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1977

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Citation of this paper:

Robson, Arthur J.. "Rawls and Uncertainty." Department of Economics Research Reports, 7710. London, ON: Department of Economics, University of Western Ontario (1977).

RESEARCH REPORT 7710

RAWLS AND UNCERTAINTY

by

Arthur J. Robson

Revised September, 1977

1. Introduction

This note derives the appropriate generalization of the Rawlsian welfare function to cases involving uncertainty. Under axioms due to Samuelson [5], it is shown that the attitudes toward risk and inequality can be essentially independent. However, if a plausible postulate due to Harsanyi [2] is additionally imposed, "infinite risk aversion" is implied. This dramatizes the role of this postulate in linking the attitudes towards risk and inequality.

2. Formulation of the Problem

Consider a gamble involving a finite number of distributions of income to the N members of society. Such "simple gamble" may be written

$$\tilde{c} = \{(\{c_i^1\}, p_1), \ldots, (\{c_i^n\}, p_n); \quad i = 1, \ldots, N\}$$

where the distribution of income

$$\{c_i^j\} = (c_1^j, \ldots, c_N^j)$$

obtains with probability $p_i > 0$, and

$$\sum_{i=1}^{n} p_i = 1$$

(A more general formulation would involve a general probability distribution. This would necessitate more sophisticated analysis but does not add essential content.) It will also be convenient to define

$$\tilde{c}_i = \{(c_i^1, p_1), \ldots, (c_i^n, p_n)\} \quad i = 1, \ldots, N$$

which is the probability distribution of income from the point of view of the ith individual.

A "compound gamble" is a gamble where the outcomes are, in turn,

gambles. Such a gamble will be written

$$\tilde{b} = \{(\tilde{c}^1, q_1), \ldots, (\tilde{c}^m, q_m)\}$$

and can be reduced to an "associated simple gamble" with certain outcomes.

In fact, "compound gambles" will be taken simply to be abbreviations for

the "associated simple gamble" (see Samuelson [5]).

Denote then the set of all "simple gambles" involving a finite number

of income distributions as $\mathfrak{A}$. It is desired to find a preference ordering,

$\prec$, defined on $\mathfrak{A}$ in such a way that

$$\{(\{c_i\}, 1)\} \prec \{(\{d_i\}, 1)\}$$

if and only if

$$\min_i c_i \leq \min_i d_i$$

that is, such that the preference ordering reduces to the Rawlsian case under

certainty.

The axioms needed for "consistency" of the ordering are from Samuelson

[5].

<u>Axiom 1</u> (Completeness) The preference ordering $\prec$ is a <u>complete</u> ordering of

$\mathfrak{A}$, which is <u>continuous</u> in the probabilities.

<u>Axiom 2</u> (Independence) <u>If</u>

$$\tilde{c}^j \sim \tilde{d}^j \qquad \text{where}$$

$$\tilde{c}^j, \tilde{d}^j \in \mathfrak{A}, \quad j = 1, \ldots, m$$

(and $\tilde{c} \sim \tilde{d}$ means $\tilde{c} \prec \tilde{d}$ and $\tilde{d} \prec \tilde{c}$)

<u>then</u>

$$\{(\tilde{c}^1, q_1), \ldots, (\tilde{c}^m, q_m)\} \sim \{(\tilde{d}^1, q_1), \ldots, (\tilde{d}^m, q_m)\}$$

where $q_i > 0$, $\sum_1^m q_i = 1$ are any probabilities generated independently of the outcomes of the $\tilde{c}^j$ and $\tilde{d}^j$.

3. Generalization of the Rawlsian Criterion

Consider now two arbitrary members of $\mathfrak{A}$,

$$\tilde{c} = \{(\{c_i^1\}, p_1), \ldots, (\{c_i^n\}, p_n)\}$$

and

$$\tilde{d} = \{(\{d_i^1\}, q_1), \ldots, (\{d_i^m\}, q_m)\}$$

Since a Rawlsian criterion holds under certainty,

$$(\{c_i^j\}, 1) \sim (\{\min_i c_i^j\}, 1)$$

and

$$(\{d_i^j\}, 1) \sim (\{\min_i d_i^j\}, 1)$$

That is, there is indifference between any distribution of income and a distribution assigning everyone the minimum income of the first distribution.[1] Hence by Axioms 1 and 2,

$$\tilde{c} \succcurlyeq \tilde{d} \quad \text{if and only if} \quad \underset{\sim}{\tilde{c}} \succcurlyeq \underset{\sim}{\tilde{d}} \quad \text{where}$$

$$\underset{\sim}{\tilde{c}} = \{(\{\min_i c_i^1\}, p_1), \ldots, (\{\min_i c_i^n\}, p_n)\}$$

and

$$\underset{\sim}{\tilde{d}} = \{(\{\min_i d_i^1\}, q_1), \ldots, (\{\min_i d_i^m\}, q_m)\}$$

Thus the comparison between $\tilde{c}$ and $\tilde{d}$ is equivalent to the comparison between gambles involving only __equal__ distributions of income, $\underset{\sim}{\tilde{c}}$ and $\underset{\sim}{\tilde{d}}$, respectively.

Denote the subset of $\mathfrak{A}$ which contains all such gambles as C. Given Axioms 1 and 2, the expected utility theorem holds on C. (See Samuelson [5].) Hence $\underset{\sim}{c} \succ \underset{\sim}{d}$ if and only if

$$\sum_{j=1}^{n} p_j \, W(\min_i c_i^j) \geq \sum_{j=1}^{m} q_j \, W(\min_i d_i^j)$$

for some welfare function $W(\cdot)$ expressing an arbitrary attitude towards risk _per se_. In other words,

Theorem 1. For any two gambles in $\mathfrak{A}$, $\tilde{c}$ and $\tilde{d}$ say, $\tilde{c} \succ \tilde{d}$ if and only if

$$E \min_i W(\tilde{c}_i) \geq E \min_i W(\tilde{d}_i).$$

Hence this is the desired (unique) extension of the Rawlsian criterion to include uncertainty.

Note 1: This rules out another apparent generalization,

$$\min_i EW(\tilde{c}_i)$$

which is, in fact, inconsistent with Axiom 2, as the following example shows. Take two people, and,

$$c_i^1 = 0 \quad i = 1$$
$$ = 1 \quad i = 2$$

and $$c_i^2 = 0 \quad i = 2$$
$$ = 1 \quad i = 1$$

Axiom 2 implies

$$\{(\{c_i^1\}, \tfrac{1}{2}), \; (\{c_i^2\}, \tfrac{1}{2})\} \sim \{(\{0\}, 1)\}$$

However,

$$\min_i E \, W(\tilde{c}_i) = \tfrac{1}{2}(W(0) + W(1)) > W(0),$$

which is a contradiction.

4. An "Individualistic" Postulate

Suppose now that, if two gambles are indifferent from the point of view of every individual, then they are indifferent from the social point of view. (This is "Postulate c" of Harsanyi [2].) In other words,

<u>Postulate 1</u>   If $\tilde{c}_i$ and $\tilde{d}_i$ involve the same outcomes with the same probabilities for $i = 1,\ldots,N$, then $\tilde{c} \sim \tilde{d}$.

Although this postulate has a plausible ring to it, it contradicts the existence of the welfare function, $W(\cdot)$, as the following simple example shows. Take

$$c_i^1 = 0 \quad i = 1$$
$$ = 1 \quad i = 2$$
$$c_i^2 = 1 \quad i = 1$$
$$ = 0 \quad i = 2$$

and $p_1 = p_2 = \frac{1}{2}$. Take on the other hand

$$d_i^1 = 0 \quad i = 1,2$$
$$d_i^2 = 1 \quad i = 1,2$$

and $q_1 = q_2 = \frac{1}{2}$. Postulate 1 requires indifference between $\tilde{c}$ and $\tilde{d}$. However

$$E \min_i W(\tilde{c}_i) = W(0)$$

whereas

$$E \min_i W(\tilde{d}_i) = \frac{1}{2}(W(0) + W(1))$$

However, if the requirement of continuity in the probabilities is modified to apply only for

$$p_i > 0 \quad \text{for all } i,$$

then there is a criterion satisfying the abbreviated Axiom 1, Axiom 2, and Postulate 1.

__Theorem 2.__ Under the above conditions, $\tilde{c} \succ \tilde{d}$ if and only if $\min\limits_{i,j} c_i^j \geq \min\limits_{i,j} d_i^j$

That is, the criterion reduces to the lowest income received by anyone in any outcome.

__Proof__ By Axiom 2, as in Section 3,

$$\tilde{c} \sim \underset{\sim}{\tilde{c}}$$

and this is independent of the number of people involved. Consider then an approximation for $\underset{\sim}{\tilde{c}}$,

$$\tilde{c}(N) = \{ (\{\underset{\sim}{c}^1\}, \tfrac{1}{N}), \ldots, (\{\underset{\sim}{c}^N\}, \tfrac{1}{N}) \}$$

where, arbitrarily,

$$\underset{\sim}{c}^1 \leq \underset{\sim}{c}^2, \ldots, \leq \underset{\sim}{c}^N$$

By Postulate 1,

$$\tilde{c}(N) \sim \tilde{c}^1(N)$$

where $\tilde{c}^1(N)$ is obtained by rearranging outcomes in such a way that person i receives $\underset{\sim}{c}^1$ in outcome i, and other income assignments preserve the personal probability distributions of income. Finally, by reapplication of Axiom 2,

$$\tilde{c}(N) \sim \{ (\{\min\limits_{i,j} c_i^j\}, 1) \}$$

and similarly,

$$\tilde{d}(N) \sim \{ (\{\min\limits_{i,j} d_i^j\}, 1) \}$$

By the modified assumption of continuity, then, $\tilde{c} \succ \tilde{d}$ if and only if

$$\min\limits_{i,j} c_i^j \geq \min\limits_{i,j} d_i^j .$$

Q.E.D.

Note 1: The attitude to risk here is analogous to the "minimax loss" criterion of statistics (see DeGroot [1], for example). It is continuous if

$$p_i > 0 \quad \text{for all } i,$$

being constant, in fact. It is not continuous generally when

$$p_i = 0$$

because the worst possibility might change.

Note 2: With a more general probability distribution, the criterion would become

$$\underset{i \, , \, s}{\text{ess inf}} \ c_i^s$$

where s parametrizes "states of the world" (see Rudin [4]).

FOOTNOTE

[1] A fully Rawlsian criterion would involve a lexicographic ordering. (See Rawls [3].) This is analytically intractable, but unlikely to be operationally different than the simplified criterion used here.

[2] In other words, $\underline{c}^1 = \underline{c}^2 = \ldots \underline{c}^{r_1}$, so that r_1/N approximates the probability of this outcome. A similar approximation holds for other outcomes.

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