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EFFICIENCY AND THE STRUCTURE OF PRODUCTION*

by

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I. INTRODUCTION

Economic theory has to date offered five treatments of the production of multiple outputs. The everyday theory of the firm assumes it away. Early work by R. G. D. Allen (1938), Hicks (1939), Samuelson (1947), and more recently, Laitinen (1980), analyzed the isolated behavior of profit-maximizing firms with access to a m-input/n-output technology. This analysis stopped short of analyzing market equilibrium. More recently, the "contestability" literature, summarized in Bailey and Friedlander (1982) has focused on the structure of market equilibrium under multiple-output production. This approach assumes "economies of scope" for the production of an exogenous collection of goods. Since primary interest there attaches to the number of plants in an industry, the issue of which goods will be produced together is settled in advance. The fourth approach goes to the opposite extreme. Arrow-Debreu-McKenzie general equilibrium theory allows each producer a distinct multi-dimensional production set. In a general equilibrium, some allocation of production of total output across firms is implied, but at the level of generality of the analysis no operationally meaningful restrictions result. Lastly, there is the "where there is sawdust there must be 'fire logs'" approach, dating back at least to Marshall (1920, pp. 321-2). In modern terms, this view uses production complementarities to generate economies of scope. However, these complementarities are completely unstructured and, like the full general equilibrium model, rule out next to nothing.

In sum, economic theory has not offered any falsifiable predictions about the manner in which production is divided among firms in market equilibrium. The novelty in this paper is that it does precisely that.

The approach taken is as follows. First, production complementarities are ruled out. Each good is produced according to an independent production process. These complementarities cannot be unstructured and uninformative if they do not exist. Next, the <u>definition</u> of a multiple output firm is that it is one which uses more than one of the production processes. However, multiple output firms operate subject to a restriction. While there are many restrictions that might be imposed, and one other is briefly examined below, in this paper it is assumed that if more than one production process is utilized, all must use <u>identical</u> factor proportions. This is an analytically tractable exaggeration of the notion that when more than one good is produced in a firm, none are produced with precisely the plant, materials and labor inputs which would be chosen were there no other goods produced there. For example, a plant producing table legs and cabinet knobs will utilize lathes which are larger than those which would be used were only knobs produced, but which are also more easily adjustable than if they were used to produce only table legs.

That multiple output plants are not just a series of contiguous single output plants is, of course, necessary for the distinction to have any content. The only issue is whether multiple output plants are more efficient or less efficient. In this paper, producing goods together compromises the production process.

In order for there to be any multiple output firms in equilibrium, there must be some economy associated with such production structures. This efficiency gain is assumed to be in terms of the non production component of the firm's operations. The costs of management, accounting, payroll, etc. are assumed to be less than the sum of those required to run the production processes independently. Multiple output firms, if there are any, exist because of something akin to public goods within the firm.

Below it is shown that the technological restriction of equal factor proportions implies (it is slightly less obvious than it appears) diseconomies of scope in production costs. The treatment of non production costs is identical to economies of scope in those activities. Thus, in brief, the results presented below are based on diseconomies of scope in production activities and economies in non-production activities.

It is worth noting that analyses of the sawdust-fire logs variety assume just the opposite. Therein, the factors preventing all goods from being produced together is the "loss of control" associated with problems of coordination that outweigh the public good aspect of the firm, in conjunction with transaction costs which prevent the sawdust from being hauled away and the fire logs produced in an environment where there is no loss of control (i.e., a single output firm). The predictive content of this approach is next to nil because the transactions costs (presumably unobservable to the analyst since they are not incurred) can be set to achieve any allocation desired.

The payoff to the approach adopted herein is that it offers numerous predictions, explains various curious phenomena as equilibrium behavior under perfect competition, and even sheds light on applied policy debates. As an example of the first, focusing on two goods, the equilibrium outcome can be "polymorphic", to borrow from Hallagan and Joerding (1983). Some firms will produce just one good, and other firms that good plus another; efficiency in competitive equilibrium yields asymmetry. This stands in contrast to the fully symmetric equilibria following from economies of scope in production costs (Baumol, et al (1982)). Another prediction is that, starting from a situation wherein both goods are produced solely by single-product firms, an increase in the non-production costs of firms specializing in the production of one good can lead to an equilibrium wherein those firms remain and the other good is produced by diversified firms. Firms specializing in production of the other good are casualties of the cost increase. This unintuitive result does not require any "perverse" occurrence. It is a natural outcome of efficiency under some not unreasonable circumstances.

Curious phenomena explained by the theory are, among others: (1) the production of golf clubs and airplane parts in a Montreal firm, and the historical curiosum that sewing maching firms typically also produce bicycles; (2) why producers of video games do not produce other leisure related goods such as records. Applied policy issues include evaluation of the claim that elimination of banking regulation will yield financial supermarkets, as well as an application to the influence of sector-specific factor taxes/subsidies on the allocation of production across firms.

It should be pointed out that the analysis to follow treates firms and plants as synonymous. A firm which owns numerous single output plants is treated as numerous firms. However, the approach examined in this paper shows promise in terms of explaining these (essentially financial) arrangements as well. Some space is directed to these considerations in the concluding section.

The paper is set out as follows. Section II contains material on the multi-product technology under the equal factor proportions restriction. Next, the associated cost functions are presented and the relevant "scope" properties examined. Section IV is devoted to a brief treatment of the competitive equilibrium as a programming problem, the solution to which is developed in Section V. Section VI details the predictions implied by the theory and offers a discussion of the applied phenomena mentioned above. Recapitulation is contained in Section VII.

II. PRODUCTION TECHNOLOGIES

In the sections to follow, the case in which each of two outputs requires inputs of two factors of production is examined. This section first presents the assumptions made concerning the independent production processes used to produce each good. It is then shown that the restriction to equal factor proportions when both goods are produced in the same plant generates a multiple output technology which is well behaved in an appropriate sense. Finally, necessary and sufficient conditions for this multiple output technology to differ from the multiple output technology which would obtain in the absence of the equal proportions restriction are provided.

The notation to be used below is as follows:[1]

$q = (q_\alpha, q_\beta) \in \mathbb{R}^2_+$	- a vector of outputs indexed by $j \in \{\alpha,\beta\}$
$X_j = (X^1_j, X^2_j) \in \mathbb{R}^2_+$	- a vector of inputs, with typical element X^i_j, used to produce q_j, $i \in \{1,2\}$;
$X^i = \sum_j X^i_j$	- total input of factor $i \in \{1,2\}$;
$X = (X^1, X^2)$	- vector of total inputs;
$L_j(\bar{q}_j) = \{(X^1_j, X^2_j) \mid q_j \geq \bar{q}_j\}$	- the input requirement set for good j. That is, the set of all input vectors which will produce at least $\bar{q}_j$, a particular q_j;
$\sigma_j(q_j, X) = \underset{s}{\text{argmin}}\{sX \mid sX \in L_j(q_j)\}$	- the smallest share of a total input vector X which will produce q_j;
$\eta_j(\xi, q_j) = \underset{X_j}{\min}\{\xi X_j \mid X_j \in L_j(q_j),\ \xi \in \mathbb{R}^2_+,\ \lVert\xi\rVert = 1\}$	- the value of the least cost factor bundle which will produce q_j when "prices" are ξ_j.

$\zeta_i(q_j,X) = \{\xi \,|\, \|\xi\| = 1,\ \xi\,\sigma_j(q_j,X)X = \eta_j(\xi_j,q_j)\}$ - the set of factor prices for which utilization of the share σ_j of total inputs X is the least cost way to produce q_j;

$L(q) = \{X \,|\, \sum_j \sigma_j(q_j,X) \le 1\}$ - the restricted multiproduct input requirement set. That is, the set of total input vectors X which will produce the output vector q under the equal factor proportions restriction;

$L^e(q) = \{X \,|\, \overline{X} \le X \Rightarrow \overline{X} \notin L(q)\}$ - lower boundary of L(q);

$\ell(q) = \{X \,|\, X_j \in L_j(q_j)\}$ - the unrestricted multiproduct input requirement set;

$\ell^e(q) = \{X \,|\, \overline{X} \le X \Rightarrow \overline{X} \notin \ell(q)\}$ - lower boundary of $\ell(q)$.

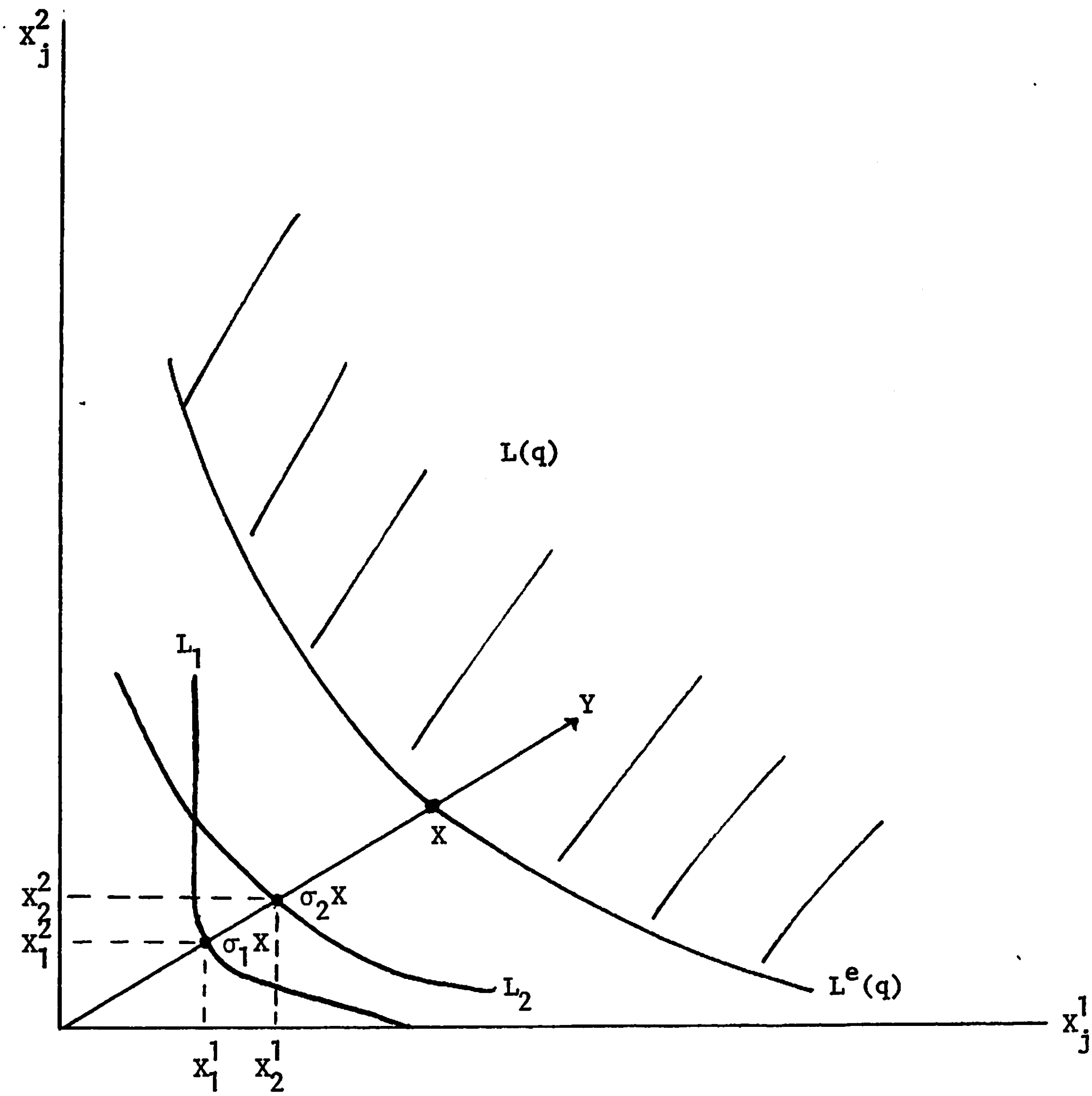
Before the relationship between the $L_j(q_j)$, L(q) and $\ell(q)$ can be examined, the assumptions made about the $L_j(q_j)$ must be specified:[2]

1. $L_j(q_j)$ is closed $\forall\, q_j$;
2. $L_j(q_j)$ is convex $\forall\, q_j$;
3. $L_j(0) = \mathbb{R}^2_+$, $q_j > 0 \Rightarrow 0 \notin L_j(q_j)$;
4. $X_j \in L(q_j)$, $\overline{X}_j \ge X_j \Rightarrow \overline{X}_j \in L_j(q_j)$;
5. If either (i) $X_j \gg 0$; or (ii) $X_j \ge 0$ and $\overline{\lambda} X_j \in L_j(\bar{q}_j)$ for some $\overline{\lambda}$ and $\bar{q}_j$, then $\forall\, q_j \ge 0$ $\{\lambda X_j \,|\, \lambda \ge 0\} \cap L_j(q_j) \ne \{\emptyset\}$;
6. $q_j \ge \bar{q}_j \ge 0 \Rightarrow L_j(q_j) \subset L_j(\bar{q}_j)$;
7. $\bigcap_{q_j \in \mathbb{R}_+} L_j(q_j) = \{\emptyset\}$.

Figure 1 depicts the manner in which the restricted input requirement set L(q) is constructed from the underlying $L_j(q_j)$. For each "direction"

FIGURE 1

Construction of Restricted Technology



vector $Y \in \mathbb{R}^2_+$, $\sigma_j X \equiv (X_j^1, X_j^2)$ is determined as the shortest vector of direction Y which will reach $L_j(q_j)$. The normalization $\Sigma_j \sigma_j = 1$ yields $X \in L^e(q)$.

The unrestricted input requirement set $\ell(q)$ is constructed from the $L_j(q_j)$ by the usual set addition as indicated in its definition. However, for the purpose at hand it will be useful to consider its construction from a different but equivalent standpoint. Let $\xi = (\xi^1, \xi^2)' \in \mathbb{R}^2_+$ be an arbitrary vector of unit length and for each j let $\bar{X}_j = \underset{X_j}{\operatorname{argmin}}\{\xi X_j | X_j \in L_j(q_j)\}$.[3] Then $\ell^e(q) = \{X | \exists \xi \text{ such that } X = \Sigma_j \bar{X}_j\}$ and $\ell(q) = \{X | X \geq \bar{X}, \bar{X} \in \ell^e(q)\}$. That is, at "prices" ξ, a point on $\ell^e(q)$ is found as the sum of the least cost input vectors which will produce q_j. Varying ξ yields all of $\ell^e(q)$.

Given the technological structure assumed above, two relevant propositions are available. First, it is shown that the restricted multi-product input requirement set is well behaved in an appropriate sense.

<u>PROPOSITION 1</u>: L(q) satisfies

(1) L(q) is closed;

(2) L(q) is convex;

(3) $L(0) = \mathbb{R}^2_+$, $q \geq 0 \Rightarrow 0 \notin L(q)$;

(4) $X \in L(q)$, $\bar{X} \geq X \Rightarrow X \in L(q)$;

(5) If either (i) $X \gg 0$; or (ii) $X \geq 0$ and $\bar{\lambda} X \in L(\bar{q})$ for $\bar{\lambda} > 0$, $\bar{q} > 0$, then

(i) $\{\lambda X | \lambda \geq 0\} \cap L(q) \neq \{\emptyset\}$ $\forall$ q;

or (ii) $\{\lambda X | \lambda \geq 0\} \cap L(q) \neq \{\emptyset\}$ $\forall$ q such that $q_j > 0$ $\Rightarrow \bar{q}_j > 0$;

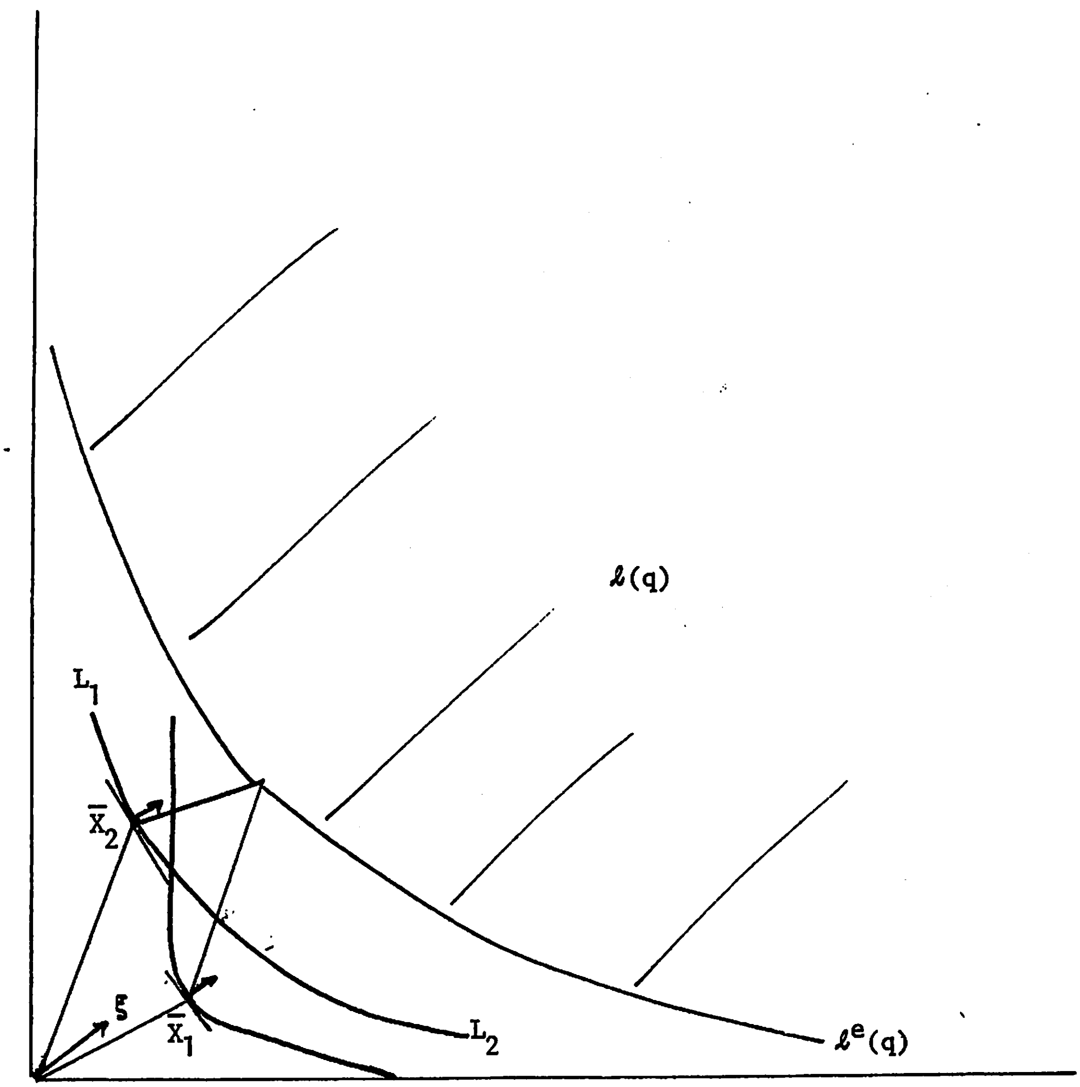
(6) $q \geq \bar{q} \geq 0 \Rightarrow L(q) \subset L(\bar{q})$;

(7) $\bigcap_{q \in \mathbb{R}^2_+} L(q) = \{\emptyset\}$.

<u>PROOF</u>: See Appendix.

FIGURE 2

Construction of Unrestricted Technology



Proposition 1 shows that the restricted multiproduct input requirement sets, which are a type of aggregate of the underlying $L_j(q_j)$, inherit the (suitably modified) properties of the $L_j(q_j)$. The proof is straightforward with the exception of part (2). Given the Proposition, the cost function analysis carried out in Section III is valid.

Of somewhat greater interest is the relationship between L(q) and $\ell(Q)$. The information relevant to the analysis below is obtained as follows. A notion of similarity between the $L_j(q_j)$ is required:

<u>DEFINITION</u>: $L_1(\cdot)$ and $L_2(\cdot)$ will be said to be <u>similar</u> with respect to (q,X) if and only if:

$$\zeta_1(q_1, X) \cap \zeta_2(q_2, X) \neq \phi.$$

Loosely, the $L_j(q_j)$ are similar if there is some set of prices for which a share of the total input vector X is the cheapest way to produce each q_j. Putting this definition to use:

<u>PROPOSITION 2</u>: Let (q,X) be such that $X \in L^e(q)$, and let $\lambda > 0$ be such that $\lambda X \in \ell^e(q)$. For any such λ, q, X, we have that $\lambda < 1 \Leftrightarrow L_1(\cdot)$ and $L_2(\cdot)$ are not similar with respect to (q,X).

<u>PROOF</u>: See Appendix.

Though the proof is arduous, the basic logic is straightforward. First of all, it is obvious that $\lambda(q,\overline{X}) \leq 1$. Thus the case of strict inequality is all that need be dealt with. To see necessity, assume

$\lambda < 1$ and that the $L_j(q_j)$ are similar for some $(q_j,\bar{X})$. Since the $L_j(q_j)$ are similar, there is some set of prices $\bar{\xi} \in \bigcap_j \zeta_j(q_j,\bar{X})$ such that a share of $\lambda\bar{X}$ is the cheapest way to produce q_j when prices are $\bar{\xi}$. But then by definition of $L(q)$, $\lambda\bar{X} \in L(q)$, contradicting $\lambda < 1$.

Sufficiency is obtained as follows. Suppose the $L_j(q_j)$ are not similar with respect to some $(q,\bar{X})$. Let $\bar{\xi}$ support $\mathcal{L}^e(q)$ at $\lambda\bar{X}$. Since the L_j are not similar for $(q,\bar{X})$, there is some $\bar{X}_j$ where $\bar{\xi}\,\bar{X}_j \leq \bar{\xi}\cdot[\sigma_j(q_j, \lambda\bar{X})\lambda\bar{X}]$, ($\sigma_j$ is the smallest share of $\lambda\bar{X}$ which will produce q_j), $\Sigma\bar{X}_j = \lambda\bar{X}$, and the inequality is strict for some j. Summing the inequalities gives

$$\bar{\xi}\,\Sigma_j\bar{X}_j < \bar{\xi}\,\lambda\bar{X}\,\Sigma_j\sigma_j(q_j,\lambda\bar{X})$$

or

$$1 < \Sigma_j\sigma_j;$$

i.e., $\lambda\bar{X} \notin L(q_j)$.

Put simply, Proposition 2 states that production under the equal factor proportions restriction invariably requires more resources if and only if the expansion paths associated with each independent production process never cross.

Given Propositions 1 and 2, the material on cost functions can be presented.

III. COST FUNCTIONS

Firms incur both production and nonproduction costs. The former include the costs of materials and services of factors utilized directly in the production of output. The latter are comprised of costs of book-keeping, management, ordering materials, etc. The operational distinction is that nonproduction costs need not vary greatly with the level of output. They operate in the manner of a fixed cost. As the analysis to follow is of the long-run competitive equilibrium type, nonproduction costs are the only fixed costs.

In this section the cost functions for single and multiple output firms are presented. Both types of firms face a vector of fixed factor prices $r = (r^1, r^2) \in \mathbb{R}^2_+$.

First consider firms producing a single product. These are referred to as specialized firms. A specialized firm producing good j faces a fixed cost $F^j > 0$. As regards variable costs, the variable cost function $C_j(q_j;r)$ is defined in the usual fashion:

$$C_j(q_j;r) \equiv \min_{X_j \geq 0}\{rX_j | X_j \in L_j(q_j)\}.$$

Turning to multiproduct firms, referred to as diversified in the two-good context, a fixed cost of F^D assumed. The notion here is again that nonproduction costs do not vary with the level of output of the goods produced in diversified firms. In addition though, it is assumed that costs of organization, management, etc. are: (i) greater in a diversified firm than in either type of specialized firm, $F^D > F^j \; \forall \; j$; but (ii) not as great in total as would occur if the goods were produced apart, $F^D < \Sigma_j F^j$. This latter restriction is effectively "economies of scope" in nonproduction costs.

The variable cost function for the diversified firm is obtained in a fashion analogous to the specialized case:

$$C^D(q;r_1,r_2) \equiv \min_{X\geq 0}\{rX|X \in L(q)\}.$$

The relationship between variable costs of production in the diversified and specialized firms is obtained as follows. First, note that

$$\sum_j C_j(q_j;r) = \min_{X_j^i}\{rX|X \in \ell(q)\}. \qquad (1)$$

This follows simply from replacing ζ with r in the construction of $\ell(q)$. Using Proposition 2:

<u>PROPOSITION 3</u> (Strong Diseconomies of Scope): Assume $L_1(\cdot)$ and $L_2(\cdot)$ are not similar with respect to <u>any</u> (q,X). Then, for any (q,r),

$$C^D(q;r) > \sum_j C_j(q_j;r).$$

<u>PROOF</u>: For any (q,r), let

$$\bar{X} = \operatorname{argmin}_{X\geq 0}\{rX|X \in L(q)\}.$$

By Proposition 2, for some $\lambda < 1$, $\lambda\bar{X} \in \ell(q)$. Thus (using (1))
$\Sigma_j C_j(q_j;r) \leq r\lambda\bar{X} < r\bar{X} = C^D(q;r)$. ■

Thus the equal factor proportions assumption coupled with dissimilarity of the underlying production processes guarantee diseconomies of scope for all factor prices and output levels.

In brief, then, product diversification yields: (i) economies of scope in nonproduction costs due to cross-good economies of scale in management, etc.; and (ii) diseconomies of scope in production costs, owing to technical inefficiencies caused by the compromising of factors which occurs when diverse goods are produced contiguously.

Given the presence of fixed costs, our assumption of competitive behavior by firms requires that the production technology exhibit strictly decreasing returns to scale. Thus, define the production set

$$Y_j = \{(X_j, q_j) \mid X_j \in L_j(q_j)\}, \qquad j=1,2.$$

We thus strengthen property 2 of $L_j(\cdot)$ to:

Y_j is strictly convex for $j=1,2$.

It is then well known that $y = \{(X,q) \mid X \in \mathcal{L}(q)\}$ is strictly convex, and straightforward to verify that $Y = \{(X,q) \mid X \in L(q)\}$ is also strictly convex. This also implies that $C^D(q;r)$ and $C^j(q_j;r)$ are strictly convex functions of q and q_j, respectively, and thus continuously differentiable almost everywhere.

Finally, there is little to be gained in carrying the behavior of specialized firms along in detail. Accordingly some simplification is undertaken at this point.

It is clear that if there are any specialized forms in equilibrium, and the "integer problem" is ignored, they will all operate at the level of output which minimizes average cost. Thus let

$$\bar{q}_j \equiv \underset{q_j}{\text{argmin}} \frac{F^j + C_j(q_j;r)}{q_j} \tag{2}$$

and

$$\bar{c}_j \equiv F^j + C_j(\bar{q}_j;r). \tag{3}$$

The relevant behavior of specialized firms is completely summarized by $\bar{q}_j$ and $\bar{c}_j$.

In summary, specialized firms produce $\bar{q}_j$ at a cost of $\bar{c}_j$. Diversified firms differ from specialized firms in that they enjoy economies of scope in nonproduction costs and diseconomies in production costs. Total cost functions for diversified firms are

$$F^D + C^D(q;r). \tag{4}$$

IV. A PROGRAMMING APPROACH TO COMPETITIVE EQUILIBRIUM

In the analysis of Sections V and VI, the competitive equilibrium configuration of specialized and diversified firms is found as the least aggregate cost method of producing given quantities of the two outputs.

The analytical economy involved in this procedure is obvious: the entire demand side of the economy can be ignored. Moreover when making predictions concerning the impact of the pattern of demand on the equilibrium configuration the laborious (but not impossible) task of altering preferences in an appropriate fashion need not be spelled out.

The purpose of this section is to briefly explain why this procedure is an appropriate one. Since rigorous demonstrations of the proposition are already available in the standard Arrow-Debreu-McKenzie general equilibrium literature, the discussion proceeds at a slightly less formal level than would be appropriate otherwise. In particular, though we do not do so below, existence of a solution is assumed. Finally, as the proposition has nothing especially to do with multiple outputs, and simplicity is gained, the case of heterogeneous firms (where all types are in infinitely elastic supply) producing a single type of good is examined.

Suppose firms of type k (k=1,...,n) produce output of a good q subject to the total cost function $F^k + c_k(q)$; $c_k^1 > 0$, $c_k^z > 0$. Let the number of firms of type j be N_k. Now assume a competitive equilibrium, and that in this equilibrium $N_k > 0$ for $k \in K \subset \{1,\ldots,n\}$, firms of type k produce q_k, and let $Q = \Sigma_k N_k q_k$. The aggregate cost of production is

$$C^* = \Sigma_k N_k [F_k + c_k(q_k)] \ .$$

Conditions necessary for an interior (w.r.t. N_i, q_k) minimum for C* subject to $\Sigma_k N_k q_k \geqq q$ are

$$F_k + c_k(q_k) - \lambda q_k = 0,$$

$$c_k' - \lambda = 0,$$

where λ is the multiplier on the aggregate output constraint.

If the N_k and q_k of competitive equilibrium do not minimize aggregate cost, it follows that one of the above conditions is violated. Since all firms face the same price, profit maximization implies the second set of conditions does not fail, and λ is the price of output. The failure must therefore be in the first set of conditions. That is, some type of firm is making non zero profits. As all firm types are elastically supplied, this too is inconsistent with equilibrium. Thus, under free entry, competitive equilibrium yields any given aggregate production at least aggregate cost.

V. SOLUTION

In accordance with the argument above, we will proceed as if the economy solves the programming problem of minimizing the total cost of producing a given aggregate bundle of the two goods.

To keep the notation as suggestive and simple as possible, we shall henceforth let α and β designate the two goods, while a numerical subscript i will indicate partial derivatives with respect to the i^{th} argument of a function. Thus, the aggregate bundle of goods to be produced is designated as (Q_α, Q_β). N_j is the number of firms of "type j", for $j=\alpha,\beta,D$, while q_j is the level of output of good $j=\alpha,\beta$, by a diversified firm. $\bar{C}_j, \bar{q}_j$ are as defined in (2) and (3).

The Lagrangian for the cost-min problem is thus:

$$= N_D[F^D + C^D(q_\alpha, q_\beta)] + N_\alpha \bar{C}_\alpha + N_\beta \bar{C}_\beta$$

$$+ \lambda_\alpha[Q_\alpha - N_D q_\alpha - N_\alpha \bar{q}_\alpha] + \lambda_\beta[Q_\beta - N_D q_\beta - N_\beta \bar{q}_\beta]$$

with λ_α and λ_β being the as yet undetermined multipliers.

The Kuhn-Tucker necessary conditions for a minimum are then as follows, with subscripts indicating partial derivatives:

$$F^D + C^D(q_\alpha, q_\beta) - \lambda_\alpha q_\alpha - \lambda_\beta q_\beta \geqq 0 \tag{5}$$

$$\bar{C}_\alpha - \lambda_\alpha \bar{q}_\alpha \geqq 0 \tag{6}$$

$$\bar{C}_\beta - \lambda_\beta \bar{q}_\beta \geqq 0 \tag{7}$$

$$C_1^D(q_\alpha, q_\beta) - \lambda_\alpha \geqq 0 \tag{8}$$

$$C_2^D(q_\alpha, q_\beta) - \lambda_\beta \geqq 0 \tag{9}$$

$$Q_\alpha - N_D q_\alpha - N_\alpha \bar{q}_\alpha = 0 \tag{10}$$

$$Q_\beta - N_D q_\beta - N_\beta \bar{q}_\beta = 0 \tag{11}$$

These conditions are individually familiar, (5)-(7) simply stating that all three types of firms must earn non-positive profits, given that outputs are valued at the shadow prices λ_j. (6)-(9) also require that all producing firms' marginal costs be equal to these shadow prices, while (10) and (11) are of course just the aggregate production constraints.

Before proceeding, we should note that, given the usual sort of technical assumptions about minimum average cost being attained at a sufficiently low output, it is straightforward, if tedious, to show that a solution satisfying (5)-(11) must exist and this is done in an Appendix.

The first preliminary result for the characterization of equilibrium is the following.

PROPOSITION 4 At most two of N_D, N_α, N_β, can be positive.

This "spanning theorem" follows from the simple fact that the necessity of zero profits and the equality of marginal costs across operating firms imply five equations (5-9) to determine only four variables: outputs by diversified firms q_α, q_β, and the common levels of marginal costs, $\lambda_\alpha, \lambda_\beta$.

Since Q_α, Q_β are both positive, Proposition 4 implies that the solution, which we will denote by $z = (q_\alpha, q_\beta, N_D, N_\alpha, N_\beta, \lambda_\alpha, \lambda_\beta)$, can take one of four forms: We will say that

$$z \in D \Leftrightarrow N_D > 0,\ N_\alpha = N_\beta = 0$$

$$z \in S \Leftrightarrow N_\alpha > 0,\ N_\beta > 0,\ N_D = 0$$

$$z \in M_j \Leftrightarrow N_D > 0,\ N_j > 0,\ N_k = 0, \quad j = \alpha,\beta;\ k \neq j\ .$$

Our present task is to examine the determinants of the form of z; that is, to determine for what parameter configurations z falls into each of the above four sets. Let $C^*(\cdot)$ be the minimized level of aggregate production costs in the economy, which therefore is a function of all the parameters of the problem. This last preliminary result greatly simplifies what follows.

PROPOSITION 5 $C^*(\cdot)$ is homogeneous of degree one in (Q_α, Q_β).

Propositions 4 and 5 highlight the result that the present model is in many ways a multi-product analog of the traditional long-run competitive equilibrium model. Proposition 4 follows from the observation that the set of firms which operate in equilibrium can be determined by considering only the zero-profit conditions for an individual firm of each type. Proposition 5 follows from the fact that proportional changes in aggregate output can always be met by proportional changes in the number of all operating firms, as in the traditional one product/firm case.

In light of Proposition 5, we henceforth consider all changes in the aggregate output vector as changes in Q_α utilizing the convenient normalization $Q_\beta = 1$. The magnitudes of (Q_α, Q_β) determine only the number of firms of each (operating) type, so long as they are large enough for at least one firm to operate. We assume this to always be so, as well as assuming away the singularly uninteresting integer problem.

We now introduce an additional restriction on the diversified firm's variable cost function:

$$\begin{aligned} C_1^D(0,\bar{q}_\beta) &< C_1^D(\bar{q}_\alpha,0) = \bar{C}_\alpha/\bar{q}_a \equiv \bar{\lambda}_\alpha \\ C_2^D(\bar{q}_2,0) &< C_2^D(0,\bar{q}_\beta) = \bar{C}_\beta/\bar{q}_\beta \equiv \bar{\lambda}_\beta \end{aligned} \tag{12}$$

The reason for (12) will be clear as we proceed. Its first role is in (i) of the following Proposition

<u>PROPOSITION 6</u> (i) If $F^D \leqq F^j$, then $N_j = 0$, $j = \alpha,\beta$

(ii) If $F^D \geqq F^\alpha + F^\beta$, then $N_D = 0$.

Part (ii) is obvious enough, since $F^D \geqq F^\alpha + F^\beta$, diversified production always carries an unambiguous total cost disadvantage. As to (i), suppose that $F^D = F^\alpha$, for example, and $N_\alpha > 0$. Then good β cannot be produced by diversified firms, since they and specialized α-producing firms have "the same" strictly convex cost function. That is, <u>both</u> types of firms could not be earning zero profits at "prices" $(\bar{\lambda}_\alpha, \lambda_\beta)$, for any λ_β. If good β is being produced by specialized firms, then its shadow price would be $\bar{\lambda}_\beta$, and a diversified firm can earn positive profits given prices $(\bar{\lambda}_\alpha, \bar{\lambda}_\beta)$, from the second inequality in (12).

A convenient notation is to let $M_\alpha \succ D$ mean that the lowest cost $z \in M_\alpha$ dominates the lowest cost $z \in D$, for example. The central result for our eventual classification of equilibria can now be given.

<u>PROPOSITION 7</u> Let F^α, F^β be given.

(i) For any $F^D > F^\alpha$, there exists a unique $Q_\alpha > 0$ denoted as $\bar{Q}_\alpha(F^D)$, such that $M_\alpha \succ D \Leftrightarrow Q_\alpha > \bar{Q}_\alpha(F^D)$.

(ii) For any $F^D > F^\beta$, there exists a unqiue $Q^\alpha > 0$, denoted as $\underline{Q}_\alpha(F^D)$, such that $M_\beta \succ D \Leftrightarrow Q_\alpha < \underline{Q}_\alpha(F^D)$.

The following characterization of $\bar{Q}_\alpha(F^D)$ is evident from the proof of Proposition 7, which is left to the Appendix: If one solves the aggregate cost-minimization problem, with the additional constraint that $N_\beta = 0$, then for given F^α, F^β, $\bar{Q}_\alpha(F^D)$ is the value of Q_α at which the solution switches between M_α and D. Symmetrically, $\underline{Q}_\alpha(F^D)$ is defined from the sub-problem with $N_\alpha = 0$. Thus, the functions $\bar{Q}_\alpha(\cdot)$, $\underline{Q}_\alpha(\cdot)$ imply nothing <u>directly</u> about the solution to the actual cost-min problem.

However, the artificial $N_\beta = 0$ problem does generate a value for λ_β. If $\lambda_\beta \geq \bar{\lambda}_\beta$, then the $N_\beta = 0$ constraint is binding, and the given (Q_α, Q_β) can always be produced more cheaply by eliminating the diversified firms and replacing them with specialized β producers. (12) guarantees that at some level of F^D (strictly between $F^\alpha + F^\beta$ and the larger of (F^α, F^β)), $\lambda_\beta < \bar{\lambda}_\beta$, and the artificial problem gives the "right" solution. The role of (12) is now clear. Without it, only $z \in D$ or $z \in S$ are possible, with the result hinging only on the relative size of F^D and (F^α, F^β). (12) allows us to prove the following

PROPOSITION 8 There exists a unique $\tilde{F}$ such that $\max\{F^\alpha, F^\beta\} \equiv \bar{F} < \tilde{F} < F^\alpha + F^\beta$ and $z \in S \Leftrightarrow F^D \geqq \tilde{F}$.

$\tilde{F}$ is precisely the level of F^D at which $\bar{Q}_\alpha$ and $\underline{Q}_\alpha$ are equal. The reasoning behind this result that the possibility of a completely specialized equilibrium hinges only on the magnitude of F^D is as follows. That once $z \in S$, an increase in F^D alone keeps $z \in S$ is clear enough if one considers the form that the system (5)-(11) must take if $z \in S$. An increase in F^D affects only (5), keeping it a strict inequality, insuring that $\dot{N}_D = 0$.

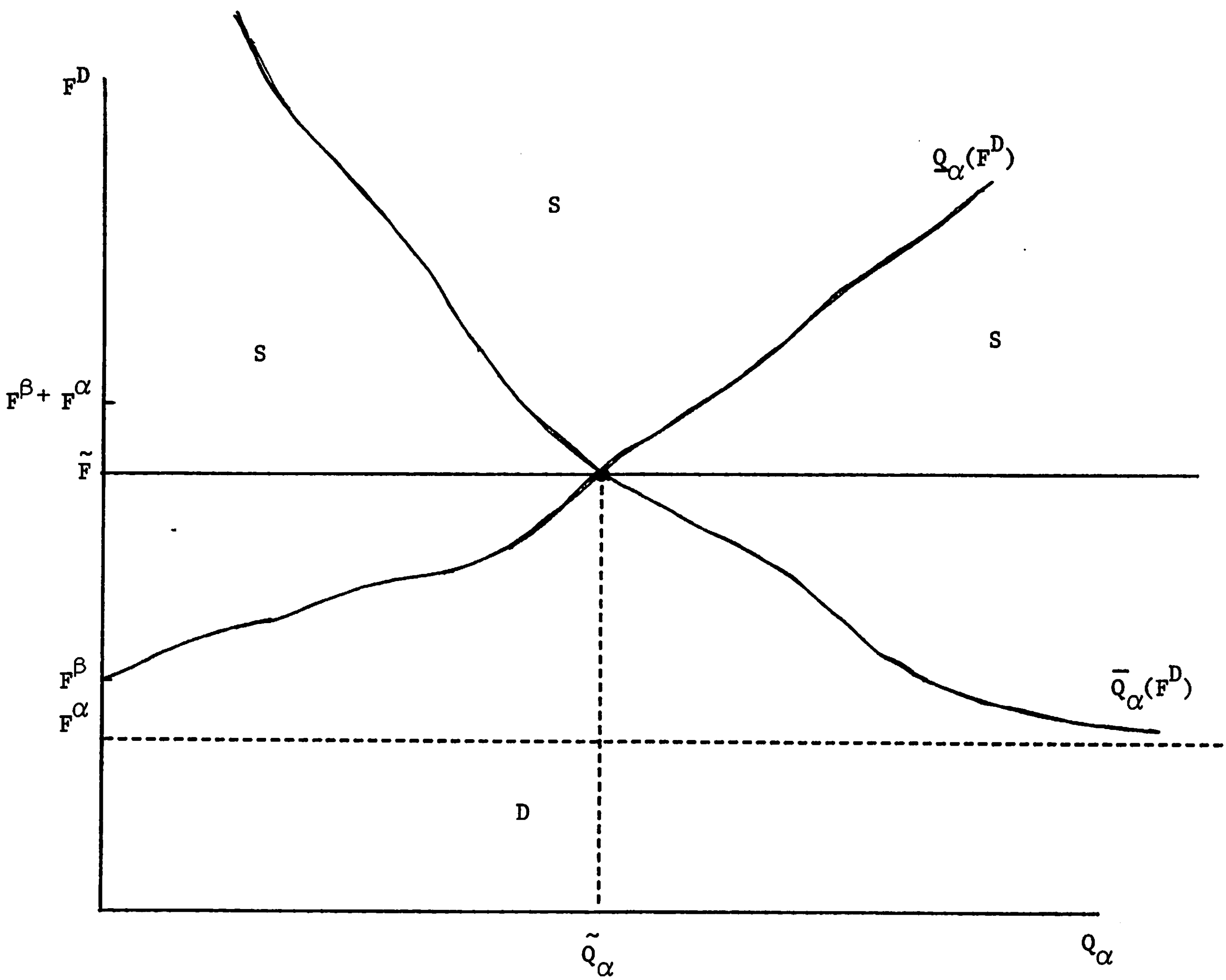
As to why changes in Q_α do not affect the fact that $z \in S$, recall that in a completely specialized equilibrium, specialized firms are of course operating such that their marginal costs equal $\bar{\lambda}_j = \bar{C}_j/\bar{q}_j$. If $z \in S$, a diversified firm producing the (q_α, q_β) combination such that his marginal costs also equal $\bar{\lambda}_\alpha, \bar{\lambda}_\beta$ is producing his output at a total cost greater than $q_\alpha\bar{\lambda}_\alpha + q_\beta\bar{\lambda}_\alpha$ (i.e., diversified production is unprofitable at "prices" $\bar{\lambda}_\alpha, \bar{\lambda}_\beta$). Changes in Q_α can always be met by changes in N_α and N_β, leaving $\lambda_j = \bar{\lambda}_j$, and thus can never make it possible for diversified firms to enter.

$\tilde{F}$ is thus the level of fixed costs just sufficient to prevent diversified production from occurring. The difference $F^\alpha + F^\beta - \tilde{F}$ (or ratio $\tilde{F}/(F^\alpha + F^\beta)$) can thus be viewed as an output-independent measure of the extent of diseconomies of scope in production.

Propositions 6, 7 and 8 are summarized in Figure 3, in which the letters S and D refer to the region into which z must fall if (F^D, Q_α) fall into the region so labelled. The diagram is rather more well-behaved than we have so far shown it must be. However, it is a direct corollary to the proof of Proposition 8 that

$$\lim_{F^D \to F^\alpha} \bar{Q}_\alpha(F^D) = \infty, \qquad \lim_{F^D \to F^\beta} \underline{Q}_\alpha(F^D) = 0 \tag{13}$$

FIGURE 3



F^D
$\underline{Q}_\alpha(F^D)$
S
S
S
$F^\beta + F^\alpha$
$\tilde{F}$
F^β
F^α
$\bar{Q}_\alpha(F^D)$
D
$\tilde{Q}_\alpha$
Q_α

Further, we can prove the following result.

PROPOSITION 9 For all $F^D \in]\bar{F},\tilde{F}[$;

(i) $\bar{Q}_\alpha(F^D) > \underline{Q}_\alpha(F^D)$

(ii) $z \in M_\alpha \Leftrightarrow Q_\alpha > \bar{Q}_\alpha(F^D)$

(iii) $z \in D \Leftrightarrow \bar{Q}_\alpha(F^D) > Q_\alpha > \underline{Q}_\alpha(F^D)$

(iv) $z \in M_\beta \Leftrightarrow Q_\alpha < \underline{Q}_\alpha(F^D)$.

This proposition allows us to label the remaining regions of the diagram. $\bar{Q}_\alpha(\cdot)$ and $\underline{Q}_\alpha(\cdot)$ have no significance for the structure of the solution for $F^D > \tilde{F}$, as shown by Proposition 8, though they are nevertheless well defined. (13) shows that for $F^D \leqq \tilde{F}$, $\bar{Q}_\alpha(\cdot)$ must be, "on average" downward sloped, and $\underline{Q}_\alpha(\cdot)$ on average of positive slope. To guarantee that these slopes do not change signs we need the following assumption

$$q_\alpha C^D_{11}(q_\alpha,q_\beta) + q_\beta C^D_{12}(q_\alpha,q_\beta) > 0, \; q_\beta C^D_{22}(q_\alpha,q_\beta) + q_\alpha C^D_{21}(q_\alpha,q_\beta) > 0 \tag{14}$$

This is not particularly restrictive, as the convexity of C^D alone implies that at most one of these can fail for any given (q_α,q_β). Also, $C^D_{12} > 0$ is clearly sufficient for (14) to hold, and this is obviously the leading case, given our construction of the diversified firm's technology. We can now present a final result

PROPOSITION 10 Given (14), for all F^D, we have[4]

$$\frac{\partial \bar{Q}_\alpha(F^D)}{\partial F^D} \le 0, \; \frac{\partial \underline{Q}_\alpha(F^D)}{\partial F^D} \ge 0$$

Under (14) then, Diagram 3 is an accurate illustration of the categorization of equilibria, and can be completely labelled utilizing Proposition 9. It will no doubt prove useful to provide some intuition as to how the structure of production is altered as (F^D,Q_α) change, when $F^D \in]\bar{F}, F^\alpha + F^\beta[$.

First, suppose (F^D,Q_α) are such that $z \in M_\beta$, and consider an increase in Q_α, F^D remaining fixed. First of all, the (q_α,q_β) combination produced by diversified firms in M_β is independent of Q_α, this being the analog of $q_\alpha = \bar{q}_\alpha$ for single product firms. If Q_α increasing is due to an increase in the production of good α, this must occur through an increase in the number of diversified firms, and thus a consequent decrease in N_β. If, on the other hand Q_α is increasing as a result of a decrease in β production, then as noted previously this must occur through a fall in N_β. Eventually $N_\beta = 0$ (when $Q_\alpha = q_\alpha/q_\beta = \underline{Q}_\alpha(F^D)$), and production becomes completely diversified. Then as Q_α continues to increase, since only diversified firms are producing, their outputs must satisfy $q_\alpha/q_\beta = Q_\alpha$. Throughout this region, $C_j^D < \bar{\lambda}_j$ for $j=\alpha,\beta$, but as Q_α (and therefore q_α/q_β) rises, C_α^D rises until it reaches $\bar{\lambda}_\alpha$. At this point, increases in Q_α, whether due to increases in α output or decreases in β output, cause an increase in N_α (and in the case of β output falling, a fall in N_D). For the diversified firms, q_α/q_β is again fixed independently of output, although at a level different from that when $z \in M_\beta$.

Note that the above discussion in no way hinged on the validity of Proposition 10 and so did not require (14). All that was assumed was (i) of Proposition 9. We now consider starting with a (Q_α,F^D) such that $z \in D$, and consider the result as F^D rises, assuming that (14) holds.

We know from (13) that there is some (Q_α,F^D) with $F^D > F^\alpha$ such that $z \in D$. Consider an increase in F^D, with Q_α fixed. It is easy enough to see that $q_\alpha/q_\beta = Q_\alpha$ independently of F^D if $z \in D$ so that an increase in F^D simply reduces N_D, and increases both q_α and q_β. This must cause a rise in the level of marginal cost for at least one of the goods. (13) implies that we could always choose Q_α high enough so that it is C_α^D which first attains the value $\bar{\lambda}_\alpha$, and thus $z \in M_\alpha$. Still, none of this requires (14).

As F^D continues to increase, this now has an affect on q_α, q_β and thus on N_D, N_α which is indeterminate. Without Proposition 10, we could not be sure that the alterations in (q_α, q_β) did not result in both $C^D_j < \bar{\lambda}_j$ again with the result that $z \in D$ once more. With (14), however, increases in F^D must leave $z \in M_\alpha$ until the marginal costs of production of both goods for a diversified firm reach the $\bar{\lambda}_j$ levels, when $F^D = \tilde{F}$, and Proposition 8 then applies.

VI. PREDICTIONS

This Section explores the predictions of the model presented in Section V which are additional to those inherent in the classification results. Throughout it is assumed that (12) and (14) hold.

There are three basic predictions about the character of equilibrium. First, as explained above, the number of types of firms will not exceed the number of types of goods; again this is a type of spanning result familiar from linear programming. Efficiency places severe constraints on the types of firms which can arise and be viable in a general equilibrium.

At the same time though, the model predicts that under various circumstances, firms specialized in the production of one good will exist alongside diversified firms producing that good and another; $z \in M_\alpha$ is an example. The behavior, known more generally as polymorphism, arises when the pattern of aggregate production is particularly intensive in one good: $Q_\alpha/Q_\beta \leq \underline{Q}_\alpha$ or $Q_\alpha/Q_\beta \geq \bar{Q}_\alpha$. In the $z \in M_\alpha$ case, diversified firms produce all of good β, and good α such that $q_\alpha/q_\beta = \bar{Q}_\alpha$. $Q_\alpha - \bar{Q}_\alpha$ is produced in aggregate by firms specializing in production of good α. Polymorphism per se is of interest because it is usually only generated where a subset of firms has some advantage (e.g. superior location). In the present case, all have access to the same opportunities.

Closely related is the prediction that when the equilibrium is polymorphic $z \in M_j$), alterations in the structure of aggregate production will have very different effects depending on the manner in which aggregate production changes. In particular, an increase in aggregate production of the good which is produced by specialized firms yields entry of more specialized firms and no adjustment whatsoever involving diversified firms. However, an increase in production of the other good leaves unchanged the q_α/q_β ratio produced by diversified firms, but raises their number at the expense of some of the specialized firms. In a sense, specialized firms form the industry "fringe".

The spanning result, polymorphism and the fringe behavior of specialized forms are the theory's coarse and unqualified predictions on the efficient structure of production.

Turning to the effects of parameter changes, first consider F^D and the F^j. As is obvious from Figure 3, sufficiently low F^D always results in $z \in D$; raising F^D eventually generates $z \in M_j$ for some j, with still greater F^D implying $z \in S$.

To consider the impact of an increase in one of the F^j, focus on raising F^α. Recall that $\overline{Q}_\alpha(F^D)$ is the value of Q_α for which the equilibrium configuration switches from $z \in D$ to $z \in M_\alpha$, if $N_\beta = 0$ is assumed.[5] Indeed $\overline{Q}_\alpha$ is found as part of the solution to

$$F^D + C^D(q_\alpha, q_\beta) - \lambda_\alpha q_\alpha - \lambda_\beta q_\beta = 0, \tag{15}$$

$$\overline{C}_\alpha - \lambda_\alpha \bar{q}_\alpha = 0, \tag{16}$$

$$C_1^D(q_\alpha, q_\beta) - \lambda_\alpha = 0, \tag{17}$$

$$C_2^D(q_\alpha, q_\beta) - \lambda_\beta = 0, \tag{18}$$

$$N_D = 1/q_\beta, \tag{19}$$

and $\bar{Q}_\alpha = N_D q_\alpha.$ (20)

That is, $\bar{Q}_\alpha$ is aggregate production of good α such that any increase in Q_α will be met by firms specialized in production of good α. Now (16) defines $\bar{\lambda}_\alpha$ (recall (12)), and (17) can be solved as $q_\alpha = \phi(\bar{\lambda}_\alpha, q_\beta)$. Using these facts and (19),

$$\bar{Q}_\alpha = \phi/q_\beta \quad (21)$$

where q_β solves

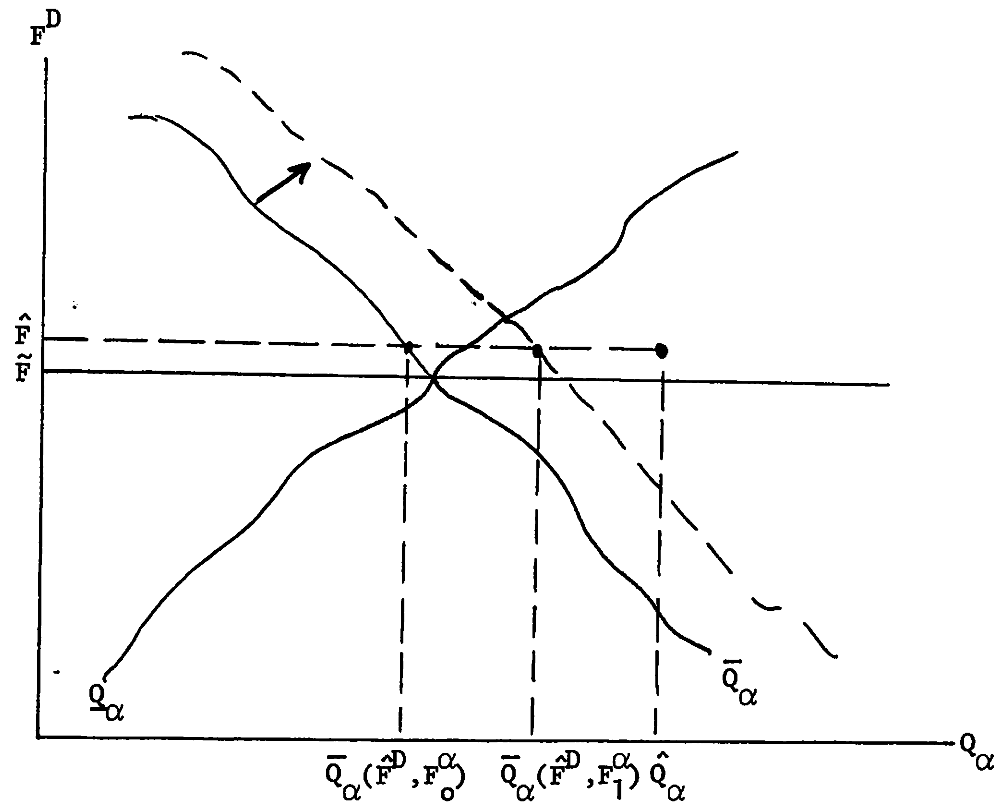
$$F^D + C^D(\phi,q_\beta) - \bar{\lambda}_\alpha\phi - C_2^D(\phi,q_\beta)q_\beta = 0. \quad (22)$$

F^α enters only through $\bar{\lambda}_\alpha$. (Indeed $d\bar{\lambda}_\alpha/dF^\alpha = 1/\bar{q}_\alpha$.) Note that since F^β does not appear here $d\bar{Q}_\alpha/dF^\beta = 0$ trivially.

Minor manipulation gives both $dq_\beta/dF^\alpha < 0$ and $d\bar{Q}_\alpha/dF^\alpha > 0$ under (14).

The logic is easy. $\bar{Q}_\alpha$ is the critical q_α/q_β ratio such that $Q_\alpha/Q_\beta > \bar{Q}_\alpha$ yields specialized firms producing good α. An increase in F^α raises the level of minimum average cost at which the latter firms can viably supply good α. Thus it would be expected that the critical q_α/q_β ratio will rise with F^α. This could only fail if a decrease in q_β raised C_1^D to such a great extent that the adjustment in q_α required to achieve $C_1^D = \bar{\lambda}_\alpha$ (at the new higher level of $\bar{\lambda}_\alpha$) is not only negative but large enough to yield q_α/q_β falling. (14) rules this out. (Convexity alone does not.) Analogous manipulations give $d\underline{Q}_\alpha/dF^\alpha = 0$ and $d\underline{Q}_\alpha/dF^\beta < 0$. Refer to Figure 4. The regions for which $z \in D$ and $z \in M_\beta$ become unambiguously larger when F^α rises; that for which $z \in S$ shrinks. The impact on region M_α is ambiguous because while some (Q_α, F^D) pairs which would have generated $z \in M_\alpha$ now yield $z \in D$, there are also (Q_α, F^D) pairs which formerly implied $z \in S$, but now yield

Figure 4



$z \in M_\alpha$. ($(\hat{Q}_\alpha, \hat{F}^D)$ in the figure is an example.) Notice that for these latter (Q_α, F^D) pairs, the rise in F^α has led to diversified firms replacing type β firms and leaving type α firms in the market. At first, this last result appears bizarre. The reasoning is as follows. Consider $(\hat{Q}_\alpha, \hat{F}^D)$. For $F^\alpha = F_o^\alpha$ (its initial value), if it were assumed $(\hat{Q}_\alpha, \hat{F}^D)$ generated $z \in M_\alpha$, diversified firms produce (in aggregate) $\bar{Q}_\alpha(\hat{F}^D, F_o^\alpha)$ of good α with $\hat{Q}_\alpha - \bar{Q}_\alpha(\hat{F}^D, F_o^\alpha)$ being produced by specialized firms of type α. Now focus on the production of $\bar{Q}_\alpha(\hat{F}^D, F_o^\alpha)$ by diversified firms. The point $(\bar{Q}_\alpha(\hat{F}^D, F_o^\alpha), \hat{F}^D)$ is such that $z \in M_\beta$; that is, the q_α/q_β ratio is so low that introduction of some type β firms would allow their portion of aggregate output to be produced at lower aggregate cost. Given the spanning result, this means that $\hat{Q}_\alpha$ is most cheaply produced by pure specialization.

In contrast, for $F^\alpha = F_1^\alpha$, the assumption that $(\hat{Q}_\alpha, \hat{F}^D)$ implies $z \in M_\alpha$, generates aggregate production by diversified firms of $\bar{Q}_\alpha(\hat{F}^D, F_1^\alpha)$. Again focusing on whether it would pay to introduce type β firms, that $\bar{Q}_\alpha(\hat{F}^D, F_1^\alpha) > \underline{Q}_\alpha$ implies the q_α/q_β ratio is such that it does <u>not</u> pay to do so. $(\hat{Q}_\alpha, \hat{F}^D)$ does not yield $z \in S$ for $F^\alpha = F_1^\alpha$.

Further insight can be obtained by imposing more structure on the relationship among F^D and the F^j. Suppose the nonproduction costs are comprised of the cost (P) of a factor which operates as a pure public input, as well as a product-specific nonproduction cost K^j. Then

$$F^j = P + K^j \tag{23}$$

and

$$F^D = P + \Sigma_j K^j. \tag{24}$$[6]

Consider a change in K^α, and focus on the induced changes in the loci $\underline{Q}_\alpha$ and $\bar{Q}_\alpha$. Raising K^α involves a movement along $\underline{Q}_\alpha$ as the only change relevant to the experiment which generates $\underline{Q}_\alpha$ is a variation in F^D. Regarding $\bar{Q}_\alpha$, raising K^α induces a shift to the right (as F^j rises) as well as a movement along $\bar{Q}_\alpha$. Given this, the consequences of changing K^α and K^β independently are straightforward though somewhat tedious. The interesting result comes from raising P, which can be analyzed as a simultaneous and equal increase in K^α and K^β. Analysis very similar to that required above to obtain the influence of an increase in F^j yields:

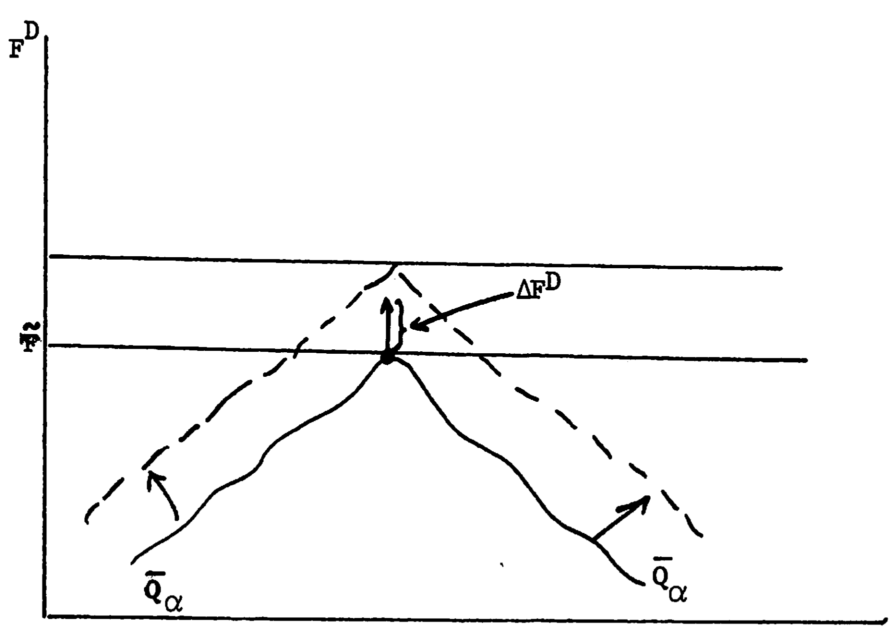
<u>PROPOSITION 11</u> Assume $C_1^D(q_\alpha, q_\beta)$ and $C_2^D(q_\alpha, q_\beta)$ are converse. Then for all (Q_α, F^D) pairs such that $D \sim M_j$;[7] an increase in $P \Rightarrow D \succ M_j$.

Thus, according to Proposition 11, when C_1^D and C_2^D are convex, raising the cost of the public good portion of nonproduction inputs implies that pure diversification becomes cheaper than partial specialization if the two were equally costly prior to the change.

The intuition behind the convexity part of the proposition is not obvious, but consider the following. Raising F^{α} alone augments the critical q_{α}/q_{β} above which $M_{\alpha} \succ D$. In particular, it was shown that such a change reduced q_{β} and did not reduce q_{α} enough to cause q_{α}/q_{β} to fall. When F^D rises along with F^{α}, a general expansion is required to spread the higher fixed cost. Convexity of C_1^D is merely sufficient to guarantee that this general expansion does not cause q_{β} to rise quickly enough to cause the critical q_{α}/q_{β} associated with the higher level of F^D to be below that associated with the lower level.

Referring to Figure 5, under the assumptions in Proposition 11, points previously on $\underline{Q}_{\alpha}$ and $\bar{Q}_{\alpha}$, shifted vertically by the addition to F^D,

Figure 5



lie inside the parameter configurations generating $z \in D$. In particular, since this also applies at point A in the figure, increases in P generate less diversification in the sense that there are no parameter configurations which, once large F^D has been accounted for, (i) previously generated some diversification which do not do so now, or (ii) generated pure diversification which do not do so now.

Particular interest is attached to this prediction for the following reason. A familiar argument is that increments to the "extent of the market" foster the development of factors specialized to or more specifically suited for utilization in the firms producing the goods in question. To the extent that these resources are nonproduction inputs (i.e. being used for accounting, product design, management training, etc.) the theory offered here suggests that "balanced" (i.e. Q_α/Q_β not varying greatly) increases in the extent of the market, which generate reductions in P (or both K^α and K^β, which will do equally well), increase the degree of specialization. Loosely, the division of production increases with the extent of the market.

Recent attempts to reduce government regulation in various ways also provide an example to which Proposition 11 might be applied. To the extent that reductions in regulation simply lower the fixed costs (K^j) of doing business, it clearly fosters specialization. On the other hand, if regulation requires superfluous duplication of paperwork which could equally well apply to two or more goods (in which case eliminating it yields $F^D = P + E = F^j$), its reduction generates greater diversification. Apparently the latter characterization typifies regulation in the banking industry. The argument that reduced regulation will generate "financial supermarkets" does not seem misplaced.

Next, the underlying restriction which generates diseconomies of scope in production costs is that diversified firms are required to utilize the same factor proportions in the production of both goods. This suggests the proposition that if two goods become "more similar" in terms of their input requirements, it is more likely that they will be produced in a diversified firm. Or to put it differently, the goods which are most likely to be produced in a single plant are those whose input requirements are "most similar".

While it is difficult to make these notions precise, the following is one parameterization for which a reasonable result can be shown. Suppose the technologies $L_j(q_j)$ can be represented by the production functions

$$q_\alpha = f(X_\alpha^1, X_\alpha^2)$$

and

$$q_\beta = h(aX_\beta^1, bX_\beta^2)$$

where a and b are positive constants, $f(\cdot)$ and $h(\cdot)$ are concave and twice differentiable, and for all y_1 and y_2

$$\frac{f_1(y_1,y_2)}{f_2(y_1,y_2)} > \frac{h_1(y_1,y_2)}{h_2(y_1,y_2)} .$$

Now consider an increase in a, and adjustment of b, such that for any given q_β, the minimum cost of producing q_β is fixed. It is easy to check that for given da > 0,

$$db = \frac{b}{a} \frac{r_1 \hat{X}^1_\beta}{r_2 \hat{X}^2_\beta} da > 0$$

where $\hat{X}_\beta = \underset{X_\beta}{\operatorname{argmin}}\{rX_\beta | X_\beta \in L_\beta(q_\beta)\}$. A given da thus rotates the entire isoquant map for good β clockwise, but leaves unchanged the cost fuctions for both types of specialized firms for all output levels. However, because the factor proportions used by a diversified firm are always intermediate with respect to those used by specialized firms (this is both obvious and easy to show), costs rise under the restricted technology, again for all output levels. Diseconomies of scope become more severe. Given this, it is not difficult to obtain

PROPOSITION 12 $da > 0 \Rightarrow \tilde{F}$ falls.

This is, as noted, the simple result of the fact that the only effect of such a da > 0 is to increase the costs of diversified production only (at the given factor prices).

The Proposition that those goods which are similar in unrestricted factor proportions will tend to be jointly produced is the source of the examples cited in the introduction. As it turns out, golf clubs and airplane parts require very similar labor and capital inputs, as did sewing machines and bicycles. Sewing machines and bicycles are no longer produced together, for the factor similarity diminished as the goods became more sophisticated. Further, even though video games and records are both recreation goods, despite their hedonic similarity they are not efficiently produced jointly because of their underlying factor dissimilarity.

These predictions have an appealing flavor, and it is for this reason that the equal factor proportions restriction was chosen. Exact equality of factor proportions can be relaxed in various ways without altering Proposition 12. For example, equality of factor proportions only within a subset of factors, or restriction of factor proportions to a cone, can be accommodated. The results will only be materially altered if the restriction used to generate diseconomies of scope requires firms producing more than one good to use factor proportions more extreme than the unrestricted choices. In such an instance, Propositions 1-11 will not change, and Proposition 12 will be reversed. This does not appear terribly plausible, as the ACME Leisure Suit and Fiber Optics Corp. has yet to appear.

Two small points conclude this section. First, differential taxation of factors depending on their use can either exaggerate or ameliorate the diseconomies of scope in production costs used to generate the above predictions. If, for example, good α (in unrestricted production) is more X^1 intensive than good β at a common set of factor prices, a subsidy to X^2 used in the production of good α can virtually eliminate the diseconomies of scope, causing diversification where none would be viable otherwise. The degree of specialization in the economy is another avenue through which such distortions operate.

Finally, non-convexities in production are one of the great dull topics in economics. Efficiency implies they will not be observed even if they are present. However, in the present context non-convexities are of some interest. The reasoning is as follows. It is easy to show that the underlying $L_j(q_j)$ can be to some degree non-convex and still yield a convex L(q). As a consequence, diversified firms may operate factor proportions such that one (at most one in the two good case) of the technologies is non-convex. As such, overall efficiency may imply what appear to be inefficiencies within the diversified firm. This provides an alternative, and efficiency-based, view of the "inefficiency observations" underlying the literature on X-efficiency.

VII. SUMMARY AND EXTENSIONS

This paper has addressed itself to the problem of determining the manner in which production is divided among producers in a competitive economy. Its basic premises can be summarized as economies of scope in nonproduction activities, and diseconomies of scope in production activities. The former arise as a result of the public good attributes of planning, accounting, and management, etc.; the latter is due to the compromises which are made in order to produce more than one good in a single plant. The particular compromise utilized in this analysis was an exaggeration of the notion that in order to produce more than one good in a single plant, factor proportions intermediate to those which would be chosen were only one good produced will be chosen by the firm. The analysis was simplified by strengthening this to the assumption that all goods produced in a single plant use identical factor proportions. Other than that, production of each good was assumed to occur independently.

The first task was to analyze the technological structure of production. It was first shown that it is possible to derive a restricted technology from the underlying independent production processes coupled with the assumption of equal factor proportions. Proposition 1 showed that this restricted multiple product production process inherited the properties of the underlying processes. It was next shown (Proposition 2) that provided the underlying technologies are such that factor proportions would be distinct across goods for all factor prices in the absence of the equal proportions restriction, that production utilizing the restricted technology always uses more resources than independent operation of the unrestricted technologies. Turning to the analysis of cost functions, the analogue to Proposition 2 is strong diseconomies of scope in production costs (Proposition 3). That is to say, the production costs associated with obtaining two goods within a single plant exceed those which follow from production in two separate plants.

Having obtained the cost structure for the multi-product plant, and knowing the cost functions for this single plant firm from the standard theory, the well-known result that a competitive economy produces any aggregate output at least aggregate cost was used to characterize competitive equilibrium. There followed a set of "classification theorems" which may be summarized as follows: For any given collection of nonproduction costs associated with single plant production, there is a critical level of non-production costs for the multi-product firm such that if nonproduction costs exceed the critical level all goods will be produced in firms specializing in the production of a single good. For nonproduction costs below the critical level, there will be some diversified firms. When aggregate production is skewed towards the production of one good, diversified firms will exist in conjunction with some specialized firms producing the good towards which aggregate production is skewed.

The theory was shown to have a rich and diverse set of predictions. Perhaps most interesting is the prediction that a change in technology which renders the unrestricted optimal factor proportions more diverse, increases the extent of specialization in the market. A surprising result is that an increase in the fixed cost for one type of specialized firm may cause the equilibrium configuration of production to change from purely specialized to partially diversified, where the specialized firm which remains in the market is the one whose cost has risen. These and other predictions were put to use explaining some applied issues.

The most interesting area to which this kind of analysis might be extended is that of the financial structure of the firm. The reasoning is basically as follows. Organization of a collection of plants under a single "managerial umbrella" presumably causes a different managerial structure than if the plants were operated independently. In a manner similar to that utilized above, diseconomies of scope in multiple plant organization are implied. On the other hand, there are obvious public good aspects to operation in financial markets. Economies of scope in the nonmanagerial component of nonproduction costs are implied. The above analysis can be applied with little modification.

Footnotes

[1]We adopt the following notation for vector inequalities: (i) $X \geq Y \Leftrightarrow X_i \geq Y_i$ for all i, and $X_i \neq Y_i$ for some i. (ii) $X \gg Y \Leftrightarrow X_i > Y_i$ for all i.

[2]These assumptions are standard, and represent a slight modification of those in Shephard (1970).

[3]$\bar{X}_j$ need not be unique.

[4]It is clear from the proof of Proposition 10 that (14) is also necessary for the result.

[5]The dependence of $\bar{Q}_\alpha$ on F^α is explicit here.

[6]It is easy to include $F^D = P + \sum_j \sigma_j K^j$, $\sigma_j > 0$.

[7]"$\sim$" means "equally costly".

APPENDIX

Proof of Proposition 1

(1) $L(q) = \{X \in R_+^m | \Sigma\sigma_j(q_j,X) \leqq 1\} = f^{-1}([0,1)$,

where $f: R_+^m \to R_+$ is defined by $f(X) = \Sigma\sigma_j(q_j,X)$.

Since $\sigma_j(q_j,X) = \min\{\lambda \geqq 0 | \lambda X \in L_j(q_j)\}$, then properties 1 and 5 imply that each σ_j, and thus $f(\cdot)$, are continuous. Hence $L(q)$, as the pre-image of $[0,1]$ under f, is closed.

(2) Let $X,\bar{X} \in L(q)$, so that $\Sigma\sigma_j(q_j,X) \leqq 1$ and $\Sigma\sigma_j(q_j,\bar{X}) \leqq 1$, and let $\sigma_j(q_j,X) = a_j$, $\sigma_j(q_j,\bar{X}) = \bar{a}_j$. Let $X^t = tX + (1-t)\bar{X}$ for some $t \in [0,1]$. We show that there exist a_j^t such that $a_j^t X^t \in L(q)$ and $\Sigma a_j^t \leqq 1$.

Define $a_j^t = 1 \Big/ \left[\frac{t}{a_j} + \frac{(1-t)}{\bar{a}_j}\right] = a_j\bar{a}_j \Big/ [t\bar{a}_j + (1-t)a_j]$.

$$\text{Then } a_j^t X^t = \frac{a_j X}{1 + \frac{(1-t)a_j}{t\bar{a}_j}} + \frac{\bar{a}_j\bar{X}}{1 + \frac{t\bar{a}_j}{(1-t)\bar{a}_j}}$$

$a_j X, \bar{a}_j\bar{X} \in L_j(q_j)$ by definition, and

$$\left(1 + \frac{(1-t)a_j}{t\bar{a}_j}\right)^{-1} + \left(1 + \frac{t\bar{a}_j}{(1-t)a_j}\right)^{-1} = \frac{t\bar{a}_j}{t\bar{a}_j + (1-t)a_j} + \frac{(1-t)a_j}{t\bar{a}_j + (1-t)a_j} = 1.$$

Further,

$$\Sigma a_j^t = \Sigma\left[\frac{t}{a_j} + \frac{(1-t)}{\bar{a}_j}\right]^{-1}$$

$$= \Sigma f\left[\frac{t}{a_j} + \frac{(1-t)}{\bar{a}_j}\right], \text{ for } f(z) \equiv 1/z$$

$\leqq \sum_j tf(1/a_j) + (1-t)f(1/\bar{a}_j)$ since $f(\cdot)$ is convex

$= t\Sigma a_j + (1-t)\Sigma\bar{a}_j \leqq 1.$ ∎

(3) $\sigma_j(0,X) = 0$, from 3. for all $X \in \Re^m_+$,

so $L(0) = \Re^m_+$.

If $q \geq 0$, and $0 \in L(q)$, then let $q_j > 0$.

Thus $\sigma_j(q_j,0) < 1 \Rightarrow \sigma_j(q_j,0) \cdot 0 = 0 \in L_j(q_j)$

contradicting 3.

(4) $X \in L(q) \Rightarrow \Sigma\sigma_j(q_j,X) \leqq 1$ and $\sigma_j(q_j,X)X \in L_j(q_j)$ for all j.

Thus, $\sigma_j(q_j,X)\bar{X} \in L_j(q_j)$ for all j by 4, and $\sigma_j(q_j,\bar{X}) \leqq \sigma_j(q_j,X)$

for all j, thus, $\Sigma\sigma_j(q_j,\bar{X}) \leqq 1$, so $\bar{X} \in L(q)$.

(5) i) Let $X > 0$. Then 5 implies that for all j,

$\bar{\lambda}_j X \in L_j(q_j)$ for <u>some</u> $\bar{\lambda}_j$. If we let

$\bar{\lambda} = \max\{\bar{\lambda}_j | j \in N\}$, $\bar{\lambda}X \in L_j(Q_j)$ for all j, and thus

$\sigma_j(n\bar{\lambda}X,q) \leqq 1/n$, so $\Sigma\sigma_j(n\bar{\lambda}x,q) \leqq 1$, and $n\bar{\lambda}X \in L(q)$.

ii) Let $X \geq 0$, and $\bar{\lambda}X \in L(\bar{q})$ with $N_o = \{j \in N | \bar{q}_j > 0\}$. Let

q be such that $q_j > 0 \Rightarrow j \in N_o$. The construction now

proceeds as in i).

(6) Let $X \in L(q)$, $q \geqq \bar{q}$. Then, since for all j, $\sigma_j(\bar{q}_j,X) \leqq \sigma_j(q_j,X)$

by 6, we have $\Sigma\sigma_j(\bar{q}_j,X) \leqq \Sigma\sigma_j(q_j,X) \leqq 1$.

(7) Let $X \in \bigcap L(q)$. Then $X \in \bigcap_{\lambda \geqq 0} L(\lambda e^1) = \bigcap_{q_1 \geqq 0} L_1(q_1)$, $e^1 = (1,0,\ldots,0) \in \Re^n$

contradicting (5) and (6). ∎

Proof of Proposition 2

(⇒) Assume $\lambda < 1$, and that the $L_j(\cdot)$ are similar wrt. (q,X). Let $\bar{r} \in \bigcap_j \xi_j(q_j,X) \neq 0$. Then $X \in L^e(q) \Rightarrow \Sigma\sigma_j(q_j,X) = 1$. But

$$\bar{r} \in \bigcap_j \xi_j(q_j,X) \Rightarrow \bar{r} \cdot \sigma_j(q_j,X)X = \eta_j(q_j,\bar{r}) = \bar{r} \cdot X; \text{ for all } j$$

$$\Rightarrow \bar{r} \cdot X = \Sigma\eta_j(q_j,\bar{r}) = \eta(q,\bar{r}).$$

But $\lambda X \in \ell^e(q)$ for $\lambda < 1 \Rightarrow \bar{r} \cdot \lambda X < \bar{r} \cdot X$

$$\Rightarrow \bar{r} \cdot X \neq \eta(q,\bar{r}).$$

(⇐) Suppose the $L_j(\cdot)$ are not similar wrt some (q,X). Let $\bar{r} \cdot \lambda X = \eta(q,\bar{r})$ (i.e., $\bar{r}$ supports $\ell^e(q)$ at $\lambda X \in \ell^e(q)$). Since the $L_j(\cdot)$ are not similar, there exist $X_j \in L_j(q_j)$ for all j such that $\Sigma X_j = \lambda X$ and $\bar{r} \cdot X_j \leqq \bar{r} \cdot [\sigma_j(q_j,\lambda X)]\lambda X$ with at least one strict inequality. Summing then, $\Sigma\bar{r} \cdot X_j = \bar{r} \cdot \lambda X < \bar{r} \cdot \lambda X \, \Sigma\sigma_j(q_j,\lambda X)$ $\Rightarrow \Sigma\sigma_j(q_j,X) > 1 \Rightarrow \lambda X \notin L(q)$. Since $X \in L^e(q)$, then $\lambda < 1$. ∎

PROPOSITION A: There exists a solution to the aggregate cost-min problem.

Proof: (i) If we impose the constraint $N_D = 0$, we know that the solution is $N_j = Q_j/\bar{q}_j$ for j=1,2, with $C^* = N_\alpha \bar{C}_\alpha + N_\beta \bar{C}_\beta \equiv C_S^*$.

(ii) If we impose the constraint $N_\alpha = N_\beta = 0$, the problem is

$$\min_{N_D} N_D[F^D + C_D(Q_\alpha/N_D, 1/N_D)]$$

$F^D > 0 \Rightarrow C^* \to \infty$ as $N_D \to \infty$, hence the optimal $N_D < \infty$. So long as we make the standard assumption that there exists some $\bar{N}_D > 0$ for which:

$$\lim_{N_D \to 0} N_D[F^D + C_D(Q_\alpha/N_D,\ 1/N_D)] > \bar{N}_D[F^D + C_D(Q_\alpha/\bar{N}_D,\ 1/\bar{N}_D)]$$

an optimal N_D^* must exist. Further,

$$\frac{\partial^2 C}{\partial N_D^2} = [Q_\alpha^2\ C_{11}^D + 2Q_\alpha C_{12}^D + C_{22}^D]/N_D^3 > 0$$

by the convexity of C, so $\bar{N}_D$ is unique.

(iii) Now set $N_\beta = 0$, so the problem is $\min_{q_\alpha, q_\beta} N^D[F^D + C^D(q_\alpha, q_\beta)] + N_\alpha \bar{C}_\alpha$.

We break this possibility down further into two cases:

(a) $N_\alpha = 0$. Then the problem is just that of (ii) again, and we have already seen that a solution of this form exists.

(b) $N_\alpha > 0$. Necessary and sufficient conditions for a solution of this form to exist are that the system below be satisfied:

$$F^D + C^D(q_\alpha, q_\beta) - \bar{\lambda}_\alpha q_\alpha - \lambda_\beta q_\beta = 0 \tag{A.1}$$

$$C_1^D(q_\alpha, q_\beta) = \bar{\lambda}_\alpha \tag{A.2}$$

$$C_2^D(q_\alpha, q_\beta) = \lambda_\beta \tag{A.3}$$

$$1 = N_D q_\beta \tag{A.4}$$

$$Q_\alpha = N_D q_\alpha + N_\alpha \bar{q}_\alpha \tag{A.5}$$

(A.2) defines a function: $q_\alpha = \phi(q_\beta)$. Using this and (A.3) allows us to reduce the first three equations to just

$$\frac{F^D + C^D(\phi(q_\beta), q_\beta) - \bar{\lambda}_\alpha \phi(q_\beta)}{q_\beta} = C_2^D(\phi(q_\beta), q_\beta) \tag{A.6}$$

Letting the LHS be $\hat{C}(q_\beta)/q_\beta$ and using (A.2), yields:

$$\frac{\partial \hat{C}(q_\beta)}{\partial q_\beta} = C_2^D(\phi(q_\beta), q_\beta)$$

Further, $\hat{C}(0) = F^D + C^D(\bar{q}_\alpha, 0) - \bar{\lambda}_\alpha \bar{q}_\alpha = F^D - F^\alpha$

Thus, so long as $F^D > F^\alpha$, since $\frac{\partial \hat{C}}{\partial q_\beta} > 0$ and $\frac{\partial^2 \hat{C}}{\partial q_\beta^2} > 0$, a solution $q_\beta^* > 0$ to (A.6) exists. This gives $q_\alpha^* = \phi(q_\beta^*)$, $\lambda_\beta^* = C_2^D(q_\alpha^*, q_\beta^*)$ and then $N_D^* = 1/q_\beta^*$, $N_\alpha^* = (Q_\alpha - N_D^* q_\alpha^*)/\bar{q}_\alpha$.

If $Q_\alpha \geqq N_D^* q_\alpha^*$ this is a candidate solution, if not, then it can be ignored.

(iv) This case with $N_\alpha = 0$ is symmetric with (iii) above.

Proposition 4 in the text implies that one of the solutions above must solve the overall problem. Thus, the overall solution must exist. ∎

<u>Proof of Proposition 5</u>

Consider a solution z^* to the cost-min problem, for given (Q_α, Q_β), yielding

$$C^*(Q_\alpha, Q_\beta, \cdot) = N_D^*(F^D + C^D(q_\alpha^*, q_\beta^*)) + N_\alpha^* \bar{C}_\alpha + N_\beta^* \bar{C}_\beta. \tag{A.7}$$

Then if we consider the new problem of producing $(\lambda Q_\alpha, \lambda Q_\beta)$ for $\lambda > 0$, the system (5)-(11) is clearly satisfied by $\hat{z}$, with $\hat{N}_j = \lambda N_j^*$, $j=D,\alpha,\beta$, and all other variables unchanged. This gives $C^*(\lambda Q_\alpha, \lambda Q_\beta, \cdot) = \lambda C^*(Q_\alpha, Q_\beta, \cdot)$ from (A.7) ∎

Proof of Proposition 6

(i) Suppose that $F^D \leqq F^\alpha$ say, and that z is such that $N_\alpha > 0$. Then (6) must be an equality. Since $Q_\beta > 0$, there are two possibilities

A. $N_D > 0$. Then (8) and (9) are equalities and (5) becomes

$$[F^\alpha + C^D(\bar{q}_\alpha,0) - \bar{\lambda}_\alpha \bar{q}_\alpha] + (F^D - F^\alpha) + C^D(q_\alpha,q_\beta) - C^D(\bar{q}_\alpha,0) - \lambda_\beta q_\beta + \bar{\lambda}_\alpha(\bar{q}_\alpha - q_\alpha)$$
$$= C^D(q_\alpha,q_\beta) - C_2^D(q_\alpha,q_\beta)q_\beta - C_1^D(q_\alpha,q_\beta)q_\beta - C_1^D(q_\alpha,q_\beta)(q_\alpha - \bar{q}_\alpha) + (F^D - F^\alpha) - C^D(\bar{q}_\alpha,0)$$

A seond-order Taylor expansion of $C^D(\bar{q}_\alpha,0)$ around (q_α,q_β) yields:

$$C^D(\bar{q}_\alpha,0) = C^D(q_\alpha,q_\beta) + C_1^D(q_\alpha,q_\beta)(\bar{q}_\alpha - q_\alpha) - C_2^D(q_\alpha,q_\beta)q_\beta + A$$

where A is a positive quadratic form, due to the convexity of $C^D(\cdot)$. Thus (5) equals $(F^D - F^\alpha) - A < 0$, and $N_\alpha > 0$ cannot be part of the solution.

B. $N_\beta > 0$. Then evaluating (5) at $(\bar{q}_\alpha,\epsilon)$ yields

$$F^D + C^D(\bar{q}_\alpha,\epsilon) - \bar{\lambda}_\alpha \bar{q}_\alpha - \bar{\lambda}_\beta \epsilon = F^\alpha + C^D(\bar{q}_\alpha,0) - \bar{\lambda}_\alpha \bar{q}_\alpha + (F^D - F^\alpha) + C^D(\bar{q}_\alpha,\epsilon) - C^D(\bar{q}_\alpha,0) - \bar{\lambda}_\beta \epsilon$$

Expanding $C^D(\bar{q}_\alpha,0)$ around $(\bar{q}_\alpha,\epsilon)$ yields:

$$C^D(\bar{q}_\alpha,0) = C^D(\bar{q}_\alpha,\epsilon) - C_2^D(\bar{q}_\alpha,\epsilon)\epsilon + B$$

with $B > 0$ again, so (5) equals

$$(F^D - F^\alpha) - B - \epsilon(\bar{\lambda}_\beta - C_2^D(\bar{q}_\alpha,\epsilon)).$$

(12) then implies that for $\epsilon > 0$ small enough, this is negative.

(ii) As noted in the text, if $N_D > 0$, then for any $q_\alpha, q_\beta > 0$,

$$N_D[F^D + C^D(q_\alpha,q_\beta)] > N_D[F^\alpha + C^D(q_\alpha,0) + F^\beta + C^D(0,q_\beta)]$$

Proof of Proposition 7

(i) Let F^D, F^α, F^β be as hypothesized and consider the system:

$$F^D + C^D(q_\alpha, q_\beta) - \bar{\lambda}_\alpha q_\alpha - \lambda_\beta q_\beta = 0$$

$$C_1^D(q_\alpha, q_\beta) = \bar{\lambda}_\alpha$$

$$C_2^D(q_\alpha, q_\beta) = \lambda_\beta .$$

The argument from Proposition A shows that if $F^D > F^\alpha$, this system has an interior solution $q_\alpha^*, q_\beta^*, \lambda_\beta^*$. Now define $N_D^* = 1/q_\beta^*$ and $\bar{Q}_\alpha = q_\alpha^*/q_\beta^*$. The resulting $z^* = (q_\alpha^*, q_\beta^*, \bar{\lambda}_\alpha, \lambda_\beta^*, N_D^*, 0, 0)$ then satisfies the FOC for the artificial cost-min problem with the constraint $N_\beta = 0$, and with $Q_\alpha = \bar{Q}_\alpha$. Strict convexity implies that these conditions are in fact sufficient, and that the solution to the above system is unique. This defines $\bar{Q}_\alpha(F^D)$.

Consider the above cost-min problem with $N_\beta = 0$, and $Q_\alpha > \bar{Q}_\alpha$. Then the FOC are satisfied by the same values of all variables, with the exception that $N_\alpha = \dfrac{Q_\alpha - \bar{Q}_\alpha}{\bar{q}_\alpha} = \dfrac{Q_\alpha - N_D^* q_\alpha^*}{\bar{q}_\alpha} > 0$. Thus, $M_\alpha \succ D$.

Suppose now that $Q_\alpha < \bar{Q}_\alpha$, and still $M_\alpha \succ D$. Then we have that $Q_\alpha = \hat{N}_D \hat{q}_\alpha + \hat{N}_\alpha \bar{q}_\alpha$ with $N_\alpha > 0$. But then as above, solving the problem for $Q_\alpha = \bar{Q}_\alpha$ yields FOC which can be satisfied by the same values, except that $\tilde{N}_\alpha = \dfrac{\bar{Q}_\alpha - N_D q_\alpha}{\bar{q}_\alpha} > \hat{N}_\alpha$, which contradicts our definition of $\bar{Q}_\alpha$.

(ii) The proof is symmetric, using the artificial cost-min problem with the constraint $N_\alpha = 0$. ∎

Proof of Proposition 8

Suppose that $F^D = F^\alpha + F^\beta$. Then diseconomies of scope in production imply that

$$F^\alpha + C^D(q_\alpha,0) - \bar{\lambda}_\alpha q_\alpha + F^\beta + C^D(0,q_\beta) - \lambda_\beta q_\beta < F^D + C^D(q_\alpha,q_\beta) - \bar{\lambda}_\alpha q_\alpha - \lambda_\beta q_\beta.$$

Consider now $\bar{Q}_\alpha(F^\alpha + F^\beta)$. This is defined by the system laid out in the proof of Proposition 7 above, in which the RHS of the inequality above is equal to 0.

The definition of $\bar{\lambda}_\alpha$ implies that:

$$F^\alpha + C^D(q_\alpha,0) - \bar{\lambda}_\alpha q_\alpha \geqq 0,$$

so that

$$F^\beta + C^D(0,q_\beta) - \lambda_\beta q_\beta < 0 \leqq F^\beta + C^D(0,q_\beta) - \bar{\lambda}_\beta q_\beta.$$

Therefore, $\lambda_\beta > \bar{\lambda}_\beta$ when $Q_\alpha = \bar{Q}_\alpha(F^\alpha + F^\beta)$.

On the other hand, as $F^D \to F^\alpha$, it is straightforward to show that in the solution to the above-mentioned system, $q_\beta \to 0$. Then, since $C_1^D(q_\alpha,q_\beta) = \bar{\lambda}_\alpha$, we must have $q_\alpha \to \bar{q}_\alpha$, and thus,

$$\lambda_\beta = C_2^D(q_\alpha,q_\beta) \to C_2^D(\bar{q}_\alpha,0) < C_2^D(0,\bar{q}_\beta) = \bar{\lambda}_\beta$$

from (12).

Thus, there exists some $F^D \in\,]F^\alpha, F^\alpha + F^\beta[$ such that $\lambda_\beta = \bar{\lambda}_\beta$ in the above system.

A symmetric argument then yields the existence of some $F^D \in\,]F^\beta, F^\alpha + F^\beta[$ such that

$$F^D + C^D(q_\alpha,q_\beta) - \bar{\lambda}_\alpha q_\beta - \bar{\lambda}_\alpha q_\alpha = 0$$
$$C_1^D(q_\alpha,q_\beta) = \bar{\lambda}_\alpha$$
$$C_2^D(q_\alpha,q_\beta) = \bar{\lambda}_\beta$$
$$N_D = 1/q_\beta$$
$$Q_\alpha = q_\alpha / q_\beta.$$

Clearly, the $(F^D,\overline{Q}^{\alpha})$ pairs which solve these two systems can differ iff the (q_α,q_β) pairs do. But (q_α,q_β) are determined by the same two equations in each system, and thus we have the existence of a $\tilde{Q}_\alpha,\tilde{F}\in\,]\overline{F},F^\alpha+F^\beta[$ satisfying both systems with

$$\overline{Q}_\alpha(\tilde{F}) = \underline{Q}_\alpha(\tilde{F}) = \tilde{Q}_\alpha.$$

We now show that $z\in S \Leftrightarrow F^D \geqq \tilde{F}$.

Let $(F^D,Q_\alpha) = (\tilde{F},\tilde{Q}_\alpha)$ and consider the aggregate cost-min problem. We will show that its solution yields the same z^* as in the two sub-problems used to define $\overline{Q}_\alpha(\tilde{F})$ and $\underline{Q}_\alpha(\tilde{F})$.

The problem defining $\overline{Q}_\alpha(\cdot)$ had the general form,

$$\min C(z) \quad \text{s.t. } g(z) = 0 \text{ and } N_\beta = 0,$$

and that defining $\underline{Q}_\alpha(\cdot)$ had the form,

$$\min C(z) \quad \text{s.t. } g(z) = 0 \text{ and } N_\alpha = 0$$

with the solution to both being the same z^*. Suppose now that in the actual cost-min problem:

$$\min C(z) \quad \text{s.t. } g(z) = 0$$

a different solution $\hat{z}$ were to emerge, with $C(\hat{z}) < C(z^*)$. Then it must be that both $\hat{N}_\alpha > 0$ and $\hat{N}_\beta > 0$, or else $\hat{z}$ would have been the solution to the first two problems, also.

However, this implies that in $\hat{z}$, we have $C_1^D(\hat{q}_\alpha,\hat{q}_\beta) = \overline{\lambda}_\alpha$ and $C_2^D(\hat{q}_\alpha,\hat{q}_\beta) = \overline{\lambda}_\beta$, which in turn implies $(\hat{q}_\alpha,\hat{q}_\beta) = (q_\alpha^*,q_\beta^*)$. Thus, if $\hat{z} \neq z^*$, it can only be because $\hat{N}_D q_\alpha^* + \hat{N}_\alpha\bar{q}_\alpha = \overline{Q}^\alpha = N_D^* q_\alpha^*$ and $\hat{N}_D q_\beta^* + \hat{N}_\beta\bar{q}_\beta = 1 = N_D^* q_\beta^*$.

Noting that $F^D + C^D(q_\alpha, q_\beta) \equiv \bar{C}$ is the same in both solutions, this yields:

$$C(z) = \hat{N}_D\bar{C} + (\overset{*}{N}_D\overset{*}{q}_\alpha - \hat{N}_D\overset{*}{q}_\alpha)\bar{C}_\alpha/\bar{q}_\alpha + (\overset{*}{N}_D\overset{*}{q}_\beta - \hat{N}_D\overset{*}{q}_\beta)\bar{C}_\beta/q_\beta$$

$$= \hat{N}_D[\bar{C} - \overset{*}{q}_\alpha\bar{\lambda}_\alpha - \overset{*}{q}_\alpha\bar{\lambda}_\beta] + \overset{*}{N}_D(\overset{*}{q}_\alpha\bar{\lambda}_\alpha + \overset{*}{q}_\beta\bar{\lambda}_\beta)$$

$$= \overset{*}{N}_D\bar{C} = C(z^*)$$

Thus, z^* solves the actual cost-min problem. Further, if one solves the problem with the constraint $N_D = 0$, the above shows that the result, $\bar{z}$, is such that $C(\bar{z}) = C(z^*)$, also. Thus, the solution can be characterized by the system:

$$\tilde{F} + C^D(q_\alpha, q_\beta) - \bar{\lambda}_\alpha q_\alpha - \bar{\lambda}_\beta q_\beta = 0$$

$$C_1^D(q_\alpha, q_\beta) = \bar{\lambda}_\alpha$$

$$C_\beta^D(q_\alpha, q_\beta) = \bar{\lambda}_\beta$$

$$1 = N_\beta \bar{q}_\beta$$

$$\tilde{Q}_\alpha = N_\alpha \bar{q}_\alpha.$$

Clearly, changes in $\tilde{Q}_\alpha$ have no effect on anything but N_α, while increases in $\tilde{F}$ only make the first expression strictly positive, leaving $z \in S$. This proves the ($\Leftarrow$) part of the claim.

As to ($\Rightarrow$) suppose that $F^D < \tilde{F}$ and $z \in S$, for any Q_α. The resulting FOC would be the same as above, with $(\tilde{F}, \tilde{Q}_\alpha)$ replaced by (F^D, Q_α), and the = in the first condition now a $\geqq$. But again, as F^D increased to $\tilde{F}$, the only effect would be to make the first condition a strict inequality. But from above, we know that this must be an equality when $F^D = \tilde{F}$, for any Q_α. ∎

Proof of Proposition 9

(i) Let $F^D \in]\bar{F},\hat{F}^D[$ be arbitrary, and suppose $\underline{Q}_\alpha(F^D) > \bar{Q}_\alpha(F^D)$. From (13) however, we know that for some $\hat{F}^D < F^D$, we have the reverse inequality. This implies another intersection $\underline{Q}_\alpha(\dot{F}^D) = \bar{Q}_\alpha(\dot{F}^D)$ for some $\dot{F}^D \in]\hat{F}^D,F^D[$, which contradicts Proposition 8. The ($\Leftarrow$) parts of (ii), (iii) and (iv) follow from Proposition 7. The ($\Rightarrow$) parts then follow from the fact that (i) implies these possibilities are exhaustive, and that $z \notin S$.

Proof of Proposition 10

We will show that $\frac{\partial \bar{Q}_\alpha(F^D)}{\partial F^D} \leq 0$. Consider again the system defining $\bar{Q}_\alpha(F^D)$. If we make use of the fact that $C_1^D(q_\alpha,q_\beta) = \bar{\lambda}_\alpha$ for any F^D, $C_{11}^D > 0$ allows us to define $q_\alpha = \Phi(q_\beta)$. Using this, and making some obvious substitutions yields the system:

$$F^D + C^D(\phi(q_\beta),q_\beta) - \bar{\lambda}_\alpha \phi(q_\beta) - C_2^D(\phi(q_\beta),q_\beta)q_\beta = 0 \tag{A.8}$$

$$\bar{Q}_\alpha = N_D \phi(q_\beta) \tag{A.9}$$

$$1 = N_D q_\beta \tag{A.10}$$

From (A.10) we have

$$0 = N_D \frac{\partial q_\beta}{\partial F^D} + q_\beta \frac{\partial N_D}{\partial F^D} \Rightarrow \frac{\partial N_D}{\partial F^D} = - \frac{N_D}{q_\beta} \frac{\partial q_\beta}{\partial F^D}$$

Then from (A.9)

$$\frac{\partial \bar{Q}_\alpha}{\partial F^D} = \phi(q_\beta) \frac{\partial N_D}{\partial F^D} + N_D \phi' \frac{\partial q_\beta}{\partial F^D}$$

$$= N_D \left(- \frac{\phi(q_\beta)}{q_\beta} + \phi'(q_\beta)\right) \frac{\partial q_\beta}{\partial F^D}$$

(A.8) then gives

$$\frac{\partial q_\beta}{\partial F^D} = - \frac{1}{C_1^D \phi' + C_2^D - \bar{\lambda}_1 \phi' - (C_{21}^D \phi' + C_{22}^D) - C_2^D}$$

$$= \frac{1}{C_{21}^D \phi' + C_{22}^D}$$

But $C_1^D(\phi(q_\beta), q_\beta) = \bar{\lambda}_\alpha \Rightarrow 0 = C_{11}^D \phi' + C_{12}^D$, so we have $\phi' = - C_{12}^D / C_{11}^D$, and

$$\frac{\partial \bar{Q}_\alpha}{\partial F^D} = - N_D \left(\frac{C_{12}^D}{C_{11}^D} + \frac{q_\alpha}{q_\beta} \right) \Bigg/ \left(- \frac{{C_{21}^D}^2}{C_{11}^D} + C_{22}^D \right)$$

$$= \frac{- N_D \left(\frac{C_{12}^D}{C_{11}^D} + \frac{q_\alpha}{q_\beta} \right)}{\frac{-(C_{21}^D)^2}{C_{11}^D} + C_{22}^D} = - \left(\frac{q_\beta C_{12}^D + q_\alpha C_{11}^D}{C_{22}^D C_{11}^D - (C_{12}^D)^2} \right)$$

since the denominator is positive, by the strict convexity of C^D, the result follows from the first inequality of (14).

A similar proof works for $\frac{\partial \bar{Q}_\alpha}{\partial F^D} > 0$. ∎

<u>Proof of Proposition 11</u>

First, recall that $\bar{Q}_\alpha(\cdot)$ (and thus the (Q_α, F^D) pairs for which $M_\alpha \sim D$) are defined by (15)-(20). Also,

$$\bar{\lambda}_\alpha = \frac{F^\alpha + C^D(\bar{q}_\alpha, 0)}{\bar{q}_\alpha} \Rightarrow \frac{\partial \bar{\lambda}_\alpha}{\partial F^\alpha} = \frac{1}{\bar{q}_\alpha} = \frac{\partial \bar{\lambda}_\alpha}{\partial P}$$

We will show that $\frac{\partial \bar{Q}_\alpha}{\partial P} > 0$.

Since ϕ is defined implicitly by $C_1^D(\phi, q_\beta) = \bar{\lambda}_\alpha$, then

$$\frac{\partial\phi}{\partial\bar{\lambda}_\alpha} = \frac{1}{C^D_{11}}, \quad \frac{\partial\phi}{\partial P} = \frac{1}{\bar{q}_\alpha C^D_{11}}$$

As before, we reduce the system to two equations, so

$$F^D + C^D - \bar{\lambda}_\alpha\phi - C^D_2 q_\beta = 0$$

$$\bar{Q}_\alpha - \phi/q_\beta = 0$$

Differentiating

$$[1 + C^D_1/\bar{q}_\alpha C^D_{11} - \phi/\bar{q}_\alpha - \bar{\lambda}_\alpha/\bar{q}_\alpha C^D_{11} - q_\beta C^D_{21}/\bar{q}_\alpha C^D_{11}]dP$$

$$+ [C^D_1\,\phi' + C^D_2 - \bar{\lambda}_\alpha\phi' - C^D_2 - q_\beta(\phi' C^D_{21} + C^D_{22})]dq_\beta = 0$$

$$d\bar{Q}_\alpha - \frac{dP}{q_\beta\bar{q}_\alpha C^D_{11}} - \left(\frac{q_\beta\phi' - \phi}{q_\beta^2}\right)dq_\beta = 0$$

so:

$$\frac{\partial q_\beta}{\partial P} = \frac{1 - \phi/\bar{q}_\alpha - q_\beta C^D_{21}/C^D_{11}\bar{q}_\alpha}{q_\beta(C^D_{22}C^D_{11} - (C^D_{12})^2/C^D_{11}}$$

$$\frac{d\bar{Q}_\alpha}{dP} = \frac{1}{q_\beta\bar{q}_\alpha C^D_{11}} - \frac{(q_\alpha + q_\beta\frac{C^D_{12}}{C^D_{11}})}{q_\beta^2}\left(\frac{1 - \phi/\bar{q}_\alpha\bar{q}_\beta C^D_{21}/C^D_{11}\;\bar{q}_\alpha}{q_\beta(C^D_{22}C^D_{11} - (C^D_{12})^2)/C^D_{11}}\right)$$

$$= \frac{1}{\bar{q}_\alpha q_\beta C^D_{11}} - \frac{(q_\alpha C^D_{11} + q_\beta C^D_{12})}{q_\beta^3}\left(\frac{1 - \phi/\bar{q}_\alpha - q_\beta C^D_{12}/C^D_{11}\bar{q}_\alpha}{C^D_{22}C^D_{11} - (C^D_{12})^2}\right)$$

Clearly, given the convexity of C^D and (14), a sufficient condition for this to be positive is that

$$1 - \phi/\bar{q}_\alpha - q_\beta C^D_{12}/C^D_{11}\ \bar{q}_\alpha < 0$$

$$\Leftrightarrow \bar{q}_\alpha C^D_{11} - q_\alpha C^D_{11} - q_\beta C^D_{12} < 0.$$

A second-order Taylor expansion gives

$$C^D_1(\bar{q}_\alpha,0) = C^D_1(q_\alpha,q_\beta) + C^D_{11}(q_\alpha,q_\beta)(\bar{q}_\alpha - q_\alpha)$$

$$- q_\beta C^D_{12}(q_\alpha,q_\beta) + A$$

where $A > 0$, if C^D_1 is convex. Since $C^D_1(\bar{q}_\alpha,0) = \bar{\lambda}_\alpha = C^D_1(q_\alpha,q_\beta)$, we have

$$C^D_{11}(\bar{q}_\alpha - q_\alpha) - C^D_{12}q_\beta = -A < 0.$$

The proof for $\underline{Q}_\alpha$ is symmetric. ∎

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