\phi}{1-\delta})^2 CV^2(E)$.

[15] Determinants of the critical γ are explored below.

[16] Since θ^2, δv, and δ^2 will all normally be relatively small, from (31)

$$\log CV^2(R) \approx 2\log\theta + 2\delta v - \delta^2$$

Noting that $\delta = (1-\theta)(1-u)$ with $r = q$

$$\frac{\partial^2 \log[(V^2 R)]}{\partial u^2} \approx -2(1-\theta)^2 < 0.$$

[17] An annual growth rate of real wages of 1.5% is somewhat high in terms of recent U.S. experience, but reflects the long-run secular rate of increase better. The average annual real rate of increase of hourly earnings of production workers in manufacturing in the U.S. over the period 1945-75 was, for example, precisely at this level. See Fleisher and Kneisner, 1980, p. 343.

[18] In Davies (forthcoming) I obtain $CV(H) = 0.687$ in a study of the income and wealth of Canadian families, which ignores earnings mobility. Taking mobility into account would reduce lifetime inequality by about 20%. (See, e.g., Lillard, 1977.) This gives $CV(H) = 0.55$. We take $\bar{H} = 1.0$, implying $V(H) = 0.3025$.

[19]The fact that $\overline{I}/\overline{R}$ is a low negative fraction with $\gamma = 2$ and $0.25 \leq u \leq 0.5$ should perhaps not lead us to view high values of γ as necessarily unrealistic. As argued earlier in the paper, negative intergenerational transfers are certainly observed, and there is no empirical assurance that $\overline{I}/\overline{R} > 0$.

TABLE I

CV(Ĝ)

γ	u	r - 1 = .015 v = 0.5 β = 0.5	r - 1 = .015 v = 0.5 β = 1.0	r - 1 = .015 v = .7071 β = 0.5	r - 1 = .015 v = .7071 β = 1.0	r - 1 = .030 v = 0.5 β = 0.5	r - 1 = .030 v = 0.5 β = 1.0	r - 1 = .030 v = .7071 β = 0.5	r - 1 = .030 v = .7071 β = 1.0	r - 1 = .045 v = 0.5 β = 0.5	r - 1 = .045 v = 0.5 β = 1.0	r - 1 = .045 v = .7071 β = 0.5	r - 1 = .045 v = .7071 β = 1.0
0.5	0.0	.487	.402	.512	.454	.411	.102	.460	.140	.288	n.a.	.356	n.a.
	.25	.480	.366	.494	.394	.421	.192	.449	.226	.338	.375	.378	.506
	.5	.488	.371	.495	.386	.448	.268	.462	.290	.395	.131	.417	.148
	.75	.510	.421	.512	.426	.490	.365	.494	.374	.463	.299	.470	.311
1.0	0.0	.471	.428	.502	.473	.416	.323	.464	.388	.371	.189	.429	.249
	.25	.438	.374	.459	.402	.393	.288	.426	.327	.358	.204	.397	.242
	.5	.411	.333	.423	.349	.379	.275	.399	.298	.356	.228	.380	.253
	.75	.387	.300	.393	.308	.371	.272	.380	.283	.359	.251	.370	.263
2.0	0.0	.462	.441	.496	.481	.418	.379	.466	.436	.400	.346	.452	.409
	.25	.412	.379	.436	.406	.378	.327	.412	.365	.368	.306	.406	.348
	.5	.356	.314	.371	.330	.334	.279	.356	.302	.333	.271	.358	.296
	.75	.280	.234	.287	.241	.276	.222	.286	.232	.285	.227	.298	.239

Note: q - 1 is maintained at .015 (annual rate) throughout the table. r - 1 is also expressed at an annual rate. Results are based on generations of 25 years' length. Where the model does not converge there is an "n.a.". For other details, see text.

TABLE II

$$\bar{I}/R$$

γ	u	r - 1 = .015		r - 1 = .030		r - 1 = .045	
		β = 0.5	β = 1.0	β = 0.5	β = 1.0	β = 0.5	β = 1.0
	0.0	-1.874	.195	-.468	.965	.501	n.a.
0.5	.25	-4.025	-.340	-1.950	.594	-.534	-1.240
	.5	-10.111	-1.874	-6.194	-.468	-3.484	.501
	.75	-42.478	-10.111	-29.303	-6.194	-19.408	-3.484
	0.0	-1.114	-.116	-.558	.443	-.168	.832
1.0	.25	-1.786	-.449	-1.222	.110	-.835	.498
	.5	-3.115	-1.114	-2.559	-.558	-2.165	-.168
	.75	-7.130	-3.115	-6.576	-2.559	-6.143	-2.164
	0.0	-.818	-.321	-.603	-.004	-.621	.098
2.0	.25	-1.081	-.506	-.919	-.228	-1.000	-.170
	.5	-1.525	-.818	-1.451	-.603	-1.636	-.621
	.75	-2.521	-1.525	-2.650	-1.451	-3.065	-1.639

Note: $\bar{I}/R$ does not depend on v. For other notes, see note to Table I.

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Table III

Analysis of Distributional Changes,

$v = 0.5, \ \beta = 1.0, \ r - 1 = .030$

	$\gamma = 0.5$				$\gamma = 1.0$			
	$u = \ldots$				$u = \ldots$			
	0.0	0.25	0.5	0.75	0.0	0.25	0.5	0.75
π_1	.977	.661	.369	.124	.722	.541	.639	.180
π_2	.023	.339	.631	.876	.278	.459	.056	.820
$CV(R)$	.102	.192	.268	.365	.323	.288	.275	.272
$CV(\underline{R}^m)$	.550	.302	.327	.388	.550	.400	.335	.298
$\pi_1^2 CV^2(R) + \pi_2^2 CV^2(\underline{R}^m)$	.010	.027	.052	.118	.078	.058	.056	.062
ρ	.003	.089	.110	.054	.075	.108	.109	.073
$\overline{R}/\overline{E}$	28.930	2.463	.681	.139	1.795	1.123	.642	.281
$G/\overline{E}$	0.0	.821	.681	.417	0.0	.374	.642	.843
$G'/\overline{E}$	0.0	.616	.341	.104	0.0	.281	.321	.211

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Appendix

This appendix details the solutions of the differential equations encountered in Section V of the paper.

To derive R_t^m as a function of past earnings shocks, and other parameters, first substitute back in (2) n times:

(A.1) $\quad R_t^m = (1-\theta)\{ \sum_{i=0}^{n} [r(1-\theta)(1-u)]^i (w_{t-i} H_{t-i} + G_{t-i})\} + [r(1-\theta)(1-u)]^n R_{t-n}^m.$

As $n \to \infty$ the second term $\to 0$, provided $r(1-\theta)(1-u) < 1.$ Now

(A.2) $\quad H_{t-i} = (1-v)\overline{H} + v H_{t-i-1}^i + \epsilon_{t-i}^i$

and, substituting back k times

(A.3) $\quad H_{t-i} = \{\sum_{j=0}^{k} v^j [(1-v)\overline{H} + \epsilon_{t-i-j}]\} + v^k H_{t-i-k}.$

We have $0 < v < 1$, so that the second term vanishes as $k \to \infty$, giving equation (24) in the text.

Substituting (24) and (25) into (A.1), as $n \to \infty$ we have

(A.5) $\quad R_t^m = (1-\theta) \sum_{i=0}^{\infty} [r(1-\theta)(1-u)]^i [w_{t-i}(\overline{H} + \sum_{j=0}^{\infty} v^j \epsilon_{t-i-j}) + u\overline{R}_t^m q^{-i}].$

Setting $\delta = \frac{r}{q}(1-\theta)(1-u)$, this gives us

(28) $\quad R_t^m = \overline{R}_t^m + (1-\theta)w_t \sum_{i=0}^{\infty} \delta^i \sum_{j=0}^{\infty} v^j \epsilon_{t-i-j}.$

Note that

(A.6) $\quad \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \delta^i v^j \epsilon_{t-i-j} = \epsilon_t + v\epsilon_{t-1} + v^2 \epsilon_{t-2} + v^3 \epsilon_{t-3} + \ldots$

$$+ \delta\epsilon_{t-1} + \delta v \epsilon_{t-2} + \delta v^2 \epsilon_{t-3} + \ldots$$

$$+ \delta^2 \epsilon_{t-2} + \delta^2 v \epsilon_{t-3} + \ldots$$

$$+ \delta^3 \epsilon_{t-3} + \ldots$$

or

$$\text{(A.7)} \qquad \sum_{i=0}^{\infty} \sum_{j=0}^{i} \delta^i v^j \varepsilon_{t-i-j} = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \delta^i v^{i-j} \varepsilon_{t-i}.$$

Now

$$\text{(A.8)} \qquad \sum_{j=0}^{i} \delta^i v^{i-j} = \begin{cases} \dfrac{\delta^{i+1} - v^{i+1}}{\delta - v} & \text{, for } \delta \neq v \\[2ex] \delta^i (i+1) & \text{, for } \delta = v \end{cases}$$

So, assuming $\delta \neq v$ we have

$$\text{(29)} \qquad R_t^m = \overline{R}_t^m + (1-\theta) w_t \sum_{i=0}^{\infty} \left(\frac{\delta^{i+1} - v^{i+1}}{\delta - v} \right) \varepsilon_{t-i}.$$

Now

$$\text{(A.9)} \qquad V(R_t^m) = (1-\theta)^2 w_t^2 V(\varepsilon) \sum_{i=0}^{\infty} \left[\frac{\delta^{i+1} - v^{i+1}}{\delta - v} \right]^2$$

$$= (1-\theta)^2 w_t^2 V(\varepsilon) \sum_{i=0}^{\infty} \left[\frac{\delta^{2(i+1)} + v^{2(i+1)} - 2v^{i+1}\delta^{i+1}}{\delta - v} \right].$$

The summation in the second term can be written

$$\left(\frac{\delta^2}{1-\delta^2} + \frac{v^2}{1-v^2} - \frac{2\delta v}{1-\delta v} \right) \frac{1}{(\delta - v)^2}.$$

As shown by Becker and Tomes (1979, p. 1186) this reduces to

$$\frac{(1 + \delta v)}{(1 - \delta v)(1 - v^2)(1 - \delta^2)}$$

so that

$$\text{(A.11)} \qquad V(R_t^m) = (1-\theta)^2 w_t^2 V(\varepsilon) \left[\frac{1 + \delta v}{(1 - \delta v)(1 - v^2)(1 - \delta^2)} \right].$$

Substituting

$$V(H) = \frac{V(\varepsilon)}{1 - v^2}$$

in (A.11) gives (30).

Figure 1

$$r > q$$

$\bar{C}_t$

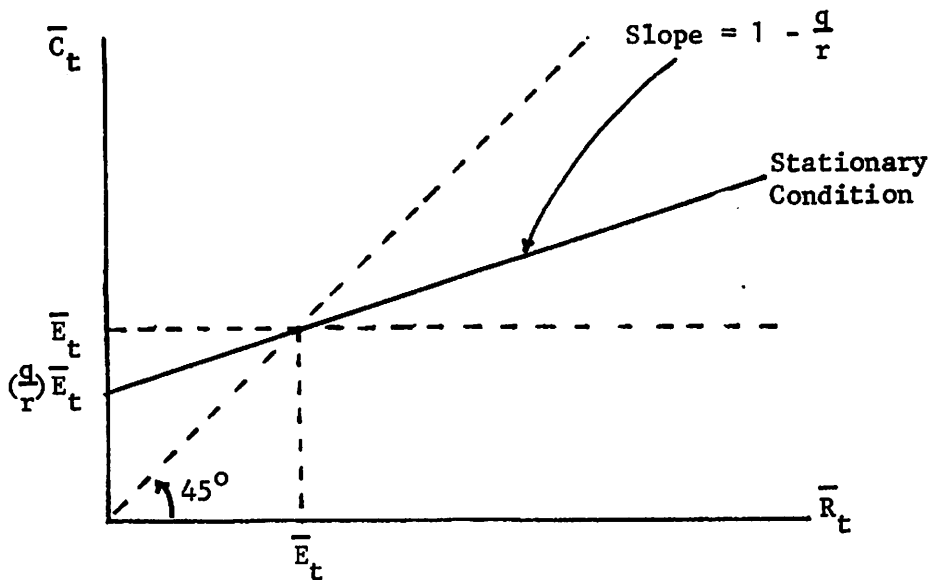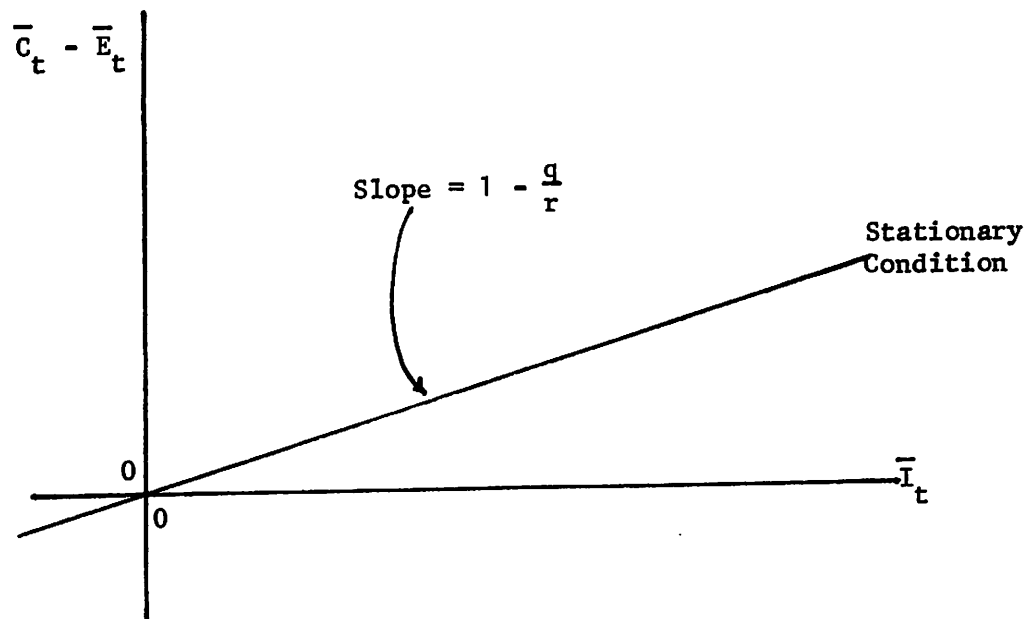Slope = $1 - \dfrac{q}{r}$

Stationary
Condition

$\bar{E}_t$

$\left(\dfrac{q}{r}\right)\bar{E}_t$

45°

$\bar{R}_t$

$\bar{E}_t$

Figure 2

$$r > q$$

$\bar{C}_t - \bar{E}_t$

Slope = $1 - \dfrac{q}{r}$

Stationary
Condition

0

$\bar{I}_t$

0

Figure 3

$$r = q, \quad \gamma = 1$$

New Cons. Fun.

Initial Cons. Fun.

Slope $= \dfrac{\theta}{1 - u'}$

Slope $= \dfrac{\theta}{1 - u^o}$

$\overline{C}_t$

$\overline{E}_t$ — Stationary Condition

$\theta \overline{E}_t$

45°

$\overline{R}'_t$ $\overline{R}^o_t$ $\overline{R}_t$

Figure 4

$$r > q, \quad \gamma = 1$$

Slope $= \theta[1 + \dfrac{qu^o}{r(1-u^o)}]$

Initial Cons. Fun.

Stationary Condition

Slope=$\theta[1 + \dfrac{qu'}{r(1-u')}]$

Slope $1 - \dfrac{q}{r}$

$\overline{C}_t$

$\overline{E}_t$

$(\dfrac{q}{r})\overline{E}_t$

$\theta(\dfrac{q}{r})\overline{E}_t$

45°

$\overline{R}'_t$ $\overline{R}^o_t$ $\overline{R}_t$