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1. Introduction

A popular model in economics is the rational distributed lag model proposed by Jorgenson (1966). In this paper we deal with the version of this model in which the polynomial in the denominator is of degree one. Popular special cases are the geometric distributed lag model of Koyck (1954) and Nerlove's (1958) adaptive expectations model. The coefficients of such models are sometimes estimated by ordinary least squares (OLS), which ignores the correlation between the right-hand lagged dependent variable and the auto-correlated disturbances and is, hence, inconsistent. Alternatively, Liviatan's (1961) instrumental variables estimator (IV) may be used to obtain consistent estimates. Some small sample properties of IV have been derived by Nagar and Gupta (1968) and by Scadding (1973). The aim of this paper is to derive exact and small σ asymptotic properties for OLS and small σ asymptotic properties of IV.

The main results of the paper can be summarized as follows. The exact, and approximate bias and mean squared error of the OLS estimator are derived and it is shown that, with the sample size fixed, OLS converges to the true value of the parameter if the noncentrality parameter of its distribution increases indefinitely; that is, as σ grows small. With regard to the IV estimator we note that its exact moments, to any order, do not exist. An approximation to the exact distribution has been obtained which is centered on the true parameter. When this approximation is valid we are able to give the distribution of the IV estimates of the coefficients of the exogenous variables.

In section 2 we present the model and its assumptions. Then in section 3 we analyze the exact and approximate moments of the OLS estimator. Finally, in section 4 we consider the distribution of the IV estimator.

2. The Model

We begin with the assumption that the values of y_t are independent drawings from normal populations with constant variances but varying means.

(2.1) $$y_t \sim N(\mu_t, \sigma^2) \text{ for } t = 1,\ldots,T$$

Next, we assume that μ_t is determined by a linear function analogous to a regression equation

(2.2) $$\mu_t = \gamma\, \mu_{t-1} + X_t'\beta$$

where X_t' is a non-random 1xK vector, β is a Kx1 vector of unknown coefficients, γ is an unknown scalar coefficient, and μ_t and μ_{t-1} are unknown means of y_t and y_{t-1}. To ensure that the process described by (2.2) is stable we assume

(2.3) $$|\gamma| < 1.$$

Now y_t can be written as

(2.4) $$\begin{aligned} y_t &= \mu_t + \eta_t = \gamma\, \mu_{t-1} + X_t'\beta + \eta_t \\ &= \gamma(\mu_{t-1} + \eta_{t-1}) + X_t'\beta + \eta_t - \gamma\, \eta_{t-1} \\ &= \gamma\, y_{t-1} + X_t'\beta + \varepsilon_t, \quad t=2,\ldots,T, \end{aligned}$$

where $\varepsilon_t = \eta_t - \gamma\, \eta_{t-1}$ and η_t is an independent drawing from $N(0,\sigma^2)$. Therefore, ε_t follows a first order moving average scheme with

(2.5) $\quad E\, \varepsilon_t = 0$ and

(2.6) $\quad \mathrm{var}(\varepsilon_t) = E(\eta_t - \gamma\, \eta_{t-1})^2 = (1 + \gamma^2)\sigma^2$

The covariance of ε_t and ε_s $(s < t)$ is

(2.7) $$E(\varepsilon_t\, \varepsilon_s) = E(\eta_t - \gamma\, \eta_{t-1})(\eta_s - \gamma\, \eta_{s-1})$$

$$= \begin{cases} -\gamma\, \sigma^2 & \text{if } s = t-1 \\ 0 & \text{if } s < t-1 \end{cases}$$

Therefore, the coefficient of autocorrelation is

(2.8) $$r(\varepsilon_t\, \varepsilon_{t-1}) = \frac{E(\varepsilon_t\, \varepsilon_{t-1})}{(1+\gamma)^2 \sigma^2} = \frac{-\gamma}{1+\gamma^2}$$

This moving average process may seem to be unduly arbitrary. However, it is solely the result of assumptions (2.1) and (2.2), and (2.2) can be obtained in several appealing ways. First, consider a Koyck (1954) type distributed lag model which has been specified in terms of μ_t (instead of in terms of the random y_t)

(2.9) $$\mu_t = \alpha_o + \alpha_1 z_t + \alpha_1 \lambda z_{t-1} + \alpha_1 \lambda^2 z_{t-2} + \dots \; .$$

This model says that the mean response, μ_t, is a function of the present value and all past values of an exogenous variable, z_t, where α_1 and λ, $0 < \lambda < 1$, are unknown coefficients. Then, applying the Koyck transformation we have

(2.10) $$\mu_t = \lambda\, \mu_{t-1} + [1, z_t] \begin{bmatrix} (1-\lambda)\alpha_o \\ \alpha_1 \end{bmatrix}$$

which is of the form (2.2) with $\gamma = \lambda$, $X_t' = [1 \;\; z_t]$ and $\beta = \begin{bmatrix} (1-\lambda)\alpha_o \\ \alpha_1 \end{bmatrix}$

A second justification for (2.2) is the adaptive expectations model of Nerlove (1958).

(2.11) $\mu_t = \alpha_o + \alpha_1 p_t^*$

(2.12) $p_t^* - p_{t-1}^* = \delta(p_{t-1} - p_{t-1}^*) \quad 0 < \delta \le 1$

which says that the mean response, μ_t, depends on expectations about the future, p_t^*, and that these expectations are adjusted by some fraction of the extent to which past expectations were in error. By substitution and transformation, this model can be reduced to

(2.13) $\mu_t = (1-\delta)\mu_{t-1} + [1 \ p_{t-1}] \begin{bmatrix} \alpha_o\delta \\ \alpha_1\delta \end{bmatrix}$

which is of the form of (2.2) with $\gamma = (1-\delta)$, $X_t' = [1 \ p_{t-1}]$ and $\beta = \begin{bmatrix} \alpha_o\delta \\ \alpha_1\delta \end{bmatrix}$.

More generally, consider the rational distributed lag model, Jorgenson (1966), in the case that the polynomial in the denominator is of degree one.

(2.14) $\mu_t = \dfrac{\sum_{j=1}^{J} \alpha_j(L) z_{tj}}{1-\gamma L}, \ 0 < \gamma < 1,$

where $\alpha_j(L)$ is a polynomial in the lag operator L and z_{tj} is observation t on exogenous variable number j. This is of the form (2.2) with $X_t\beta = \sum_{j=1}^{J} \alpha_j(L) z_{tj}$.

3. Least Squares Estimation

It is convenient, at this point, to rewrite equation (2.4) in matrix notation

(3.1) $y = \gamma \, y_{-1} + X \, \beta + \varepsilon$

where y is an nxl vector of independent random variables $y_t \sim N(\mu_t \ \sigma^2)$, $n = T - 1$, y_{-1} is an nxl vector of independent random variables $y_{t-1} \sim N(\mu_{t-1}, \sigma^2)$, X is an nxK matrix with non-random rows X_t', and ε is an nxl vector of moving average disturbances. The OLS estimates of γ and β are obtained by solving the equations

$$(3.2)\qquad c\, y'_{-1}y_{-1} + y'_{-1}X\, b = y'_{-1}y$$

$$(3.3)\qquad c\, X'y_{-1} + X'X\, b = X'y$$

OLS will not be consistent in this application because

$$(3.4)\qquad y'_{-1}\varepsilon = -\gamma\, \eta'_{-1}\eta_{-1} - \gamma\, \mu'_{-1}\, \eta_{-1} + \mu'_{-1}\eta + \eta'_{-1}\, \eta,$$

(where μ_{-1}, η and η_{-1} are nx1 vectors of elements μ_{t-1}, η_t and η_{t-1}) and we cannot reasonably expect to have plim $n^{-1}\eta'_{-1}\eta_{-1} = 0$. Nevertheless, we will consider the OLS estimator of γ, c, and present its exact moments.

Equation (3.3) can be solved for b which can then be substituted into (3.2) to obtain:[2]

$$(3.5)\qquad c = \frac{y'_{-1}\, M\, y}{y'_{-1}\, M\, y_{-1}} = \frac{z'\, N\, z}{z'\, D'_1 M D_1 z}$$

where $M = I - X(X'X)^{-1}X'$ is an idempotent matrix with rank $h = n - K$, $z_t = \frac{y_t}{\sigma} \sim N(\mu_t/\sigma, 1)$ $(t = 1, \ldots, T)$, and $N = \frac{1}{2}(D'_1\, MD_2 + D'_2\, MD_1)$, with $D_1 = [I_n\ 0]$ and $D_2 = [0\ I_n]$. That is, D_1 and D_2 are nxn identity matrices bordered by columns of zeros. N and $D'_1\, MD_1$ are both symmetric and $D'_1\, MD_1$ is idempotent of rank h so that we can find a TxT orthogonal matrix P such that

$$(3.6)\qquad P\, D'_1\, MD_1\, P' = \begin{bmatrix} I_h & 0 \\ 0 & 0 \end{bmatrix},$$

where I_h is an hxh identity matrix. Now let

$$(3.7)\qquad Pz = \begin{bmatrix} s \\ t \end{bmatrix} \sim N(P\bar{z},\ I_T) = N\left[\begin{pmatrix} \bar{s} \\ \bar{t} \end{pmatrix},\ I_T\right]$$

where $\bar{z} = Ez$, $\bar{s} = Es$, $\bar{t} = Et$ and s and t are vectors of h and $m = T - h = K + 1$, elements respectively. Also let

$$(3.8)\qquad P\,N\,P' = \begin{bmatrix} A & C' \\ C & B \end{bmatrix}$$

where A and B are square symmetric matrices of order h and m, respectively, and C is mxh. Then we can write

$$(3.9)\qquad c = \frac{z'\,P'\;\;P\,N\,P'\,P\,z}{z'\;\;P'\;\;P\,D_1'MD_1\,P'\;\;P\,z} = \frac{s'A\,s + 2t'C\,s + \bar{t}'B\,t}{s'\,s}$$

Now $s's$ has a non-central χ^2 distribution with h degrees of freedom and a non-centrality parameter

$$(3.10)\qquad \theta = \tfrac{1}{2}\bar{s}'\,\bar{s} = \tfrac{1}{2}\bar{z}'\;D_1'\;MD_1\;\bar{z} = \frac{\mu_{-1}'\;M\,\mu_{-1}}{2\sigma^2}\;.$$

Also

$$(3.11)\qquad [\bar{s}'\;\;\bar{t}']\begin{bmatrix} A & C' \\ C & B \end{bmatrix}\begin{bmatrix} \bar{s} \\ \bar{t} \end{bmatrix} = \bar{z}'\;N\,\bar{z} = \frac{\mu_{-1}'\;M\,\mu}{\sigma^2} = \frac{\gamma\,\mu_{-1}'\;M\,\mu_{-1}}{\sigma^2} = 2\gamma\theta$$

since $M\,\mu = \gamma\,M\,\mu_{-1}$.

Remembering that the elements of t are independent of those of s, we expand (3.9) to obtain

$$(3.12)\qquad Ec = \sum_i^h a_{ii}\,E\left(\frac{s_i^2}{s's}\right) + \sum_{i\neq j}^h\sum^h a_{ij}\,E\left(\frac{s_i s_j}{s's}\right) + 2\sum_i^m\sum_j^h c_{ij}\;Et_i\;E\left(\frac{s_j}{s's}\right)$$

$$+\;E\left(\frac{1}{s's}\right)\sum_i^m b_{ii}E(t_i^2) + E\left(\frac{1}{s's}\right)\sum_{i\neq j}^m\sum^m b_{ij}E(t_i\;t_j)\;.$$

Using (A.5), (A.7), (A.8) and (A.9) from Appendix A gives

$$(3.13)\qquad Ec = \tfrac{1}{2}\sum_{i}^{h} a_{ii}\,(\bar{s}_i^2\, f_{1,2} + f_{0,1}) + \tfrac{1}{2}\sum_{i\neq j}^{h}\sum^{h} a_{ij}\,\bar{s}_i\,\bar{s}_j\, f_{1,2}$$

$$+ \sum_{i}^{m}\sum_{j}^{h} c_{ij}\,\bar{t}_i\,\bar{s}_j\, f_{0,1} + \frac{f_{-1,0}}{2}\,[\sum_{i}^{m} b_{ii}(1+\bar{t}_i^2) + \sum_{i\neq j}^{m}\sum^{m} b_{ij}\,\bar{t}_i\,\bar{t}_j]$$

$$= \tfrac{1}{2}\,\bar{s}'A\,\bar{s}\, f_{1,2} + \tfrac{1}{2}(\text{tr } A + 2\bar{t}'\,C\,\bar{s})f_{0,1} + \tfrac{1}{2}(\bar{t}'\,B\,\bar{t} + \text{tr } B)f_{-1,0}$$

$$= \gamma\,\theta\, f_{1,2} + \bar{t}'\,C\,\bar{s}(f_{0,1} - f_{1,2}) + \tfrac{1}{2}\,\bar{t}'\,B\,\bar{t}(f_{-1,0} - f_{1,2})$$

$$+ \tfrac{1}{2}\text{tr } A\, f_{0,1} + \tfrac{1}{2}\text{tr } B\, f_{-1,0}.$$

Using the asymptotic expansion[3] of the confluent hypergeometric function in (A.6) we can write functions like $f_{0,1}$, $f_{1,2}$ etc., up to order $\frac{1}{\theta^2}$ (or order σ^4), as

$$(3.14)\qquad f_{\delta,\lambda} = \theta^{-(\lambda-\delta)}\left[1 + \frac{(\lambda-\delta)(1-\frac{h}{2}-\delta)}{\theta} + \frac{(\lambda-\delta)(\lambda-\delta+1)(2-\frac{h}{2}-\delta)(1-\frac{h}{2}-\delta)}{2\theta^2}\right].$$

Substituting (3.14) into (3.13) gives, up to order σ^2,

$$(3.15)\qquad E(c - \gamma) = [\tfrac{1}{2}\text{tr } N - \tfrac{h}{2}\gamma]\frac{1}{\theta} + \frac{\bar{z}'\,Q_1\bar{z}}{\theta^2}$$

where $\text{tr } A + \text{tr } B = \text{tr } P\,N\,P' = \text{tr } N$ and $Q_1 = P'\begin{bmatrix}0 & 0\\ C & B\end{bmatrix}P$. Since $\bar{z}'Q_1\bar{z}$ is of order σ^{-2}, and all other terms inside the square brackets are of order 1, $Ec \to \gamma$ as $\theta \to \infty$ by $\sigma \to 0$, with n fixed. That is, for a given n, as the random component of y shrinks Ec approaches γ.

To find Ec^2 we first expanded c^2 (see Appendix B) then took expectations and simplified to obtain

(3.16) $$Ec^2 = \gamma^2 \theta^2 f_{2,4} + [(\bar{t}' C \bar{s})(\bar{s}' A \bar{s})](f_{1,3} - f_{2,4})$$

$$+ [\tfrac{1}{2}(\bar{t}' B \bar{t})(\bar{s}' A \bar{s}) + (\bar{t}' C \bar{s})^2](f_{0,2} - f_{2,4})$$

$$+ [(\bar{t}' B \bar{t})(\bar{t}' C \bar{s})](f_{-1,1} - f_{2,4}) + [\tfrac{1}{4}(\bar{t}' B \bar{t})^2](f_{-2,0} - f_{2,4})$$

$$+ [\tfrac{1}{2}(\bar{s}' A \bar{s})(\mathrm{tr}\, A) + \bar{s}' A^2 \bar{s} + 2\bar{s}' A_* A \bar{s} - \bar{s}' A_*^2 \bar{s}] f_{1,3}$$

$$+ [2\bar{t}' C A \bar{s} + \bar{t}' C A_{**} \bar{s} + \tfrac{1}{2}(\bar{s}' A \bar{s})(\mathrm{tr}\, B) + \bar{s}' C' C \bar{s} + \tfrac{1}{2}(\mathrm{tr}\, A)^2$$
$$+ \tfrac{1}{2}(\mathrm{vec}\, A)'(\mathrm{vec}\, A)] f_{0,2}$$

$$+ [\tfrac{1}{2}(\bar{t}' B \bar{t})(\mathrm{tr}\, A) + \bar{t}' C C' \bar{t} + 2\bar{t}' B C \bar{s} + (\bar{t}' C \bar{s})(\mathrm{tr}\, B) + \mathrm{tr}(CC')$$
$$+ \tfrac{1}{2}(\mathrm{tr}\, B)(\mathrm{tr}\, A)] f_{-1,1}$$

$$+ [2(\bar{t}' B \bar{t})(\mathrm{tr}\, B) + 4\bar{t}' B B' \bar{t} + 2(\mathrm{vec}\, B)'(\mathrm{vec}\, B) + (\mathrm{tr}\, B)^2] f_{-2,0}$$

where A_* is a diagonal matrix whose non zero elements are from the main diagonal of A, A_{**} is a matrix with the vector $[a_{11}\ a_{22} \ldots a_{hh}]$ for each column and vec A (or vec B) is the h^2x1 vector of all the elements of A (or B).

Using (3.14) we can write the series expansion of Ec^2, up to order σ^2, as (see Appendix B for details)

$$(3.17)\qquad Ec^2 = \gamma^2 - \{(h+2)\gamma^2\}\frac{1}{\theta} + \{\tfrac{1}{2}(\bar{z}'\ Q_2\ \bar{z})(\mathrm{tr}\ N)$$

$$+ \bar{z}'\ N^2\ \bar{z} + (\mathrm{tr}\ B + 2\gamma)\bar{z}'\ Q_1\ \bar{z} + \bar{s}'(2A_*\ A - A_*^2)\bar{s} + \bar{t}'\ C\ A_{**}\ \bar{s}$$

$$+ \bar{t}'[3B^2 + \tfrac{1}{2}\ \mathrm{tr}(B)B]\bar{t}\}\frac{1}{\theta^2}$$

where

$$Q_2 = P'\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} P.$$

The term $(h + 2)\gamma^2$ inside the first set of curly brackets on the right side of (3.17) is of order one. The terms within the second are of order σ^{-2}. Since θ and θ^2 are of orders σ^2 and σ^4, respectively, $Ec^2 \to \gamma^2$ as $\sigma \to 0$, that is as $\theta \to \infty$.

We conclude from (3.15) and (3.17) that

$$(3.18)\qquad \lim_{\sigma\to 0} E(c - \gamma) = 0$$

and

$$(3.19)\qquad \lim_{\sigma\to 0} E(c - \gamma)^2 = 0$$

from which it follows that c converges in probability to γ as $\sigma \to 0$.

We now consider the first moment of b. Using (3.3) and (3.1) we have

$$(3.20)\qquad b = (X'X)^{-1}X'y - c(X'X)^{-1}X'y_{-1} = \beta + (\gamma - c)(X'X)^{-1}X'y_{-1} + (X'X)^{-1}X'\varepsilon$$

$$= \beta + \sigma(\gamma - c)(X'X)^{-1}X'\ D_1\ P'\ P\ z + (X'X)^{-1}X'\varepsilon$$

$$= \beta + \sigma(\gamma - c)L\begin{bmatrix} s \\ t \end{bmatrix} + (X'X)^{-1}X'\ \varepsilon$$

where $L = (X'X)^{-1}X'\ D_1\ P'$. Consider any element from this vector, say the first. Its expectation is

$$(3.21)\qquad Eb_1 = \beta_1 + \sigma\gamma\ell_1'\begin{bmatrix} \bar{s} \\ \bar{t} \end{bmatrix} - \sigma\ E\{c\ell_1'\begin{bmatrix} s \\ t \end{bmatrix}\}$$

where $\ell_1' = [d', e']$ is the first row of L which is partitioned to conform to

$\begin{bmatrix} s \\ t \end{bmatrix}$. $Ec[d', e'] \begin{bmatrix} s \\ t \end{bmatrix}$ is obtained in detail in Appendix B. The expectation which results is,

$$(3.22) \quad Eb_1 = \beta_1 + \sigma\,\gamma(d'\,\bar{s} + e'\,\bar{t}) - \sigma\,\gamma(d'\,\bar{s}\,\theta\, f_{2,3} + e'\,\bar{t}\,\theta\, f_{1,2})$$
$$- \sigma\{(\bar{t}'\,C\,\bar{s})(d'\,\bar{s})(f_{1,2} - f_{2,3}) + (\bar{t}'\,C\,\bar{s})(e'\,\bar{t})(f_{0,1} - f_{1,2})$$
$$+ \tfrac{1}{2}(\bar{t}'\,B\,\bar{t})(d'\,\bar{s})(f_{0,1} - f_{2,3}) + \tfrac{1}{2}(\bar{t}'\,B\,\bar{t})(e'\,\bar{t})(f_{-1,0} - f_{1,2})$$
$$+ [d'\,A\,\bar{s} + \tfrac{1}{2}(\text{tr } A)(d'\,\bar{s})]f_{1,2} + [\bar{t}'\,C\,d + e'\,C\,\bar{s} + \tfrac{1}{2}(\text{tr } A)(e'\,\bar{t})$$
$$+ \tfrac{1}{2}(\text{tr } B)(d'\,\bar{s})]f_{0,1} + [e'\,B\,\bar{t} + \tfrac{1}{2}(\text{tr } B)(e'\,\bar{t})]f_{-1,0}\}.$$

We can now use (A.6) to write the terms, up to order θ^{-1} of the series, like $f_{2,3}$, $f_{1,2}$ etc., which appear in (3.22). After collecting terms we have

$$(3.23) \quad Eb_1 = \beta_1 + \sigma\{\frac{\gamma h}{2}(d'\,\bar{s} + e'\,\bar{t}) + \gamma\, d'\,\bar{s} - \tfrac{1}{2}(\text{tr } N)(d'\,\bar{s} + e'\,\bar{t})$$
$$+ [d'\;\, e']\, P\, N\, P' \begin{bmatrix} \bar{s} \\ \bar{t} \end{bmatrix}\}\frac{1}{\theta} - \sigma\{(1 - h)\bar{z}'\,Q_1\,\bar{z}(d'\,\bar{s} + e'\,\bar{t})$$
$$+ 2(\bar{z}'\,Q_1\,\bar{z})(e'\,\bar{t}) + (\bar{t}'\,B\,\bar{t})(d'\,\bar{s} + e'\,\bar{t})\}\frac{1}{\theta^2} + 0(\theta^{-2}).$$

The terms inside the first curly bracket are, after multiplication by σ, of order one. The terms inside the second curly bracket are, after multiplication by σ, of order σ^{-2}, while θ^{-j} is of order σ^{2j}. Therefore if $\theta \to \infty$ by $\sigma \to 0$ $Eb_1 \to \beta_1$. That is, for a given n, as the random component of the model shrinks Eb_1 approaches β.

If we wish to consider the distribution of b as $\sigma \to 0$ we must standardize equation (3.20) to obtain

$$(3.24) \qquad \frac{1}{\sigma}(b - \beta) = (\gamma - c)(X'X)^{-1} X' D_1 z + \frac{1}{\sigma}(X' X)^{-1} X' \varepsilon \ .$$

Now $(X' X)^{-1} X' D_1 z$ is multivariate normal with means vector $(X' X)^{-1} X' \mu_{-1}$ and covariance matrix $(X' X)^{-1} X' D_1 D_1' X(X'X)^{-1} = (X' X)^{-1}$. Also $\frac{1}{\sigma}(X' X)^{-1} X' \varepsilon$ is multivariate normal with a zero means vector and a covariance matrix $(X' X)^{-1} X' V X(X' X)^{-1}$ where

$$(3.25) \qquad V = \begin{bmatrix} 1+\gamma^2 & -\gamma & 0 & & 0 \\ -\gamma & 1+\gamma^2 & -\gamma & 0 & 0 \\ & & & & \\ 0 & & & & -\gamma \\ 0 & & & -\gamma & 1+\gamma^2 \end{bmatrix}$$

Neither of these two multivariate normal distributions depend on σ^2. Furthermore, from (3.18) and (3.19), $(c - \gamma)$ converges in probability to zero as $\sigma^2 \to 0$. Therefore, $\frac{1}{\sigma}(b - \beta)$ converges in distribution to $\frac{1}{\sigma}(X' X)^{-1} X' \varepsilon$ as $\sigma^2 \to 0$. Hence, as σ^2 grows small b is approximately normal and unbiased with approximate covariance matrix $\sigma^2 (X' X)^{-1} X' V X(X' X)^{-1}$.

4. Consistent Estimation of γ

Liviatan (1963) has proposed two consistent estimators for models like (2.4). The simplest of them uses a lagged exogenous variable,[4] w, as an instrument to produce the normal equations.

$$(4.1) \qquad \hat{\gamma} w'y_{-1} + w'X\hat{\beta} = w'y$$

$$(4.2) \qquad \hat{\gamma} X'y_{-1} + X'X\hat{\beta} = X'y$$

from which

$$(4.3) \qquad \hat{\gamma} = \frac{w'My}{w'My_{-1}} = \frac{u}{u_1}$$

where $u \sim N(w'M\mu, \sigma^2 w'Mw) = N(\bar{u}, \omega^2)$, $u_1 \sim N(w'M\mu_{-1}, \sigma^2 w'Mw) = N(\bar{u}_1, \omega^2)$ and

$E(u-\bar{u})(u_1-\bar{u}_1)' = \sigma^2 w' M D_2 M w = \rho\omega^2$ where ρ is the coefficient of correlation between u and u_1. Under (2.1), this ratio has a distribution of the type described by Fieller (1932) which has no moments of any order. However, if $\frac{\omega}{\bar{u}_1} < 1/3$ the distribution of $\hat{\gamma}$ can be approximated by (Scadding (1973)).

$$(4.4) \qquad f(\hat{\gamma}) \doteq \frac{-\omega^2[(\bar{u}\rho-\bar{u}_1)+(\bar{u}_1\rho-\bar{u})\hat{\gamma}]}{\sqrt{2\pi[\omega^2(\hat{\gamma}^2-2\hat{\gamma}\rho+1)]^3}} \exp\left\{\frac{-(\bar{u}-\bar{u}_1\hat{\gamma})^2}{2\omega^2(\hat{\gamma}^2-2\hat{\gamma}\rho+1)}\right\}.$$

When this approximation is valid $\hat{\gamma}$ is nearly unbiased (Nagar and Gupta (1968), Carter (1976)). Higher moment can be obtained from Merrill (1928).

Scadding (1973) and Nagar and Gupta (1968) analyzed the distribution of $\hat{\gamma}$ only. We wish to consider the distribution of $\hat{\beta}$. So we use (4.2) to obtain

$$(4.5) \qquad \hat{\beta} = (X' X)^{-1} X' (y - \hat{\gamma} y_{-1}).$$

Its sampling error can be written as

$$(4.6) \qquad \frac{1}{\sigma}(\hat{\beta} - \beta) = (\gamma - \hat{\gamma})(X' X)^{-1} X' D_1 z_1 + \frac{1}{\sigma}(X' X)^{-1} X' \varepsilon.$$

If $\sigma \to 0$ (hence $\frac{\omega}{\bar{u}_1} \to 0$), $\hat{\gamma} \to \gamma$ and the distribution of $\hat{\beta} - \beta$ approaches to that of $\frac{1}{\sigma}(X' X)^{-1}X' \varepsilon$ which is the same as the limiting distribution of $\frac{1}{\sigma}(b - \beta)$ as seen at the end of section 3. Then, like b, as $\sigma \to 0$ the distribution of $\hat{\beta}$ is approximately normal with means vector β and covariance matrix $\sigma^2 (X' X)^{-1} X' V X(X' X)^{-1}$.

5. Conclusions

The coefficients of a rational distributed lag model with a first degree polynomial in the denominator can be estimated by least squares or by instrumental variables. This paper presents some exact and asymptotic properties of these estimators.

We find that the OLS estimator of the coefficient of the lagged dependent variable converges in probability to the true value of the coefficient as θ, the non-centrality parameter of its distribution, grows large; that is as σ, the standard deviation of the errors, grows small. Also, as σ grows small, the OLS estimator of the coefficients vector of the exogenous variables becomes unbiased. In addition, for small σ, the IV coefficient estimators possess these same properties. These findings suggest that when both the sample size and the error variance are small OLS is a useful estimator which is not inferior to IV.

APPENDIX

A. Expectations Required in Section 3

Let $z_1,\ldots,z_T$ be independent normal variates with

(A.1) $$E\, z_i = \bar{z}_i \quad \text{and} \quad \text{Var}\, z_i = 1 \qquad i = 1,\ldots,T$$

Then we know that the distribution of

(A.2) $$W = z' B z,$$

where $z' = [z_1,\ldots,z_T]$ and B is idempotent with rank h, is 'noncentral chi-square' with h degrees of freedom and the parameter of noncentrality

(A.3) $$\theta = \tfrac{1}{2} \bar{z}' B \bar{z}$$

The density function of W is given by

(A.4) $$f(W) = e^{-\theta} \sum_{m=0}^{\infty} \frac{\theta^m}{m!} \frac{W^{\frac{1}{2}(h+2m)-1} e^{-\frac{1}{2}W}}{2^{\frac{1}{2}(h+2m)}\Gamma[(h+2m)/2]}, \quad 0 < W < \infty.$$

Therefore, if $h/2 > r$ [see Ullah (1974, p. 147)]

(A.5) $$E\, W^{-r} = \int_0^{\infty} W^{-r} f(W)\, dW = 2^{-r} f_{-r,0}, \qquad r=1,2,\ldots$$

where

(A.6) $$f_{\delta,\nu} = \frac{\Gamma(h/2 + \delta)}{\Gamma(h/2 + \nu)} e^{-\theta}\, {}_1F_1(h/2 + \delta;\ h/2 + \nu;\ \theta)$$

and writing $\delta = -r$, $\nu = 0$, we get $f_{-r,0}$ and so on. The function ${}_1F_1(\ )$ represents the confluent hypergeometric function.[5]

The expectations required in Section 3 may now be stated as follows:[6]

(A.7) $$E(z_i W^{-r}) = 2^{-r} \bar{z}_i f_{-r+1,1}$$

(A.8) $$E(z_i^2 W^{-r}) = 2^{-r}[\bar{z}_i^2 f_{-r+2,2} + f_{-r+1,1}];$$

(A.9) $$E(z_i z_j W^{-r}) = 2^{-r} \bar{z}_i \bar{z}_j f_{-r+2;2}$$

(A.10) $$E(z_i^3 W^{-r}) = 2^{-r} \bar{z}_i^3 f_{-r+3,3} + 3 \times 2^{-r} \bar{z}_i f_{-r+2,2},$$

(A.11) $$\underset{i \neq j}{E}(z_i^2 z_j W^{-r}) = 2^{-r} \bar{z}_i^2 \bar{z}_j f_{-r+3,3} + 2^{-r} \bar{z}_j f_{-r+2,2}$$

(A.12) $$\underset{i \neq j \neq k}{E}(z_i z_j z_k W^{-r}) = 2^{-r} \bar{z}_i \bar{z}_j \bar{z}_k f_{-r+3,3}$$

(A.13) $$E(z_i^4 W^{-r}) = 2^{-r} \bar{z}_i^4 f_{-r+4,4} + 6 \times 2^{-r} \bar{z}_i^2 f_{-r+3,3} + 3 \times 2^{-r} f_{-r+2,2}$$

(A.14) $$\underset{i \neq j}{E}(z_i^3 z_j W^{-r}) = 2^{-r} \bar{z}_i^3 \bar{z}_j f_{-r+4,4} + 3 \times 2^{-r} \bar{z}_i \bar{z}_j f_{-r+3,3}$$

(A.15) $$\underset{i \neq j}{E}(z_i^2 z_j^2 W^{-r}) = 2^{-r} \bar{z}_i^2 \bar{z}_j^2 f_{-r+4,4} + 2^{-r}(\bar{z}_i^2 + \bar{z}_j^2) f_{-r+3,3} + 2^{-r} f_{-r+2,2}$$

(A.16) $$\underset{i \neq j \neq k}{E}(z_i^2 z_j z_k W^{-r}) = 2^{-r} \bar{z}_i^2 \bar{z}_j \bar{z}_k f_{-r+4,4} + 2^{-r} \bar{z}_j \bar{z}_k f_{-r+3,3}$$

(A.17) $$\underset{i \neq j \neq k \neq \ell}{E}(z_i z_j z_k z_\ell W^{-r}) = 2^{-r} \bar{z}_i \bar{z}_j \bar{z}_k \bar{z}_\ell f_{-r+4,4}$$

B. The Evaluation of Useful Expectations

The first step in finding Ec^2 is to expand c^2 as:

(B.1) $$c^2 = \frac{1}{(s's)^2}[(s'As)^2 + 4(t'Cs)(s'As) + 2(t'Bt)(s'As) + 4(t'Cs)^2 + 4(t'Bt)(t'Cs) + (t'Bt)^2].$$

Keeping in mind the symmetry of A and B, the terms inside the square brackets can be expanded[7] as

(B.2) $$(s'As)^2 = \sum_i^h a_{ii}^2 s_i^4 + 4\sum_{i\neq j}^{h}\sum^{h} a_{ii}a_{ij}s_i^3 s_j + 2\sum_{i\neq j\neq k}^{h}\sum^{h}\sum^{h} a_{ii}a_{jk}s_i^2 s_j s_k + 4\sum_{i\neq j\neq k}^{h}\sum^{h}\sum^{h} a_{ij}a_{ik}s_i^2 s_j s_k + \sum_{i\neq j}^{h}\sum^{h} a_{ii}a_{jj}s_i^2 s_j^2 + 2\sum_{i\neq j}^{h}\sum^{h} a_{ij}^2 s_i^2 s_j^2 + \sum_{i\neq j\neq k\neq \ell}^{h}\sum^{h}\sum^{h}\sum^{h} a_{ij}a_{k\ell}s_i s_j s_k s_\ell$$

(B.3) $$(t'Cs)(s'As) = \sum_i^m\sum_k^h c_{ik}a_{kk}t_i s_k^3 + \sum_i^m\sum_{j\neq k}^{h}\sum^{h} c_{ij}a_{kk}t_i s_k^2 s_j + 2\sum_i^m\sum_{j\neq \ell}^{h}\sum^{h} c_{ij}a_{j\ell}t_i s_j^2 s_\ell + \sum_i^m\sum_{j\neq k\neq \ell}^{h}\sum^{h}\sum^{h} c_{ij}a_{k\ell}t_i s_j s_k s_\ell$$

(B.4) $$(t'Cs)^2 = \sum_i^m\sum_k^h c_{ik}^2 t_i^2 s_k^2 + \sum_i^m\sum_{k\neq \ell}^{h}\sum^{h} c_{ik}c_{i\ell}t_i^2 s_k s_\ell + \sum_{i\neq j}^{m}\sum^{m}\sum_k^h c_{ik}c_{jk}t_i t_j s_k^2 + \sum_{i\neq j}^{m}\sum^{m}\sum_{k\neq \ell}^{h}\sum^{h} c_{ik}c_{j\ell}t_i t_j s_k s_\ell$$

(B.5) $$(t'Bt)(t'Cs) = \sum_i^m\sum_k^h b_{ii}c_{ik}t_i^3 s_k + \sum_{i\neq j}^{m}\sum^{m}\sum_k^h b_{ii}c_{jk}t_i^2 t_j s_k + 2\sum_{i\neq j}^{m}\sum^{m}\sum_\ell^h b_{ij}c_{i\ell}t_i^2 t_j s_\ell + \sum_{i\neq j\neq k}^{m}\sum^{m}\sum^{m}\sum_\ell^h b_{ij}c_{k\ell}t_i t_j t_k s_\ell .$$

Since $t_i \sim N(\bar{t}_i, 1)$, the moments of t_i are

(B.6) $$Et_i^2 = 1 + \bar{t}_i^2$$

(B.7) $$Et_i^3 = \bar{t}_i^3 + 3\bar{t}_i$$

(B.8) $$Et_i^4 = \bar{t}_i^4 + 6\bar{t}_i^2 + 3 \ .$$

Using (B.1) to (B.8) together with (A.5) to (A.17) we can write the exact second moment of c for $h > 4$.

(B.9) $$c^2 = \tfrac{1}{4} \sum_i^h a_{ii}(\bar{s}_i^4 f_{2,4} + 6\bar{s}_i^2 f_{1,3} + 3f_{0,2}) +$$

$$+ \sum_{i\neq j}^{h} \sum^{h} a_{ii} a_{ij} (\bar{s}_i^3 \bar{s}_j f_{2,4} + 3\bar{s}_i \bar{s}_j f_{1,3})$$

$$+ \tfrac{1}{2} \sum^{h}\sum^{h}\sum^{h}_{i\neq j\neq k} (a_{ii} a_{jk} + 2a_{ij} a_{ik})(\bar{s}_i^2 \bar{s}_j \bar{s}_k f_{2,4} + \bar{s}_j \bar{s}_k f_{1,3})$$

$$+ \tfrac{1}{4} \sum^{h}\sum^{h}_{i\neq j} (a_{ii} a_{jj} + 2a_{ij}^2)[\bar{s}_i^2 \bar{s}_j^2 f_{2,4} + (\bar{s}_i^2 + \bar{s}_j^2) f_{1,3} + f_{0,2}]$$

$$+ \tfrac{1}{4} \sum^{h}\sum^{h}\sum^{h}\sum^{h}_{i\neq j\neq k\neq \ell} a_{ij} a_{k\ell} \bar{s}_i \bar{s}_j \bar{s}_k \bar{s}_\ell f_{2,4} + \sum_i^m \sum_j^h c_{ij} a_{jj} \bar{t}_i (\bar{s}_j^3 f_{1,3} + 3\bar{s}_j f_{0,2}$$

$$+ \sum_i^m \sum^{h}\sum^{h}_{j\neq k} c_{ij}(a_{kk} + 2a_{jk})\bar{t}_i(\bar{s}_j^2 \bar{s}_k f_{1,3} + \bar{s}_k f_{0,2})$$

$$+ \sum_i^m \sum^{h}\sum^{h}\sum^{h}_{j\neq k\neq \ell} c_{ij} a_{k\ell} \bar{t}_i \bar{s}_j \bar{s}_k \bar{s}_\ell f_{1,3}$$

$$+ \tfrac{1}{2}[\sum_i^m b_{ii}(1 + \bar{t}_i^2) + \sum^{m}\sum^{m}_{i=j} b_{ij} \bar{t}_i \bar{t}_j][\sum_i^h a_{ii}(\bar{s}_i^2 f_{0,2} + f_{-1,1})$$

$$+ \sum_{i\neq j}^{h}\sum^{h} a_{ij}\, \bar{s}_i\, \bar{s}_j\, f_{0,2}]$$

$$+ [\sum_{i}^{m}\sum_{k}^{h} c_{ik}^2(1 + \bar{t}_i^2) + \sum_{i\neq j}^{m}\sum^{m}\sum_{k}^{h} c_{ik}\, c_{jk}\, \bar{t}_i\, \bar{t}_j](\bar{s}_k^2\, f_{0,2} + f_{-1,1})$$

$$+ [\sum_{i}^{m}\sum_{k\neq \ell}^{h}\sum^{h} c_{ik}\, c_{i\ell}(1 + \bar{t}_i^2) + \sum_{i\neq j}^{m}\sum^{m}\sum_{k\neq \ell}^{h}\sum^{h} c_{ik}\, c_{j\ell}\, \bar{t}_i\, \bar{t}_j]\bar{s}_k\, \bar{s}_\ell\, f_{0,2}$$

$$+ \sum_{\ell}^{h} [\sum_{i}^{m} b_{ii}\, c_{i\ell}(\bar{t}_i^3 + 3\bar{t}_i) + \sum_{i\neq j}^{m}\sum^{m} (b_{ii}\, c_{j\ell} + 2b_{ij}\, c_{i\ell})(1 + \bar{t}_i^2)\bar{t}_j$$

$$+ \sum_{i\neq j\neq k}^{m}\sum^{m}\sum^{m} b_{ij}\, c_{k\ell}\, \bar{t}_i\, \bar{t}_j\, \bar{t}_k]\bar{s}_\ell\, f_{-1,1} \quad .$$

$$+ \frac{f_{-2,0}}{4} \{\sum_{i}^{m} b_{ii}^2(\bar{t}_i^4 + 6\bar{t}_i^2 + 3) + 4\sum_{i\neq j}^{m}\sum^{m} b_{ii}\, b_{ij}(\bar{t}_i^3 + 3\bar{t}_i)\bar{t}_j$$

$$+ 2\sum_{i\neq j\neq k}^{m}\sum^{m}\sum^{m} [(b_{ii}\, b_{jk} + 2b_{ij}\, b_{ik})(1 + \bar{t}_i^2)\bar{t}_j\, \bar{t}_k]$$

$$+ \sum_{i\neq j}^{m}\sum^{m} [(b_{ii}\, b_{jj} + 2b_{ij}^2)(1 + \bar{t}_i^2)(1 + \bar{t}_j^2)]$$

$$+ \sum_{i\neq j\neq k\neq \ell}^{m}\sum^{m}\sum^{m}\sum^{m} b_{ij}\, b_{k\ell}\, \bar{t}_i\, \bar{t}_j\, \bar{t}_k\, \bar{t}_\ell\} \quad .$$

In moving from (B.9) to (3.16) we have used (B.2) to (B.5) as well as the following equalities:

(B.10) $$(\bar{s}' A\, \bar{s})(\mathrm{tr}\ A) = \sum_{i}^{h} a_{ii}^2\, \bar{s}_i^2 + \sum_{i\neq j}^{h}\sum^{h} a_{ii}\, a_{jj}\, \bar{s}_i^2 + 2\sum_{i\neq j}^{h}\sum^{h} a_{ij}\, a_{ii}\, \bar{s}_i\, \bar{s}_j$$

$$+ \sum_{i\neq j=k}^{h}\sum^{h}\sum^{h} a_{ij}\, a_{kk}\, \bar{s}_i\, \bar{s}_j$$

(B.11) $$\bar{s}'\, A^2\, \bar{s} = \sum_{i}^{h}\sum_{k}^{h} a_{ik}^2\, \bar{s}_i^2 + \sum_{i\neq j}^{h}\sum_{k}^{h}\sum^{h} a_{ik}\, a_{jk}\, \bar{s}_i\, \bar{s}_j$$

(B.12) $$\bar{s}'\, A_*\, A\, \bar{s} = \sum_{i}^{h} a_{ii}^2\, s_i^2 + \sum_{i\neq j}^{h}\sum^{h} a_{ii}\, a_{ij}\, s_i\, s_j$$

(B.13) $$\bar{t}'\, C\, A\, \bar{s} = \sum_{i}^{m}\sum_{j}^{h} c_{ij}\, a_{jj}\, \bar{t}_i\, \bar{s}_j + \sum_{i}^{m}\sum_{k\neq j}^{h}\sum^{h} c_{ij}\, a_{jk}\, \bar{t}_i\, \bar{s}_k$$

(B.14) $$\bar{t}'\, C\, A_{**}\, \bar{s} = \sum_{i}^{m}\sum_{j}^{h} c_{ij}\, a_{jj}\, \bar{t}_i\, \bar{s}_j + \sum_{i}^{m}\sum_{j\neq k}^{h}\sum^{h} c_{ij}\, a_{kk}\, \bar{t}_i\, \bar{s}_k$$

(B.15) $$\bar{s}'\, C'\, C\, \bar{s} = \sum_{j}^{h}\sum_{i}^{m} c_{ij}^2\, \bar{s}_j^2 + \sum_{j\neq k}^{h}\sum^{h}\sum_{i}^{m} c_{ij}\, c_{ik}\, \bar{s}_j\, \bar{s}_k$$

(B.16) $$(\mathrm{vec}\ A)'\,(\mathrm{vec}\ A) = \sum_{i}^{h} a_{ii}^2 + \sum_{i\neq j}^{h}\sum^{h} a_{ij}^2$$

(B.17) $$(\bar{t}'\, C\, \bar{s})(\mathrm{tr}\ B) = \sum_{i}^{m}\sum_{k}^{h} b_{ii}\, c_{ik}\, \bar{t}_i\, \bar{s}_k + \sum_{i\neq j}^{m}\sum^{m}\sum_{k}^{h} b_{ii}\, c_{jk}\, \bar{t}_j\, \bar{s}_k$$

The terms $(\bar{s}'\ A\ \bar{s})(\mathrm{tr}\ B)$, $(\bar{t}'\ B\ \bar{t})(\mathrm{tr}\ A)$ and $(\bar{t}'\ B\ \bar{t})(\mathrm{tr}\ B)$ are all similar in form to (B.10), $\bar{t}'\ B\ C\ \bar{s}$ is similar to (B.13), $\bar{t}'\ C\ C'\ \bar{t}$ and $\bar{t}'\ B\ B'\ \bar{t}$ are similar to (B.15) and $(\mathrm{vec}\ B)'(\mathrm{vec}\ B)$ is similar to (B.16). A_* is the diagonal matrix formed by setting all off diagonal elements of A to zero. A_{**} is an hxh matrix whose every column is the vector $[a_{11}\ a_{22} \ldots a_{hh}]'$ and vec A is the h^2x1 vector of all elements of A.

In moving from (3.16) to (3.17) the following equality was used:

(B.18) $$\bar{z}' N^2 \bar{z} = \bar{s}' A^2 \bar{s} + \bar{s}' C' C \bar{s} + 2\bar{t}' C A \bar{s} + 2\bar{t}' B C \bar{s} + \bar{t}' C C' \bar{t} + \bar{t}' B^2 \bar{t}$$

We consider now $E\, c[d' \; e'] \begin{bmatrix} s \\ t \end{bmatrix}$ which is a part of Eb_1. The first component of this expectation is

(B.19) $$E\, c\, d' s = E\left[\frac{(s' A s + 2t' C s + t' B t)(d' s)}{s' s}\right] .$$

The terms in the numerator inside the square bracket must be expanded to give:

(B.20) $$(s' A s)(d' s) = \sum_i^h a_{ii} d_i s_i^3 + \sum_{i \neq k}^h \sum^h a_{ii} d_k s_i^2 s_k + 2\sum_{i \neq j}^h \sum^h a_{ij} d_i s_i^2 s_j + \sum^h \sum^h \sum_{i \neq j \neq k}^h a_{ij} d_k s_i s_j s_k$$

(B.21) $$(\bar{t}' C s)(d' s) = \sum_i^m \sum_j^h c_{ij} d_j t_i s_j^2 + \sum_i^m \sum^h \sum_{j \neq k}^h c_{ij} d_k t_i s_j s_k$$

(B.22) $$(t' B t)(d' s) = \left(\sum_i^m b_{ii} t_i^2 + \sum_{i \neq j}^m \sum^m b_{ij} t_i t_j\right)\left(\sum_k^h d_k s_k\right)$$

When we combine the expansions with the results of Appendix A we obtain

(B.23) $$E\, c\, d' s = \tfrac{1}{2} \sum_i^h a_{ii} d_i (\bar{s}_i^3 f_{2,3} + 3\bar{s}_i f_{1,2}) + \tfrac{1}{2} \sum_{i \neq k}^h \sum^h a_{ii} d_k (\bar{s}_i^2 \bar{s}_k f_{2,3} + \bar{s}_k f_{1,2})$$
$$+ \sum_{i \neq j}^h \sum^h a_{ij} d_i (\bar{s}_i^2 \bar{s}_j f_{2,3} + \bar{s}_j f_{1,2}) + \tfrac{1}{2} \sum^h \sum^h \sum_{i \neq j \neq k}^h a_{ij} \bar{s}_i \bar{s}_j \bar{s}_k f_{2,3}$$

$$+ \sum_i^m \sum_j^h c_{ij}\, d_j\, \bar{t}_i (\bar{s}_j\, f_{1,2} + f_{0,1}) + \sum_i^m \sum_{j \neq k}^h \sum^h c_{ij}\, d_k\, \bar{t}_i\, \bar{s}_i\, \bar{s}_j\, f_{1,2}$$

$$+ \tfrac{1}{2}(\sum_i^m b_{ii}(\bar{t}_i^2 + 1) + \sum_{i \neq j}^m \sum^m b_{ij}\, \bar{t}_i\, \bar{t}_j) \sum_k^h d_k\, \bar{s}_k\, f_{0,1}.$$

Noting that

(B.24) $$d'\, A\, \bar{s} = \sum_i^h a_{ii}\, d_i\, \bar{s}_i + \sum_{i \neq j}^h \sum^h a_{ij}\, d_i\, \bar{s}_j\ ,$$

(B.25) $$(\mathrm{tr}\, A)(d'\, \bar{s}) = \sum_i^h a_{ii}\, d_i \bar{s}_i + \sum_{i \neq j}^h \sum^h a_{ii}\, d_j\, \bar{s}_j \quad \text{and}$$

(B.26) $$\bar{t}'\, C\, d = \sum_i^m \sum_j^h c_{ij}\, \bar{t}_i\, d_j$$

we can simplify (B.23) to

(B.27) $$E\, c\, d'\, s = \gamma\, \theta\, d'\, \bar{s}\, f_{2,3} + (\bar{t}'\, C\, \bar{s})(d'\bar{s})(f_{1,2} - f_{2,3}) + \tfrac{1}{2}(\bar{t}'\, B\, \bar{t})(d'\bar{s})(f_{0,1} - f_{2,3})$$

$$+[d' A\, \bar{s} + \tfrac{1}{2}(\mathrm{tr}\, A)(d'\bar{s})] f_{1,2} + [(\bar{t}'\, C\, d) + \tfrac{1}{2}(\mathrm{tr}\, B)(d' s)] f_{0,1}\ .$$

The second component of $E\, c[d'\ \ e'] \begin{bmatrix} s \\ t \end{bmatrix}$ is

(B.28) $$E\, c\, e'\, t = E[\frac{(s'\, A\, s)(e'\, t) + 2(t' C\, s)(e' t) + (t'\, B\, t)(e'.t)}{s' s}]$$

$$= [\tfrac{1}{2} \sum_i^h a_{ii}(\bar{s}_i^2\, f_{1,2} + f_{0,1}) + \tfrac{1}{2} \sum_{i \neq j}^h \sum^h a_{ij}\, \bar{s}_i\, \bar{s}_j\, f_{1,2}] e'\, \bar{t}$$

$$+ \sum_j^h [\sum_i^m c_{ij}\, e_i (1 + \bar{t}_i^2) + \sum_{i \neq k}^m \sum^m c_{ij}\, e_k\, \bar{t}_i\, \bar{t}_k] \bar{s}_j\, f_{0,1}$$

$$+ [\sum_{i}^{m} b_{ii}\ e_i(\bar{t}_i^3 + 3\bar{t}_i) + \sum_{i \neq k}^{m}\sum^{m} b_{ii}\ e_k(1 + \bar{t}_i^2)\bar{t}_k$$

$$+ 2\sum_{i \neq j}^{m}\sum^{m} b_{ij}\ e_i(1 + \bar{t}_i^2)\bar{t}_j + \sum_{i \neq j \neq k}^{m}\sum^{m}\sum^{m} b_{ij}\ e_k\ \bar{t}_i\ \bar{t}_j\ \bar{t}_k]\frac{f_{-1,0}}{2}$$

$$= \gamma\ \theta(e'\bar{t})f_{1,2} + (\bar{t}' C\ \bar{s})(e'\bar{t})(f_{0,1} - f_{1,2}) + \tfrac{1}{2}(\bar{t}' B\ \bar{t})(e'\bar{t})(f_{-1,0} - f_{1,2})$$

$$+ [\tfrac{1}{2}(\mathrm{tr}\ A)(e' t) + e' C\ \bar{s}]f_{0,1} + [e' B\ \bar{t} + \tfrac{1}{2}(\mathrm{tr}\ B)(e'\bar{t})]f_{-1,0}\ .$$

Using similar expansions and simplifications as were used in deriving E c d's, equation (3.22) is obtained by substituting (B.27) and (B.28) into (3.21).

Footnotes

[1]The authors are grateful to D. Hendry, R. D. Terrell, R. Kohn and D. F. Nicholls for extremely useful suggestions on our earlier draft of this paper which was entitled "The Finite Sample Properties of OLS and IV Estimators in Regression Models with a Lagged Dependent Variable." An earlier version of this paper was presented to the European meetings of the Econometric Society in Vienna in September 1977.

[2]At first glance equation (3.5) resembles the analogous equation for two-stage least squares and so one might hope to use the results of Richardson (1968) and Sawa (1969) to analyze the behavior of c. However, these earlier results do not apply here directly because the elements of y_{-1} are not independent of those of y and there is no cannonical form of the model for which independence holds. In fact the covariance matrix of y and y_{-1} is singular. Another complication is the presence of y_T in the numerator of (3.5) but not in the denominator.

[3]If $\theta > 0$ and a, c > 0, then, using Sawa's (1972, p. 667) results we have

$$ {}_1F_1(a;c;\theta) = \frac{\Gamma c}{\Gamma a} \; e^{\theta} \; \theta^{-(c-a)} \left[\sum_{j=0}^{p-1} \frac{(c-a)_j (1-a)_j}{j!} \; \theta^{-j} + 0(\theta^{-p}) \right] . $$

θ will grow if $\mu_1' M \mu_1 \to \infty$ or if $\sigma^2 \to 0$. Kadane (1970), (1971) has analyzed the behavior of estimators as $\sigma^2 \to 0$.

One can also obtain the bias upto order 1/n by using the following result. For large a and b, with $\theta > 0$,

$$ {}_1F_1(a;b;bx) = e^{bx}(1+x)^{a-b}\left[1 - \frac{(b-a)(b-a+1)}{2b}\left(\frac{x}{1+x}\right)^2 + 0\left(\frac{1}{|b|^2}\right)\right] $$

so long as (b-a) and x are bounded; Slater (1960, p. 66). θ is also a concentration parameter because

$$ p[|c - \gamma| > \epsilon| = 0 \text{ as } \theta \to \infty $$

using (3.18) and (3.19).

[4]Liviatan considered the case where X has only one column so the choice of w was obvious.

[5] $$ {}_1F_1(a;c;x) = \frac{\Gamma c}{\Gamma a} \sum_{n=o}^{\infty} \frac{\Gamma(a+n)}{\Gamma(c+n)} \frac{x^n}{n!} $$

[6]The results in (A.7), (A.8), (A.10) and (A.13) are given in Ullah (1974). (A.7) and (A.8) also follow by using (A.5) in the results of Bock (1975, p. 216). The remaining expectations can be obtained from Nagar and Ullah (1973).

[7]We have not expanded the product $(t' B t)(s' A s)$ because t is independent of s. Also the expansion of $(t' B t)^2$ is of the same form as (B.2).

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