1984

Risk, Income, Search and Price Stabilization

R Preston McAfee

10429

ISSN: 0318-725X
ISBN: 0-7714-0498-0

RESEARCH REPORT 8402

RISK, INCOME, SEARCH AND PRICE STABILIZATION

by

R. Preston McAfee

January 1984
Comments requested

A search model where agents maximize expected utility, rather than minimize total cost, is considered. The effect on reservation prices of increasing income, the cost of search, risk aversion and the proportion of income devoted to the good are established. It is shown that, if the individual is able to search many times, he prefers search and price dispersion to stabilized prices. Finally, a simulation is executed to illustrate the relative magnitudes of these effects and illuminate some unresolved questions.

I wish to thank John Chilton, Glenn MacDonald, John McMillan and Peter Morgan for assistance in preparing this manuscript. Any errors remain my own.

Introduction

The usual model of search suffers from two unrealistic assumptions which create an undeniable elegance in the theory. First, the individual is risk neutral, and not averse to a risk with no bound on possible expenditure. The only rationalization for such a model is the existence of insurance for search. Certainly this is incompatible with search itself, for such an insurance company would perform the search for a fee, and the price distribution would collapse.

Second, the usual model supposes that the individual will buy precisely one unit. This is a plausible model of search for durables such as cars, mattresses or television sets, and unrealistic for goods such as apples, camera film, or neckties. In addition, even for goods one buys exactly one unit of, the quantity decision may still exist. For example, cars come with options, and television sets vary in quality. We shall, however, avoid the issue of discreteness and assume the individual is choosing real quantities, subject to a budget constraint.

This paper injects finite resources, risk aversion and a quantity decision into the usual model. Otherwise, the model will be consistent with the traditional development. That is, we shall consider the distribution of prices to be known and invariant across stores. For a complete analysis of the current status of search theory, see Diamond and Rothschild [1978], Lippman and McCall [1981] and Rothschild [1973]. For related work, see Morgan [1981], and Manning and Morgan [1981]. In the latter, the analysis begins much as this paper's, in a more general framework. However, the emphasis is on sample sizes and demand effects rather than risk aversion.

Considerable interest has been focused, in recent years, on the issue of whether individuals prefer price dispersions to stabilized prices. Most of the consideration has been given to whether individuals prefer prices that vary over time. In this paper, we consider whether agents prefer search to stabilized prices at a moment in time. It is shown that individuals prefer search when their income is sufficiently large relative to the cost of search.

In addition, this paper includes simulations of the model, to illustrate the various magnitudes of the effects. One interesting feature of these simulations is that, in all cases, demand by individuals looks very much like demand facing a competitive firm, in the sense that demand is nearly horizontal at a certain price, and above that price, drops to nearly zero. Thus, despite the fact that reservation prices change as search persists, most demand is effectively at a single reservation price.

This indicates that total cost minimization does not have significantly different implications for search equilibrium than the more plausible utility maximization model presented here, in the following sense. In the latter case, most of the demand by a particular consumer is at some price $p \le p^*$, that is, with very high probability, the consumer will buy if and only if $p \le p^*$.

The Model

An expected utility maximizer has an indirect utility function $V(p)w(y)$, where prices $p = (p_1,\ldots,p_r)$ and y is income. The prices of goods $2,\ldots,r$ are known with certainty, and the distribution of p_1, $F(p)$, is known. $F'(p) = f(p)$. Let y be the amount of money the individual will spend and c the invariant cost of sequential search. Define J to satisfy $0 < y - Jc \le c$.

Two assumptions simplify the model without sacrificing the simultaneous search and quantity decisions. First, we postulate the absence of recall. If an individual passes up a price, he cannot return to buy at that price.

Second, the purchasing decisions for the n goods must be made simultaneously. Thus it is not possible to buy a little now and search further. These two assumptions combine to guarantee that a reservation price strategy is optimal, where the individual sets reservation prices $q_1,\dots,q_J$ and buys at store n if and only if the offered price does not exceed q_n. The individual cannot afford to search more than J times, by construction. We make these assumptions so that a reservation price strategy will be optimal. This also guarantees that the reservation prices are known at the outset. This model is plausible when price variation occurs across time, and not across stores.

It is of some interest to know precisely how restrictive the assumption that utility is $V(p)w(y)$ is. The following theorem answers this question.

Theorem 1: An expected utility maximizer has utility in the form $V(p)w(y)$ if and only if his preferences are homothetic and display constant relative risk aversion in income.

Proof: In Appendix.

Because $p_2,\dots,p_r$ are constant, we may write $v(p) = V(p,p_2,\dots,p_r)$. In general, very little structure is forced on v by quasiconcave preferences. V must be quasiconvex, but this does not imply v is convex. This is why we must assume v to be strictly convex. Most utility functions used in practice satisfy this assumption, and for this reason it does not appear overly restrictive. For discussion of this matter, and sufficient conditions to guarantee the convexity of v, see Turnovsky, Shalit and Schmitz [1980].

Given the structure on search previously developed, we obtain expected utility of

$$EU = \sum_{n=1}^{J} [\int_{o}^{q_n} v(p) \frac{f(p)}{F(q_n)} dp] w(y - nc)\ F(q_n) \prod_{j=1}^{n-1} (1 - F(q_j))$$

$$= \sum_{n=1}^{J} w_n\ G(q_n)\ F(q_n) \prod_{j=1]}^{n-1} (1 - F(q_j))$$

where $w_n = w(y - nc)$ and $G(q_n) = \int_{o}^{q_n} v(p) \frac{f(p)}{F(q_n)} dp$.

We assume $G(\infty) < \infty$, which in effect posits finite expected utility. Define:

$$H(q) = G(q)\,F(q) + v(q)(1 - F(q)).$$

When f has compact support, $F(q_n) = 1$ has the same effect as $q_n = \infty$. Occasionally we shall describe such a q_n as being "effectively infinite".

Theorem 2: $v(q_n) = \frac{w_{n+1}}{w_n} H(q_{n+1})$ and $q_J = \infty$.

Proof: In Appendix.

Theorem 2 establishes that $w_n V(q_n)$, the utility if one buys at reservation price q_n at store n, equals $w_{n+1}\, H(q_{n+1})$. Because $H(q_{n+1}) = G(q_{n+1})\, F(q_{n+1}) + v(q_{n+1})(1 - F(q_{n+1}))$, $w_{n+1}\, H(q_{n+1})$ can be interpreted as the expected utility of further search. This follows because $W_{n+1}G$ gives the utility if one purchases at store n+1, which occurs with probability $F(q_{n+1})$, and $v(q_{n+1})(1 - F(q_{n+1}))W_{n+1}$ yields the utility if one searches beyond store n+1. Thus Theorem 2 shows that $w_n v(q_n)$ acts as a reservation utility level, in that the utility at the reservation price equals the expected utility of further search. The reservation prices serve the dual purposes of being the most one will pay, and indicating the expected utility of further search. This parallels the usual model, where reservation prices indicate what one will pay and are the expected total cost of further search.

The next three theorems reveal some structure inherent in this model. Proofs may be found in the Appendix.

Theorem 3: $\frac{\partial q_n}{\partial y} < 0$, $\frac{\partial q_n}{\partial c} > 0$.

Theorem 4: $q_n < q_{n+1}$.

Theorem 5: $\lim_{y\to\infty} q_1 = \inf\{p|f(p) > 0\}$.

None of these results are counterintuitive. One expects to pay less if income rises, or more if search costs rise. Similarly, it is not surprising that as income diverges, one's reservation price falls to the bottom of the support of the price density.

To put these results in perspective, it is important to interpret the experiments of increasing y and c. Consider an individual who earns only labor income, and chooses to work L(w) hours, as a function of the real wage w. His income is thus $y = wL(w)$. Generally, we think of the cost of search as being composed of a fixed cost c_o and expended time t_o, valued on the margin at the wage. This is consistent with search involving driving around, where one expends some gasoline and also time. Thus, y/c is given by

$$y/c = \frac{wL(w)}{c_o + wt_o}$$

and

$$\frac{dy/c}{dw} = \frac{L'(w)(c_o/w + t_o) + c_oL(w)/w^2}{(c_o/w + t_o)^2}$$

Thus y/c is an increasing function of the wage, if labor increases with wages. Consequently, we may consider that the experiment of increasing y or decreasing c coincides with an increase in the real wage.

To analyze the effects of changes in risk aversion and the income share of good one, we consider a more specific model. This model will also be used in the simulations described in the fourth section.

Let indirect utility be in the form $p_1^{-\alpha}, p_2^{-\alpha_2}, \ldots p_r^{-\alpha_n} y^{\beta}$, where $\beta = \alpha + \sum_{i=2}^{r} \alpha_i$. This is Cobb-Douglas utility, raised perhaps to a power. If all prices are known with certainty, and y is an amount of money to be spent, the indirect utility function $U(y) \sim y^{\beta}$. Thus the Pratt [1964] measure of relative risk aversion is $1-\beta$, and an increase in β corresponds to a decrease in risk aversion.

One of the primary advantages of this model is that we can analyze the effect on reservation prices of an increase in risk aversion, and also an increase in the good's income share to the individual, which is α/β. One should note that to properly increase risk aversion, we should leave good 1's relative importance unchanged, that is, leave α/β constant.

Theorem 6: $\frac{\partial q_n}{\partial \beta} > 0$, $\left.\frac{\partial q_n}{\partial \alpha}\right|_{\alpha/\beta \text{ constant}} \leq 0$ and $\frac{\partial q_n}{\partial \alpha} < 0$.

Proof: See Appendix.

Search may properly be viewed as a money gample, for one is gambling precisely the cost of search. Thus, as we expect, increasing risk aversion increases reservation prices, meaning the risk averse person searches, or gambles, less.

When β is increased, the individual becomes less money risk averse, which would tend to decrease reservation prices. However, the relative importance of the good, α/β, is also decreased, tending to increase reservation prices. We see from the theorem that the second effect dominates in the case where α is constant.

The amount of money an individual spends on good 1, $\frac{\alpha(\hat{y}-nc)}{\beta\ p}$, is proportional to α. Thus α measures the importance of good 1 to the individual. We saw that increasing α decreases reservation prices. A moment's reflection will make sense of this. When α increases, the value of obtaining a lower price increases, while the cost of this action, more search, remains invariant. Thus one expects more expenditure on search. There is an analogous result to the third inequality expressed in a more general model in Manning and Morgan [1981]. This result is implicit in the original analysis by Stigler [1961], but is not demonstrated there. It seems unlikely that an analog for Theorem 6 holds for more general utility functions. The intuition is as follows. Analysis of risk ordinarily proceeds by considering infinitesimal risks. This method will not work with search, which entails inherently global risks. When the support of the price density function is small, the gains from search are necessarily small also. In addition, with risk averse individuals, there is concavity in income, and usually convexity in prices. Consequently, reservation prices will depend on global evaluation of income shares, risk aversion, and so forth. We considered this special case precisely because these parameters are globally constant, making the analysis relatively straightforward. Because none of these parameters enter into the usual (total cost minimizing) model of search, it is inadequate to analyze these questions. We now turn to the issue of whether individuals prefer search to stabilized prices.

Search and Price Stabilization

Prices may vary both across time and across stores at a given moment in time. In assessing the value of price stabilization, researchers have only considered the temporal variation in prices. This was first considered by Waugh [1944], and also investigated by Oi [1961], Samuelson [1972] and Turnovsky, Shalit and Schmitz [1980]. In all cases, agents were offered the opportunity to choose a price from a distribution, or accept the mean price of the distribution, and said to prefer stability if they chose the latter.

In this section, we shall address a different question. It has been well established that prices not only vary in time, but also at a moment in time across stores. We shall consider what it means to stabilize prices across stores, and who gains from this activity.

It is not precisely clear how one would go about stabilizing a price dispersion and what the outcome would be. Perhaps the best example would be government run alcoholic beverage stores. These eliminate the price dispersion found in other areas, but there is no reason to think the price outcome is the average dispersion price, or anything but the monopoly price.

However, it is certainly within the power of the government to stabilize prices at values other than the monopoly price, and we shall consider the effects of this. Define q_o by:

$$v(q_o) = \frac{w_1}{w_o} H(q_1) .$$

That is, q_o is the first reservation price of an individual with income y+c. Then we have,

Theorem 7: Expected utility of search is $w(y)v(q_o)$.

Proof: In Appendix.

Theorem 7 permits us to compare the expected utility of the search gamble with stabilization at any price p_s. The individual prefers search if and only if $q_o < p_s$. As a consequence, we see that people with large incomes prefer search, due to Theorem 5. In addition, by Theorem 4, individuals prefer search over prices stabilized at the reservation price q_1 at their first store. Thus, willingness to pay does not imply one prefers stabilized prices. We have presumed that, with stabilized prices, one need not expend search costs. If one must spend c to buy with stabilized prices, then search becomes even more attractive, as the following theorem demonstrates.

Theorem 8: If, at stabilized prices, the individual must expend c to purchase, then all individuals prefer search to prices stabilized at their expected value.

Proof: This follows immediately from the presumed convexity of v, the fact that setting $q_1 = \infty$ is within the choice set of the agent, and Jensen's inequality (Rudin [1974]).

One might think that the need to search would make price variation less attractive, in the sense that one must expend search costs. But, of course, the opposite is the case, as the ability to search offers one the freedom not to take the first draw, and increase expected utility.

It is perhaps of interest to note that, if $q_o < p_s < q_1$, then the individual will always accept the stabilized price p_s if it is offered during his search. However, given the choice of a p_s price stabilized world and a price dispersion, the individual will choose the latter. Thus, the fact that the individual agrees to buy at p_s is not sufficient to guarantee that he prefers prices stabilized there.

An interesting reinterpretation of this model permits us to consider the temporal price variation. Suppose prices fluctuate from day to day, given by the distribution F. Further suppose that the individual may choose to buy on a given day, or wait until the next day to buy, the latter with associated cost c. This could be interpreted as a rental cost. In this case, the no recall assumption makes perfect sense, as it merely posits the inability to travel backward in time. Thus, instead of searching from store to store, one searches from day to day. The number J is the number of days the individual can hold out without purchase. Theorem 7 tells us that one can determine if the individual prefers price instability to stabilization at price p_s if, after augmenting his income by c, he will not buy at p_s if p_s is offered on his first trial, and that a sufficient condition for the individual to prefer price instability is the unwillingness to buy at p_s on his first price sample.

It should be noted that, even if some stores charge more than p_s in the distribution, and hence in equilibrium some people pay more than p_s, it need not be the case that anyone prefers stability. The reason is that the demand at prices higher than p_s may be due to ex post losers in their searches, and that $q_1 < p_s$ for all individuals. Indeed, if v is convex and $p_s = \bar{p}$, this is certainly the case, by Theorem 8.

As a final note on this topic, the author considered whether the individual prefers search to the expected outcome of search. That is, will the individual pay the expected costs of search to obtain the expected price the search would yield? It seems unlikely that one always prefers search to utility of the expected outcome. However, the result of over a thousand simulations failed to reveal a single example where the agent preferred the utility of the expected value. This question hence remains an open conjecture. Indeed, it would appear that the conditions implying search is preferable are quite weak, and a single example of the agent not preferring search would be of interest.

The simulation did pay an unexpected dividend. There appears to be a substantial amount of regularity to the reservation prices which appears difficult to characterize formally. The next section illustrates the simulations, and how reservation prices and the demand function behave.

Simulation Results

The most interesting feature of the simulations was their striking uniformity. Whether F was concave or convex, and whether the individual was a risk lover or highly risk averse, doesn't seem to affect the pattern that reservation prices or expected demand take. Over a thousand examples were executed, and the only distinctions seem to be the value of the lowest reservation price.

The utility function is $(\prod_{i=1}^{r} p_i^{-\alpha_i})y^{\beta}$, where $\beta = \sum_{i=1}^{r} \alpha_i$. It is also given that $F(p) = p^R$, $\alpha_1 \le R$ and $0 \le p \le 1$. While this is admittedly a special case, a full range of values were considered. In particular, the ranges are: $0.001 \le \alpha \le \beta$, $.02 \le \beta \le 2$, $\alpha + .01 \le R \le 50$ and $2c \le y \le 1000c$ were examined. In all cases, reservation prices followed the roughly exponential pattern illustrated in Figures 1, 2 and 3.

The demand at price p by the individual is not particularly simple to calculate. Suppose $q_{n-1} < p \le q_n$, so that the individual will purchase at this price if and only if he observes it after store n-1 and does not observe another acceptable price first. Thus, if the individual does buy at price p at store $i \geq n$, his demand is:

$$D_i(p) = (\frac{\alpha}{\beta})(\frac{y-ic}{p}) = \frac{\alpha c}{\beta p}(\hat{y}-i).$$

Averaging this by the probability of reaching store i, we find, if $q_{n-1} < p \leq q_n$:

$$ED(p) = \sum_{i=n}^{J} D_i(p)(\prod_{j=1}^{i=1} 1-F(q_j))F(q_i) = \frac{\alpha c}{\beta p} \sum_{i=n}^{J} (\hat{y}-i)F(q_i) \prod_{j=1}^{i=1} (1-F(q_i))$$

It is trivial to show ED is strictly decreasing between 0 and the maximum price offered. In addition, there is a sense in which demand is "more convex" than the certainty demand, which is proportional to 1/p. The reason is that increases in price affect demand not only through the certainty demand decreases, but also through associated increases in store number, and consequent decreases in income available at this price. ED may be interpreted to be the average demand facing stores charging p, divided by f(p).

The shape of demand is quite important in procuring a market equilibrium which supports a distribution of prices. This has been analyzed in many papers, and the one with search closest to ours is Reinganum [1979], although her analysis avoids the income effects that are one of the features distinguishing ours. Carlson and McAfee [1983] have developed an equilibrium price dispersion model that avoids the assumption that some individuals have search costs arbitrarily close to zero, and explicitly calculates the equilibrium for an interesting special case. The reader is referred there for more discussion of this topic. All models of equilibrium price dispersions require calculation of the demand functions, motivating some analysis of this issue in our model.

The demand function is more convex than 1/p, but how convex is it? This depends in a complex way on F, and the results of our simulations indicate that demand makes a very sharp right angle after 5-10% of income is spent on search. The uniformity of the demand functions computed was quite striking, and nine typical examples are provided in Figures 4, 5 and 6.

This was a surprising outcome. Despite the curvature of the demand at a given store, and the distribution of prices facing the individual expected demand looks very much like demand facing a competitive firm when the price is constant and known. That is, the firm can sell arbitrarily much at one price, and nearly zero at any higher price. This indicates that, to support an equilibrium price dispersion, a distribution of individuals appears to be necessary, to eliminate the demand corner each individual has.

Whether the results of this simulation can be shown, or even formulated, in a precise sense provides an intriguing topic for further research. That demand in the price dispersion model mimics demand in the perfect competition environment is a peculiar feature of this model that perhaps indicates that the competitive model travels well when its assumptions are relaxed.

Conclusion

We have shown that even for a nice utility function, the optimal search strategy is quite different from the usual model of search. However, the structure in this case is not intractable, and we showed that reservation prices increase with search costs, the number of searches completed, and risk aversion, while reservation prices decrease with income and the importance of the good to the individual's utility. Finally, we showed that even a risk averse person acts like a risk lover with regards to search, because his expected utility through search exceeds his utility at the average price. This shows that the price aspect of the search gamble dominates the cost of search (money gamble) aspect.

The indirect utility function is generally convex in prices, and this provides an interesting topic for further research. Define a search lover to be any individual for whom the expected utility of search exceeds utility of the expected price and income. The conditions characterizing risk averse search lovers would elucidate both risk aversion and search.

Due to the complexity of this model, the author has not investigated search for more than one good. However, in the total cost minimizing case, this topic has been explored in Burdett and Malueg [1981] and Carlson and McAfee [1983]. It seems plausible that the theorems would hold for this case, although the reservation price structure is rather complex, involving finding "reservation sums" for each subset of the goods.

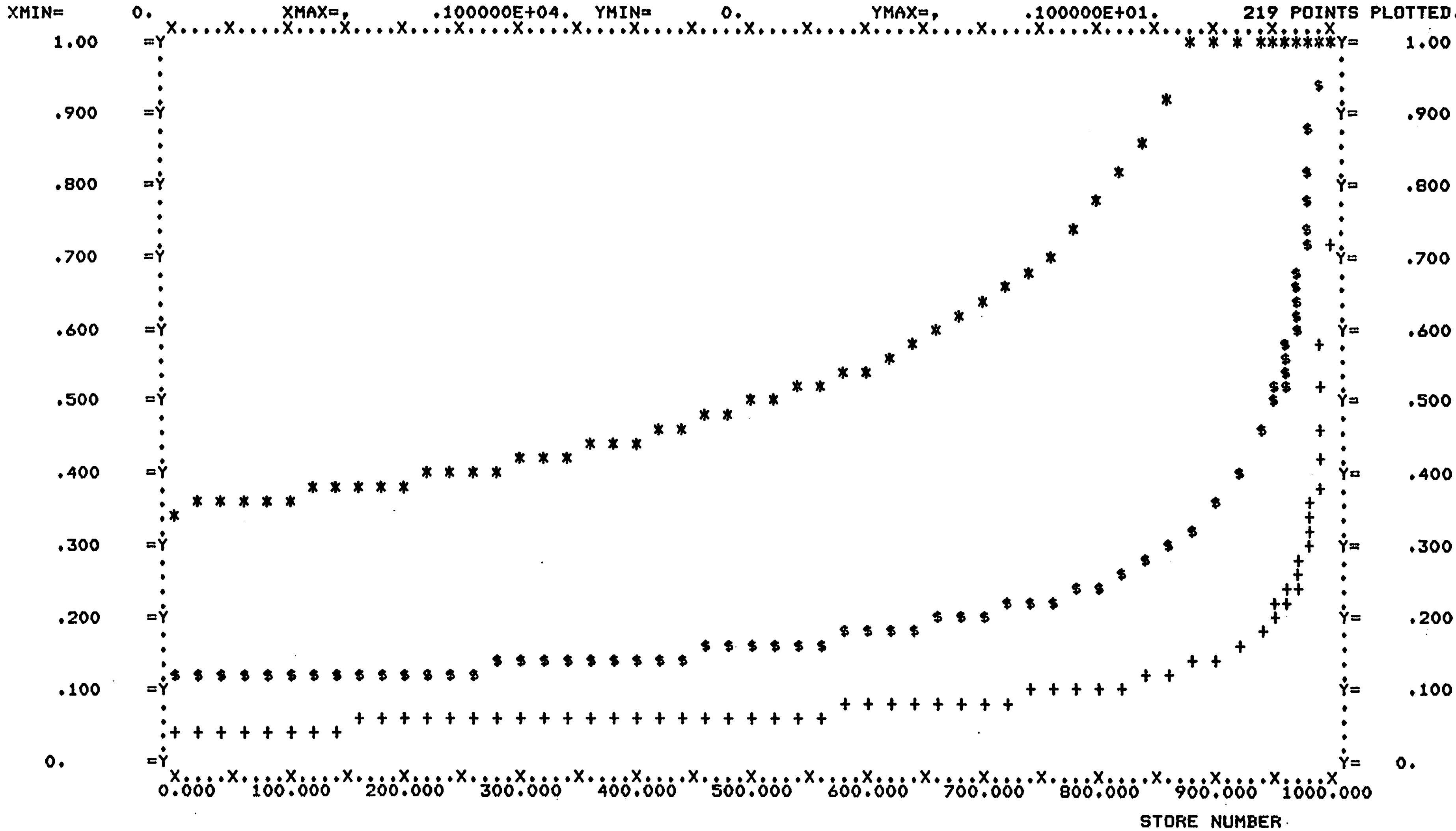

FIGURE 1: RESERVATION PRICES FOR VARIOUS VALUES OF ALPHA
BETA = 0.6, Y = 1000C, AND R = 2
ALPHA = 0.01, 0.1 AND 0.6 GRAPHED WITH A *, $, AND A +, RESPECTIVELY

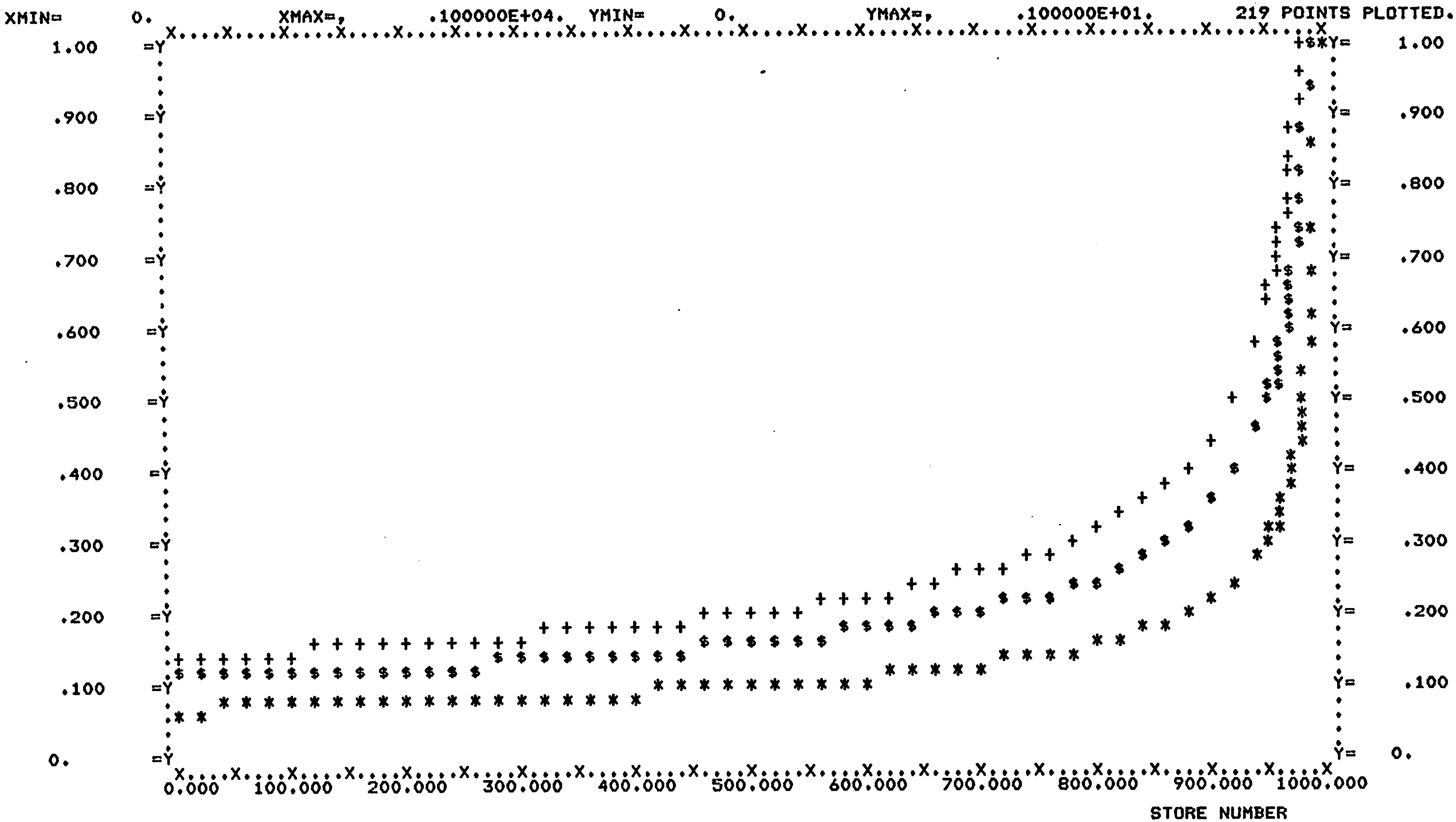

FIGURE 2: RESERVATION PRICES FOR VARIOUS VALUES OF BETA
ALPHA = 0.1, Y = 1000C, AND R = 2
BETA = 0.2, 0.6, AND 1, GRAPHED WITH A *, $, AND A +, RESPECTIVELY

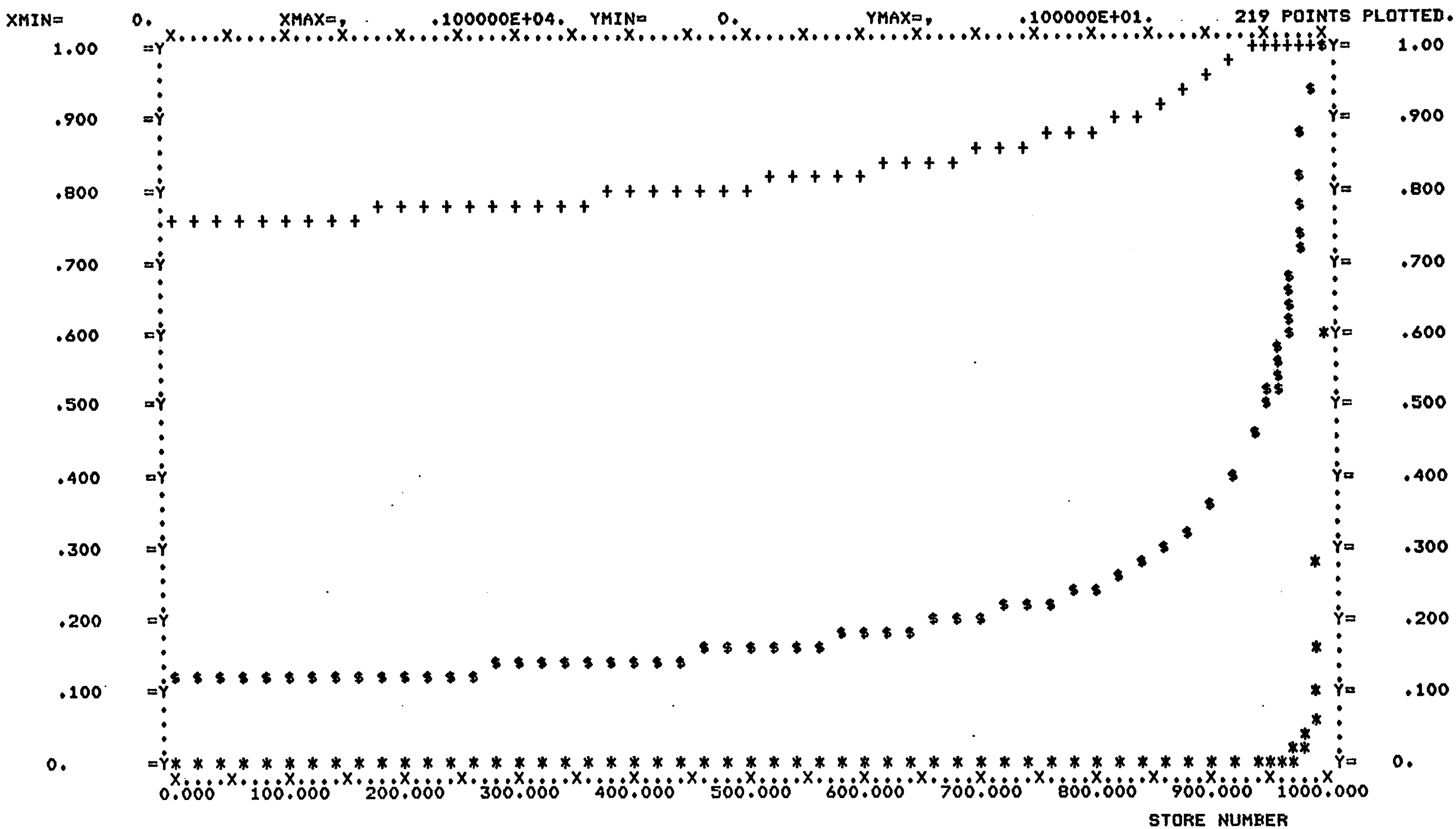

FIGURE 3: RESERVATION PRICES FOR VARIOUS DISTRIBUTION POWERS, DENOTED R
ALPHA = 0.1, BETA = 0.6 AND Y = 1000C
R = 0.5, 2, AND 10 GRAPHED WITH A *, $ AND A +, RESPECTIVELY

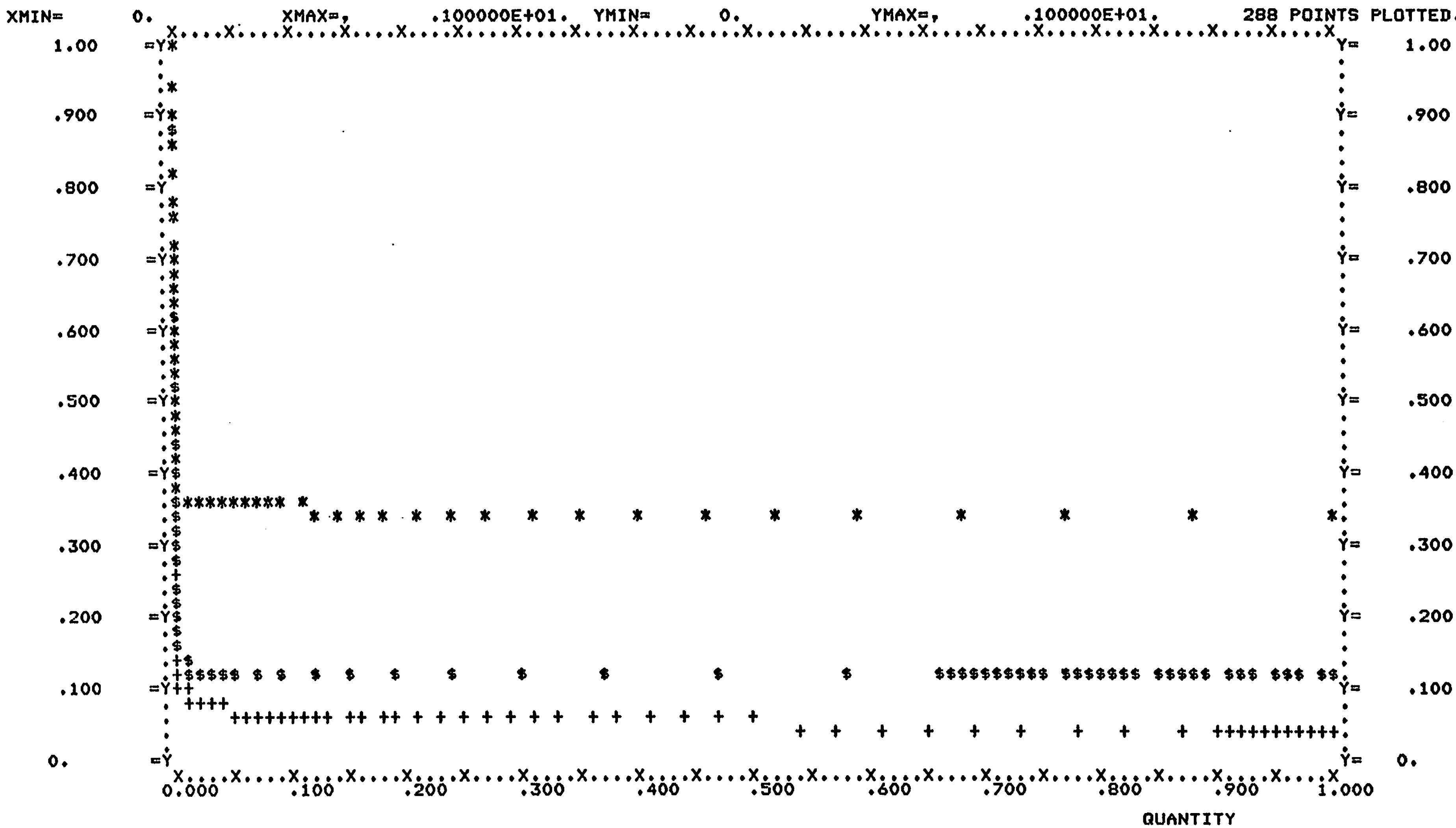

FIGURE 4: DEMAND, AS A PROPORTION OF D(Q1), FOR VARIOUS VALUES OF ALPHA
BETA = 0.6, Y = 1000C, AND R = 2
ALPHA = 0.01, 0.1, AND 0.6, GRAPHED WITH A *,$ AND A +, RESPECTIVELY

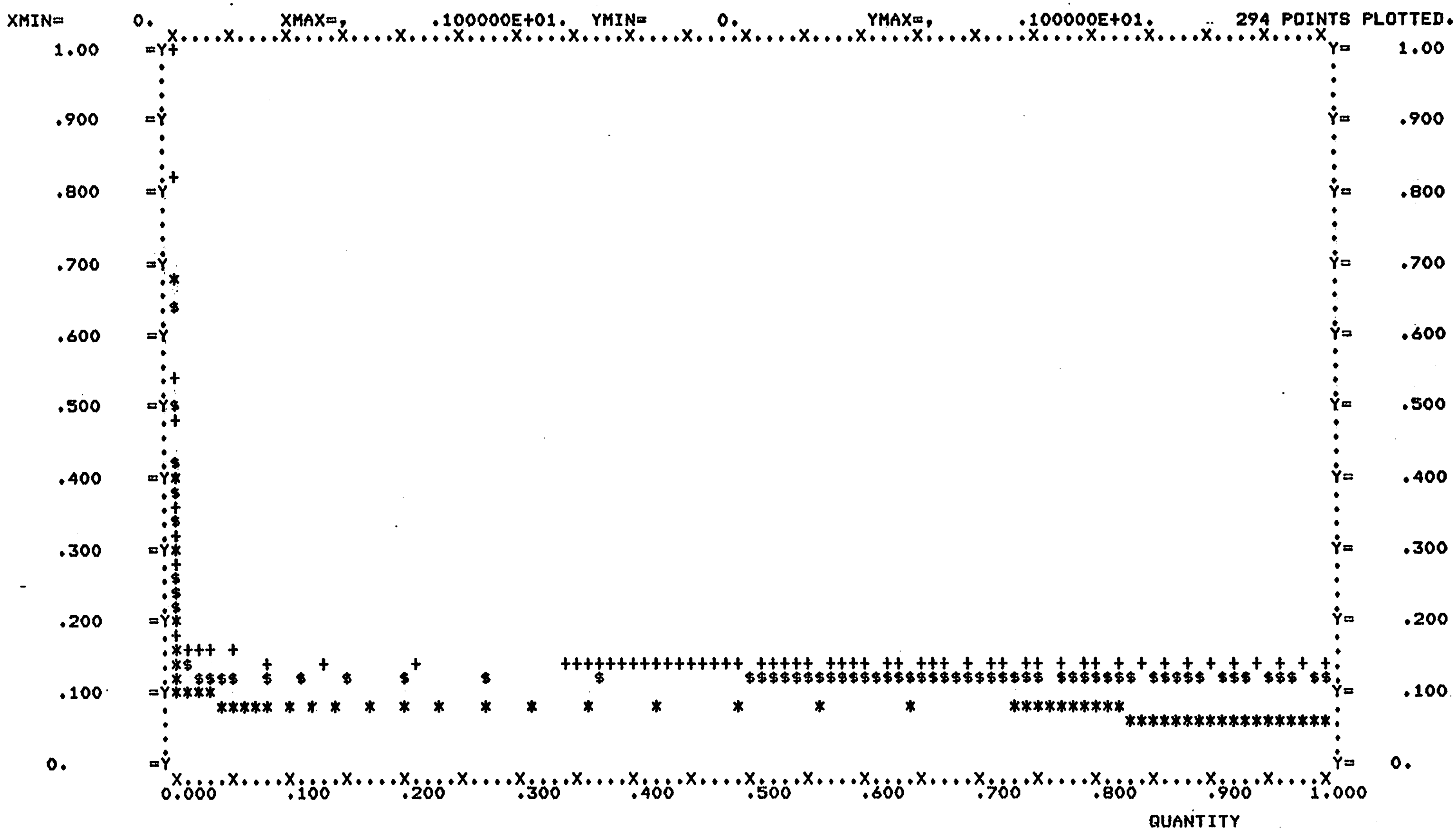

FIGURE 5: DEMAND, AS A PROPORTION OF D(Q1), FOR VARIOUS VALUES OF BETA
ALPHA = 0.1, Y = 1000C, AND R = 2
BETA = 0.2, 0.6 AND 1, GRAPHED WITH A *, $ AND A +, RESPECTIVELY

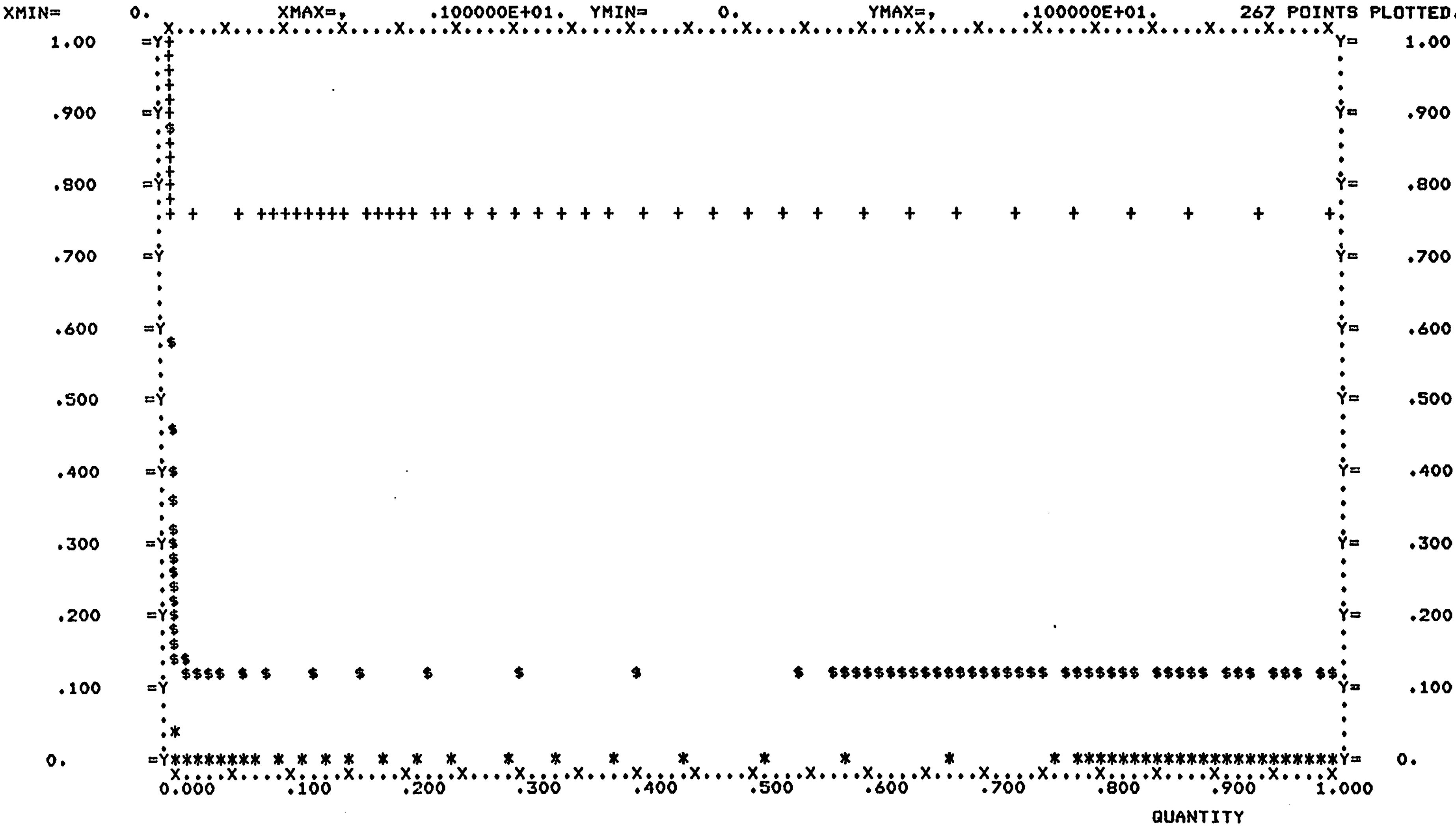

FIGURE 6: DEMAND, AS A PROPORTION OF D(Q1), FOR VARIOUS DISTRIBUTION POWERS, DENOTED R
ALPHA = 0.1, BETA = 0.6, AND Y = 1000C
R = 0.5, 2, AND 10, GRAPHED WITH A *, $, AND A +, RESPECTIVELY

REFERENCES

Burdett, K. and Malueg, D. [1981]. "The Theory of Search for Several Goods," Journal of Economic Theory 24, pp. 362-76.

Carlson, J., and McAfee, R. [1983]. "Joint Search for Several Goods," Journal of Economic Theory, forthcoming.

Carlson, J., and McAfee, R. [1983]. "Discrete Equilibrium Price Dispersion," Journal of Political Economy 91, June 1983.

Diamond, P. and Rothschild, M., eds. [1978]. Uncertainty in Economics, Academic Press, New York, pp. 425-515.

Lippman, S. and McCall, J. [1981]. "The Economics of Uncertainty: Selected Topics and Probabilistic Methods" in Handbook of Mathematical Economics, Volume 1, Arrow, K. and Intrilligator, M., eds., North-Holland, New York, pp. 217-227.

Manning, R. and Morgan, P. [1981]. "Search and Consumer Theory," Review of Economic Studies 49, pp. 203-216.

Morgan, P. [1982]. "Search and Optimal Sample Size," presented at the Economic Theory Workshop, University of Western Ontario, February 3, 1982.

Oi, W. Y. [1961]. "The Desirability of Price Instability Under Perfect Competition," Econometrica 29, pp. 58-64.

Pratt, J. W. [1964]. "Risk Aversion in the Small and the Large," Econometrica 32, pp. 122-136.

Reinganum, J. [1979]. "A Simple Model of Equilibrium Price Dispersion," Journal of Political Economy 87.

Rothschild, M. [1973]. "Models of Market Organization With Imperfect Information: A Survey," Journal of Political Economy 81, pp. 1283-1308.

Rudin, W. [1974]. Real and Complex Analysis, McGraw-Hill, New York, pp. 62-78.

Samuelson, P. A. [1972]. "The Consumer Does Benefit From Feasible Price Stability," Quarterly Journal of Economics 86, pp. 476-493.

Stigler, G. J. [1961]. "The Economics of Information," Journal of Political Economy 69, p. 219.

Turnovsky, S., Shalit, H., and Schmitz, A. [1980]. "Consumer's Surplus, Price Instability and Consumer Welfare," Econometrica 48, pp. 135-152.

Waugh, F. V. [1944]. "Does the Consumer Benefit from Price Instability," Quarterly Journal of Economics 58, pp. 602-614.

APPENDIX

Proof of Theorem 1:

(only if) V(p)w(y) is already in homothetic form, implying hometheticity. Obtain demand by

$$X_i(p,y) = \frac{\partial V/\partial p_i \; w(y)}{V(p)\; w'(y)} \quad .$$

From homotheticity, we know X_i is linear in y. Thus $\frac{w(y)}{w'(y)}$ is linear in y. This is equivalent to constant relative risk aversion.

(if) By homotheticity, utility may be written as $\Psi(V(p)y)$, for some Ψ. There are two constant relative risk aversion cases $\Psi(V(p)y) = Ky^{\beta}$ and $\Psi(V(p)y) = K\ln y$. To establish that β is invariant to p, observe that

$$X_i = \frac{y^{\beta}\frac{\partial K}{\partial p_i} + K y^{\beta} K \ln y \frac{\partial \beta}{\partial p_i}}{K\beta\; y^{\beta-1}}$$

Because homotheticity guarantees X_i is linear in y, $\frac{\partial \beta}{\partial p_i} = 0.$

Proof of Theorem 2:

From optimization,

$$w_n v(q_n) = \sum_{K>n}^{J} w_K [\int_0^{q_K} v(p)f(p)dp] \prod_{j=n+1}^{K-1} (1 - F(q_j))$$

$$= w_{n+1}\int_0^{q_{n+1}} v(p)f(p)dp + [\sum_{K>n+1} w_K \int_0^{q_K} v(p)f(p)dp \prod_{j=n+2}^{K-1} (1 - F(q_j))] \\ (1 - F(q_{n+1}))$$

$$= w_{n+1}\int_0^{q_{n+1}} v(p)f(p)dp + w_{n+1}V(q_{n+1})(1 - F(q_{n+1})) \; .$$

To prove that this characterizes a global maximum, it is sufficient to show that q_n defined by $v(q_n) = \frac{w_{n+1}}{w_n} H(q_{n+1})$ dominates $q_n = \infty$, as the zero endpoint is clearly suboptimal. This is trivial.

Proof of Theorem 3:

To prove this, we can presume $w(y) = y^\beta$ or $w(y) = \ell ny$. Inductively assume $\frac{\partial q_{n+1}}{\partial y} \leq 0$, this is true for $\frac{\partial q_J}{\partial y}$. As $H' < 0$, the result follows immediately. Observe $\operatorname{sgn}(\frac{\partial q_n}{\partial y}) = -\operatorname{sgn}(\frac{\partial q_n}{\partial c})$ to finish the theorem.

Proof of Theorem 4:

Claim: $\frac{w_{n+2}}{w_{n+1}} < \frac{w_n}{w_{n+1}}$.

Proof of Claim: For the case $w(y) = y^\beta$, this follows as $(y - nc)(y - (n+Z)c) < (y - (n+1)c)^2$. For the case $w(y) = \ell ny$, we have $\frac{d}{dy}[\frac{w'(y)}{w(y)}] < 0$, thus $\frac{w_{n+1} - w_{n+2}}{w_{n+1}} > \frac{w_n - w_{n+1}}{w_n}$. Therefore, $\frac{w_{n+2}\, w_n}{w_{n+1}^2} < 1$, and thus $(\frac{w_n\, w_{n+2}}{(w_{n+1})^2} - 1)(H(q_{n+2})) <$

$0 \leq H(q_{n+1}) - H(q_{n+2})$, as $H' < 0$. Thus,

$$\frac{w_n\, w_{n+2}}{(w_{n+1})^2} H(q_{n+2}) < H(q_{n+1})$$

or $$\frac{w_{n+2}}{w_{n+1}} H(q_{n+2}) < \frac{w_{n+1}}{w_n} H(q_{n+1})$$

or $$v(q_{n+1}) < v(q_n) .$$

Proof of Theorem 5:

Let $\underline{p} = \inf\{p | f(p) > 0\}$

show $\lim_{y\to\infty} q_1 = \underline{p}$.

By contradiction, suppose $\lim_{y\to\infty} q_1 = p_o > \underline{p}$. p_o exists by virtue of q_1 being monotone in y, bounded below by $\underline{p}$. As y gets large, expected utility gets arbitrarily close to:

$$EU_o = \sum_{n=1}^{J} w_n\, G(p_o)\, F(p_o) \prod_{j=1}^{n-1} (1 - F(p_o))$$

and thus $\frac{\partial EU}{\partial p_o}$ gets arbitrarily close to:

$$\sum_{n=1}^{J} w_n (1 - F(p_o))^{n-2} f(p_o)[V(p_o)(1 - F(p_o)) - (n-1)\int_{\underline{p}}^{p_o} v(p)\, f(p)\, dp] <$$

$$\sum_{n=1}^{J} w_n (1 - F(p_o))^{n-2} f(p_o)[V(p_o)(1 - F(p_o)) - (n-1)V(p_o)\, F(p_o)] =$$

$$\sum_{n=1}^{J} w_n (1 - F(p_o))^{n-2} f(p_o)\ V(p_o)[1 - n\, F(p_o)] \to 0, \quad \text{as } y\to\infty.$$

Thus, as y gets large, it becomes optimal to decrease p_o, as claimed. If $f(p_o) = 0$, choose a $p'_o < p_o$, such that $f(p'_o) \neq 0 \neq F(p'_o)$ and the argument goes through.

Proof of Theorem 6:

Let ln be the natural logarithm, $K_n = \frac{w_{n+1}}{w_n} = (\frac{y - (n+1)c}{y - nc})^{\beta}$. Then $q_n = [K_n\, H(q_{n+1})]^{-\frac{1}{\alpha}}$.

(i) $$\frac{\partial q_n}{\partial \beta} = -\frac{1}{\alpha}\,[K_n\, H(q_{n+1})]^{\frac{1}{\alpha}-1}\,[H(q_{n+1})\frac{\partial K_n}{\partial \beta} + K_n\, H'(q_{n+1})\,\frac{\partial q_{n+1}}{\partial \beta}].$$

$$\frac{\partial K_n}{\partial \beta} = K_n \ln\left(\frac{y-(n+1)c}{y-nc}\right) < 0\ . \quad H'(q_{n+1}) = -\alpha\, q_{n+1}^{-\alpha-1}\,(1-F(q_{n+1})) \le 0\ .$$

Thus, since $\frac{\partial q_J}{\partial \beta} = 0$, backward recursion yields $\frac{\partial q_n}{\partial \beta} > 1$.

(ii) Case I: f has compact support. Let $\bar{p}$ satisfy $F(\bar{p}) = 1$. Using a transformation of variables, letting f be the pdf of $p/\bar{p}$ and redefining the value of q_n as $q_n/\bar{p}$, we may presume $\bar{p} = 1$. If α/β is constant, $K_n^{-1/\alpha}$ is constant. Thus:

$$\frac{\partial q_n}{\partial \alpha} = -\frac{1}{\alpha}\, K_n^{-1/\alpha}(H(q_{n+1}))^{-1/\alpha-1}[H'(q_{n+1})\,\frac{\partial q_{n+1}}{\partial \alpha} + \frac{\partial H}{\partial \alpha}\Big|_{q_{n+1}\text{ constant}}]\ .$$

Using backward recursion, we assume $\frac{\partial q_{n+1}}{\partial \alpha} \le 0$, which is true for $n+1 = J$. $H' \le 0$, so it remains to verify that

$$\frac{\partial H}{\partial \alpha}\Big|_{q_{n+1}\text{ constant}} \ge 0.$$

$$\frac{\partial H}{\partial \alpha}\Big|_{q_{n+1}\text{ constant}} = -\int_0^q \ln p(p^{-\alpha})f(p)dp - \ln q(q^{-\alpha})(1-F(q)) \ge$$

$$\ge -\int_0^{\min\{q,1)} \ln p)p^{-\alpha})f(p)dp \ge 0$$

Thus, if f has compact support, $\frac{\partial q_n/\bar{p}}{\partial \alpha} < 0$ for $n < J$.

Case II: Let $\bar{p}\to\infty$, then $\lim_{\bar{p}\to\infty} \frac{\partial q_n/\bar{p}}{\partial \alpha} \le 0$, as desired.

(iii) This follows from (i) and (ii), as $\frac{\partial q_n}{\partial \alpha} = \frac{1}{\alpha}\frac{\partial q_n}{\partial \alpha}\Big|_{\alpha/\beta \text{ constant}} - \frac{\beta}{\alpha}\frac{\partial q_n}{\partial \beta}$.

Proof of Theorem 7:

$$\text{Expected utility} = \sum_{n=1}^{J} w_n G(q_n) F(q_n) \prod_{j=1}^{n-1} (1 - F(q_j))$$

$$= \sum_{n=1}^{J-1} w_n G(q_n) F(q_n) \prod_{j=1}^{n-1} (1 - F(q_j)) + v(q_{J-1}) w_{J-1} \prod_{j=1}^{J-1} (1 - F(q_j))$$

$$= \sum_{n=1}^{J-2} w_n G(q_n) F(q_n) \prod_{j=1}^{n-1} (1 - F(q_j)) + w_{J-1} [G(q_{J-1}) F(q_{J-1})$$

$$+ (1 - F(q_{J-1})) v(q_{J-1})] \prod_{j=1}^{J-2} (1 - F(q_j))$$

$$= \sum_{n=1}^{J-2} w_n G(q_n) F(q_n) \prod_{j=1}^{n-1} (1 - F(q_j)) + w_{J-2} v(q_{J-2}) \prod_{j=1}^{J-2} (1 - F(q_j))$$

$$= \ldots$$

$$= w_1 [G(q_1) F(q_1) + (1 - F(q_1)) v(q_1)]$$

$$= w_o V(q_o)$$