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A THEORY OF OPTIMAL RETIREMENT

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ABSTRACT

The paper develops a theory of retirement behavior based on a lifetime labor supply model. The analysis is somewhat less general than previous optimizing models, but a number of interesting predictions are derived. Particular attention is given to the impact of the structure of pension benefits on retirement age.

## A Theory of Optimal Retirement

### I. Introduction

What determines the age at which an individual chooses to retire? How is this decision influenced by the structure of pension benefits? These are questions worthy of the increased attention they are currently receiving.

A theoretically sound model of retirement must recognize that retirement is one segment of the time path of lifetime labor supply. This step has been taken by several authors probing the lifetime pattern of participation and hours of work.<sup>1</sup> These analyses provide a meager (frequently empty) set of predictions about retirement behavior, the reason being that retirement is treated incidentally as any period when optimal hours of work are zero. As such, it is very difficult to show that retirement (so defined) exists, or is unique, or even occurs at the end of working life.

To obtain predictions concerning retirement behavior, a less general model is necessary. One set of restrictions is to retain freely variable hours of work, but constrain preferences, the time path of wages and the structure of pensions so as to yield an optimal time path of hours that is positive until some date and zero thereafter. An alternative approach is to take account of imperfect substitutability between bodies and hours per body within the firm, implying a preference by firms for a working period of given length, and assume that when individuals work, they must provide a fixed number of hours per unit of time.

We take the latter route.<sup>2</sup> While it is evident that predictions may be obtained either way, the fixed hours assumption is roughly correct empirically and allows a straightforward and intuitive analysis of what is essentially a dichotomous decision.<sup>3</sup> The results can be traced to a single assumption; the question of which assumption (or interaction among which

assumptions) underlies the results does not arise. One cost of the assumption is that the analysis is really only appropriate for men.

Given fixed hours, individuals optimize by choosing a consumption path and a retirement age. The maximization is subject to a lifetime wealth constraint dependent upon wages, non-labor income and pension benefits. Pensions are specified herein as a continuous payment beginning at the retirement age, and are comprised of a fixed sum plus an amount which depends on retirement age (penalizing early retirement) and a weighted average of past wages.

The major results are, that under plausible assumptions, all of the following induce earlier retirement: increases in non-labor income; reductions in length of life; a shortening of the period over which past wages are averaged in the formation of pension benefits; weighting more recent wages (with length of the averaging period fixed) more heavily in the construction of pension benefits; increases in the fixed portion of the pension; and increases in the (either average or marginal) penalty for early retirement.

Section II discusses optimal choice of retirement age. The results in the preceding paragraph are discussed in Section III. Section IV is a summary and discussion of the results.

## II. Optimal Choice of Retirement Date

This section analyzes optimal choice of retirement date. We first derive the optimal consumption path, with retirement age taken as given. Then, retirement age is derived (using ordinary marginal methods) with consumption varying optimally.

Individuals maximize lifetime utility,  $V^0$ . Instantaneous utility,  $u[\cdot]$ , depends on the flow of a composite good,  $c(t) \geq 0$  and leisure,  $l(t) \in [0,1]$ . It

is assumed that  $u[\cdot]$  is time invariant, monotone increasing and concave with  $u_c[0, l(t)] = \infty$ .

As mentioned above, we assume that hours of work  $(1 - l(t))$  are fixed and that there is a unique retirement age  $T$ . Denoting the length of life by  $N$ ,

$$l(t) = \begin{cases} \bar{l} \in (0,1) & t \in [0, T] \\ 1 & t \in (T, N] \end{cases} .$$

Assuming a fixed rate of discount on future utility,  $\rho$ ,

$$V^0 = \int_0^T u[c^W(t), \bar{l}] e^{-\rho t} dt + \int_T^N u[c^R(t), 1] e^{-\rho t} dt \quad (1)$$

where  $c^W(t)$  and  $c^R(t)$  denote consumption when working and retired respectively.<sup>4</sup>

When working, the individual receives wage income of  $w(t)$ . Non-labor income is a fixed constant  $y$ .<sup>5</sup> Once retired, a pension benefit of  $B(T)$  is received.  $B(T)$  does not depend on  $t$ .<sup>6</sup> The price of the composite good is assumed to be unity for all  $t$ , and a perfect capital market exists with constant interest rate  $r$ . The life wealth constraint is therefore

$$\begin{aligned} & \int_0^T [y + w(t)] e^{-rt} dt + [y + B(T)] \int_T^N e^{-rt} dt \\ & = \int_0^T c^W(t) e^{-rt} dt + \int_T^N c^R(t) e^{-rt} dt \end{aligned} \quad (2)$$

Life wealth, which we shall call  $F(T)$ , is the left-hand side of (2).

The full problem is to choose  $T$ ,  $c^W(t)$  and  $c^R(t)$  to maximize (1) subject to (2). Since  $T$  is the object of attention it is useful to choose the optimal  $c^W(t)$  and  $c^R(t)$ , holding  $T$  fixed, as a first step. Then we may focus on the choice of  $T$  with  $c^W(t)$  and  $c^R(t)$  varying optimally.

Conditional Consumption Paths  $\hat{c}^W(t)$ ,  $\hat{c}^R(t)$

For a given value of  $T$ , necessary and sufficient conditions characterizing the optimal consumption path (denoting optimal values with a " $\hat{\phantom{x}}$ ") are<sup>7</sup>

$$F(T) = \int_0^T \hat{c}^W e^{-rt} dt + \int_T^N \hat{c}^R e^{-rt} dt \quad (3a)$$

$$u_c(\hat{c}^W, \bar{l}) = \lambda e^{-(r-\rho)t}, \quad t \in [0, T] \quad (3b)$$

$$u_c(\hat{c}^R, 1) = \lambda e^{-(r-\rho)t}, \quad t \in (T, N] \quad (3c)$$

$$\lambda > 0, \quad (3d)$$

where  $\lambda$  is a time-invariant dynamic multiplier on the wealth constraint. Notice that (3b) and (3c) imply

$$u_c[\hat{c}^W(T), \bar{l}] = u_c[\hat{c}^R(T), 1].$$

It follows that<sup>8</sup>

$$\hat{c}^R(T) \gtrless \hat{c}^W(T) \Leftrightarrow u_{c\bar{l}} \gtrless 0. \quad (4)$$

That is, if increased leisure raises (lowers)  $u_c$ , retirement must be accompanied by an increase (decrease) in consumption.

When choosing  $T$ , we will require  $\partial \hat{c}^W / \partial T$  and  $\partial \hat{c}^R / \partial T$ . It is easily shown that<sup>9</sup>

$$\text{sign} \left\{ \frac{\partial \hat{c}^W}{\partial T} \right\} = \text{sign} \left\{ \frac{\partial \hat{c}^R}{\partial T} \right\} = - \text{sign} \left\{ \frac{\partial \lambda}{\partial T} \right\} = \text{sign} \{ F_T - [\hat{c}^W(T) - \hat{c}^R(T)] e^{-rT} \}. \quad (5)$$

An increase in  $T$  affects the optimal consumption path by changing wealth  $F$  and by changing the length of time over which  $\hat{c}^W$  and  $\hat{c}^R$  must be financed. In the typical case in which the market wage exceeds the pension benefit and there are penalties for early retirement, later retirement will increase  $F$ . If  $u_{c\bar{l}} > 0$ , (4) implies that  $\hat{c}^R(T) > \hat{c}^W(T)$  and an increase in  $T$  will shorten the period over which higher consumption occurs. Both these changes will have



the effect of raising consumption for every  $t$ . However, if  $u_{c\bar{l}} < 0$ , the sign of  $\partial c/\partial T$  depends on which effect dominates.

### The Choice of T

Given optimal conditional consumption paths  $\hat{c}^W$  and  $\hat{c}^R$ ,  $T$  is chosen to maximize

$$V(\hat{c}^W, \hat{c}^R) = \int_0^T u(\hat{c}^W, \bar{l}) e^{-\rho t} dt + \int_T^N u(\hat{c}^R, 1) e^{-\rho t} dt,$$

the sum of discounted utility achieved while working plus that achieved while retired. Assuming an interior solution for  $T$ <sup>10</sup> the optimal  $T$  satisfies the necessary and sufficient conditions

$$V_T = \left\{ u(\hat{c}^W, \bar{l}) e^{-\rho T} + \int_0^T u_c \frac{\partial \hat{c}^W}{\partial T} e^{-\rho t} dt \right\} + \left\{ -u(\hat{c}^R, 1) e^{-\rho T} + \int_T^N u_c \frac{\partial \hat{c}^R}{\partial T} e^{-\rho t} dt \right\} = 0 \quad (6)$$

and

$$V_{TT} < 0. \quad (7)$$

The interpretation of (6) is straightforward. Incrementing  $T$  has two effects on the discounted utility attained during the working period. First, the utility flow  $u(\hat{c}^W, \bar{l})$  extends for a longer period. Second  $\hat{c}^W$  is altered at every  $t$ . The resulting change in utility is  $u_c \partial \hat{c}^W / \partial T$ . These two effects are contained in the first set of braces. Balancing these effects is the impact of  $T$  on discounted utility achieved during retirement, contained in the second set of braces. Incrementing  $T$  implies that the flow  $u[\hat{c}^R(T), 1]$  is no longer received and that  $\hat{c}^R$  changes for each  $t$ , which alters utility by  $u_c \partial \hat{c}^R / \partial T$ .

Using (3) and the expressions in footnote 9, (6) is readily reduced to

$$\{u[\hat{c}^W(T), \bar{l}] - u[\hat{c}^R(T), 1]\}e^{-\rho T} + \lambda\{F_T - [\hat{c}^W(T) - \hat{c}^R(T)]e^{-rT}\} = 0 \quad (8)$$

where the second term reflects the impact of  $T$  on both  $\hat{c}^W$  and  $\hat{c}^R$ . Equation (8) follows from the fact that the utility value of the change in  $c^W$  and  $c^R$  can be represented by the utility value of an appropriate change in wealth (recall  $\lambda$  is the marginal utility of wealth).

Using (8), it follows immediately that a wealth maximization approach (choosing  $T$  such that  $F_T = 0$ ) cannot be used to approximate or place bounds on the utility maximizing  $T$ . That is,  $u_{c^l} > 0$  implies  $\hat{c}^R(T) > \hat{c}^W(T)$ , whence the first term (of (8)) in braces is negative, yielding  $F_T - [\hat{c}^W(T) - \hat{c}^R(T)]e^{-rT} > 0$ . This tells us nothing about  $F_T$  since the second term is positive. If  $u_{c^l} < 0$ ,  $F_T$  is also unsigned, since the first term of (8) in braces is unsigned. Thus, while wealth maximization may seem a useful approximation to the problem, the wealth maximizing  $T$  bears no simple relation to the utility maximizing optimal retirement date.<sup>11</sup>

### III. The Response of Retirement Date to Changes in Parameters

In this section we consider the manner in which the optimal retirement date responds to changes in  $y, \rho, r, N$  and parameters of the benefit structure.  $T$  is a continuously differentiable function defined by (6),  $V_T = 0$ . It follows that for an arbitrary parameter  $v$

$$\frac{dT}{dv} = - \frac{V_{Tv}}{V_{TT}}$$

which, using (7), has the same sign as  $V_{Tv}$ . Accordingly, only the latter need be investigated. Further, for those parameters that operate solely upon the wealth constraint ( $y$  and the parameters of the benefit structure) a Slutsky-type decomposition is easily obtained. That is,  $V_{Tv}$  may be written as a wealth effect, expressing the impact of  $v$  on  $T$  through variations in  $F$ , and a quasi-substitution effect, giving the impact of  $v$  on  $T$  through changes in  $F_T$ .<sup>12</sup>

Consider changes in non-labor income  $y$ .  $V_{Ty}$  may be written as

$$V_{Ty} = \frac{\partial \lambda}{\partial T} \int_0^N e^{-rt} dt.$$

It follows that retirement time is a normal good ( $dT/dy < 0$ ) if  $\partial \lambda / \partial T < 0$ .

Recalling (5), this is equivalent to  $F_T - [\hat{c}^R(T) - \hat{c}^W(T)]e^{-rT} > 0$ . Equations (4) and (8)

then jointly imply that  $u_{c\ell} > 0$  is sufficient (though not necessary) for

increases in unearned income to lower retirement age.<sup>13</sup> For the sequel,

$V_{Ty} > 0$  is assumed. As retirement involves greater leisure, this is simply the

lifetime analogue of the static normality of leisure.

The impact of  $\rho$  is generally ambiguous:

$$V_{T\rho} = -T \{ [u[\hat{c}^W(T), \bar{\ell}] - u[\hat{c}^R(T), 1]] e^{-\rho T} + \frac{T}{\theta} \frac{t}{T} u_c \frac{\partial \hat{c}^W}{\partial T} e^{-\rho t} dt \\ + \int_0^N \frac{t}{T} u_c \frac{\partial \hat{c}^R}{\partial T} e^{-\rho t} dt \}$$

If the  $t/T$  factors were not in the integrals,  $V_{T\rho} = 0$  (i.e., absent the  $t/T$ ,

$V_{T\rho}$  is proportional to (6)). A little manipulation shows that the sign of

$V_{T\rho}$  is the same as that of

$$1 - \frac{\int_0^T \frac{e^{-(2r-\rho)t}}{u_{cc}} \cdot \frac{t}{T} dt + \int_0^N \frac{e^{-(2r-\rho)t}}{u_{cc}} \frac{t}{T} dt}{\int_0^T \frac{e^{-(2r-\rho)t}}{u_{cc}} dt + \int_0^N \frac{e^{-(2r-\rho)t}}{u_{cc}} dt}$$

the sign of which depends on both (a) whether  $c^W$  and  $c^R$  are rising or falling; and (b) whether  $u_{cc}$  is concave or convex in  $c$ . Somewhat contrary to intuition,  $dT/d\rho$  is indeterminate, since it is not clear which effects dominate.

Increases in  $r$  have two generally conflicting effects:

$$V_{Tr} = -\lambda \left\{ T[w(T) - B(T)] + B_T \int_T^N te^{-rt} dt \right\} \\ + \frac{\partial \lambda}{\partial T} \int_0^N -ts(t)e^{-rt} dt$$

where  $s(t)$  represents flow savings at date  $t$ .<sup>14</sup> The first term captures the effect of  $r$  on  $F_T$ . If  $B_T > 0$  and  $w(T) > B(T)$ , then  $F_T > 0$ . Increases in  $r$  reduce this "substitution" incentive to retire later, tending to reduce  $T$ . On the other hand, changes in  $r$  have wealth effects operating through the savings plan. These are represented by the second term. Suppose, for example, that  $\partial s / \partial t > 0$  so that the individual is first a net borrower, then a net lender.<sup>15</sup> Increases in  $r$  imply that any given amount of early borrowing requires a greater sacrifice of consumption later, reducing lifetime utility. This reduction in "real wealth" induces later retirement. More generally ( $\partial s / \partial t \gtrless 0$ ), the sign of this effect is ambiguous.

Increasing the length of life may be expected to raise  $T$ . With some manipulation

$$V_{TN} = \lambda B_T e^{-rN} + \frac{\partial \lambda}{\partial T} e^{-rN} \left\{ \left[ \frac{y+B(T)}{\hat{c}^R(N)} - 1 \right] + \frac{u_c [\hat{c}^R(N), 1]}{\hat{c}^R(N) u_{cc} [\hat{c}^R(N), 1]} \right\}$$

The first term accounts for the fact that if delaying retirement raises one's pension, a longer life makes this effect more important, tending to raise  $T$ . The second term is again a type of wealth effect. If there is dissaving at  $t=N$ ,  $\hat{c}^R(N) > y+B(T)$ . A lengthening of lifetime is then equivalent to an extension of a period of dissaving, and requires greater sacrifice of earlier consumption. This tends to raise  $T$ .

The second portion of the wealth effect represents the fact that raising  $N$  reduces the marginal effect of  $T$  on discounted utility obtained while retired. Accordingly,  $T$  must be raised (the retirement period shortened) to regain the equality required by (6). Inclusion of this term in the wealth effects reflects the duality relation that an increment to marginal utility may equivalently be attained via a reduction in wealth. Note that  $u_c / c^R u_{cc}$  is the reciprocal of the elasticity of marginal utility. Accordingly, the less elastic is marginal utility, the greater the required adjustment in  $T$  when  $N$  rises.

#### The Structure of Benefits: $B(T)$

Our specification of  $B(T)$  is as follows:

$$B(T) = M + \alpha(T)\bar{w} \quad (9)$$

$M$  is a constant minimum benefit.  $\alpha(T)$  is an arbitrary positive function that penalizes early retirement ( $\alpha_T > 0$ ).  $\bar{w}$  is an average of wages received over a period preceding retirement. In particular, let  $T_0$  be a negative number and let  $\phi(\tau)$  be an arbitrary averaging function on  $[T_0, 0]$ . That is

$$\int_{T_0}^0 \phi(\tau) d\tau = 1$$

$$\phi(\tau) > 0 \Leftrightarrow \tau \in [T_0, 0].$$

Thus, we define  $\bar{w}$  by average wages over the period  $t=T+T_0$  to  $t=T$ :

$$\bar{w} = \int_{T_0}^0 w(T+\tau) \phi(\tau) d\tau.$$

We assume below that  $w(t)$  is monotonically increasing and concave. It follows immediately that<sup>16</sup>

$$\frac{\partial \bar{w}}{\partial T}, \frac{\partial \bar{w}}{\partial T_0} > 0 \quad (10a)$$

and

$$\frac{\partial^2 \bar{w}}{\partial T \partial T_0} < 0 \quad (10b)$$

That is, raising either  $T$  or  $T_0$  involves averaging over a period for which wages are higher. Also, raising  $T_0$  implies that  $w$  (being concave in  $t$ ) is averaged over a period during which  $w$  is flatter on average.

Several of the comparative statics results to follow are unambiguous if the following condition holds:

$$\alpha_T < r\alpha / [1 - e^{-r(N-T)}]. \quad (11)$$

That is, (11) is sufficient for all of the terms in the comparative statics expressions to have the same sign. It is not a necessary condition for any of the results. Under the assumptions made above it is not possible to show that (11) must be satisfied. If  $M=0$  and  $w$  is constant,  $\alpha_T = r\alpha / [1 - e^{-r(N-T)}]$  is the necessary condition for maximization of discounted pension benefits.<sup>17</sup>

Both wage growth and the fact that one gets paid for working ( $w(T) > 0$ ) induce retirement after the pension maximizing date (tending to produce (11)). However,  $M > 0$  and the fact that leisure is greater once retired ( $\bar{l} < 1$ ) work in the opposite direction. Thus we shall assume that (11) holds for the purposes of discussion, bearing in mind however that while it may not hold it is also not necessary.

We are now in a position to consider changes in a parameter,  $v$ , of the pension structure, (9). As mentioned above, the Slutsky-type decomposition

$$V_{Tv} = \lambda \frac{\partial F_T}{\partial v} + \frac{\partial \lambda}{\partial T} \frac{\partial F}{\partial B} \frac{\partial B}{\partial v} \quad (12)$$

holds for these experiments. Note that factors lowering  $F_T$  or raising  $B$  tend to reduce  $T$ . For future reference, (9) implies that we may write

$$F_T = [w(T) - M - \alpha \bar{w}] e^{-rT} + (\alpha_T \bar{w} + \alpha \frac{\partial \bar{w}}{\partial T}) \frac{e^{-rT}}{r} [1 - e^{-r(N-T)}] \quad (13)$$

An increase in  $M$  has two effects.  $B$  rises and, by (13),  $F_T$  falls. Therefore an increase in the minimum pension unambiguously reduces  $T$ .

Consider shortening the averaging period ( $dT_0 > 0$ ). Changes in  $T_0$  operate through  $\bar{w}$ . Increments to  $T_0$  raise  $\bar{w}$ , and hence  $B$ , so the wealth effect reduces  $T$ . There are three effects on  $F_T$ . Raising  $T_0$  reduces  $\partial \bar{w} / \partial T$ , tending to lower  $T$ . Second,  $\bar{w}$  rises. This raises the pension, inducing earlier retirement, but also accentuates the impact of the penalty for early retirement. Under (11), the former dominates. Formally,

$$\begin{aligned} V_{TT_0} &= \lambda \frac{e^{-rT}}{r} \left[ \alpha \frac{\partial \bar{w}}{\partial T_0} \left\{ -r + \frac{\alpha T}{\alpha} [1 - e^{-r(N-T)}] \right\} \right. \\ &\quad \left. + \alpha \frac{\partial^2 \bar{w}}{\partial T \partial T_0} [1 - e^{-r(N-T)}] \right] \\ &\quad + \frac{\partial \lambda}{\partial T} \frac{\partial F}{\partial B} \alpha \frac{\partial \bar{w}}{\partial T_0} < 0. \end{aligned}$$

More generally, shifting more weight (holding  $T_0$  fixed) to time nearer retirement when forming the average,  $\bar{w}$ , has the same effects as increasing  $T_0$ . Specifically, if  $\phi(\tau)$  is parameterized by  $\gamma$  such that (see Figure 1)

$$\int_{T_0}^0 \frac{\partial \phi}{\partial \gamma}(\tau) d\tau = 0$$

$$\text{sign} \left\{ \frac{\partial \phi}{\partial \gamma} \right\} = \text{sign} \{ \tau - \tau_\gamma \} \text{ for some } \tau_\gamma \in [T_0, 0],$$

then changes in  $\gamma$  operate in exactly the same fashion as increases in  $T_0$ .

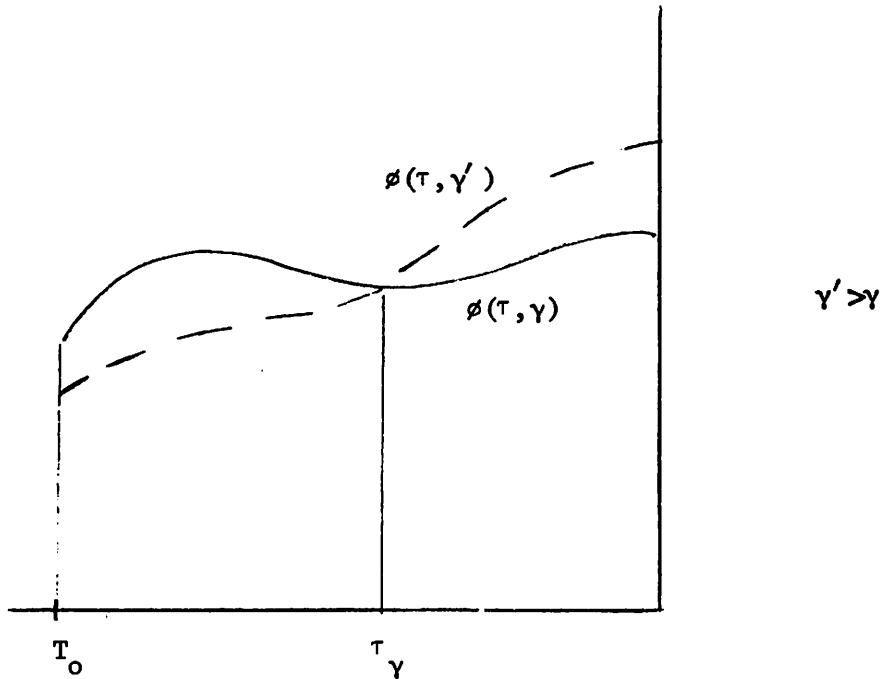


Figure 1



To investigate changes in the function penalizing early retirement,  $\alpha(T)$ , parameterize  $\alpha(T)$  by<sup>18</sup>

$$\alpha(T) = a_0 + a_1 A(T), \quad A' > 0.$$

Increments to  $a_0$  (raising the average penalty for early retirement) shift  $\alpha(T)$  vertically. This raises  $B(T)$ , producing a wealth effect that depresses  $T$ . There are two effects on  $F_T$ . First, the cost of putting off receipt of  $B(T)$  rises, tending to reduce  $T$ . On the other hand, the effect of increments to  $T$  on  $B(T)$  rises, tending to increase  $T$ . Formally

$$V_{Ta_0} = \lambda \bar{w} \frac{e^{-rT}}{r} \left\{ -r + \frac{1}{\bar{w}} \frac{\partial \bar{w}}{\partial T} [1 - e^{-r(N-T)}] \right\} + \frac{\partial \lambda}{\partial T} \frac{\partial F}{\partial B} \bar{w}$$

which is strictly negative so long as delaying retirement does not raise the average wage at a rate faster than the rate of interest. This is a plausible restriction given that wage growth near retirement is typically low.

Increasing  $a_1$  raises the effect of  $T$  on  $B$  (the marginal penalty for early retirement). As usual the wealth effect works to reduce  $T$ . There are three effects on  $F_T$ . First, as  $B(T)$  is larger, the cost of forestalling receipt rises. However, raising  $T$  now has a greater effect on  $B$  for two reasons. Given  $\bar{w}$ , raising  $T$  has a larger impact on  $\alpha \bar{w}$ . Also, the impact of  $T$  on  $B$  through changes in  $\bar{w}$  is larger once  $a_1$  is increased. Formally,

$$V_{Ta_1} = \lambda \bar{w} \frac{e^{-rT}}{r} \left\{ -r + \left[ \frac{A_T}{A} + \frac{1}{\bar{w}} \frac{\partial \bar{w}}{\partial T} \right] [1 - e^{-r(N-T)}] \right\} + \frac{\partial \lambda}{\partial T} \frac{\partial F}{\partial B} A \bar{w}$$

which is unambiguously negative under (11) if wage growth is small.

#### IV. Summary and Discussion

This paper has presented a utility maximizing model of retirement behavior. In order to derive predictions concerning retirement, it was assumed that hours of work are fixed. This assumption allows us to present a straightforward analysis of the problem. The optimal retirement date is that which equalizes the marginal contribution to lifetime utility of time spent in the employed and retired states.

The most interesting results of the paper relate to the impact of the benefit structure on the retirement decision. In particular, changes in the structure that place more emphasis on recent wages are generally expected to induce earlier retirement. This is especially relevant if the agency determining the pension focuses on nominal wages or follows a partial indexing procedure. Inflation, and the consequent nominal wage increases then effectively shift greater weight to current wages in the determination of pension benefits, inducing earlier retirement.

Second, increases in the average or marginal penalty for early retirement will probably induce more early retirement unless accompanied by a decline in the base pension. This occurs because uncompensated increases in the reward for continued labor force participation also raise the benefit level. Individuals may retire earlier to obtain the greater total benefits despite the contrary marginal incentives.

Finally, in focusing on the microeconomic aspects of the problem, we have ignored constraints imposed by balanced budget or pay-as-you-go pension schemes. However, labor market equilibrium requires that all else the same, individuals be indifferent between jobs. This should not be ignored and is easily handled. If the structure of the pension is viewed as a characteristic of the job, our comparative statics experiments may be viewed as asking how

individuals' behavior will differ across pension plans, provided we examine the compensated results. These results are obtained by examination of (12) with the final term omitted. As the income effects support the compensated effects in our analysis, our predictions are unchanged, though not so strong. The difference is that (11) becomes both necessary and sufficient if one suppresses the income effects.

Footnotes

<sup>1</sup>See Blinder and Weiss (1976), Heckman (1975, 1976) and Ryder, Stafford and Stephan (1976).

<sup>2</sup>Having made the fixed hours assumption, it is straightforward to provide conditions under which labor supply is characterized by an uninterrupted period of work followed by retirement. Interruptions in the working period (during which a pension is not received) are easily incorporated. We ignore them for the sake of exposition. Given the existence of pensions that penalize early retirement, the period of non-participation may be taken to occur at the end of working life. The crucial assumption is fixed hours.

<sup>3</sup>There is no doubt that there is a good deal of cross-section variation in hours supplied. However, a substantial portion of the variation is a result of permanent differences between individuals. Lillard (1979) for example, finds that about 60% of the unexplained variation in the labor supply of male heads of households (Michigan Panel Study of Income Dynamics Data 1967-1973) is accounted for by permanent individual-specific factors. Accordingly there is less variation in individual labor supply than in the cross section.

<sup>4</sup>A number of other considerations can be included relatively easily. For example, provided leisure remains fixed at  $\bar{l}$ , optimal accumulation of human capital can be treated in much the same fashion as consumption; that is, optimized in a first stage, conditional on T.

<sup>5</sup>Nonlabor income  $y$  does not refer to returns on savings.

<sup>6</sup>The exact structure of  $B(T)$  is detailed below. For the moment all that is required is that  $B$  depends on  $T$ .

<sup>7</sup>The solution to the consumption problem is a straightforward extension of the standard known model (see, for example, Takayama (1974), pp. 652-654). For the present case, let

$$I(t) = \begin{cases} 1 & t \in [0, T] \\ 0 & t \in (T, N] \end{cases} .$$

Then, the Hamiltonian, conditioned on T, is

$$\mathcal{H} = \{I(t)u[c^W(t), \bar{l}] + [1 - I(t)]u[c^R(t), 1]\}e^{-\rho t} \\ + \lambda\{y + I(t)[w(t) - c^W(t)] + [1 - I(t)][B(T) - c^R(t)]\}e^{-rt} .$$

For notational convenience, the dependence of  $c^W$  and  $c^R$  on  $t$  will be suppressed where possible.

<sup>8</sup>More precisely

$$\lim_{t \rightarrow T^+} c^R(t) \geq c^W(T) \Leftrightarrow u_{c^W} \geq 0 .$$

<sup>9</sup>Differentiation of the necessary conditions yields

$$u_{cc}(\hat{c}^W, \bar{l}) \frac{\partial \hat{c}^W}{\partial T} = \frac{\partial \lambda}{\partial T} e^{-(r-\rho)t} \quad t \in [0, T] \quad (1')$$

$$u_{cc}(\hat{c}^R, 1) \frac{\partial \hat{c}^R}{\partial T} = \frac{\partial \lambda}{\partial T} e^{-(r-\rho)t} \quad t \in (T, N] \quad (2')$$

$$F_T = \frac{\partial F}{\partial T} = [\hat{c}^W(T) - \hat{c}^R(T)]e^{-rT} + \int_0^T \frac{\partial \hat{c}^W}{\partial T} e^{-rt} dt + \int_T^N \frac{\partial \hat{c}^R}{\partial T} e^{-rt} dt . \quad (3')$$

Solving for  $\partial \lambda / \partial T$  yields

$$\frac{\partial \lambda}{\partial T} = \frac{F_T - [\hat{c}^W(T) - \hat{c}^R(T)]e^{-rT}}{\int_0^T \frac{e^{-(2r-\rho)t}}{u_{cc}} dt + \int_T^N \frac{e^{-(2r-\rho)t}}{u_{cc}} dt} . \quad (4')$$

Substituting (4') into (1') and (2') gives

$$\frac{\partial \hat{c}^W}{\partial T} = \frac{e^{-(t-\rho)t}}{u_{cc}} \frac{\partial \lambda}{\partial T}$$

and

(5')

$$\frac{\partial \hat{c}^R}{\partial T} = \frac{e^{-(r-\rho)t}}{u_{cc}} \frac{\partial \lambda}{\partial T},$$

whence (5) is immediate (recall  $u_{cc} < 0$ ).

<sup>10</sup>It is of course possible for  $T=0$  or  $T=N$  to occur. A sufficiently large pension guarantees the former, while a large wage guarantees the latter. There is nothing to analyze in these cases, so we assume an interior solution.

<sup>11</sup>Indeed, in light of the expressions for  $F_T$  given below, plausible  $w(T) > B(T)$  and  $\partial B/\partial T > 0$  imply that  $F_T > 0$  for all  $T$ . Thus the wealth maximizing individual is unlikely to retire at all. By contrast, the utility maximizing individual retires to obtain more leisure.

<sup>12</sup>The Slutsky-type expressions can be derived by (i) total differentiation of  $V_T$ ; (ii) substitution from (3); and (iii) simplification using (4') and (5') from footnote 9.

<sup>13</sup>Whether the restriction  $u_{cl} > 0$  is plausible is unclear. The typical assumption is  $u_{cl} = 0$  (see Heckman (1975), for example). In the context of a household production framework where  $Z$  is a home produced good, with linear homogeneous production function  $z(c, l)$ , and utility is  $v(Z)$ , we may obtain  $u(c, l)$  as  $v[z(c, l)]$ . In this case it follows that

$$u_{cl} \stackrel{>}{<} 0 \text{ as } \sigma \eta \stackrel{\leq}{\geq} 1$$

where  $\sigma$  is the Allen elasticity of substitution between  $c$  and  $l$  in  $z(\cdot)$ , and  $\eta$  is the elasticity of marginal utility. Becker and Ghez (1975, p. 138,

Table 4.2) estimate  $\sigma \geq .56$ . Savings studies, seeking to explain the observed interest inelasticity of savings, estimate  $\eta \geq 1$  (see Davies (1980);  $\eta$  is approximately the reciprocal of the intertemporal substitution elasticity in consumption). Accordingly  $\sigma\eta > 1/2$ , implying  $u_{c\ell} < 0$  is possible. In any case  $u_{c\ell} > 0$  is merely sufficient for normality of retirement time.

<sup>14</sup>That is

$$s(t) = \begin{cases} y + w(t) - c^W(t) & t \in [0, T] \\ y + B(T) - c^R(t) & t \in (T, N] \end{cases} .$$

<sup>15</sup>In the absence of a bequest motive  $\int_0^N s(t)e^{-rt} dt = 0$  must hold.

<sup>16</sup>The results for  $T_0$  require that

$$\frac{1}{\phi(\tau)} \frac{\partial \phi(\tau)}{\partial T_0} = \phi(T_0) .$$

<sup>17</sup>The intuition is precisely that a finite-life version of the famous Austrian capital theory problem: when to cut down a tree. If the tree has current value  $\alpha$ , and this value grows with  $T$ , the tree should be cut down when the present value of the increment to  $\alpha$  obtained by letting it grow (corrected for finite horizon)

$$\alpha_T \cdot \frac{1}{r} [1 - e^{-r(N-T)}]$$

equals the cost of leaving it there (its current value)  $\alpha$ . For our case, the pension takes on the role of the tree.

<sup>18</sup>That is,  $A(T)$  gives  $\alpha(T)$  its shape, with  $a_0$  locating  $\alpha(T)$  vertically and  $a_1$  rotating it.

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