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THE NATURE OF RISK AVERSION*

BY

R. Preston McAfee

ABSTRACT

Risk aversion is normally considered to be exogenous, like preferences. This paper supposes that risk aversion is generated by natural selection maximizing expected life span in the face of risky gambles concerning food acquisition. Two stylized facts emerge: only poor people play lotteries, and people display decreasing absolute risk aversion.
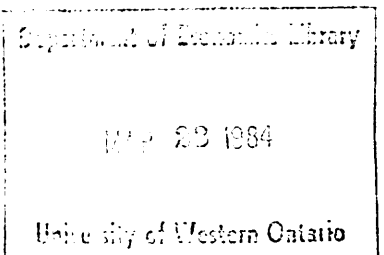
Introduction

In the presence of perfect capital markets, it is clear that a risk neutral firm tends to prosper over a risk averse or risk loving firm. Thus, the long-run evolution of firms should yield risk neutral firms. However, it is rare to observe risk neutral people. Indeed, most people insure wealth, that is, they pay a risk premium thus acting risk averse. Furthermore, many play lotteries, where they appear to act as risk lovers. I will argue that both of these facts can be explained by evolution. In addition, two testable predictions will emerge: that individuals have decreasing absolute risk aversion, and only poor people play lotteries. Both of these are "stylized facts" about people's behavior.

Most economic models presume that risk aversion is similar to preferences, in that it is exogenously given to the agent. I wish to amend that assumption in a minor way. Let us suppose that, while agents cannot vary their own risk aversion, our agents' ancestors were forced to compete sometime in the distant past and that today's risk aversion is the result of this competition. Thus, I am presuming that heredity explains today's risk aversion. It is the purpose of this paper to explain heredity.

This construction permits us to plausibly dismiss the assumption of perfect capital markets. In addition, we may virtually ignore the pleasure of the agent, for evolution generates species that persist in time, rather than happy individuals. Perpetuating the species generally involves providing the individuals enough time to procreate, which suggests a utility function for nature: the life expectancy of the individual. Thus, we consider that evolution is designing animals which, ceteris paribus, maximize their life span.

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While this is not an accurate model of the effect of natural selection on people today, it is a plausible model of the influences on mankind's ancestors. Evolution will also alter the physical characteristics of the animal to increase the likelihood of perpetuating the species, of course. Given two animals that differ only in their attitude toward risk, the one with the longer life span will produce more offspring, and ultimately predominate.[1] Thus, I argue that it makes evolutionary sense to argue that preference toward risk (today) is explained by a life expectancy maximization procedure.

The commodity that plays the role of money in risk preference for animals is food. Hunting for food is inherently risky, as there is always the possibility that the "food" will kill the hunter, or that another animal will appear and kill the hunter. In addition, obtaining some food is riskier than obtaining other food. The risk depends on the circumstances as well as the food; a hunter would prefer an environment less likely to leave him exposed to predators. To some extent, the hunter can control the risk, that is, he can choose not to pursue very risky game. For example, only a starving tiger attacks an elephant.

Thus, we may plausibly argue that there is a distribution of risky food acquisitions, and that the animal can choose to take on a risk or not. One expects, _ceteris paribus_, that an animal that has just eaten will not take on a risky food acquisition, while a hungry animal might.

This work is most closely related to an interesting paper by Farrell (1970). Farrell's model has speculators faced with gambles as in this paper, in which the speculator faces a monetary gain or ruin (death) with probability p_i. However, the similarity ends at this point. Farrell's agents are endowed

with a given p_i (that may be different for distinct agents) and are not allowed to choose among gambles. Thus, no implications about risk aversion could possibly be deduced. Second, Farrell's agents have different p_i, so that his result concerns the survival of better endowed individuals, and not the process generating the endowment. That is to say, although there is a superficial similarity of the gambles facing the agents in this model and Farrell's model, Farrell is addressing a completely different question. For an overview of the literature concerning the application of natural selection to economics, see Lippman and McCall (1981). For a discussion of the applicability of natural selection to economics, see Alchian (1950).

Food Acquisition

We introduce some simplifications to the preceding discussion to achieve a tractable result. First, let the animal enjoy perfect storage of food. Without loss of generality, let the consumption of food stores occur at rate 1, so that the animal can survive y days on a store of y.

Food acquisition gambles appear stochastically in time, according to a Poisson distribution. By using appropriate units, we can let the rate at which gambles appear be unity. The gambles are assumed to take the form:

$$\begin{cases} \text{increase food stock by c} & \text{probability p} \\ \text{death} & \text{probability 1-p} \end{cases} \tag{1}$$

The gambles are not created equal, and thus we suppose p takes on a uniform distribution on $[0,1]$. Evidently the animal will set a reservation safety level q (which depends on y) so that he takes the gamble if $p \geq q$, and otherwise stays in his lair.

Finally, let there by an exogenous death rate $\delta > 0$, so that the probability the animal remains alive at time t, if he takes no food gambles, is $e^{-\delta t}$.

This model may not seem to suit some actual animals, but I think, with suitable reinterpretation, it does. First, there seems to be a startling lack of incidents in which tigers, alligators, and wolves are killed as food. However, they are, on occasion, injured by their prey, and thus they do face risks in food acquisition. Indeed, a wolverine is likely to injure a wolf who tries to eat it, while a rabbit injures a wolf only rarely. Second, herbivores are not ever injured by their 'prey'. However, some feeding grounds offer more risk of predators than others.

Finally, of course, the real case we are interested in is some human ancestor. Human ancestors hunted everything from berries to wooly mammoths, with all degrees of risk in between. Thus, it is not implausible to consider natural selection determining optimal risk aversion in the face of circumstances stylized by this model.

Given the animal is still alive at time t, the probability he finds an acceptable gamble at that time is

$$e^{-u} \equiv e^{-\int_{0}^{t}(1-q(s))ds}(1-q(t)).$$

Let y index the animal's stock, and $N_y = N(y)$ be the expected life span, given this stock. If the animal finds an acceptable gamble at time t, his life span becomes:

$$t + \frac{1+q}{2} N_{y-t+c}$$

as he has already survived t, and $\frac{1+q}{2}$ is the probability he survives the gamble, and N_{y-t+c} is his life span if he does survive.

Thus, the probability the animal either dies or gambles at time t is

$$e^{-u+\delta t}(\dot{u}+\delta).^2$$

This results in a gamble $\frac{\dot{u}}{\dot{u}+\delta}$ of the time, and death by natural causes, with life span t, the other $\frac{\delta}{\dot{u}+\delta}$ of the time. The animal starves to death with probability:

$$e^{-\delta y-u}\bigg|_{t=y} = 1 - \int_0^y e^{-\delta t-u}(\delta+\dot{u})dt$$

yielding a life span of y. Thus

$$N(y) = \int_0^y e^{-\delta t-u}[\delta+\dot{u}] \; [\frac{\dot{u}}{\delta+\dot{u}}(t + \frac{1+q}{2} N_{y-t+c}) + \frac{\delta}{\delta+\dot{u}} t]dt + ye^{-\delta t-u}\bigg|_{t=y}$$

$$= y + \int_0^y e^{-\delta t-u}[\dot{u} \frac{1+q}{2} N_{y-t+c} + (\delta+\dot{u})(t-y)]dt$$

$$= y + \int_0^y e^{-\delta t-u}[\frac{2\dot{u}-\dot{u}^2}{2} N_{y-t+c} + (\delta+\dot{u})(t-y)]dt \qquad (2)$$

$$\equiv y + \int_0^y H(u,\dot{u},t)dt.$$

We may employ the Euler equation to (2) to solve for $N(y)$.[3] Because this sort of search model has been employed extensively in discrete time (see Lippman and McCall (1981)), it may be of interest to some readers to see the continuous time solution method. As a result, this follows in exten- sive detail. Time has been suppressed as a variable where confusion can not occur.

$$\frac{\partial H}{\partial u} = - e^{-\delta t - u}[\frac{2\mathring{u}-\mathring{u}^2}{2} N_{y-t+c} + (\delta+\mathring{u})(t-y)]dt \ .$$

$$\frac{\partial H}{\partial \mathring{u}} = e^{-\delta t - u}[(1-\mathring{u})N_{y-t+c} + (t-y)]$$

$$= e^{-\delta t - u}[q \ N_{y-t+c} + t - y] \ .$$

$$\frac{d}{dt}\frac{\partial H}{\partial \mathring{u}} = e^{-\delta t - u}[- (\delta+\mathring{u})((1-\mathring{u})N_{y-t+c} + t - y) + \mathring{q} \ N_{y-t+c} - q \ N'_{y-t+c} + 1] \ .$$

From the Euler equation,

$$\frac{\partial H}{\partial u} = \frac{d}{dt}\frac{\partial H}{\partial \mathring{u}}, \ \text{or,}$$

$$[\frac{2\mathring{u}-\mathring{u}^2}{2} - \mathring{u} + \mathring{u}^2 - \delta(1-\mathring{u}) + \mathring{q}]N_{y-t+c} - q \ N'_{y-t+c} + 1 = 0 \ ,$$

or,

$$[\frac{\mathring{u}^2}{2} - \delta(1-\mathring{u}) + \mathring{q}]N_{y-t+c} - q \ N'_{y-t+c} + 1 = 0 \ .$$

Substituting $\mathring{u} = 1 - q$, we have:

$$[\frac{(1-q)^2}{2} - \delta q + \mathring{q}]N_{y-t+c} - q \ N'_{y-t+c} + 1 = 0 \ . \tag{3}$$

By transversality,

$$\left.\frac{\partial H}{\partial \mathring{u}}\right|_{t=y} = e^{-\delta t - u}[q \ N_{y-t+c} + t - y]\bigg|_{t=y} = 0 \ .$$

Therefore, $q(y)N_c = 0$, or,

$$q(y) = 0 \ . \tag{4}$$

Furthermore,

$$N(y)-y = - \int_o^y \frac{\partial H}{\partial u} dt = - \int_o^y \frac{d}{dt}\frac{\partial H}{\partial \mathring{u}}dt = -\frac{\partial H}{\partial \mathring{u}}\bigg|_o^y = \frac{\partial H}{\partial \mathring{u}}\bigg|_{t=0}$$

$$= q(o)N_{y+c} - y.$$

Thus

$$N_y = q(o)N_{y+c} \, .$$
(5)

The Bellman Principle of Optimality guarantees q is a function of income alone. Thus we may write $p(z) = q(y-t)$ for $z = y - t$, and $p'(z) = -\overset{\bullet}{q}$.[4] Thus, by (4) and (5),

$$p(o) = 0 \, ,$$
(6)

$$p(z) = \frac{N_z}{N_{z+c}} \, , \text{ and,}$$
(7)

$$p'(z)N_{z+c} = N_z' - p(z)N_{z+c}' \, .$$

Hence, from (3),

$$N_{z+c}[\frac{[1-p(z)]^2}{2} - \delta p(z)] - N_z' + p(z)N_{z+c}' + 1 = 0, \text{ or,}$$

$$\frac{[N_{z+c}-N_z]^2}{2 \, N_{z+c}} - \delta N_z + 1 = N_z' \, .$$
(8)

From (6) and (7),

$$N(o) = 0$$
(9)

$$N'(o) = \frac{1+N(c)}{2}$$
(10)

We may rewrite (8) to procure:

$$N_z' = -\delta N_z + \frac{1-p^2}{2} N_{z+c} + 1 + (p^2-p)N_{z+c}$$

$$= -\delta N_z + [1-p][\frac{1+p}{2} N_{z+c} - N_z] + 1$$
(11)

Equation (6) has the obvious interpretation that a starving animal takes any gamble. If applied to people, a person with little money is more likely to take a dangerous job, for his threshold p will be small.

Equation (11) shows that the marginal value of food is 1 (the time this food buys) plus the gain if he gambles $(\frac{1+p}{2} N_{z+c} - N_z)$ times the probability he gambles, minus the depreciation of food by death from natural causes. Thus (11) is just the marginal benefit of food in an obvious way.

Also, we see from (8) and integration that, if $c = 0$, $N_z = \frac{1}{\delta}(1-e^{-\delta z})$. Thus, if food gambles drop out of the problem, the individual has constant absolute risk aversion.

To characterize the solution to (8), note

$$\frac{1 - e^{-\delta z}}{\delta} < N_z < \frac{1}{\delta} .$$

The left inequality occurs because N_z must exceed the suboptimal strategy of $p = 1$, while the right inequality occurs because $N_\infty = \frac{1}{\delta}$. This, in conjunction with (8), forces $N_z' > 0$. The existence and uniqueness of N is demonstrated in Appendix 2.

Furthermore,

$$N_z'' = - \delta N_z' + (1-p(z))(N_{z+c}'-N_z') - \frac{(1-p(z))^2}{2} N_{z+c}' . \tag{12}$$

All three terms of this expression are negative if $N_{z+c}' - N_z' \leq 0$. Since $\lim_{z \to \infty} N_{z+c}' - N_z' = 0$, we obtain $N_z'' < 0$. Thus, the animal is risk averse, that is,

$$EN(x) < N(Ex)$$

for a random variable $x \geq 0$.

In addition, $\frac{N(c)}{c} > N'(c) = 1 - \delta N_c + \frac{(N_{2c}-N_c)^2}{.2\, N_{2c}} > 1 - \delta(N(c))$.

or

$$N(c) \geq \frac{c}{1+c\delta} = \frac{1}{\delta + \frac{1}{c}} .$$

Furthermore, as $N_{z+c}' < N_z'$

$$p'(z) = \frac{N_{z+c} N_z' - N_z N_{z+c}'}{N_{z+c}^2} > 0 . \tag{13}$$

This equation is a formal statements of the claim that richer individuals are less apt to buy lottery tickets than poor ones.

Finally, it remains to be shown that the animal displays decreasing absolute risk aversion.[5]

From (12),

$$\frac{N_z''}{N_z'} = - \delta + [1-p(z)][\frac{N_{z+c}'}{N_z'} - 1] - \frac{(1-p(z))^2}{2} \frac{N_{z+c}'}{N_z'}$$

$$= - \delta - (1-p(z)) + \frac{1-p^2(z)}{2}[\frac{N_{z+c}'}{N_z'}] . \tag{14}$$

Thus

$$\frac{d}{dz} \frac{N_z''}{N_z'} = p'(z) - (p)p'(z) \frac{N_{z+c}'}{N_z'} + \frac{1-p^2}{2}[\frac{d}{dz} \frac{N_{z+c}'}{N_z'}]$$

$$= p'(z)[1 - p(z) \frac{N_{z+c}'}{N_z'}] + \frac{1-p^2}{2} \frac{d}{dz} \frac{N_{z+c}'}{N_z'} \tag{15}$$

The details of the proof that (15) is positive are relegated to Appendix 3. However, we may show $N_z''' > 0$ quite easily, and the value of N_z''' is necessary to the proof. This shows the reservation safety level is concave, for

$$p''(z) = - \frac{2N_z'}{N_{z+c}} p'(z) + \frac{N_{z+c}N_z'' - N_z N_{z+c}''}{N_{z+c}^2} < 0,$$

if $N_z''' > 0$, as this implies $N_z'' < N_{z+c}'' < 0$.

$$N_z''' = - \delta N_z'' + (1-p)[N_{z+c}'' - N_z''] - p'(z)[N_{z+c}' - N_z'] \tag{16}$$

$$+ p(z)p'(z)N_{z+c}' - \frac{(1-p(z))^2}{2} N_{z+c}''.$$

If $N_y''' \geq 0$ for all $y > z$, then $N_z''' > 0$, as every term in (16) is positive. Since $\lim_{z \to \infty} N_z''' = 0$, $N_z''' > 0$, as desired.

Conclusion

A model of natural selection over agents faced with gambles that either extend their life span or kill them was presented. The agents were endowed with fixed risk preference and natural selection favors those whose risk preference maximizes their life span. This yields an indirect utility function over food or wealth. Two facts emerged. First, with respect to wealth gambles that do not kill the agent, the agents display decreasing absolute risk aversion. Second, they display increasing aversion to lotteries involving death as an outcome. The first of these two has unambiguous economic meaning and is certainly testable in principle. The second warrants further discussion.

Consider again a lottery like (1). Such a lottery is faced by any person working a risky job. Individuals set a reservation risk level $p(z)$ and accept the risky job if p exceeds $p(z)$. From (13), $p(z)$ is increasing in wealth z. Thus, ceteris paribus, the model predicts that poor people will take riskier jobs. This is a formal statement of the notion that a person's wealth measures the value of his life. Indeed, the agent in this model values his at precisely $\frac{p(z)c}{1-p(z)}$, owing to his indifference between gaining c with probability $p(z)$ and death with probability $1-p(z)$. The indifference follows from (5).

I think this model may be applied in broader circumstances to predict that only poor people buy lottery tickets. Consider a lottery ticket that pays c with probability p, and costs k. Let z_o satisfy $p = p(z_o)$. Then the expected utility of the lottery ticket to an individual with wealth z_o is:

$$p(z_o)N(z_o+c) + (1-p(z_o))N(z_o-k) \geq$$
$$p(z_o)N(z_o+c) + (1-p(z_o))N(death) = N_{z_o} .$$

Thus, some poor individuals will purchase lottery tickets. This is graphically illustrated in Figure 1.

Thus, the model predicts that very poor people will play lotteries, not because they are risk lovers (at least in the small), but because bankruptcy establishes a lower bound to their utility of money. In particular, we expect to observe lottery players to be those individuals with zero or negative current net worth.[6] Essentially, we can understand actuarially unfair lottery playing as the result of equating negative current net worth to zero.[7]

Note, of course, that an individual doesn't have to actually die to drive this phenomenon. Because the utility of money is given to the individual exogenously, the individual will act as if death is the result of negative net wealth, even if it is not. That is to say, natural selection has given him a utility function that is not in accord with the welfare state, and the individual is not able to alter it to make it more appropriate to his circumstances, as, by hypothesis, it is exogenous to the individual.

Virtually every result of this model is, in principle, testable. Although we are not generally allowed to offer experimental subjects gambles including death as an outcome, indirect evidence of attitude to the risk of death could be obtained by examining the net wealth of mercenaries and others in risky occupations.

The following example would permit an experimental test of the results using animals. I observed squirrels coming down from trees to gather nuts from the ground. There were cats about, thus making this a risky activity for the squirrels. The squirrels enjoyed virtually perfect storage, although seasonality entered into the distribution of gambles. I noted that the squirrels would venture down as long as no cat was within 50 feet: this seemed to be a reservation risk level. Experimentally, one
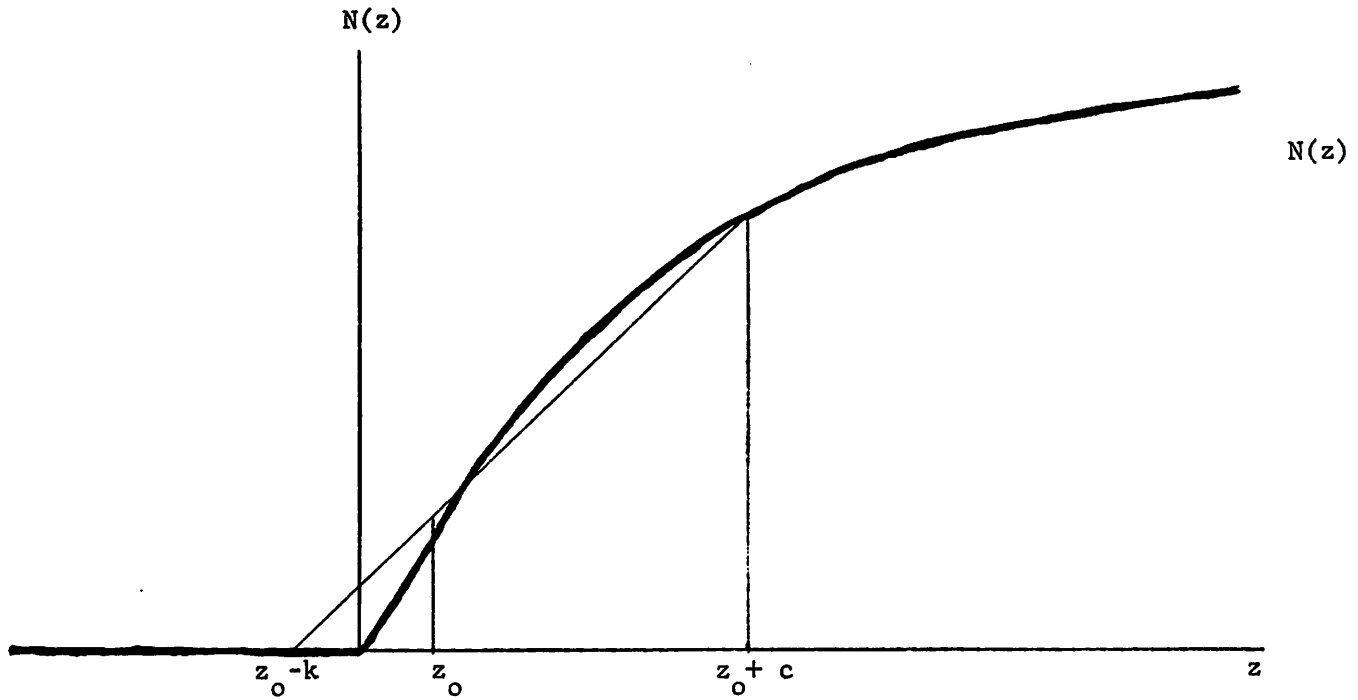
Figure 1: A person of wealth z_0 will bankrupt himself to obtain a lottery ticket with probability $p = p(z_0)$.

could test this model by establishing whether this distance varied with the number of nuts stored, according to equations (7) and (8).

A number of obvious extensions of this model may merit exploration. In particular, one might allow the rate gambles appear to vary, permit c to be random, and inject noise into the ex ante observations of p and c. Further, one could consider arbitrary distributions of p, allow for depreciation of storage, and add a third outcome to (1), wherein food is expended with some probability (an unsuccessful, nonfatal hunt). I find it unlikely that any of these ultimately alter the flavor of the model presented here.

This approach, however, may be profitably considered in areas outside of risk aversion. Natural selection optimizes those characteristics that are exogenous to the individual. This simple 'economy of nature' may be exploited to explain where conjectural variations originate, to justify the assumption of rational expectations, and to understand why people have a preference for brussels sprouts. That is, whenever something that could vary is fixed for each individual, it is reasonable to ask whether this is advantageous for the individual, and optimize over it, invoking natural selection. As in this paper, of course, nature's objective does not necessarily coincide with the individuals' objectives.

Perhaps the most profitable line of research spawned by this paper involves determination of the optimal discount rate on future wealth. Consider an infinitesimal investment A to receive B, t_o units of time in the future if one is still alive at this time. As A and B are infinitesimal, there will be no change in the value of p(z) as a result of the investment. The death rate, by (2), is

$$r(y) = \delta + \frac{(1 - p(y))^2}{2}$$

which arises from death by natural causes plus the rate of dying from gambles.[8] This last is the rate of taking gambles $(1-p(y))$ multiplied by the probability a gamble is fatal, $(1-p(y))/2$. Thus, for infinitesimal gambles, the discount rate is $r(y)$. Differentiating r, we find that the discount rate for infinitesimal gambles is a convex strictly decreasing function of wealth. This explains why poor people borrow and rich people save, at a given interest rate, as people will borrow when their rate of time preference is less than the prevailing interest rate.

However, this analysis applies only to infinitesimal gambles. In general, we expect large gambles to affect the optimal value of p. The mathematics involved in determining the optimal value of p, given that one will receive a lump sum return in the future, is quite complex. The characterization of this discount rate, in the case of lumpy investment, provides the most intruiguing line of inquiry generated from this model and is left as an open problem. I hypothesize that in general the discount rate is convex in wealth and converges to δ as wealth diverges. The resolution of this issue would suggest a functional form which is, in principle, testable. In addition, it would lead to an understanding of how risk aversion and time preference are related, an issue which, to my knowledge, has never been examined.

Another topic meriting further research emerges from the very existence of risk aversion. Risk aversion provides incentives to pool resources (food store), while free riding produces disincentives of this behavior. Presumably, comparison of these two yields an optimal tribal size. However, once a tribe has been established, the optimal utility function may change, given the tribe's existence. Thus, one may be able to solve simultaneously for the

optimal tribe size, n, and utility function. Of course, tribes also offer gains from specialization. One might adopt this model by letting a tribe of size n obtain $c(n)$ in a successful hunt, with $c' > 0$. In this way, one might be able to examine altruism from the standpoint of natural selection. Indeed, altruism may be nature's way of equating the social benefit of providing food for a tribe to the privately paid cost.

Footnotes

[1] For example, if one offspring per year is produced, then maximizing expected life span maximizes the probability the genes persist in time.

[2] For a reference, see Ross (1970).

[3] This is more subtle than it may appear at first, as the optimized function N appears in the definition of N. Worse still, the values of N appearing on the right-hand side include values of the argument both greater and less than the argument value on the left. This rules out any sort of inductive argument. However, the usual Euler equation holds, as is demonstrated in Appendix 1.

[4] What is being invoked here is that the optimal value of q does not depend on the value of y at some time in the past. Thus, the q that prevails at time t when one started with y is the same as that prevailing at time $t + \Delta$ when one started with $y + \Delta$. This occurs because, given one reached a point of having food store y, the path of q must still be optimal.

[5] For the definition of absolute risk aversion, its derivation and a discussion of its interpretation, see Pratt (1964). Note that I have ignored the negative sign on r(z). For a discussion of the evidence for decreasing absolute risk aversion, see Sinn (1983), Ch. V.

[6] It is not obvious, of course, exactly what belongs as an argument in the objective function of the individual. I have chosen current net wealth, as opposed to the expected present value, because this seems to be the closest to the idea of a food store. In the model, z is the current food stock and N(z) is the expected life span given that food stock. Thus, N(z) takes into account expected future returns, and hence those

returns do not belong as the argument to N.

[7]This is basically the same argument advanced by Sinn (1983).
Indeed, his Figure 9, p. 165 is the same as my Figure 1. In both cases,
the diagram follows because, in his terminology, you can't get "blood out
of a stone" (Sinn calls it the BLOOS rule). The difference, however,
is that Sinn's analysis applies to the present value of net worth, while
mine applies to the current value. Even if the present value of net
worth is positive, if current net worth is negative, my model predicts
people will behave as in Figure 1.

[8]This follows because one gets the investment return B as long as
one doesn't die. Since A and B are infinitesimal, the curvature of N
does not enter into the discount.

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Appendix 1:

Equations, such as (2), where the maximized value of the function appear as an argument, present a certain difficulty in solving. Consider

$$N(y) = \max_{\dot{x}} \int_0^y f(x,\dot{x},t)N(t+c) + g(x,\dot{x},t)dt \tag{T1}$$

as a typical such equation, where $x(0) = 0$ is given.

PROPOSITION: A necessary condition for $x(t)$ to solve (T1) is

$$f_1 N + g_1 = \frac{d}{dt} [f_2 N + g_2], \tag{T2}$$

almost every t.

PROOF: Consider a variation $x + \beta v$ of x, $\beta \in \Re$, $v(0) = v(y) = 0$.

Write

$$J(y,\beta) = \int_0^y F(x+\beta v, \dot{x}+\beta\dot{v}, t) J(t+c, \beta) + g(x+\beta v, \dot{x}+\beta\dot{v}, t)dt$$

Then maximization requires, $\forall v$,

$$0 = \frac{\partial J}{\partial \beta} (y,0) = \int_0^y J(t+c, 0) [vf_1 + \dot{v}f_2] + g_1 v + g_2 \dot{v} + f \frac{\partial J}{\partial \beta} (t+c,0)dt$$

$$= \int_0^y v[N(t+c)f_1 + g_1 - \frac{d}{dt} [N(t+c)f_2 + g_2] + f \frac{\partial J}{\partial \beta} (t+c,0)dt .$$

For $v = 0$, we observe $\int_0^y f \frac{\partial J}{\partial \beta} (t+c,0)dt = 0$.

For $v = (N(t+c)f_1 + g_1 - \frac{d}{dt} (N(t+c)f_2 + g_2))S(t)$,

$$S(t) = \begin{cases} 1/\epsilon & t_0 - \epsilon \leq t \leq t_0 + \epsilon \\ 0 & \text{otherwise} \end{cases}, \text{ yields (T2), taking } \lim_{\epsilon \to 0} .$$

An intuitive way of observing that (T2) follows from (T1) exploits the observation that (T1) is a "renewal function". Note that x does not, per se, appear in $N(t+c)$ at all. We may think of x as a function of two arguments t and y, so that

$$N(y) = \max \int_0^y f(x(t,y), \dot{x}(t,y), t) N(t+c) + g(x(t,y), \dot{x}(t,y), t) dt$$

and thus changing $x(t,y)$ has no influence on the value of $N(t+c)$ (which depends on $x(s, t+c)$) except in the measure zero case of $t = y-c$. This is not, of course, a formal proof, but provides, I think, a clear intuition of the maximization procedure for this sort of equation.

Appendix 2:

EXISTENCE:

Define $F_N(z)$ by

$$F_N(z) = N + \frac{1}{c}[\frac{1}{\delta} - N](z - \hat{z}) \quad \text{for } \hat{z} \le z \le \hat{z}+c$$

and

$$F_N'(z) = 1 - \delta F_N(z) + \frac{(F_N(z+c) - F_N(z))^2}{2F_N(z+c)} \tag{T3}$$

for $z < \hat{z}$.

Fix $N_o > N_1$, as in Figure 2.

For any $N \le N_1$, define $Z_1(N)$ and $Z_o(N)$ by:

$$N = F_{n_1}(z_1) = F_{N_o}(z_o) .$$

Claim: $Z_o(N) - Z_1(N)$ is decreasing in $N \le N_1$. $\tag{T4}$

Proof of Claim: As $Z_i(N)$ is the inverse function of F, we need only show that

$$F_{N_1}'(z_1) > F_{N_o}'(z_o)$$

This is clearly true at N_1, as

$$F_{N_o}'(z_o) = 1 - \delta N_1 + \frac{(F_{N_o}(z_o+c) - N_1)^2}{2F_{N_o}(z_o+c)} < 1 - \delta N_1 + \frac{(1/\delta - N_1)^2}{2/\delta} = F_{N_1}'(z_1) \tag{T5}$$
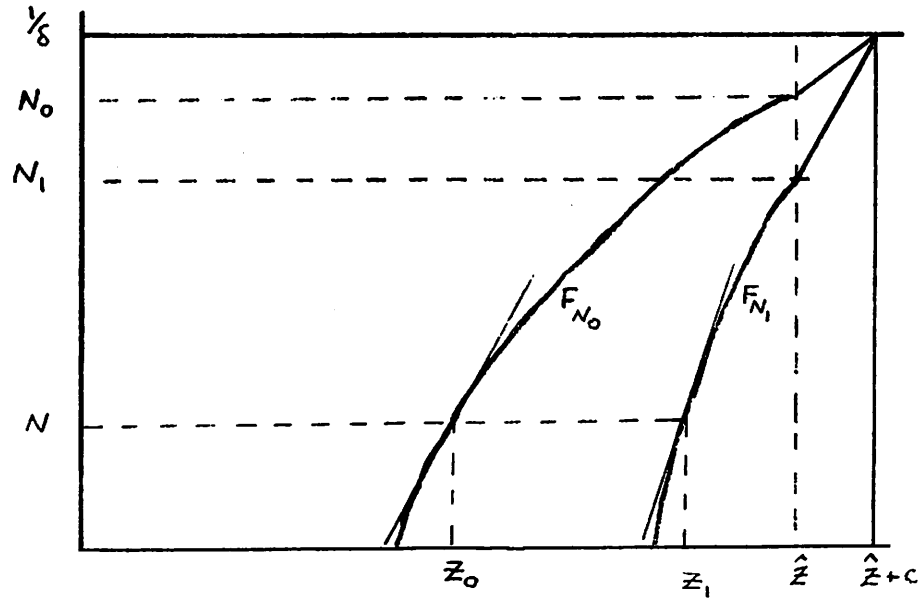
Figure 2: The slope of F_{N_o} is less than that of F_{N_1} for a given value of $N = F_{N_o} = F_{N_1}$.
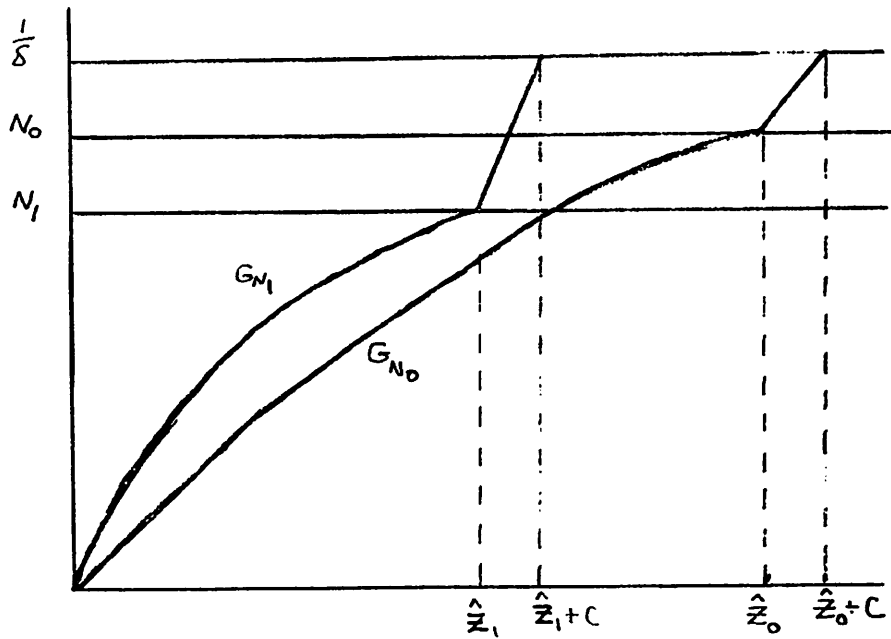

Figure 3: Shifting the value of $\hat{z}$ so that $F_N(0) = 0$ yields a decreasing sequence of functions.

Because $\frac{(x-N)^2}{2x}$ is increasing in x for $x \geq N$, $F'_{N_o}(z_o(N)) < F'_{N_1}(z_1(N))$, for

when one substitutes $F_{n_1}(z_1+c)$ for $F_{N_o}(z_1+c)$, (T5) continues to hold. []

Now consider shifting $\hat{z}$ so that $F_n(0) = 0$, as in Figure 3. From (T4), it
is clear that the shifted function, given by

$$G_{N_i}(z) = F_{N_i}(z - z_i(0))$$

satisfies $G_{N_o}(z) < G_{N_1}(z)$ if $N_o > N_1$ and $z \leq z_1(N_1) - z_1(0)$. This occurs as

$F'_{N_1} > F'_{N_o}$ under these conditions.

Thus, as $G_N(z)$ is a decreasing sequence of functions, a pointwise limit

N(z) exists. It remains to be shown that G_N converges uniformly. However,

note that G_N is concave, indeed, the same proof as appears in the text works,

with the exception that G''_{N_i} is not defined at $z_i(N_i) - z_i(0) - c$. Thus

$\{G'_{N_i}\}$ $N_o \leq N_i < 1/\delta$ takes on a maximum, and it is G'_{N_o}. This forces G to converge

uniformly, by virtue of being uniformly continuous.

NOTE from (T3) that if $G_N(z)$ has a kth derivative, then $G_N(z-c)$ has a $k+1^{st}$

derivative. Thus, since $G'' < 0$ and $G''' > 0$ (the same proofs in the text work),

N has continuous first and second derivatives. This forces N to satisfy T3,

and hence (8), in the limit.

Finally, we show next that the limit of G_N is the only differentiable

function with these properties.

Uniqueness Among Differentiable N:

Suppose N_o and N_1 satisfy (8), $N_i(0) = 0$, $\lim_{z \to \infty} N_i(z) = 1/\delta$, i $\in \{0,1\}$. Suppose,

by way of contradiction, that $(\exists z)$ $N_o(z) \neq N_1(z)$.

Then let $z^* = \arg \max_z \left| N_1(z) - N_0(z) \right|$. $\hfill$ (T6)

Without loss of generality, let $N_0(z^*) \leq N_1(z^*)$. $\hfill$ (T7)

Then $N_1'(z^*) - N_0'(z^*) = 0$.

Thus,

$$0 = N_1'(z^*) - N_0'(z^*) = -\delta(N_1(z^*) - N_0(z^*)) + \frac{(N_1(z^*+c) - N_1(z^*))^2}{2N_1(z^*+c)} - \frac{(N_0(z^*+c) - N_0(z^*))^2}{2N_1(z^*+c)} \ .$$

Thus,

$$0 < \frac{(N_1(z^*+c) - N_1(z^*))^2}{N_1(z^*+c)} - \frac{(N_0(z^*+c) - N_0(z^*))^2}{N_0(z^*+c)} \ . \hfill \text{(T8)}$$

CASE I: $N_1(z^*+c) \geq N_0(z^*+c)$

Then, by (T8),

$$0 < \frac{(N_1(z^*+c) - N_1(z^*))^2}{N_0(z^*+c)} - \frac{(N_0(z^*+c) - N_0(z^*))^2}{N_0(z^*+c)} \ ,$$

or

$$N_1(z^*+c) - N_1(z^*) > N_0(z^*+c) - N_0(z^*) \ ,$$

or

$$N_1(z^*+c) - N_0(z^*+c) > N_1(z^*) - N_0(z^*) \ ,$$

contradicting (T6).

CASE II: $N_1(z^*+c) - N_0(z^*+c) < 0$

Note that $\frac{(x-N)^2}{x}$ is increasing in x if $x > N$. Thus from (T8), substituting $N_0(z^*+c)$ for $N_1(z^*+c)$, we find

$$0 < \frac{(N_0(z^*+c) - N_1(z^*))^2}{N_0(z^*+c)} - \frac{(N_0(z^*+c) - N_0(z^*))^2}{N_0(z^*+c)}$$

forcing $N_1(z^*) < N_0(z^*)$. This contradicts (T7).

Appendix 3:

Note that, if $\dfrac{d}{dz}\dfrac{N''_z}{N'_z} \geq 0$ for $z > y$, then $\dfrac{d}{dz}\dfrac{N'_{z+c}}{N'_z} = \dfrac{N'_{z+c}}{N'_z}[\dfrac{N''_{z+c}}{N'_{z+c}} - \dfrac{N''_z}{N'_z}] \geq 0$, implying

by (15) that $\dfrac{d}{dz}\dfrac{N''_z}{N'_z} > 0$. Thus, if $\dfrac{N''_z}{N'_z}$ is decreasing at y, it is decreasing for

$z > y$, as is $\dfrac{N'_{z+c}}{N'_z}$.

So suppose, by way of contradiction, that $\dfrac{N''_z}{N'_z}$ is nonincreasing for $z > y$.

From (15), $\dfrac{N'_{z+c}}{N'_z}$ is strictly decreasing for $z > y$. Thus, since $1 > \dfrac{N'_{z+c}}{N'_z} > 0$,

$\lim\limits_{z\to\infty}\dfrac{N'_{z+c}}{N'_z} = \alpha \in [0,1]$, and from (14), $\lim\limits_{z\to\infty}\dfrac{N''_z}{N'_z} = -\delta$. Consequently, for large

enough z, greater than y,

$(1-p)\dfrac{N''_{z+c}}{N'_{z+c}} > (1+p)\dfrac{N''_z}{N'_z}$, or,

$(1+p)\dfrac{N'_{z+c}}{N'_z} > (1-p)\dfrac{N''_{z+c}}{N''_z}$. This implies

$\dfrac{1-p^2}{2}\dfrac{N'_{z+c}}{N'_z} > (1-p)\dfrac{N''_{z+c}}{N''_z} - \dfrac{1-p^2}{2}\dfrac{N''_{z+c}}{N''_z}$

Thus

$$\frac{1-p^2}{2}\frac{N'_{z+c}}{N'_z} > (1-p)\frac{N''_{z+c}}{N''_z} - p'(z)\frac{(N'_{z+c} - N'_z)}{N''_z} - \frac{1-p^2}{2}\frac{N''_{z+c}}{N''_z} + p(z)p'(z)\frac{N'_{z+c}}{N''_z}$$

Thus

$$-\delta - (1-p) + \frac{1-p^2}{2}\frac{N'_{z+c}}{N'_z} > -\delta + (1-p)[\frac{N''_{z+c}}{N''_z} - 1] - p'[\frac{N'_{z+c} - N'_z}{N''_z}]$$

$$- \frac{1-p^2}{2}\frac{N''_{z+c}}{N''_z} + pp'\frac{N'_{z+c}}{N''_z} \ .$$

From (14) and (16), this is

$$\frac{N''_z}{N'_z} > \frac{N'''_z}{N''_z} \ , \text{ or}$$

$$\frac{d}{dz}\frac{N''_z}{N'_z} = \frac{N'''_z}{N'_z} - [\frac{N''_z}{N'_z}] > 0 \ .$$

This contradicts the hypothesis, proving

$$\frac{d}{dz}\frac{N''_z}{N'_z} > 0 \ . \quad \blacksquare$$