1993

# A Model of Political Competition with Citizen-Candidates

Martin Osborne

Al Slivinski

POLITICAL
ECONOMY
RESEARCH
GROUP

# PAPERS IN POLITICAL ECONOMY

**Paper No. 36**

# "A Model of Political Competition with Citizen-Candidates"

**Martin Osborne and Al Slivinski**

#35469

#35470

#35471

#35472



The UNIVERSITY of WESTERN ONTARIO

The Political Economy Research Group was established in the faculty of Social Science at the University of Western Ontario in 1988. Its purpose is to foster scholarship, teaching and interdisciplinary research in political economy, with a focus on:

1. the application of economic models and methods to the study of political processes and institutions,
2. the economic impact of political processes and institutions,
3. the influence of economic factors on the formation of public policy and on institutional change,
4. the politics of economic policy making,
5. the political, social, and economic effects of public policy.

*Co-directors:*
Ronald Wintrobe (Economics)
Robert Young (Political Science)

*Board of Directors:*
Peter Howitt (Economics)
B.B. Kymlicka (Political Science)
John N. McDougall (Political Science)
Peter Neary (History)
John Whalley (Economics)

*Staff:*
Jayne Dewar

*For further information:*
Political Economy Research Group,
Department of Economics,
Social Science Centre,
London, Ontario, Canada N6A 5C2
phone: (519) 661-3877
fax: (519) 661-3292

ISSN: 1181-7151
ISBN: 0-7714-1532-X



# A MODEL OF POLITICAL COMPETITION WITH CITIZEN–CANDIDATES

MARTIN J. OSBORNE[1] AND AL SLIVINSKI[2]

May 1993

ABSTRACT

We develop and analyze the Nash equilibria of a model of electoral competition in which each citizen chooses whether or not to run as a candidate. Citizens care about the policy position of the winning candidate; it is costly to become a candidate and there is a benefit from being the winner of the competition that goes beyond the returns to being able to implement one's preferred policies. We consider both plurality rule and majority rule with runoffs. Depending on the parameters an equilibrium in which there are one, two, or more candidates may exist. Under plurality rule we find, for some parameter values, three-candidate equilibria in which one candidate is sure to lose. We show that under either system the number of candidates depends positively on the ratio of the benefits of winning to the costs of entry and identify a sense in which a two-candidate equilibrium is more likely under plurality rule than under a runoff system.

**Preliminary Draft.** Please do not quote. Comments welcome.

[1]Department of Economics, McMaster University, Hamilton, Canada, L8S 4M4. Thanks to the Social Sciences and Humanities Research Council of Canada for financial support.
[2]Department of Economics, University of Western Ontario, London, Canada, N6A 5C2.

## 1. INTRODUCTION

The number of candidates and the dispersion in their positions varies widely across political competitions. How can we account for this diversity? A prominent hypothesis, known as "Duverger's Law", is that the form of the electoral system plays a central explanatory role: plurality rule fosters a two-party system while both proportional representation and a two-ballot system with a "runoff" favor the existence of many parties (Duverger (1964, pp. 217, 239)). We study a model that gives some support for this hypothesis and at the same time illuminates the role of two other explanatory variables: the costs and benefits of winning an election and the distribution of the citizens' preferences.

In our model there is a population of citizens, each of whom has preferences over policies. Each citizen chooses whether to become a candidate in the competition; running as a candidate costs $c$. The winner of the election implements her favorite policy and also reaps a direct benefit $b$ (the "spoils of office"). Our results are broadly consistent with some stylized facts about electoral outcomes. Under plurality rule we find that when $b$ is small relative to $c$ (as in an election for the chair of an academic department, or a Republican gubernatorial primary in an overwhelmingly Democratic U.S. state?) only one candidate runs. As $b$ increases relative to $c$ the number of candidates that can coexist in an equilibrium increases; for some range of values of $b$ all equilibria involve precisely two candidates, and in any such equilibrium the candidates' positions are distinct. Among the equilibria that may exist for larger values of $b$ are ones in which there are three candidates, one of whom (Perot? the U.K. Liberal Party/SDP?) is bound to lose; such a candidate enters in order to affect the identity of the winner. In any equilibrium at most two candidates whose probability of winning is positive take the same position. Under a runoff system also the number of candidates who can coexist in equilibrium increases as $b$ increases relative to $c$. However, under a runoff system, in contrast to plurality rule, equilibria in which many candidates take the same position are possible. Further, there is a sense in which a two-candidate election is less likely and the candidates' positions are less dispersed under a runoff system than under plurality rule.

The dependence of the number of candidates on the size of $b$ relative to $c$ is very natural. However, to our knowledge there is no other model in which such a result emerges. Our result that there may exist equilibria in which candidates with no chance of wining enter simply to affect the identity of the winner is also, to our knowledge, novel.

Our model differs from most of those stemming from the work of Hotelling (1929) in that the set of players is the set of *citizens*, not a set of exogenously given *candidates*. Starting with the work of Wittman (1977, 1983), several models in the Hotelling mold allow the candidates to be concerned not simply with winning the election but also with the policy of the winning candidate. In all these models, however, the existence of the candidates and the character of their preferences are given exogenously (although, under the assumption that candidates can commit to policies, the policies that are offered in an equilibrium are determined in the models). In our model, by contrast, the set of candidates who enter is determined within the model (and in particular the number of candidates is not restricted to be two, as in Wittman's model).

The strategic calculations that drive our model concern the desirability of entry into a political competition. Two other ideas for explaining the number of candidates in political competitions focus on the strategic calculations of voters. The first is formalized by Palfrey (1989) (in a model that draws upon the work of Cox (1987)), who captures the idea that under plurality rule a vote for a candidate other than the two most likely to win is "wasted". Palfrey's model, however, does not rule out equilibria with more than two candidates entirely: there still exist such equilibria, in which all candidates are tied for first place. Moreover, such ties may tend to arise in equilibrium if the candidates are able to choose their positions (in Palfrey's model the parties' positions are fixed).

The second idea is formalized by Feddersen (1992) who draws upon a feature of the model of Feddersen, Sened, and Wright (1990) to capture the following idea. If information is perfect and voting is costly then every voter must be pivotal in equilibrium (otherwise she would not vote); thus for each citizen there can be no more than one candidate who is preferred to the lottery over all winning candidates (since if there are two or more such candidates then the citizen can switch to voting for one of these candidates and cause that candidate to win outright). If all citizens' utility functions are concave then we conclude that there are at most two candidates in any equilibrium. In Feddersen's model, as in ours, the only strategic actors are citizens. The models differ in that the strategic decision in our model is whether to run as a candidate while the decision in Feddersen's model is the position for which to vote.

In related work Feddersen (1993) modifies his earlier model by allowing deviations by groups of citizens, a group that supports a given position being interpreted as a "party". This assumption reduces the size of the set of equilibria, significantly limiting the difference that may exist between the two parties' positions. It also introduces into the model "parties" as strategic entities, a party being simply the set of citizens who vote for a position. A party takes an action only if all its members approve the action; there is no independent role for a party leader (political entrepreneur). By contrast, a party in our model is identified with a single citizen who takes decisions unilaterally. Our model and that of Feddersen differ in another significant respect: the strategic reasoning of the parties. In Feddersen's model a party (i.e. the set of citizens voting for some position) that considers changing its position assumes that if it does so the voting behavior of all other citizens will be unchanged (although its new position may be more attractive than the one for which they are voting). By contrast, in our model a party that decides to deviate takes into account the effect of its action on the citizens' voting behavior. Finally, in both models there is perfect information, though this assumption appears to play a more critical role in Feddersen's argument.

Our model yields a richer range of equilibrium outcomes than does Feddersen's: more than two candidates may enter an election for a lucrative office, and some of these may be sure to lose. This richer range of equilibria seems desirable since the number of candidates in plurality rule elections varies so widely. For example, Wright and Riker (1989, Table 2) report that more than 25% of plurality-rule Democratic gubernatorial primaries in the U.S. between

1950 and 1982 involved four or more candidates and more than 25% also involved just one candidate.[3] Even in U.S. presidential elections, in which the two major parties are legally entitled to significant advantages over minor parties (they receive maximal funding for their election campaigns and grants for holding their national conventions, for example) there have been at least 11 candidates in all of the last seven elections and in three of these elections (1992, 1980, and 1968) a third party has received more than 5% of the popular vote. To take a final example, in the six general elections in Canada between 1962 and 1974 *four* parties each received at least 5% of the popular vote (and, except in 1974, at least 5% of the seats in the parliament).

Unlike Feddersen, we retain the two-stage structure of Hotelling's model and, unlike either Feddersen or Palfrey, assume that citizens *qua* voters do not act strategically. Certainly the assumption that citizens cast votes rationally, weighing expected costs and benefits, is attractive. Further, there is some evidence (see for example Riker (1982, pp. 762–764) and Abramson et al. (1992)) that some voting is strategic in the sense of the simple static model. However, whether this simple static model captures the behavior of voters better than the model of sincere voting is unclear. Voters in the world may be motivated by considerations that extend beyond the current election: they may wish to signal their support for a cause in order to increase its (rational) support in future elections. Further, even under plurality rule the winner of an election is frequently not a dictator; the fact that the support of a losing party is substantial could have a significant effect on the post-election outcome. Finally, a vote for a sure loser may be rational if it prevents that candidate from withdrawing from the competition and such a withdrawal would cause the remaining parties to take even less attractive positions. That is, "sincere" voting may be a reasonable model of rational behavior in a larger model. In addition (as argued, for example, by Riker (1982, p. 764)), there is some doubt that the chance of a citizen's vote affecting the outcome of some elections is large enough to explain voting behavior to begin with; if this is so then arguments that rest on

[3] By contrast, more than 25% of primaries under the runoff system had more than *seven* candidates and less than 4% had only one candidate.

strategic voting may be suspect. In any case we follow Riker's (1982, p. 764) suggestion to "turn attention away from the expected utility calculus of the individual voter and to the expected utility calculus of the politician and other more substantial participants in the system": we seek to isolate the extent to which such strategic calculations can limit the number of participants in political competitions *independently* of the strategic calculations of voters.

We maintain throughout the assumption that a citizen who enters the competition as a candidate must espouse her favorite position. This assumption is reasonable if the possibilities for commitment are limited, since in such a case voters can correctly anticipate that the candidate who wins will indeed carry out her favorite policy regardless of the announcements she made during the campaign. In situations in which commitments are possible—either explicitly or implicitly (perhaps using future behavior as a commitment device)—a candidate may be able credibly to claim that she will carry out a policy different from her favorite policy. Though we have not investigated the question formally, it appears that the possibility of a small amount of commitment does not significantly change our results.

## 2. The Model

Each of a continuum $I$ of *citizens* has single-peaked preferences over the set of *policy positions*, which we take to be the real line $\mathbf{R}$. The distribution function of the citizens' favorite (ideal) positions on $\mathbf{R}$ is $F$, which we assume to be continuous; we denote its median by $m$. Each citizen can choose to enter the competition ($E$) or not ($N$). A citizen who chooses $E$ is referred to as a *candidate*. After all citizens have simultaneously made their entry decisions they cast their votes. Voting is "sincere": a candidate whose position $x_j$ is occupied by $k$ candidates (including herself) attracts the fraction $1/k$ of the votes of the citizens whose ideal points are closer to $x_j$ than to any other occupied position. Under *plurality rule* the winner of the election is the candidate who obtains the most votes; if two or more candidates tie for first place then each wins with equal probability. Under a *runoff system* the winner is determined as follows. If some candidate obtains a majority (more than half the votes) then that candidate is the winner. If no candidate obtains a majority then the winner is the candidate who obtains a majority in a second election between the two candidates who obtained the most votes in the first round. In both cases ties are dealt with via an equal-probability rule.

Each citizen's payoff depends on the distance between her ideal point and that of the winner of the election, on whether she is a candidate or not, and on her probability of winning. The preferences over policies of a citizen with ideal point $a$ are represented by the function $-|x-a|$; a citizen who chooses $E$ incurs the (utility) cost $c > 0$ and, if she wins, derives the benefit $b \geq 0$. Thus if a citizen with ideal position $a$ chooses $N$ and the ideal position of the winner is $w$ then her payoff is

$$-|w-a|.$$

A citizen who chooses $E$ obtains the payoff

$$b-c$$

if she wins outright and the payoff

$$-|w-a|-c$$

if she loses outright to a candidate with ideal position $w$. If no citizen enters then all obtain the payoff of $-\infty$. Each citizen's preferences over lotteries are represented by her expected payoff.

In summary, we study the strategic game in which the set of players is $I$, the set of actions of each player is $\{E, N\}$, and the preferences of each player are those given above. The solution notion that we use is Nash equilibrium, which we henceforth refer to simply as "equilibrium". We refer to a distribution of candidates' ideal positions on $I$ as a "configuration".

## 3. Results for Plurality Rule

We begin by studying the case of plurality rule. Our first result rules out as equilibria some, though not all, configurations in which some candidate loses with certainty. (The proofs of all results are given in the appendix.)

**Lemma 1** *In equilibrium a candidate does not lose with certainty if either (i) there are other candidates with the same ideal position as hers or (ii) the ideal positions of all other candidates are on the same side of her ideal position.*

A call for individuals to run for some elected office sometimes results in a single citizen offering herself as a candidate and thus winning the election by acclamation. (As we have noted this was so, for example, in over 25% of the plurality-rule gubernatorial Democratic primaries in the U.S. between 1950 and 1982 (see Wright and Riker (1989, p. 161).) The next result shows that such an outcome is consistent with our model: if $b$ is small enough relative to $c$ then regardless of the nature of the distribution $F$ of the citizens' ideal points there is an equilibrium in which a single candidate runs unopposed. The result shows further that if $b$ is smaller than $c$ then this candidate's ideal position need not be the median $m$ of $F$.

**Proposition 2 (One-candidate equilibria under plurality rule)** *There is a one-candidate equilibrium if and only if $b \leq 2c$. If $c \leq b \leq 2c$ then the candidate's ideal position is $m$ while if $b < c$ then it may be any position within the distance $(c - b)/2$ of $m$.*

Many elections are contested. The next result completely characterizes the set of parameters for which a two-candidate election is an equilibrium outcome. To state the result we need the following definitions. Suppose that there are two candidates, with ideal positions $m - \epsilon$ and $m + \epsilon$ for some $\epsilon > 0$, so that each receives half of the votes. Let $s(\epsilon, F)$ be the position between $m - \epsilon$ and $m + \epsilon$ with the property that if a citizen with this ideal position enters the competition then each of the candidates still receives the same fraction of the votes:

$$F\left(\tfrac{1}{2}(m - \epsilon + s(\epsilon, F))\right) = 1 - F\left(\tfrac{1}{2}(m + \epsilon + s(\epsilon, F))\right).$$

If $\epsilon$ is small then no citizen whose ideal position is in $(m - \epsilon, m + \epsilon)$ can enter the competition and win, while if $\epsilon$ is large there is such a citizen who can win. Let $e_p(F)$ be the critical value of $\epsilon$ below which all such entrants lose and above which some such entrant wins. (Note that if the density of

$F$ is single-peaked and symmetric about its median then $s(\epsilon, F) = m$ and $e_p(F) = 2(m - F^{-1}(\frac{1}{3}))$.)

**Proposition 3 (Two-candidate equilibria under plurality rule)**

**a.** *Two-candidate equilibria exist if and only if* $b \geq 2(c - e_p(F))$.

**b.** *In any two-candidate equilibrium the candidates' ideal positions are* $m - \epsilon$ *and* $m + \epsilon$ *for some* $\epsilon \in (0, e_p(F)]$.

**c.** *An equilibrium in which the candidates' positions are* $m - \epsilon$ *and* $m + \epsilon$ *exists if and only if* $\epsilon > 0$, $\epsilon \geq c - b/2$, *and one of the following conditions is satisfied.*

i. $\epsilon < e_p(F)$ *and* $c \geq |m - s(\epsilon, F)|$.

ii. $\epsilon = e_p(F) \leq 3c - b$.

*In particular, the set of equilibrium values of* $\epsilon$ *contains the interval* $(\max\{0, c - b/2\}, \min\{c, e_p(F)\})$ *and is a subset of* $[\max\{0, c - b/2\}, e_p(F)]$.

By Lemma 1 each candidate must win with probability one half in any two-candidate equilibrium, from which it follows that the candidates' positions must be symmetric about the median. That there is no equilibrium in which two candidates take the same position follows from recognizing that entry by some third citizen with an appropriate ideal position always results in a win for the entrant. Since $b \geq 2c$ is necessary to keep two identically-positioned candidates willing to enter, a third citizen who could win for sure would always enter, so that no such equilibrium is possible. The condition $\epsilon \geq c - b/2$ ensures that the payoff to winning is large enough that each of the candidates finds to worthwhile to enter. Finally, any citizen whose ideal point lies between those of the two candidates obtains a payoff of $b - c$ if she wins as opposed to $-\epsilon$ if she stays out of the competition. Since $\epsilon \geq c - b/2$ we have $b \geq 2(c - \epsilon) > c - \epsilon$, so that in equilibrium no such citizen can win if she enters, giving the requirement $\epsilon \leq e_p(F)$. Such a citizen who loses when she enters may still prefer to enter since she may change the outcome of the election from one in which each candidate has an equal chance of winning to one in which her favorite candidate wins with certainty.[4] The second part of condition (i) ensures that any such deviation is in fact not profitable. Condition (ii) ensures that any citizen who can enter and tie for first place is no better off doing so than she is staying out of the competition.

Note that $e_p(F) > 0$ and $|m - s(\epsilon, F)| < \epsilon$ for any distribution $F$. Note also that if the density of $F$ is single-peaked and symmetric about $m$ then, since $m = s(\epsilon, F)$, the second part of condition (i) is never binding.

Equilibria involving more than two candidates are possible. Since there have been many significant examples of three-party competitions (e.g. national elections in the U.K. and Canada, and several U.S. presidential elections) it is of particular interest to determine when a three-candidate equilibrium can occur. The next result shows that in any such equilibrium there is at least some dispersion in the candidates' positions.

**Lemma 4** *In any equilibrium at most two winning candidates share any given ideal position.*

A complete characterization of the conditions under which three-candidate equilibria exist is complex. The next result gives some features of these equilibria. Lemma 4 implies that not all three candidates can have the same ideal position, so there remain two possibilities. If two candidates share one position and a third has a different position then by Lemma 1 the paired candidates must split two thirds of the vote. The other possibility is that all three candidates have different ideal positions, in which case the two extreme candidates must each have a positive probability of winning and so must obtain the same fraction of the vote, which must be not less than that of the center candidate.

**Proposition 5 (Three-candidate equilibria under plurality rule)**

**a.** *Every three-candidate equilibrium takes one of the following three forms.*

i. *The positions of the three candidates are all different; each candidate obtains one third of the votes.*

[4]The fact that a citizen may contemplate entry in order to affect the outcome of the election, even though she has no chance of winning herself, is of independent interest, as we see below.

ii. *The positions of two of the candidates are the same (either $F^{-1}(\frac{1}{3})$ or $F^{-1}(\frac{2}{3})$) while that of the third is different; each candidate obtains one third of the votes.*

iii. *The positions of the three candidates are all different; each of the extreme candidates obtains the same fraction of the votes while the central candidate obtains a smaller fraction (and hence surely loses).*

**b.** *A necessary condition for any such equilibrium to exist is $b \geq 3c$.*

**c.** *If a configuration of type (i) or type (ii) is an equilibrium for the parameters $(b, c, F)$ then the same configuration is an equilibrium for the parameters $(b', c', F)$ whenever $c' > c$ and $b' - 3c' \geq b - 3c$.*

To illustrate the considerations that arise when determining the conditions under which each type of equilibrium exists suppose first that the density of $F$ is symmetric and single-peaked. Consider the configuration in which one candidate is located at the median $m$ of $F$ and the other two candidates are located symmetrically about $m$, at $m - \epsilon$ and $m + \epsilon$, in such a way that each candidate receives a third of the votes. If a citizen whose ideal point is at most $m - \epsilon$ enters then she certainly loses and causes the election to be a tie between the candidates at $m$ and $m+\epsilon$; thus she is better off staying out of the competition. Similarly a citizen whose ideal point is in $(m - \epsilon, m]$ who enters certainly loses and causes the outright winner to be the candidate at $m + \epsilon$, and is thus better off staying out of the competition. Symmetric arguments show that it is optimal for every citizen whose ideal point is greater than $m$ to stay out of the competition. Finally, the candidate at $m - \epsilon$ obtains the payoff $\frac{1}{3}b - c - \frac{1}{3}\epsilon - \frac{1}{3}(2\epsilon)$; if she withdraws she obtains $-\epsilon$ (since the candidate at $m$ is then the outright winner). Thus her decision to enter is optimal if $b \geq 3c$. This condition is more than sufficient to guarantee that the entry of the candidate at $m$ is optimal. Thus the configuration is an equilibrium if and only if $b \geq 3c$.

If the middle candidate it located at a point different from $m$ and the three candidates tie for first place then it may be possible for some citizen whose ideal point is between the two extreme candidates to enter and win outright.

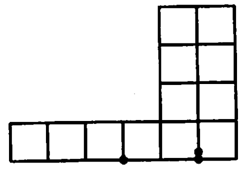


**Figure 1.** An example of a three-candidate equilibrium of type (ii). One candidate is at $a_1$ and two are at $t_2 = F^{-1}(\frac{2}{3})$. The area of each square is 1. We require that $c \geq \frac{4}{9}$ and $b \geq 3c + 4$.

If this is so then, given the condition on $b$ and $c$ required to make it optimal for the candidates to enter, the deviant citizen is better off. Thus in equilibrium it must not be possible for such a citizen to enter and win outright. In addition, even if such a citizen does not win her entry may change the outcome of the election; to ensure that this change is not advantageous the cost $c$ must be large enough.

If the density of the distribution of ideal points is symmetric about its median then no equilibrium of type (iii) exists: in any configuration in which the extreme candidates tie (as they must in equilibrium) it is desirable for the middle candidate to withdraw since this leads to victory for the candidate whom she likes best. If in addition the distribution of ideal points is single-peaked then there is no equilibrium of type (ii) either, since a citizen with ideal point just closer to the median than the position at which there are two candidates can enter and win outright. To see how asymmetric distributions can support such equilibria, consider the two examples in Figures 1 and 2. In the case of Figure 1 any citizen who enters between $a_1$ and $t_2$ loses and, given that $c \geq \frac{4}{9}$, generates an outcome that is worse for her than that which occurs when she stays out. The condition $b \geq 3c + 4$ guarantees that it is optimal for each of the candidates to enter. In the case in Figure 2 the middle candidate loses, but is indifferent between entering and staying out of the competition: she obtains a payoff of $-c - \frac{1}{2} \cdot 4 - \frac{1}{2} \cdot 2 = -4$ as a candidate and the same payoff if she withdraws (since the candidate at $a_1$ is then the outright winner). The condition $b \geq 4$ is required to make it optimal for each of the other candidates

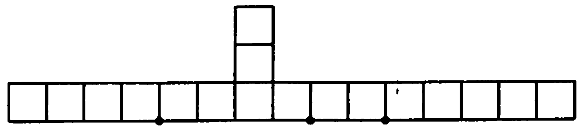


**Figure 2.** An example of a three-candidate equilibrium of type (iii). One candidate is at each of the points $a_1$, $a_2$, and $a_3$. The area of each square is 1. We require that $c = 1$ and $b \geq 4$.

to enter.

There are distributions of the citizens' ideal points for which *no* 3-candidate equilibrium exists for *any* values of $b$ and $c$. An example is a distribution $F$ whose density is symmetric about its median and has its mass concentrated at $t_1 = F^{-1}(\frac{1}{3})$ and $t_2 = F^{-1}(\frac{2}{3})$. We argued above that for such a distribution there is no equilibrium of type (iii); there is no equilibrium of type (i) or type (ii) since an entrant at either $t_1$ or $t_2$ can win outright.

We do not have a characterization of the conditions under which an $n$-candidate equilibrium exists for an arbitrary value of $n$. However, we can show that an equilibrium exists in which there are $k \geq 3$ *winners* only if $b \geq kc$.

**Proposition 6** *A necessary condition for the existence of an equilibrium in which $k \geq 3$ candidates tie for first place is $b \geq kc$.*

Further, equilibria in which there are three or more candidates can exist only if $b \geq 3c$. In particular, if $2c < b < 3c$ then in *all* equilibria there are exactly two candidates.

**Proposition 7** *A necessary condition for the existence of an equilibrium in which there are three or more candidates is $b \geq 3c$.*

Some features of the possible equilibria for the case that the density of $F$ is single-peaked and symmetric about its median are summarized in Figure 3.

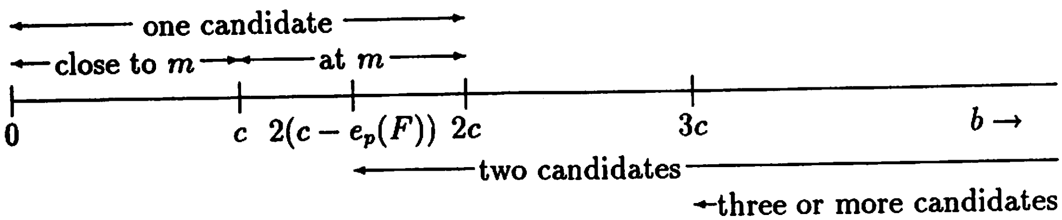


**Figure 3.** Equilibria under plurality rule for a symmetric single-peaked density of ideal points. (The value of $2(c - e_p(F))$ may be less than $c$.)

## 4. Results for a Runoff System

Proposition 2, which gives the conditions for a one-candidate equilibrium under plurality rule, remains true under a runoff system, since the entry of one additional candidate cannot induce a runoff. However, the structure of the set of multicandidate equilibria differs between the two electoral systems. Under plurality rule there is no two-candidate equilibrium in which the candidates' ideal positions are the same (Proposition 3) and in no equilibrium are more than two candidates' positions the same (Lemma 4). Under a runoff system, on the other hand, there are equilibria in which any number of candidates have the same ideal position.

**Proposition 8 (Multicandidate equilibria under a runoff system)** *For $k \geq 2$ there is a $k$-candidate equilibrium in which the ideal position of every candidate is $m$ if and only if $kc \leq b \leq (k+1)c$.*

At such an equilibrium a citizen whose ideal position is different but not too distant from $m$ obtains the most votes in the first round if she enters but then surely loses the runoff. The two inequalities in the result guarantee that none of the candidates prefers to withdraw and no further citizen with ideal position $m$ wishes to enter.

Other two-candidate equilibria are possible under a runoff system. Such equilibria have features in common with those that exist under plurality rule: the candidates must receive the same fraction of the votes and hence take the positions $m - \epsilon$ and $m + \epsilon$ for some $\epsilon > 0$, and $\epsilon$ must be small enough that no citizen with ideal position in $(m-\epsilon, m+\epsilon)$ can enter and win outright (since she would then prefer to do so). However, the limit on the size of $\epsilon$ under a runoff system is different than it is under plurality rule: under a runoff system a citizen with ideal position in $(m-\epsilon, m+\epsilon)$ who obtains a larger fraction of the votes than just *one* of the other candidates gets into a runoff, which she is certain to win. (By contrast, under plurality rule such a citizen needs to obtain a larger fraction of the votes than *both* of the other candidates in order to win outright.) To be precise, let $e_r(F)$ be the supremum of the values of $\epsilon$ for which there is a position $d \in (m-\epsilon, m+\epsilon)$ such that a citizen who enters at $d$ obtains a smaller fraction of the votes than *both* of the existing candidates. If $\epsilon > e_r(F)$ then the configuration in which one candidate is at $m-\epsilon$ and one is at $m+\epsilon$ is not an equilibrium since there is a citizen with ideal point in $(m-\epsilon, m+\epsilon)$ who wins outright if she enters (and hence prefers to do so). Note that from the definition it is immediate that $e_r(F) \leq e_p(F)$: there can be less separation between the candidates' positions under a runoff system. (If the density of $F$ is single-peaked and symmetric then $e_r(F) = \min\{m - F^{-1}(\frac{1}{4}), 2(m - F^{-1}(\frac{1}{3}))\}$.)

**Proposition 9 (Two-candidate equilibria under a runoff system)**

**a.** *Two-candidate equilibria exist if and only if* $2(c - e_r(F)) \leq b \leq 4c$.

**b.** *In any two-candidate equilibrium the candidates' ideal positions are* $m-\epsilon$ *and* $m+\epsilon$ *for some* $\epsilon \in [0, e_r(F)]$.

**c.** *An equilibrium in which the candidates' positions are* $m-\epsilon$ *and* $m+\epsilon$ *exists if and only if either (1)* $\epsilon = 0$ *and* $2c \leq b \leq 3c$ *or (2)* $\epsilon > 0$, $\epsilon \geq c - b/2$, $b \leq 4c$, *and one of the following conditions is satisfied.*

i. $\epsilon < e_r(F)$.

ii. $\epsilon = e_r(F) \leq 2c - b$.

*In particular, the set of equilibrium values of* $\epsilon$ *contains the interval* $(\max\{0, c - b/2\}, e_r(F))$ *and is a subset of* $[\max\{0, c - b/2\}, e_r(F)]$.

If a two-candidate equilibrium exists under plurality rule then there is no bound on the value of $b$ for which the equilibrium exists. The same is not true under a runoff system since a citizen whose ideal point is the same as that of one of the candidates has a positive probability of getting into a runoff, and of ultimately winning, if she enters the competition. The condition $b \le 4c$ is necessary to make the entry of such a citizen unattractive. (The value of this upper bound on $b$ depends on our assumption that an election in which one candidate obtains exactly one-half of the votes precipitates a runoff. If in such an election the candidate with one-half of the votes wins on the first round with some positive probability then the upper bound on $b$ is higher.)

By Proposition 3 a two-candidate equilibrium exists under plurality rule if and only if $b \ge 2(c - e_p(F))$. By contrast Proposition 9 shows that a two-candidate equilibrium exists under a runoff system whenever $2(c - e_r(F)) \le b \le 4c$. Thus given the distribution $F$ of ideal points the set of values of $(b, c)$ for which a two-candidate equilibrium exists under a runoff system is a subset of the set of values that result in a two-candidate equilibrium under plurality rule.

We have seen that under a runoff system there can exist two-candidate equilibria in which both candidates choose the same position while no such equilibrium exists under plurality rule. For values of the parameters for which there exists a two-candidate equilibrium under both electoral systems we can compare the maximal amount of dispersion in the candidates' positions that can exist. If $c \ge e_r(F)$ then since $e_r(F) \le e_p(F)$ the comparison is unambiguous: the maximal amount of dispersion in the candidates' positions is at least as large under plurality rule as it is under a runoff system. If $c < e_r(F)$ then because the requirement $c \ge |m - s(\epsilon, F)|$ in Proposition 3 may rule out equilibria under plurality rule in which $\epsilon > c$ the maximal degree of dispersion in the candidates' positions *may* be larger under a runoff system than under plurality rule. However, if the distribution $F$ is single-peaked and symmetric about its median, or is not too different from a distribution with these properties, then $|m - s(\epsilon, F)| < c$ for *all* values of $\epsilon$, so that the maximal degree of dispersion is definitely greater under plurality rule.

We found that under plurality rule there are distributions $F$ of ideal points for which no three-candidate equilibria exist for any values of $b$ and $c$; under a runoff system such equilibria exist for any distribution $F$ if $3c \leq b \leq 4c$. In this sense three-candidate equilibria are more likely under a runoff system. However, this leaves open the possibility that there are combinations of $b$, $c$, and $F$ that give rise to three-candidate elections under plurality rule but not under a runoff system. Thus we need a complete characterization of the parameters for which a three-candidate equilibrium exists under a runoff system. Note first that under a runoff system there is never a three-candidate equilibrium in which one candidate is sure to lose since such a candidate does not make it into the runoff and cannot have any effect on the ultimate outcome, and thus does not wish to incur the cost $c$. Nonetheless, the possibility of entry by a (fourth) citizen who cannot win, but who can alter who does, does influence the structure of the three-candidate equilibrium configurations.

Suppose that there is one candidate at $a_1 = m + t_1 - t_2$, one at $a_2 = t_1 + t_2 - m$, and one at $a_3 = t_2 + m - t_1$. If a citizen with ideal point in $(a_1, a_2)$ enters the competition then she gets into a runoff (with the candidate at $a_3$) if she obtains a larger fraction of the votes than both the candidate at $a_1$ and the candidate at $a_2$. Let $\lambda_1(F)$ be the maximal amount by which the fraction of the votes received by any such citizen who enters the competition exceeds the larger of the fractions received by the candidates at $a_1$ and $a_2$:

$$\begin{aligned}\lambda_1(F) \;=\; \max_{a\in(a_1,a_2)} \Big\{ & F(\tfrac{1}{2}(a_2+a)) - F(\tfrac{1}{2}(a_1+a)) - \\ & \max\big\{F(\tfrac{1}{2}(a_1+a)), \tfrac{2}{3} - F(\tfrac{1}{2}(a_2+a))\big\}\Big\}\end{aligned}$$

Then we need $\lambda_1(F) \leq 0$ in order that no citizen with ideal point in $(a_1, a_2)$ can enter and certainly get into a runoff. Similarly define $\lambda_2(F)$ (for citizens with ideal points in $(a_2, a_3)$).

**Proposition 10 (Three-candidate equilibria under a runoff system)**

**a.** *If $b \neq 4c$ then there is a unique three-candidate equilibrium in which not all of the candidates take the same position. In this equilibrium the positions of the three candidates are all different, equal to $a_1 = m+t_1-t_2$, $a_2 = t_1 + t_2 - m$, and $a_3 = t_2 + m - t_1$; each candidate obtains one third of the votes in the first ballot.*

**b.** *There exists $\bar{c}$ such that this equilibrium exists if and only if the following three conditions are satisfied.*

i. $b \geq 6c + 2|t_1 + t_2 - 2m|$.

ii. $c \geq \bar{c}$.

iii. $\max\{\lambda_1(F), \lambda_2(F)\} \leq 0$ *(for generic $F$).*

The reason that only this single configuration is a candidate for an equilibrium is that all three candidates must have a positive probability of being the ultimate winner, else they prefer not to enter. Thus each must obtain one third of the first-round vote and each must have a positive probability of winning in the second round if they reach it. In any configuration that satisfies these conditions and in which two candidates share an ideal position it is profitable for a fourth candidate who shares the lone candidate's ideal position to enter unless $b = 4c$. Thus all three must have distinct ideal positions. The two extreme candidates surely lose a runoff with the center candidate so they must have a positive probability of winning against one another in a runoff. The only configuration with these properties is the one defined in (a) of the proposition.

As to (b), condition (i) is necessary and sufficient for all three candidates to want to enter while (ii) guarantees that no fourth citizen wishes to enter for the sole purpose of altering the ultimate outcome by causing one candidate (as well as herself) to not make it to the runoff. Finally, (iii) captures the requirement that it must not be possible for any citizen with an ideal position between $a_1$ and $a_3$ to enter and get into the runoff, which she will surely win.

This results shows that indeed there are parameters that give rise to three-candidate elections under plurality rule but not under a runoff system, so that the comparison between the two systems with respect to the "likelihood" of three-candidate elections is ambiguous.


**Figure 4.** An example of a equilibrium under a runoff system in which an extremist loses. One candidate is at each of the points $a_1$, $a_2$, $a_3$, and $a_4$. The area of each square is 1. The dashed line is the midpoint of $[a_2, a_3]$. We require $b = 0$ and $c = 1$.

## 5. Concluding Remarks

### *Losing Extremists*

Under plurality rule a losing candidate $i$ who withdraws gives her votes to her two neighbors. If the result is that one of the neighbors wins then unless this neighbor's position is further from $i$'s position than the average winning position if $i$ stays in the competition (which requires the distribution of winning positions to be very asymmetric), $i$ is better off withdrawing. If $i$'s withdrawal has no effect on the outcome then she is also better off (since she saves the cost of entry). Thus in our model plurality rule gives citizens little incentive to enter if they lose; in particular Lemma 1 shows that an extremist never loses. (Nevertheless, equilibria in which candidates lose may exist, as we have seen.)

Under a runoff system the incentives are different. A candidate who withdraws increases the chances (roughly speaking) that her neighbors get into a runoff; but these neighbors may lose the runoff and may have displaced a more moderate candidate who would have had some chance of winning. To illustrate, consider the example in Figure 4, in which an extreme candidate loses in an equilibrium under a runoff system. If either candidate 1 withdraws then candidate 2 gets into a runoff against candidate 4, which she loses. Thus candidate 1's withdrawal changes the outcome from one in which 3 and 4 each win with probability $\frac{1}{2}$ to one in which 4 wins for sure. The change due to the exit of candidate 2 is similar, and the values of the parameters ensure that candidates 3 and 4 find it worthwhile to enter and no other citizen finds it worthwhile to do so.

While a general result eludes us our model thus captures a respect in which the incentives inherent in plurality and runoff systems differ: under a runoff system there may be an incentive for a citizen with an extreme position to enter in order to splinter the vote among other extreme parties of the same persuasion and change the outcome from one in which one of these parties gets into a runoff that she loses to an extremist of the opposite persuasion to one in which a more moderate candidate gets into such a runoff and wins. Under plurality rule the incentive is in the opposite direction: splintering the vote tends to decrease the chance of a like-minded candidate being elected and hence is not desirable.

### *The Cost of Entry*

We have assumed that the cost of entry $c$ is independent of both the number of candidates in the competition and the electoral system. The latter assumption may be unreasonable. A runoff system *could* operate with a single ballot, in which citizens both cast a "first-round" vote and express their preferences between every pair of candidates. However, this is not how runoff systems are usually operated; rather, there are two separate elections. The cost of running in both of these elections may exceed the cost of running in a single plurality-rule election, or perhaps the total cost may be approximately the same, the cost of participating in the first round of the runoff system being less than the cost of being a candidate in a plurality-rule election. We have not explored the implications that these various possibilities may have in our model.

### *Uncertainty*

The exact ties that occur in most of the equilibria that we discuss are an artifact of our assumption of perfect information. If candidates are uncertain about the distribution of the citizens preferences (or about $b$ or $c$) then it seems that equilibria may occur in which many candidates have different, positive probabilities of winning. The presence of uncertainty may have other significant implications that are absent from our model.

## References


Abramson, P. R., J. H. Aldrich, P. Paolino, and D. W. Rohde (1992), "'Sophisticated' Voting in the 1988 Presidential Primaries", *American Political Science Review* **86**, 55–69.

Cox, G. W. (1987), "Duverger's Law and Strategic Voting", Unpublished paper, Department of Government, University of Texas, Austin.

Duverger, M. (1964), *Political Parties.* London: Methuen.

Feddersen, T. J. (1992), "A Voting Model Implying Duverger's Law and Positive Turnout", *American Journal of Political Science* **36**, 938–962.

Feddersen, T. J. (1993), "Coalition-Proof Nash Equilibria in a Model of Costly Voting Under Plurality Rule", unpublished paper.

Feddersen, T. J., I. Sened, and S. G. Wright (1990), "Rational Voting and Candidate Entry under Plurality Rule", *American Journal of Political Science* **34**, 1005–1016.

Hotelling, H. (1929), "Stability in Competition", *Economic Journal* **39**, 41–57.

Palfrey, T. R. (1989), "A Mathematical Proof of Duverger's Law", pp. 69–91 in *Models of Strategic Choice in Politics* (P. C. Ordeshook, ed.), Ann Arbor: University of Michigan Press.

Riker, W. H. (1982), "The Two-Party System and Duverger's Law: An Essay on the History of Political Science", *American Political Science Review* **76**, 753–766.

Wittman, D. (1977), "Candidates with Policy Preferences: A Dynamic Model", *Journal of Economic Theory* **14**, 180–189.

Wittman, D. (1983), "Candidate Motivation: A Synthesis of Alternative Theories", *American Political Science Review* **77**, 142–157.

Wright, S. G., and W. H. Riker (1989), "Plurality and Runoff Systems and Numbers of Candidates", *Public Choice* **60**, 155–175.

## APPENDIX

This appendix contains the proofs of all of the results stated in the text.

**Proof of Lemma 1.** Suppose that a candidate who loses with certainty deviates and withdraws. In case (i) the fraction of votes received by the other candidates whose ideal position is the same as hers increases and the fraction of votes received by every other candidate remains the same. In case (ii) the fraction of the votes received by the candidate (or candidates) whose position is closest to hers increases and the fraction of votes received by every other candidate remains the same. Thus in both cases her deviation either has no effect on the outcome of the election or causes the set of winners to be the set of candidates whose ideal position is closest to hers (rather than a set of candidates with more distant ideal positions). Since the deviation saves her the cost $c$ it is thus profitable. □

**Proof of Proposition 2.** A one-candidate equilibrium requires that no other citizen with the same ideal position wishes also to enter, which implies that $\frac{1}{2}b - c \leq 0$. Further, $\frac{1}{2}b \leq c$ implies that it is an equilibrium for a single citizen with ideal position $m$ to choose $E$, since any entrant with a different ideal position who enters must lose and withdrawal by the single entrant yields her $-\infty$.

If there is a single entrant with ideal position $a \neq m$ then a citizen with any ideal position $d \in (a, 2m - a)$ can win outright by entering, getting a payoff of $b - c$ rather than $-|a - d|$. Thus we must have $-|a - d| \geq b - c$ for any such $d$, which implies that $b \leq c$ and $|m - a| \leq (c - b)/2$. Finally, if $b \leq c$ and $|m - a| \leq (c - b)/2$ then entry by a single citizen with ideal position $a$ is an equilibrium by the following argument. For a citizen with ideal position $d \in (a, 2m - a)$ it is best to choose $N$, as we have argued. A citizen whose ideal position is either $a$ or $2m - a$ wins with probability $\frac{1}{2}$ if she enters, and hence does not enter either. A citizen with ideal position outside $(a, 2m - a)$ surely loses if she enters, and hence does not enter by Lemma 1. Finally, the citizen who enters at $a$ obtains $-\infty$ if she withdraws. □

Before proving Proposition 3 we establish a result that is useful in the sequel.

**Lemma 11** *If $b > c$ then in any equilibrium no citizen can enter the competition and win outright.*

*Proof.* A citizen who wins outright obtains the payoff $b - c > 0$ while one who stays out of the competition obtains a nonpositive payoff. □

**Proof of Proposition 3.** If there are two candidates and they have the same ideal position $a$ then each obtains a payoff of $\frac{1}{2}b - c$. If either of them withdraws then she obtains a payoff of 0, so we require $b \geq 2c$ in order for the configuration to be an equilibrium. But now a citizen whose ideal position $d \neq a$ is close enough to $a$ wins outright if she enters the competition, so by Lemma 11 the configuration is not an equilibrium. Hence in any two-candidate equilibrium the candidates have different ideal positions. By Lemma 1 each candidate has a positive probability of winning, so that the candidates' positions are $m - \epsilon$ and $m + \epsilon$ for some $\epsilon > 0$.

We now establish **c**. In order for such a configuration in which one candidate is at $m - \epsilon$ and one is at $m + \epsilon$ to be an equilibrium we require that each citizen's action be optimal, with the following implications.

- *Citizens who enter*: For such a citizen the payoff to $E$ is $\frac{1}{2}b - \frac{1}{2} \cdot 2\epsilon - c$ while the payoff to $N$ is $-2\epsilon$. Thus for equilibrium we require $\epsilon \leq c - b/2$.
- *Citizens with ideal points outside $(m - \epsilon, m + \epsilon)$ (other than those who enter)*: If such a citizen chooses $E$ then she loses and either does not affect the set of winners or causes this set to consist only of the more remote candidate. Hence $N$ is optimal for her.
- *Citizens with ideal points in $(m - \epsilon, m + \epsilon)$*: If by entering such a citizen wins outright then she obtains the payoff $b - c$. If she stays out of the competition she obtains $-\epsilon$. By the first point above we have $b \geq 2(c - \epsilon)$, so that in equilibrium no such citizen can win outright if she enters: $\epsilon \leq e_p(F)$.

If $\epsilon = e_p(F)$ then a citizen who enters at $d = s(\epsilon, F)$ ties for first place with the two existing candidates, obtaining a payoff of $\frac{1}{3}b - c - \frac{1}{3}(d - m + \epsilon) - \frac{1}{3}(m + \epsilon - d) = \frac{1}{3}b - c - \frac{2}{3}\epsilon$. If she stays out of the competition she obtains $-\epsilon$. Thus for equilibrium we require $\epsilon \leq 3c - b$.

If $\epsilon < e_p(F)$ then the entry of a citizen with ideal position $d \in (m - \epsilon, s(\epsilon, F))$ causes the candidate at $m + \epsilon$ to win, resulting in a payoff for the citizen of $d - m - \epsilon - c$ rather than $\frac{1}{2}(d - m - \epsilon) + \frac{1}{2}(m - \epsilon - d)$; thus in equilibrium we require $d - m \leq c$ for all such $d$, or $s(\epsilon, F) - m \leq c$. Symmetrically the condition that a citizen with ideal position in $(s(\epsilon, F), m + \epsilon)$ not be better off by entering is equivalent to $m - s(\epsilon, F) \leq c$. Finally, if a citizen with ideal position $s(\epsilon, F)$ enters then she does not affect the outcome, and hence is definitely worse off.

Part **b** and the first part of **c** now follow. To complete the proof of **c** note that $|m - s(\epsilon, F)| < \epsilon$ for all $\epsilon > 0$ so that if $\epsilon \leq c$ then certainly $|m - s(\epsilon, F)| \leq c$. Finally, it follows from **c** that and equilibrium exists if and only if either $e_p(F) > c - b/2$ or $e_p(F) = c - b/2 \leq 3c - b$. But if $c - b/2 > 0$ then $b < 2c$, so that certainly $c - b/2 \leq 3c - b$. Thus **a** follows. □

**Proof of Lemma 4.** If in equilibrium more than two candidates take the same position then Lemma 1 implies that each has a positive probability of winning, which is thus the same for all of them. Each of their payoffs is at most $\frac{1}{3}b - c$, while if one of them withdraws then she obtains 0 (since then the set of winners is the set of candidates remaining at that position); thus we require $b \geq 3c$ in equilibrium. But now in order for it to be optimal for a citizen with ideal point just to either side to choose $N$ we need $b - c \leq 0$ (since she wins outright if she enters, obtaining the payoff $b - c$, and obtains a negative payoff if she chooses $N$). These two inequalities are incompatible, so no such equilibrium exists. □

**Proof of Proposition 5.** From Lemma 4 there is no equilibrium in which all three candidates have the same ideal point. The remaining possibilities are that the three candidates are at different positions or two are at one position and the third is at a different position. By Lemma 1 all three candidates must win with positive probability in the second case, so that part (a) follows, except for the claim about the precise positions of the candidates in part (ii). To establish the remainder of the result we give a complete characterization of the three-candidate equilibria.

Throughout we let $t_1 = F^{-1}(\frac{1}{3})$ and $t_2 = F^{-1}(\frac{2}{3})$. For any position $t$ and $\epsilon > 0$ let $\mu(\epsilon, t, F)$ be the maximal fraction of votes received by a candidate who locates at some point in $(t-\epsilon, t+\epsilon)$ when there is one candidate at $t-\epsilon$, one at $t+\epsilon$, and no other candidates in $[t-\epsilon, t+\epsilon]$:

$$\mu(\epsilon, t, F) = \max_{d \in (t-\epsilon, t+\epsilon)} \left\{ F\left(\tfrac{1}{2}(t+\epsilon+d)\right) - F\left(\tfrac{1}{2}(t-\epsilon+d)\right) \right\}.$$

Further let $h(t, \alpha, F)$ be the largest value of $\epsilon$ for which this maximal fraction is at most $\alpha$: $h(t, \alpha, F) = \max\{\epsilon \colon \mu(\epsilon, t, F) \leq \alpha\}$.

**Lemma 12** *There is a three-candidate equilibrium in which the positions taken are $a_1$, $a_2$, and $a_3$ with $a_1 < a_2 < a_3$ and every candidate receives a third of the votes if and only if for some $\epsilon_1 > 0$ and $\epsilon_2 > 0$ the following four conditions are satisfied.*

i. $a_1 = t_1 - \epsilon_1$, $a_2 = t_1 + \epsilon_1 = t_2 - \epsilon_2$, *and* $a_3 = t_2 + \epsilon_2$.

ii. $b \geq 3c + 2|\epsilon_1 - \epsilon_2|$.

iii. $c \geq \frac{2}{3}\max\{\epsilon_1 - 2\epsilon_2, \epsilon_2 - 2\epsilon_1\}$.

iv. $\epsilon_i < h(t_i, \frac{1}{3}, F)$ *for* $i = 1$, $2$.

*Proof.* Condition (i) is necessary and sufficient for each candidate to receive a third of the votes. The necessity and sufficiency of the remaining conditions are implications of the requirement that each citizen's action be optimal, as follows.

- *Citizens who enter*: For the candidate at $a_1$ the payoff to $E$ is $\frac{1}{3}b - c - \frac{2}{3}\epsilon_1 - \frac{2}{3}(\epsilon_1 + \epsilon_2)$ while the payoff to $N$ is $-2\epsilon_1$. Thus for equilibrium we require $b \geq 3c + 2(\epsilon_2 - \epsilon_1)$. The analogous condition implied by the optimality of the decision of the candidate at $a_3$ is $b \geq 3c + 2(\epsilon_1 - \epsilon_2)$, so that (ii) follows. For the candidate at $a_2$ the payoff to $E$ is $\frac{1}{3}b - c - \frac{2}{3}\epsilon_1 - \frac{2}{3}\epsilon_2$ while that to $N$ is $-2\epsilon_j$ for $j = 1$ or $j = 2$, so that (ii) implies that $E$ is optimal for her.

- *Citizens with ideal points outside* $(a_1, a_3)$ *(other than those who enter)*: If such a citizen chooses $E$ then she loses and causes the set of winners to be no closer to her ideal point than they are when she chooses $N$. Hence $N$ is optimal.

- *Citizens with ideal points in* $(a_1, a_3)$: Since by the first point above we have $b \geq 3c$ if follows from Lemma 11 that in an equilibrium we require $\epsilon_i \leq h(t_i, \frac{1}{3}, F)$ for $i = 1, 2$.

  If $\epsilon_1 = h(t_1, \frac{1}{3}, F)$ then a citizen with some position $d \in (a_1, a_2)$ can enter and cause the election to be a tie between her and the candidate at $a_3$. In order for such a deviation to be undesirable we need $\frac{1}{2}b - c - \frac{1}{2}(a_3 - d) \leq -\frac{1}{3}(d - a_1) - \frac{1}{3}(a_2 - d) - \frac{1}{3}(a_3 - d)$, or $d \leq a_3 - 4\epsilon_1 - 3b + 6c$. Using (ii) this implies that $d \leq a_3 - 4\epsilon_1 - 3c - 6|\epsilon_1 - \epsilon_2| < a_3 - 4\max\{\epsilon_1, \epsilon_2\}$, contradicting $d \geq a_1$. Thus if $\epsilon_1 = h(t_1, \frac{1}{3}, F)$ then a citizen with ideal point in $(a_1, a_2)$ can profitably deviate; similarly if $\epsilon_2 = h(t_2, \frac{1}{3}, F)$ then a citizen with ideal point in $(a_2, a_3)$ can profitably deviate. Thus we need $\epsilon_i < h(t_i, \frac{1}{3}, F)$ for $i = 1, 2$.

  Now, if under this condition a citizen with ideal position $d \in (a_1, a_2)$ enters then the candidate at $a_3$ wins, resulting in a payoff for the citizen of $d - a_3 - c$ rather than $-\frac{1}{3}(d - a_1) - \frac{1}{3}(a_2 - d) - \frac{1}{3}(a_3 - d)$; thus in equilibrium we require $3c \geq 2d - 2a_3 + a_2 - a_1$ for all such $d$, or $3c \geq 3a_2 - 2a_3 - a_1 = 2\epsilon_1 - 4\epsilon_2$. Symmetrically the condition that a citizen with ideal position in $(a_2, a_3)$ not be better off by entering is equivalent to $3c \geq 2\epsilon_2 - 4\epsilon_1$.

This completes the proof. □

We now consider the possibility of an equilibrium in which two candidates take the same position; we restrict attention to the case in which this position greater than the position of the remaining candidate. Let $a_1 = 2t_1 - t_2$ and define $\phi(F)$ to be the maximum plurality of a candidate at some point $d \in (a_1, t_2)$ when there is one candidate at $a_1$ and two at $t_2$:

$$\begin{aligned}\phi(F) = \max_{d \in (a_1, a_2)} & \left[F\left(\tfrac{1}{2}(d+t_2)\right) - F\left(\tfrac{1}{2}(d+a_1)\right) - \right. \\ & \left. \max\left\{F\left(\tfrac{1}{2}(d+a_1)\right), \tfrac{1}{2}\left(1 - F\left(\tfrac{1}{2}(d+t_2)\right)\right)\right\}\right].\end{aligned}$$

For a generic distribution $F$ we have $\phi(F) \neq 0$, and it is to this case that we restrict attention. Finally, define $r(F)$ to be the position $d \in (a_1, t_2)$ with the property that the entry of a candidate at $d$ causes each of the three original candidates to receive the same fraction of the votes: $F(\frac{1}{2}(r(F)+a_1)) = \frac{1}{2}[1 - F(\frac{1}{2}(r(F) + t_2))]$.

**Lemma 13** *For a distribution $F$ with $\phi(F) \neq 0$ there is a three-equilibrium in which one candidate takes the position $a_1$ and two candidates take the same position $a_2 > a_1$ if and only if the following four conditions are satisfied.*

i. $a_1 = 2t_1 - t_2$ *and* $a_2 = t_2$.

ii. $b \geq 3c + 2(t_2 - t_1)$.

iii. $c \geq \max\{\frac{4}{3}(t_1 - r(F)), \frac{2}{3}(r(F) - t_1)\}$.

iv. $\phi(F) < 0$.

*Proof.* First note that a necessary and sufficient condition for each candidate to obtain a third of the votes is $\frac{1}{2}(a_1 + a_2) = t_1$; let $a_2 - t_1 = t_1 - a_1 = \epsilon$. Now consider the conditions implied by the optimality of each citizen's action.

- *Candidates at $a_2$*: Each of these candidates obtains 0 if she withdraws (since the other candidate at $a_2$ wins in this case), so for equilibrium we need $\frac{1}{3}b - c - \frac{2}{3}\epsilon \geq 0$, or $b \geq 3c + 2\epsilon$.

- *Candidate at $a_1$*: This candidate obtains $-2\epsilon$ if she withdraws, so for equilibrium we need $\frac{1}{3}b - c - \frac{4}{3}\epsilon \geq -2\epsilon$, or $b \geq 3c - 2\epsilon$, which follows from the condition in the previous point.

- *Citizens who are not candidates*: By the first point above we have $b \geq 3c$ so it follows from Lemma 11 that in an equilibrium no citizen can win outright if she enters.

Now, if $a_2 < t_2$ then any citizen whose ideal point is in $(a_2, t_2)$ wins outright if she enters, while if $a_2 > t_2$ then any citizen whose ideal point is in $(t_2, a_2)$ wins outright if she enters. Thus we need $a_2 = t_2$, so that $\epsilon = t_2 - t_1$. Combined with the condition in the first point above this gives us (i) and (ii).

Under these conditions any citizen whose ideal point is outside $(a_1, t_2)$ loses if she enters. In order to ensure that no citizen in $(a_1, t_2)$ can win outright if she enters we need $\phi(F) \leq 0$, which under our assumption that $\phi(F) \neq 0$ yields (iv).

Finally we need to find the conditions under which a citizen whose ideal point is in $(a_1, t_2)$ and who loses if she enters is worse off if she enters than if she stays out. Consider a citizen whose ideal point is $d$. There are three cases.

- $d = r(F)$: In this case the entry of the citizen has no effect on the outcome, so certainly the entrant is worse off.
- $d < r(F)$: In this case the entry of the citizen causes the candidates at $t_2$ to be the winners. Thus for an equilibrium we require $-\frac{1}{3}(d - a_1) - \frac{2}{3}(t_2 - d) \geq -(t_2 - d) - c$, or $c \geq \frac{2}{3}(d - t_1)$. This condition must hold for any $d < r(F)$, so we need $c \geq \frac{2}{3}(r(F) - t_1)$.
- $d > r(F)$: In this case the entry of the citizen causes the candidate at $a_1$ to be the winner. Thus for an equilibrium we require $-\frac{1}{3}(d - a_1) - \frac{2}{3}(t_2 - d) \geq -(d - a_1) - c$, or $c \geq \frac{4}{3}(t_1 - d)$. This condition must hold for any $d < r(F)$, so we need $c \geq \frac{4}{3}(t_1 - r(F))$.

This completes the proof. □

Finally we consider the possibility of an equilibrium in which the positions of the three candidates are different and the middle candidate loses.

**Lemma 14** *There is an equilibrium in which three candidates take the positions $a_1$, $a_2$, and $a_3$ with $a_1 < a_2 < a_3$ and the candidates at $a_1$ and $a_2$ receive* *the same fraction of the votes while the candidate at $a_3$ receives a smaller fraction if and only if there exist $m_1 \in (t_1, m)$, $m_2 \in (m, t_2)$, $\epsilon_1 > 0$, and $\epsilon_2 > 0$ such that $F(m_1) = 1 - F(m_2)$ and the following five conditions are satisfied.*

i. $a_1 = m_1 - \epsilon_1$, $a_2 = m_1 + \epsilon_1 = m_2 - \epsilon_2$, *and* $a_3 = m_2 + \epsilon_2$.

ii. $b \geq 4c$.

iii. *Either* $\epsilon_1 > \epsilon_2$ *and* $(a_1 + a_3)/2 > m$, *or* $\epsilon_2 > \epsilon_1$ *and* $(a_1 + a_3)/2 < m$.

iv. $|\epsilon_1 - \epsilon_2| = c$.

v. $\epsilon_1 \leq h(m_1, 1 - F(m_2), F)$ *and* $\epsilon_2 \leq h(m_2, F(m_1), F)$.

*Proof.* Condition (i) is necessary and sufficient for the candidates at $a_1$ and $a_3$ to tie for first place and the candidate at $a_2$ to lose. The necessity and sufficiency of the remaining conditions are implications of the requirement that each citizen's action be optimal, as follows.

- *Citizens who enter*: For the candidate at $a_1$ the payoff to $E$ is $\frac{1}{2}b - c - (\epsilon_1 + \epsilon_2)$ while the payoff to $N$ is $-2\epsilon_1$ (since the candidate at $a_2$ wins in this case). Thus for equilibrium we require $b \geq 2c + 2(\epsilon_2 - \epsilon_1)$. The analogous condition implied by the optimality of the decision of the candidate at $a_3$ is $b \geq 2c + 2(\epsilon_1 - \epsilon_2)$. Thus we need $b \geq 2c + 2|\epsilon_1 - \epsilon_2|$.

  Now suppose that the candidate at $a_2$ withdraws. If $(a_1 + a_3)/2 = m$, so that the other two candidates still tie for first place, then certainly the withdrawal is advantageous. Suppose that $(a_1 + a_3)/2 > m$, so that the candidate at $a_1$ wins. Then in order for the entry of the candidate at $a_2$ to be optimal we need $-c - \epsilon_1 - \epsilon_2 \geq -2\epsilon_1$, or $c \leq \epsilon_1 - \epsilon_2$. Since $c > 0$ this implies that $\epsilon_1 > \epsilon_2$ and hence $(a_1 + a_3)/2 < a_2$. Similarly if $(a_1 + a_3)/2 < m$ then we need $c \leq \epsilon_2 - \epsilon_1$ and hence $\epsilon_2 > \epsilon_1$ and $(a_1 + a_3)/2 > a_2$.

- *Citizens with ideal points outside $(a_1, a_3)$ (other than those who enter)*: If such a citizen chooses $E$ then she loses and causes the set of winners to be no closer to her ideal point than they are when she chooses $N$. Hence $N$ is optimal.

- *Citizens with ideal points in* $(a_1, a_3)$: By the first point above we have $b \geq 2c$ so from Lemma 11 we require $\epsilon_1 \leq h(m_1, 1 - F(m_2), F)$ and $\epsilon_2 \leq h(m_2, F(m_1), F)$ in an equilibrium.

  Suppose that a citizen with ideal point $d \in (a_1, a_2)$ enters. Then the candidate at $a_3$ wins and the citizen obtains the payoff $-c - a_3 - d$ rather than $-\frac{1}{2}(d - a_1) - \frac{1}{2}(a_3 - d)$. Thus we need $a_1 - a_3 \geq 2(d - a_3 - c)$ for all $d \in (a_1, a_2)$, or $a_1 - a_3 \geq 2(a_2 - a_3 - c)$, which is equivalent to $c \geq \epsilon_1 - \epsilon_2$. The analogous condition for a citizen with ideal point in $(a_2, a_3)$ is $c \geq \epsilon_2 - \epsilon_1$. Thus we need $c \geq |\epsilon_1 - \epsilon_2|$.

  In the first point above we concluded that if $(a_1 + a_3)/2 > m$ then $c \leq \epsilon_1 - \epsilon_2$, while if $(a_1 + a_3)/2 < m$ then $c \leq \epsilon_2 - \epsilon_1$. It follows that $c = |\epsilon_1 - \epsilon_2|$.

This completes the proof. □

**Proof of Proposition 6.** Let the candidates' positions be $a_1 \leq a_2 \leq \cdots \leq a_n$. By Lemma 1 the candidates at $a_1$ and $a_n$ are winners. First suppose that $a_1 = a_2$. Then by Lemma 4 we have $a_3 > a_1$. Let one of the candidates at $a_1$ be $i$. By Lemma 1 the candidates at $a_1$ are winners, so that if $i$ withdraws then the other candidate at $a_1$ is the winner. Thus $i$'s payoff to $N$ is 0 while her payoff to $E$ is less than $(1/k)b - c$. Hence for an equilibrium of this type we require that $b > kc$. A similar argument can be made for an equilibrium in which $a_{n-1} = a_n$.

Now suppose that $a_1 < a_2$ and $a_{n-1} < a_n$. We must have either $a_n - a_1 \geq 2(a_2 - a_1)$ or $a_n - a_1 \geq 2(a_n - a_{n-1})$. In the former case we claim that a candidate at $a_1$, say $i$, can profitably withdraw unless $b \geq kc$. If she does so then only the fraction of the votes received by the candidates at $a_2$ changes. If these candidates were originally winners then the withdrawal of $i$ makes them the only winners. If there is only one candidate at $a_2$ and she originally lost then, since she obtains all of $i$'s votes, she becomes the outright winner. Thus in each case $i$'s withdrawal yields her a payoff of $-(a_2 - a_1)$ as opposed to at most $(1/k)b - [(k-2)/k](a_2 - a_1) - (1/k)(a_n - a_1) - c$ when she enters. Since

$a_n - a_1 \geq 2(a_2 - a_1)$ this latter payoff is at most $(1/k)b - (a_2 - a_1) - c$, which is at least $-(a_2 - a_1)$ only when $b \geq kc$. □

**Proof of Proposition 7.** By Proposition 5b the condition $b \geq 3c$ is necessary for an equilibrium to exist in which there are three candidates. We now show that if there are four or more candidates then there are at least three winners, so that the result follows from Proposition 6.

We need to show that there is no equilibrium in which there are two winners and at least four candidates. By Lemma 1 in any such equilibrium all the candidates take different positions, and the two whose positions are extreme are the winners. The cases of four and five or more candidates require different arguments, as follows.

First consider the case of four candidates. Let the candidates be 1, 2, 3, and 4 and let their ideal points be $a_1$, $a_2 = a_1 + \epsilon_1$, $a_3 = a_2 + \epsilon_2$, and $a_4 = a_3 + \epsilon_3$. If the withdrawal of candidate 2 leads to a tie for first place between candidates 1 and 3 then certainly 2's withdrawal is beneficial. Thus 2's withdrawal must lead to an outright victory for either 1 or 3. Similarly the withdrawal of candidate 3 must lead to an outright victory for either 2 or 4. Suppose that the withdrawal of 2 leads to a win for 1 and the withdrawal of 3 leads to a win for 2. The optimality of the entry decision of candidate 2 then requires $\epsilon_1 \geq \frac{1}{2}\epsilon_1 + \frac{1}{2}(\epsilon_1 + \epsilon_3)$ or $\epsilon_1 \geq \epsilon_2 + \epsilon_3$ and the optimality of the entry decision of candidate 3 requires $\epsilon_2 \geq \frac{1}{2}(\epsilon_1 + \epsilon_2) + \frac{1}{2}\epsilon_3$ or $\epsilon_2 \geq \epsilon_1 + \epsilon_3$. Since these two inequalities are incompatible this pattern of winners in the event of the withdrawals of candidates 2 and 3 is not possible. Similar arguments eliminate two of the three other possible patterns, leaving the possibility that candidate 3 wins when candidate 2 withdraws and vice versa, which implies that $\epsilon_2 \geq \epsilon_2 + \epsilon_3$. Now, we claim that in this case there is a point in $(a_2, a_3)$ at which an entrant can win for sure. To see this, first note that there is a point in $(a_2, a_3)$ at which an entrant can receive the votes of all citizens whose ideal points lie in the interval $(\frac{1}{2}(a_1 + a_3), \frac{1}{2}(a_2 + a_4))$, since $\frac{1}{2}(a_2 + a_4) - \frac{1}{2}(a_1 + a_3) = \frac{1}{2}(\epsilon_1 + \epsilon_3) \leq \frac{1}{2}\epsilon_2$. To complete the argument we show that the votes of these citizens are enough to win. Let $\alpha = F\left(\frac{1}{2}(a_1 + a_2)\right) = 1 - F\left(\frac{1}{2}(a_3 + a_4)\right)$, $\beta_1 = F\left(\frac{1}{2}(a_1 + a_3)\right) - F\left(\frac{1}{2}(a_1 + a_2)\right)$, $\beta_2 = F\left(\frac{1}{2}(a_3 + a_4)\right) - F\left(\frac{1}{2}(a_2 + a_4)\right)$, and $\gamma = F\left(\frac{1}{2}(a_2 + a_4)\right) - F\left(\frac{1}{2}(a_1 + a_3)\right)$. The fact that candidate 2 wins when candidate 3 withdraws, and vice versa, means that $\gamma + \beta_1 > \alpha + \beta_2$ and $\gamma + \beta_2 > \alpha + \beta_1$, so that $\gamma > \alpha$, completing the argument.

Now suppose that there are $n \geq 5$ candidates. Call them candidates 1 through $n$ and let the position of candidate $i$ be $a_i$ for $i = 1, \ldots, n$; let $a_i = a_{i-1} + \epsilon_{i-1}$ for $i = 1, \ldots, n-1$ and let $\sum_{j=3}^{n-3} \epsilon_j = \delta$. (If $n = 5$ then $\delta = 0$.) Suppose that candidate 2 withdraws. If this results in a tie for first place between candidates 1 and 3 then the withdrawal is profitable, so instead either candidate 1 or candidate 3 must win outright. If candidate 1 wins then in order for the entry of 2 to be optimal we require $\epsilon_1 \geq \frac{1}{2}\epsilon_1 + \frac{1}{2}(\epsilon_2 + \delta + \epsilon_{n-2} + \epsilon_{n-1})$, or $\epsilon_1 \geq \epsilon_2 + \delta + \epsilon_{n-2} + \epsilon_{n-1}$; if candidate 3 wins then the analogous condition is $\epsilon_2 \geq \epsilon_1 + \delta + \epsilon_{n-2} + \epsilon_{n-1}$. Similarly, if candidate $n$ wins when $n-1$ withdraws then in order for the entry of $n-1$ to be optimal we require $\epsilon_{n-1} \geq \epsilon_1 + \epsilon_2 + \delta + \epsilon_{n-2}$; if candidate $n-2$ wins then the analogous condition is $\epsilon_{n-2} \geq \epsilon_1 + \epsilon_2 + \delta + \epsilon_{n-1}$. It is easy to see that no combination of these conditions is possible. □

**Proof of Proposition 8.** If there are $k \geq 2$ candidates with ideal position $m$ then entry by a citizen with a different ideal position results in certain defeat: if $k = 2$ or $k \geq 3$ and the position of the entrant is far from $m$ then the entrant fails to reach the runoff, and otherwise she loses in the runoff. Thus it is optimal for such a citizen not to enter. It is optimal for another citizen with ideal position $m$ not to enter since $b/(k+1) \geq c$, and it is optimal for the $k$ candidates to enter since $b/k \leq c$. □

**Proof of Proposition 9.** It is immediate that in any equilibrium the positions of the candidates are $m - \epsilon$ and $m + \epsilon$ for some $\epsilon \geq 0$. The case in which $\epsilon = 0$ is covered in Proposition 8. If $\epsilon > 0$ then such a configuration is an equilibrium if and only if each citizen's action is optimal, with the following implications.

- *Citizens who enter*: For such a citizen the payoff to $E$ is $\frac{1}{2}b - \frac{1}{2} \cdot 2\epsilon - c$ while the payoff to $N$ is $-2\epsilon$. Thus for equilibrium we require $b \geq 2(c - \epsilon)$.
- *Citizens with ideal points outside* $[m - \epsilon, m + \epsilon]$: If such a citizen chooses

$E$ then she loses (either in the first or second ballots) and either does not affect the set of winners (if she does not get into the runoff) or causes this set to consist only of the more remote candidate. Hence $N$ is optimal for her.

- *Citizens with ideal points $m - \epsilon$ or $m + \epsilon$ (other than those who enter)*: If such a citizen chooses $E$ then she gets into a runoff with probability $\frac{1}{2}$ and wins this runoff with probability $\frac{1}{2}$. Thus her payoff is $\frac{1}{2}(\frac{1}{2}b - \frac{1}{2} \cdot 2\epsilon) + \frac{1}{2} \cdot \frac{1}{2}(-2\epsilon) - c = \frac{1}{4}b - \epsilon - c$. If she chooses $N$ then her payoff is $-\epsilon$. Thus it is optimal for her to stay out if and only if $b \leq 4c$.

- *Citizens with ideal points in* $(m-\epsilon, m+\epsilon)$: If by choosing $E$ such a citizen wins outright then she obtains the payoff $b-c$ rather than the payoff $-\epsilon$ that she obtains when she chooses $N$. Since by the first point above we have $b \geq 2(c-\epsilon)$ we deduce that no such citizen can win outright if she enters: $\epsilon \leq e_r(F)$.

  If $\epsilon = e_r(F)$ then there is a citizen with ideal position in $(m-\epsilon, m+\epsilon)$ who, if she becomes a candidate, ties for second place in the first ballot and hence gets into a runoff with probability $\frac{1}{2}$; in a runoff she wins for sure. Her payoff to $E$ is thus $\frac{1}{2}b - \frac{1}{2}\epsilon - c$, while her payoff to $N$ is $-\epsilon$. Thus for equilibrium we require $b \leq 2c - \epsilon$.

  If $\epsilon < e_r(F)$ then any citizen with ideal position in $(m-\epsilon, m+\epsilon)$ who becomes a candidate loses for sure and has no effect on the set of winners. Thus it is optimal for any such citizen to choose $N$.

Parts **b** and **c** follow. To show **a** note that if $e_r(F) = c - b/2$ then $b < 2c$, so that $e_r(F) = c - b/2 < 2c - b$. □

**Proof of Proposition 10. a.** First consider the possibility of an equilibrium in which two candidates are at the same position and the third is at a different position. Suppose, without loss of generality, that candidate 1 is at $a_1$ and candidates 2 and 3 are at $a_2 > a_1$. If candidate 1 receives less than a third of the votes then she certainly loses, and hence is better off withdrawing. If candidate 1 receives at least a third of the votes then she must receive at most half of the votes otherwise she wins outright on the first ballot and candidates 2 and 3 are better off withdrawing. Further, she must have a positive probability of winning in the runoff (otherwise she prefers to withdraw), so that she must receive exactly half of the votes; this means that $a_1 = m - \epsilon$ and $a_2 = m + \epsilon$ for some $\epsilon > 0$. Now, in order for candidates 2 and 3 to prefer $E$ to $N$ we need $\frac{1}{4}b - \frac{1}{2} \cdot 2\epsilon - c \geq \frac{1}{2} \cdot (-2\epsilon)$, or $b \geq 4c$. But if another candidate with ideal position $a_1$ enters then each of the four wins with probability $\frac{1}{4}$, so to prevent this possibility it must be that $b \leq 4c$. Hence for such an equilibrium we require $b = 4c$.

Now consider the possibility of an equilibrium in which the three candidates' ideal positions are $a_1 < a_2 < a_3$. If any of the three has no chance of winning in the first round then she prefers to withdraw. Thus each must obtain one third of the votes in the first round, so that the positions satisfy the condition in Lemma 12(i). This implies that candidate 2 surely wins if she makes it to the runoff. Each of the other two candidates prefers to withdraw unless she has a positive probability of winning in a runoff; since she can win only if she faces the other extreme candidate we must have $m - a_1 = a_3 - m$, which implies that the positions are those given in part (a) of the result.

**b.** First note that for this configuration candidates 1 and 3 win with probability $\frac{1}{6}$ and candidate 2 wins with probability $\frac{2}{3}$. In order for the configuration to be an equilibrium we require that each citizen's action be optimal, with the following implications.

- *Citizens who enter*: With probability $\frac{1}{3}$ candidate 1 is in a runoff with candidate 2, which candidate 2 certainly wins; with probability $\frac{1}{3}$ she is in a runoff with candidate 3, which she wins with probability $\frac{1}{2}$; and with probability $\frac{1}{3}$ she does not make it to the runoff, which candidate 2 wins. Thus her payoff is $\frac{1}{3}(-2\epsilon_1) + \frac{1}{3}[\frac{1}{2}b - \frac{1}{2}(2\epsilon_1 + 2\epsilon_2)] + \frac{1}{3}(-2\epsilon_1) - c$. If she withdraws then she obtains $-2\epsilon_1$ (since candidate 2 then certainly wins). Thus to make it optimal for her to enter we need $b \geq 6c + 2(\epsilon_2 - \epsilon_1)$. A similar calculation for candidate 3 yields the condition $b \geq 6c+2(\epsilon_1 - \epsilon_2)$, so that we need $b \geq 6c+2|\epsilon_2-\epsilon_1| = 6c+2|t_1+t_2-2m|$. Finally, in order for the entry of candidate 2 to be optimal we require $\frac{1}{3}b+\frac{1}{3}b-\frac{1}{3}(\epsilon_1+\epsilon_2)-c \geq$

$-(\epsilon_1+\epsilon_2)$, or $b \geq \frac{3}{2}c-(\epsilon_1+\epsilon_2)$, which is implied by the previous condition.

- *Citizens with ideal points outside* $(a_1, a_3)$ *(other than those who enter)*: The entry of any such citizen either has no effect on the outcome (if the citizen's position is very extreme) or causes candidate 2 to be the outright winner. Thus for it to be optimal for such a citizen with ideal position $d \leq a_1$ not to enter we require $-\frac{1}{6}(a_1-d)-\frac{2}{3}(a_2-d)-\frac{1}{6}(a_3-d) \geq -(a_2-d)-c$, or $3c \geq 2m-(t_1+t_2)$. The analogous condition for a citizen with ideal position $d \geq a_3$ is $3c \geq t_1+t_2-2m$, so that we require $3c \geq |t_1+t_2-2m|$.

- *Citizens with ideal points in* $(a_1, a_2)$: If such a citizen who enters the competition obtains a larger fraction of the votes than candidates 1 and 2 then she gets into the runoff and wins for sure. Thus in equilibrium we need $\lambda_1(F) \leq 0$ for $i = 1, 2$.

  If $\lambda_1(F) < 0$, the generic case, then a citizen with ideal point in $(a_1, a_2)$ who enters certainly loses, but affects the identity of the winner. In order to ensure that the net effect of the change is not beneficial we require $c$ to be large enough.

- *Citizens with ideal points in* $(a_2, a_3)$: Symmetric considerations apply to these citizens, resulting in the requirements that $\lambda_2(F) \leq 0$ and $c$ be large enough.

- *Citizens with ideal point* $a_2$ *(other than the one who enters)*: The entry of such a citizen leads to candidates 1 and 3 tying in a runoff, and outcome that is worse for the citizen than the equilibrium one (in which candidates 1 and 3 each win with probability $\frac{1}{6}$ and candidate 2 wins with probability $\frac{2}{3}$.

This completes the proof. $\square$