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STABLE LOW-LEVEL EQUILIBRIUM

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This paper analyzes a simple dynamic model of an economy in which search and recruiting are costly in the labor market and the cost per unit of selling goods depends upon the level of aggregate demand. As in the models of Diamond (1982), Howitt (1984) and Howitt and McAfee (1983) there are multiple stationary equilibria, which can be Pareto-ranked, and the low-level (Pareto-inferior) equilibria are similar in many respects to the persistent states of unemployment depicted by Keynes, with no wage- or price-rigidity. The novel feature of the present model is that it admits the possibility that all stationary equilibria are locally stable. In particular, in the simplest case of two non-degenerate stationary equilibria there may exist a perfect-foresight equilibrium path attainable from any initial position in the neighborhood of the low-level stationary equilibrium and converging asymptotically to that equilibrium. The paper also discusses applications of the same formal model to various other problems.

1. The Model

There are three types of tradable objects; homogeneous labor services, money, and goods; and two types of agents; workers and entrepreneurs. The entrepreneurs hire labor services from workers in exchange for money wages, produce goods from the labor services, sell the goods for money, and spend their net receipts buying goods from other entrepreneurs. There is a continuum of entrepreneurs, and a continuum of workers in each entrepreneur's labor force. Other than the absolute aversion each agent has to acquiring goods which he or his workers have produced,[1] all goods are identical.

Workers can supply labor services at a fixed rate at no utility cost.[2] When they enter the labor market they search at a fixed speed until they contact an entrepreneur, at which time they negotiate a wage contract. There is no cost to searching at this fixed speed or slower, but an infinite cost of searching any faster. Each entrepreneur has a fixed location and broadcasts a "recruiting net" that attracts the attention of searching workers. As described in our earlier paper the rate at which an entrepreneur thereby contacts workers is $\alpha\theta U$, where α is the (fixed) speed of search by workers, U is the mass of searching workers, and θ measures the size of the firm's recruiting net. An entrepreneur's cost of recruiting depends only upon the size of his net, according to the smooth cost function $G(\theta)$. Assume that:

(1) $$G'(\theta) > 0, \quad G''(\theta) > 0 \ \forall\ \theta > 0; \ G(0) = G'(0) = 0.$$

Each entrepreneur takes as given the number of searching workers. Thus the only way he can increase his rate of contacting is to increase the size of his net; in particular, he cannot attract more workers by offering to pay higher wages since, by assumption, the workers cannot receive that offer or make any other communication with the entrepreneur until they are already in his net.[3]

When a labor-market contact has been made a wage-contract is agreed upon. Each agent is assumed to be risk-neutral, with the same positive rate of time-discount, r; thus they are bargaining in a situation of bilateral monopoly with no opportunity for gains from risk-sharing or intertemporal trade. Assume that the bargaining always results in an agreement to pay a real wage at each date equal to the fixed positive fraction w of the worker's current marginal revenue product, net of selling costs, for as long as the match lasts. The sense in which this agreement is efficient will be discussed in Section 3.

Each match terminates exogenously according to a Poisson process with separation rate $\delta > 0$. When a separation occurs the worker reenters the pool of searching workers and the entrepreneur continues producing with his remaining labor force. Let n denote the size of an entrepreneur's labor force. It follows from our discussion so far that n varies over time according to the differential equation:

(2) $$\dot{n} = \alpha\theta U - \delta n$$

Let N denote the total mass of workers, E the total mass of entrepreneurs, and $\bar{n}$ the average size of labor force across all entrepreneurs at some date. Then the number unemployed (the mass of searching workers) at that date is

(3) $$U \equiv N - E\bar{n}$$

To avoid extraneous monetary complications assume that all money is a pure record-keeping device. Each worker has an account that is continuously credited at a rate equal to his current wage income and debited at a rate equal to his purchases of goods. The unit of account is goods, and no interest accrues on the accounts. There is no positive aggregate stock of money and the rules of exchange require each agent always to maintain a non-negative balance. The assumptions of risk neutrality and identical constant rates of time preference rule out any gain from the opening of a credit market, so none is assumed to exist.

Goods are not storeable, so the only use to which a worker's current purchases can be put is immediate consumption. Thus the demand for goods by a worker will be identically equal to his wage income, since any delay in spending income would reduce the present utility of such purchases. He will therefore voluntarily maintain a zero balance continuously in his account.

Each entrepreneur has a similar account, which is credited at a rate equal to his current sales and debited at a rate equal to his wage bill plus his purchases of goods from other entrepreneurs. All costs other than the wage bill are purely psychic, so the only use to which an entrepreneur's current purchases of goods can be put is current consumption, and he too will choose to maintain a zero balance in his account, by purchasing goods at a rate equal to his net receipts (sales minus wage bill).

Thus the aggregate demand for goods will be continuously equal to aggregate income (gross of psychic costs) and hence equal to the aggregate supply of goods. There are, however, costs of selling goods. An entrepreneur's selling cost is assumed to be proportional to the mass of customers he must sell to in order to execute his sales.

Assume that each worker buys from only one entrepreneur but each entrepreneur buys with uniform density from a mass γE of other entrepreneurs, where $0 < \gamma \leq 1$. Then the total mass of distinct transactions in the goods market is $N + \gamma E^2$, and the total quantity of goods sold per transaction is $\frac{Y}{N+\gamma E^2}$, where Y denotes the aggregate demand (and aggregate supply) of goods. Assume that each entrepreneur is assigned customers of each type in the same fixed proportion, $N:\gamma E^2$. Then he faces a quantity demanded per customer of $\frac{Y}{N+\gamma E^2}$ no matter how much he wants to sell. If he wants to sell y units then the number of customers he must serve is $\frac{\text{total quantity}}{\text{quantity per customer}} = \frac{y}{Y/(N+\gamma E^2)}$. Let the selling cost per customer be $\sigma' > 0$. Then the cost (measured in units of goods) of selling y units is $\sigma' \frac{y}{Y/(N+\gamma E^2)} = (\sigma/\bar{y})y$, where $\sigma \equiv \sigma' (N+\gamma E^2)/E$, and where $\bar{y} \equiv Y/E$ is the level of aggregate demand per entrepreneur.

The representative entrepreneur will take the per-unit cost $(\sigma/\bar{y})$ as given, since σ is determined by technology and $\bar{y}$ depends upon the collective decisions of all other agents. However, this cost embodies an important external economy of the sort emphasized by Diamond (1982) and Howitt (1984); the greater the level of aggregate demand the lower the per-unit cost of selling goods.

Output per worker is a positive constant f, given by technology. Thus if all output is offered for sale, $y = fn$ and $\bar{y} = f\bar{n}$. However, it will be offered for sale if and only if the sales receipts cover the selling cost, i.e., $1 - \sigma/\bar{y} \geqq 0$. In order to rule out extraneous self-fulfilling prophecies of $\bar{y} = 0$, assume that all output will be sold if and only if $1 - \sigma/f\bar{n} \geqq 0$. Thus:

$$(4) \qquad y \quad (\bar{y}) = \begin{Bmatrix} fn & (f\bar{n}) & \text{if} & 1 - \sigma/f\bar{n} \geqq 0 \\ 0 & (0) & \text{if} & 1 - \sigma/f\bar{n} < 0 \end{Bmatrix}$$

where $1 - \sigma/f\bar{n}$ denotes a negative number when $f\bar{n} = 0$.

For any number x, let x^+ denote max (x,0). It follows directly from (4) that an entrepreneur's sales revenue net of selling costs will be:

$$(5) \qquad y(1-\sigma/\bar{y}) = \begin{Bmatrix} fn(1-\sigma/f\bar{n}) & \text{if} & 1-\sigma/f\bar{n} \geqq 0 \\ 0 & \text{if} & 1-\sigma/f\bar{n} < 0 \end{Bmatrix} = fn(1-\sigma/f\bar{n})^+ = n(f-\sigma/\bar{n})^+$$

Accordingly, he will pay a real wage rate equal to $w(f-\sigma/\bar{n})^+$, and his consumption will equal $y - wn(f-\sigma/\bar{n})^+$. Thus his instantaneous utility will be (according to a permissible linear transformation):

$$\begin{aligned} & \text{consumption - selling cost - recruiting cost} \\ = \; & y - wn(f-\sigma/\bar{n})^+ - y(\sigma/\bar{y}) - G(\theta) \\ = \; & y(1-\sigma/\bar{y}) - wn(f-\sigma/\bar{n})^+ - G(\theta) \\ = \; & (1-w)n(f-\sigma/\bar{n})^+ - G(\theta) \qquad \text{(by (5)).} \end{aligned}$$

2. Equilibrium

An entrepreneur's decision problem is to choose a time-path of employment $\{n(t)\}_{t=o}^{t=\infty}$ and recruiting effort $\{\theta(t)\}_{t=o}^{t=\infty}$ so as to maximize the present value:

$$\int_{o}^{\infty} e^{-rt}((1-w)n(t)(f-\sigma/\bar{n}(t))^{+} - G(\theta(t)))dt$$

subject to an initial condition $n(0) = n_o$, the law of motion (2), the definition (3), and the given time path of average employment $\{\bar{n}(t)\}_{t=o}^{t=\infty}$, which he foresees perfectly.

The necessary and sufficient conditions for a regular maximum are (2), (3), and:

(6) $$G'(\theta) = \alpha\lambda(N-E\bar{n}) \quad \forall t \geqq 0$$

(7) $$\dot{\lambda} = (r+\delta)\lambda - (1-w)(f-\sigma/\bar{n})^{+} \quad \forall t \geqq 0$$

(8) $$\lim_{t\to\infty} e^{-rt}\lambda(t)n(t) = 0$$

where $\lambda(t)$ is the current shadow value to the entrepreneur of recruiting another worker.

In defining equilibrium time paths we will restrict attention to perfectly symmetrical situations, in which all entrepreneurs choose $n(t) = \bar{n}(t)\ \forall t \geqq 0$. Substituting this equality into the maximum conditions and simplifying yields a formal definition of equilibrium as a pair of time-paths $\{n(t),\lambda(t)\}_{t=o}^{t=\infty}$ such that

(9) $$\dot{n} = \alpha(N-En)F(\lambda\alpha(N-En)) - \delta n \quad \forall t \geqq 0$$

(10) $$\dot{\lambda} = (r+\delta)\lambda - (1-w)(f-\sigma/n)^{+} \quad \forall t \geqq 0$$

(11) $$\lim_{t\to\infty} e^{-rt}\lambda(t)n(t) = 0$$

(12) $$0 \leqq n \leqq N/E, \quad 0 \leqq \lambda \quad \forall t \geqq 0$$

where the function $F(\cdot)$ is the inverse of $G'(\cdot)$. The inequalities on n in (12) are required for the definition to make economic sense; the non-negativity of λ can actually be derived from (10) and (11).

Equation (9) can be interpreted as identifying the (gross of attrition) demand for labor per entrepreneur as $\alpha(N-En)\theta$, which is increased by any increase in the shadow value of employment to the entrepreneur, λ, and by any increase in the number unemployed, $N-En$. An increase in the number unemployed has two effects upon the demand for labor: first it raises the rate at which searching workers are attracted to the entrepreneur even with no increase in recruiting effort θ, and second it raises the number of additional workers that would be attracted by any increase in recruiting effort, thereby encouraging such an increase. Integrating (10) identifies the shadow value of employment to the entrepreneur as the expected present value of a worker's real marginal revenue product net of selling costs and wages: $\lambda(t) = \int_0^\infty e^{-(r+\delta)s}(1-w)(f-\sigma/n(t+s))^+ds$, where the appropriate rate of discount is the pure rate of time preference, r, plus the rate of attrition, δ.

The phase diagram for the system (9)~(12) is illustrated in Figure 1. The locus $\dot{n} = 0$, derived from (9), can be expressed as: $\lambda = \lambda^1(n) \equiv \frac{1}{\alpha(N-En)} G'(\frac{\delta n}{\alpha(N-En)})$. From (1) $\lambda^1(n)$ is a strictly increasing positive-valued function on the domain $(0,N/E)$, with $\lim_{n\uparrow N/E} \lambda^1(n) = +\infty$. The locus $\dot{\lambda} = 0$, derived from (10), can be expressed as $\lambda = \lambda^2(n) \equiv (r+\delta)^{-1}(1-w)(f-\sigma/n)^+$, which equals zero on the domain $[0,\sigma/f]$, and is bounded on $[0,N/E]$. Both $\lambda^1(n)$ and $\lambda^2(n)$ are continuous on $(0,N/E)$.

The Jacobean of the system defined by (9) and (10) is:

$$(13) \qquad J = \begin{bmatrix} \partial\dot{n}/\partial n \; , & \partial\dot{n}/\partial\lambda \\ (-) & (+) \\ \partial\dot{\lambda}/\partial n & \partial\dot{\lambda}/\partial\lambda \\ (-) & (+) \end{bmatrix} = \begin{bmatrix} -E\alpha(F+\lambda\alpha(N-En)F') - \delta \; , & \alpha^2(N-En)^2F' \\ -(1-w)\sigma/n^2 \; , & r+\delta \end{bmatrix}$$

where, from (1)

$$(14) \qquad F(x) > 0, \; F'(x) > 0 \quad \forall x > 0.$$

Hence the directional arrows in Figure 1.

A nondegenerate stationary equilibrium is a pair $(n^*,\lambda^*) > 0$ such that $\dot{\lambda} = \dot{n} = 0$. Generally there exist an even number (possibly zero) of stationary-equilibrium values of employment, because they are identical to the solutions on $(0,N/E)$ of the equation $\lambda^1(n) - \lambda^2(n) = 0$, whereas $\lambda^1(n) - \lambda^2(n) > 0$ for values of n close enough to either end of this interval. We shall assume that exactly two stationary equilibria exist, as depicted by H and L (high- and low-level) in Figure 1.

The low-level equilibrium is similar in some respects to the persistent state of unemployment envisioned by Keynes. For example, employment remains low because of a reciprocal interaction between the goods and labor markets, similar to that involved in the Keynesian multiplier process. More specifically, low employment (n_L) causes low output through the production function; low output causes a low level of aggregate demand (since the two must be equal in equilibrium); this low level of aggregate demand depresses the shadow value of employment; $\lambda_L = (r+\delta)^{-1}(1-w)(f-\sigma/n_L)$, which in turn depresses the demand for labor, despite the large number unemployed, thereby keeping employment low. In particular, this direct effect of aggregate demand for goods upon the demand for labor, not mediated by wages or prices, similar to the spillover effect emphasized by Barro and Grossman (1971) in their exposition of the multiplier process.[4]

Furthermore, unemployment in this model is involuntary in the everyday sense of the word. The unemployed are doing everything they can to regain employment; namely searching at the speed α. They cannot improve their prospects by offering to work for less than the going wage because in order to make such a communication to a prospective employer a worker must first make contact, by which time his unemployment will have been eliminated even without the offer. Nor would the offer do anything to hasten the arrival of a contract, which occurs according to a Poisson process at the rate $\alpha\theta$. He is powerless to affect α, which is given by technology, or θ, which the entrepreneurs choose according to (6) and (7) on the basis of the economy-wide averages $\bar{n}$ and $\bar{w}$.

Finally, as Keynes argued in Chapter 19 of the General Theory, this state of unemployment persists despite the absence of any price- or wage-rigidity. The output market clears in the usual sense, wage bargains are struck in real terms with no money-illusion, and, as we shall see in Section 3, the assumed outcome of the wage bargain is efficient in the sense that no potential gains from trade attainable by the parties to the bargain are left unexploited.

Any bounded solution to (9) and (10) statisfying the inequalities (12) will be a (dynamic) equilibrium. The high-level stationary equilibrium H is a saddle-point for (9) and (10), because det J = $(\partial\dot{n}/\partial\lambda)(\partial\dot{\lambda}/\partial\lambda)\left(\frac{\partial\dot{n}/\partial n}{\partial\dot{n}/\partial\lambda} - \frac{\partial\dot{\lambda}/\partial n}{\partial\dot{\lambda}/\partial\lambda}\right)$ $= \alpha^2(N-En^2)F' \cdot (r+\delta)\left(\frac{d\lambda^2}{dn} - \frac{d\lambda^1}{dn}\right) < 0$ at H. Therefore, for any initial n_o in a neighborhood of n_H there exists a unique equilibrium path starting with employment equal to n_o and converging monotonically upon H. In this sense the high-level equilibrium is stable.

Under robust conditions the low-level equilibrium is also stable, in the stronger sense that for any n_o in a neighborhood of n_L there are continuum many equilibrium paths starting with employment equal to n_o and converging upon L. To show this it suffices to show that L is a regular sink for (9) and (10). Since $\det|J| = \alpha^2(N-En)^2F'\cdot(r+\delta)(\frac{d\lambda^2}{dn} - \frac{d\lambda^1}{dn}) > 0$ at L, it suffices to show conditions under which the trace of J, $r - E\alpha(F+\lambda\alpha(N-En)F')$ is negative. This will be the case, according to (14), if $r=0$. Since J is continuous in r, this will also be the case for all values of r in some interval $(0,\bar{r})$.

For the sake of welfare comparisons note that the expected present value of utility of the representative entrepreneur as of time zero along the equilibrium path $\{n(t),\lambda(t)\}$ is:

$$U^e \equiv \int_0^\infty e^{-rt}((1-w)(fn-\sigma)^+ - \theta G'(\theta))dt + \int_0^\infty e^{-rt}h(\theta)dt \tag{15}$$

where $h(\theta) \equiv \theta G'(\theta) - G(\theta)$. From (1),

$$h'(\theta) > 0 \quad \text{if } \theta > 0. \tag{16}$$

Note that $\int_0^\infty e^{-rt}\theta G'(\theta)dt$

$$= \int_0^\infty e^{-rt}\theta\alpha\lambda(N-En)dt \quad \text{(from (6))}$$

$$= \int_0^\infty e^{-rt}\lambda(\dot{n}+\delta n)dt \quad \text{(from (2) and (3))}$$

$$= [e^{-rt}\lambda n]_0^\infty - \int_0^\infty e^{-rt}n(\dot{\lambda} - r\lambda)dt + \int_0^\infty e^{-rt}\lambda\delta n dt \quad \text{(from integration by parts)}$$

$$= -\lambda(0)n(0) + \int_0^\infty e^{-rt}(1-w)(fn-\sigma)^+dt \quad \text{(from (10) and (11)).}$$

From this last result and (15):

(17) $$U^e = \lambda(0)n(0) + \int_0^\infty e^{-rt}h(\theta)dt.$$

Next, note that the expected present value of utility of the randomly selected worker as of time zero equals the present value of the aggregate wage bill divided by the mass of workers:

(18) $$U^w \equiv \frac{1}{N}\int_0^\infty e^{-rt}Ew(fn(t) - \sigma)^+dt.$$

To verify that the low-level stationary equilibrium L is indeed Pareto-inferior to H in some meaningful sense, note first that the stable branch approaching H must slope up, and must lie above $\lambda^2(n)$ for all $n \in (n_L, n_H)$. This is because if the branch is followed backwards, from right to left, starting at a point like A close to but to the left of n_H, where it must lie above $\lambda^2(n)$, it can be continued down to the left as long as it remains above $\lambda^2(n)$ and $n > n_L$. Nor can it cross $\lambda^2(n)$ when $n > n_L$ because all trajectories touching $\lambda^2(n)$ in that range lie underneath $\lambda^2(n)$ immediately to the right of the point of contact, which could not be the case the first time the stable branch crossed $\lambda^2(n)$ to the left of n_H.

Thus there exists an equilibrium $\{\hat{n}(t), \hat{\lambda}(t)\}$ starting with $\hat{n}(0)$ greater than but arbitrarily close to n_L, with:[5]

(19) $$(\hat{n}(t), \hat{\lambda}(t)) \{\stackrel{>}{\geqq}\} (n_L, \lambda_L) > (\sigma/f, 0) \ \forall t \{\stackrel{>}{\geqq}\} 0$$

(20) $$(\dot{\hat{n}}(t), \dot{\hat{\lambda}}(t)) > 0$$

which converges on H. Furthermore, such paths can start with $\hat{n}(0) = n_L$ in the case where L is a sink, for otherwise the stable branch would have to cross $\lambda^2(n)$ at L, implying that L was not in fact a sink.

Any such path Pareto-dominates L in terms of U^e and U^w. According to (18), $\hat{U}^w - U_L^w > 0$ because, from (19), $(f\hat{n}(t) - \sigma)^+ - (fn_L - \sigma)^+ = f(\hat{n}(t) - n_L) > 0$ $\forall t > 0$. According to (17), and (19), $\hat{U}^e - U_L^e > 0$ if $\int_0^\infty e^{-rt}(h(\hat{\theta}(t)) - h(\theta_L))dt > 0$. According to (15) this will be the case if $\hat{\theta}(t) - \theta_L > 0$ $\forall t > 0$. This in turn is the case because for $t > 0$:

$$\hat{\theta}(t) - \theta_L$$

$$= \frac{\dot{\hat{n}}(t) + \delta\hat{n}(t)}{\alpha(N - E\hat{n}(t))} - \frac{\delta n_L}{\alpha(N - En_L)} \qquad \text{(by (2) and (3))}$$

$$> \frac{\delta\hat{n}(t)}{\alpha(N - E\hat{n}(t))} - \frac{\delta n_L}{\alpha(N - En_L)} \qquad \text{(by (20))}$$

$$> 0 \qquad \text{(by (19))}.$$

Thus in the case where L is a sink, there exists an equilibrium starting at n_L and converging to H, which dominates L. Furthermore, by continuity, any equilibrium starting in a neighborhood of L which converges asymptotically to L is dominated by an equilibrium starting at the same level of employment and converging to H.

3. Efficiency of Contracts

The wage-contract specified above is efficient in the sense that no other contract could make both parties better off, taking as given all the other contracts of the entrepreneur, the time paths $\bar{n}(t)$ and $\bar{\lambda}(t)$, and the fact that all other entrepreneurs are offering the specified contract. In fact, any contract which ensures that the match will last as long as possible, no matter what the wage profile, will be efficient in this sense. To prove this let any other contract take the general form $W(t,x)$, $Q(t,x)$, where $x(t)$ is any set of (possibly random) variables upon which the contract can be made contingent.

If an exogenous separation has not occurred by date t and if $x(t)=x$ then the worker will work the fraction $Q(t,x)$ of full time for the wage $W(t,x)$. The contract specified in Section 1 is $(W(t,x),Q(t,x)) = (wy(t),1)$ where $y(t) \equiv (f-\sigma/\bar{n}(t))^{+}$.

This general specification allows the possibility that the worker might quit to accept an offer at some (random) date T, after having initiated search under the terms of his contract. Because of constant returns the gain to the entrepreneur from entering into the contract is independent of all his other hiring decisions, and equals:

$$\Delta^{e} \equiv E\int_{0}^{T} e^{-(r+\delta)t}(Q(t,x)y(t) - W(t,x))dt.$$

Note that we are treating all recruiting costs as sunk.

The gain to the worker is more complicated. Let $V(t)$ be the value of beginning to search for a job at date t while unemployed, and $U(t)$ be the value of accepting a job at date t. Then:

$$U(t) = \int_{t}^{\infty} e^{-(r+\delta)(\tau-t)}(wy(\tau)+\delta V(\tau))d\tau^{6}$$

and the gain to the worker from the contract is:

$$\Delta^{w} = E\{\int_{0}^{T} e^{-(r+\delta)t}(W(t,x)+\delta V(t))dt + e^{-(r+\delta)T}U(T)\}$$

The total gain is thus:

$$\Delta \equiv \Delta^{e}+\Delta^{w} = E\{\int_{0}^{T} e^{-(r+\delta)t}(Q(t,x)y(t)+\delta V(t))dt + e^{-(r+\delta)T}U(T)\}$$

The gain from any contract, like the one specified in Section 1, which guarantees that a match will last as long as possible, is found by setting $Q(t,x)\equiv 1$ and $T=\infty$ with probability one:

$$\Delta' = \int_0^\infty e^{-(r+\delta)t}(y(t) + \delta V(t))dt$$

Therefore the sum of gains under such a contract exceeds that of any other contract by the amount:

$$\begin{aligned}
&\Delta' - \Delta \\
&= E\int_0^T e^{-(r+\delta)t}(1 - Q(x,t))y(t)dt + E\{\int_T^\infty e^{-(r+\delta)t}(y(t) + \delta V(t))dt - e^{-(r+\delta)T}U(T)\} \\
&\geqq e^{-(r+\delta)T}E\{\int_T^\infty e^{-(r+\delta)(t-T)}(y(t) + \delta V(t))dt - U(T)\} \quad \text{(because } Q<1\text{)} \\
&= e^{-(r+\delta)T}E\int_T^\infty e^{-(r+\delta)(t-T)}(1 - w)y(t)dt \qquad \text{(by the definition of U)} \\
&\geqq 0 \qquad \text{(because } w<1\text{).}
\end{aligned}$$

4. Generalizations and Extensions

Suppose that the current profit of an entrepreneur, instead of being $(1 - w)(f - \sigma/\bar{n})^+ n$, were given by the more general function $\pi(n,\bar{n})$ with:

$$(21) \qquad \begin{cases} \pi(n,\bar{n}) \text{ continuous and concave in n on } R_+^2,\ h(n) \equiv \pi_1(n,n) \\ \text{continuous on } [0,N/E] \text{ with } h(n) = 0 \text{ on } [0,\underline{n}] \text{ for some } \underline{n} > 0. \end{cases}$$

Then the differential equation (10) would be replaced by:

$$(10') \qquad \dot{\lambda} = (r + \delta)\lambda - h(n) \qquad \forall t \geqq 0 .$$

In Figure 1 $\lambda^1(n)$ would be exactly as before but $\lambda^2(n)$ would be $(r + \delta)^{-1}h(n)$, which would still equal zero on $[0,\underline{n}]$ and be bounded on $[0,N/E]$, although it would not necessarily be monotonic and concave on $[\underline{n},N/E]$. The generic existence of multiple stationary equilibria would follow exactly as before. The Jacobean of (9) and (10′) would be as in (13) except with $d\dot{\lambda}/dn = -h'(n)$.

Thus in the two-equilibrium case of Figure 1 H would still be a saddle-point. For $r \in (0,\bar{r})$ L would still be a sink and there would still exist an equilibrium path leading from n_L to H.

Thus as far as the positive analysis of the system's dynamics are concerned our analysis is more general than the previous sections might indicate.[7] This suggests that the same analysis might be useful in a variety of other contexts, in which n is an entrepreneur's scale of operations, $\bar{n}$ the scale of his average rival, $G(\frac{\dot{n}+\delta n}{N-E\bar{n}})$ the cost of expanding at the gross rate $\dot{n}+\delta n$, and $\pi(n,\bar{n})$ the current profit rate, gross of expansion costs.

In order for the analysis to apply to some context, two key features must be present. First there must be an external economy of scale in current profits, in the sense that the marginal profitability of scale be positively affected by the rivals' scale. For in order to get $\lambda^2(n)$ to rise above the horizontal axis, at some point beyond $\underline{n}$ we need $h'(n) = \pi_{11} + \pi_{12} > 0$, which by the concavity assumption of (21) implies $\pi_{12} > 0$. (This concavity assumption is needed in order for solutions (9) and (10$'$) to qualify as equilibria.) Second, there must be some natural limit (N) to the overall scale of this activity; the cost of expanding at any given gross rate $\dot{n}+\delta n$ becomes infinite as $\bar{n}E$ approaches this limit. This is what makes $\lambda^1(n)$ necessarily rise above $\lambda^2(n)$ at the upper end of [0,N/E] as well as the lower end.

Although this diseconomy of scale in the cost of expansion is not strictly required in order for multiple equilibria to exist, it is required for the low-level equilibrium of Figure 1 to be stable. For suppose that the cost, instead of being $G(\frac{\dot{n}+\delta n}{N-E\bar{n}})$, depended only upon the gross rate of expansion: $\tilde{G}(\dot{n}+\delta n)$. Then instead of (6) we would have

(6′) $\tilde{G}'(\dot{n} + \delta n) = \lambda$

The differential equation (9) would become:

(9′) $\dot{n} = \tilde{F}(\lambda) - \delta n$

where $\tilde{F}$ is the inverse function of $\tilde{G}'$, and (10′) would remain unchanged. In Figure 1, $\lambda^2(n)$ would be unaffected but $\lambda^1(n)$ would become $\tilde{G}'(\delta n)$. If $\tilde{G}$ satisfied (1) then $\lambda^1(n)$ would be increasing, with $\lambda^1(0) = 0$. Instead of $\lim_{n \to N/E} \lambda^1(n) = \infty$ we would have $\lim_{n \to \infty} \lambda^1(n) = \infty$. If h(n) were bounded on R_+ (as in the example of the previous sections), generic multiplicity would follow as before, since $\lambda^1(n)$ would be above $\lambda^2(n)$ on $(0, \bar{n}]$ and again for sufficiently large values of n. But none of the equilibria could be sinks because the trade of the Jacobean of (9′) and (10′) is $\frac{\partial \dot{n}}{\partial n} + \frac{\partial \dot{\lambda}}{\partial \lambda} = r > 0$.

One context where the analysis might apply is that of the growth of a city. Agglomeration economies coexist with the natural limits imposed by the availability of land. Thus not only might there be multiple equilibrium sizes to the city but each one might be stable (in a partial equilibrium sense).

Another example might be the market for any new product whose demand is ultimately limited by the availability of potential customers. The economy of scale could arise for a variety of reasons. Imitators can free ride on the increasing familiarity that raises demand when other firms operate on a larger scale, as IBM has been accused of doing with several products (see, for example, Burstein, 1984). The likelihood that a service network or an auxillary-product market will develop could depend upon the rivals' scale, as in the market for cars that operate on propane gas, for quadraphonic hi-fis that require special records, for video cassette recorders that require special tapes, or for turbo-engined cars that require servicing by specially trained mechanics. In all these cases the analysis suggests that one is likely to find stable low-level equilibria.

As a final example consider the process of economic growth, which, according to countless writers from Adam Smith through Schumpeter is intimately connected with external economies of scale. Roemer (1983) has shown how a representative-firm analysis like ours can handle the dynamics of growth by characterizing the description of equilibrium trajectories as solutions to a surrogate decision problem, and using familiar Hamiltonian dynamics. His analysis is similar to the system (9′), (10′) in which we have seen that not all equilibria can be stable because there is no limit to expansion. This analysis suggests that combining the limitations of finite natural resources together with external economies in a model of growth gives rise to stable low-level equilibria.[8]

In all these examples, including the unemployment example of the previous sections, the multiplicity of equilibria obviously depends upon the inability of agents to internalize the external economy of scale. This seems to make most sense in the macro-examples of unemployment and economic growth, where internal diseconomies are likely to discourage the large-scale organization of the market in question under a few entrepreneurs. Casual empiricism suggests that internalization does occur in some of the micro-examples cited, as where companies like Betamax and VHS jointly produce recorders and tapes, although there is no a priori reason to believe that it is so extensive as to vitiate completely our analysis.

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Footnotes

[1]One might think of backscratching services or academic papers as goods fitting this description.

[2]A constant average cost could be included with no substantive change in the analysis.

[3]The effects of relaxing this assumption are examined, in a somewhat different context, by Howitt and McAfee (1983).

[4]The fact that reduced aggregate supply causes a one-for-one reduction in aggregate demand is, however, quite unKeynesian, being nothing other than Keynes's version of Say's Law.

[5]The notation $>$ denotes strict inequality for each component.

[6]To derive this, note that the value in the event that the job terminates at date T is $\int_t^T e^{-r(\tau-t)} wy(\tau)d\tau + e^{-r(T-t)}V(T)$, whereas the probability (density) of that event is $\delta\, e^{-\delta(T-t)}$. Then the expected value is $\int_{T=t}^{\infty} \{\int_t^T e^{-r(\tau-t)} wy(\tau)d\tau + e^{-r(T-t)}V(T)\}\delta\, e^{-\delta(T-t)}dT$, which yields the expression in the text if the first term is integrated by parts. The next expression in the text can be derived analogously.

[7]In addition to this more general profit function we could also change U from (3) to $N - E\bar{n} - \beta n$, $\beta > 0$. This would allow for the small-numbers case where the firm's own rate of hiring would perceptibly influence the number of remaining searchers. If we continued to assume that workers' search behavior will be unaffected by the unilateral decision by one firm to offer an above-market wage, the results of the previous paragraph would continue to hold. (The λ^1 and λ^2 curves would continue to have the indicated shapes, and although the Jacobean of the system would be somewhat more

complicated its trace would still be negative when $r = 0$.)

[8]With these limits the equilibria cannot solve any surrogate decision problem, since that would require the trace of the Jacobean to equal the rate of interest (see Levhari and Liviatan, 1972).

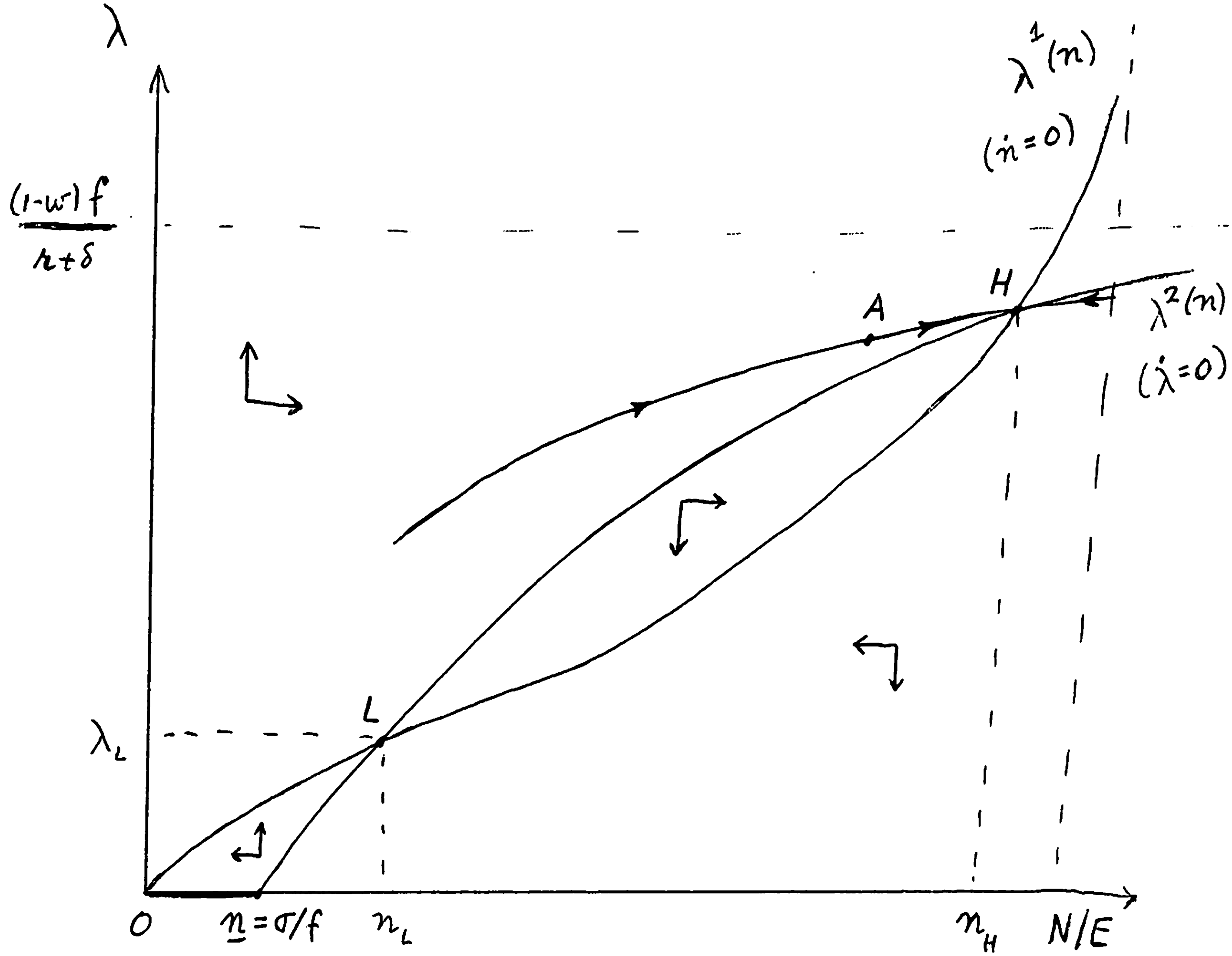

Figure 1