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A CONTINUOUS-TIME MODEL OF A STOCK MARKET

VALUE MAXIMIZING FIRM

by

D. T. Scheffman

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A Continuous-Time Model of a Stock Market
Value Maximizing Firm

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I. Introduction

In recent years there has been an increasing amount of literature concerned with developing a theory of the firm's optimal investment policy in a world of uncertainty. One avenue of investigation has built directly on the neoclassical theory of investment, á la Irving Fisher. This approach (which I shall call the neoclassical literature), has concentrated on a model of a perfectly competitive firm which faces prices which are generated by Markov processes.¹ The firm is assumed to maximize expected discounted net cash flow, where the interest (discount) rate is an exogenous, deterministic function of time. Under these assumptions, various properties of the optimal investment policy are derived. The obvious weakness of this approach lies in the ad hoc assumption of expected value maximization. Thus the literature completely sidesteps the issue concerning the relationship between the firm's behavior and its shareholders' well-being.

A parallel, but largely separate literature has arisen out of finance theory. The finance literature has long argued that the appropriate criterion for a firm is to maximize its stock market value.² This seems intuitively to be more reasonable than expected discounted profits maximization. With the development of the Capital Asset Pricing Model (CAPM), it has been shown that within the framework of the CAPM, if competitive conditions are assumed to hold, stock market value maximization leads to (Pareto) optimal investment behavior,³ which strengthens the case for value maximization. Most of this finance literature has considered one-period models, and therefore properties of the optimal investment policy were of very limited interest.

In this paper we develop a simple model which attempts to synthesize the best features of the neoclassical and finance models. Consistent with the neoclassical literature, we use a disaggregated approach to the specification of the firm's net cash flow, starting with the basic technology and market prices faced by the firm. This allows us, as in the neoclassical literature, to analyze the effects of price uncertainty on the firm's investment policy and to gain insight into the appropriate specification of investment demand equations. Consistent with the finance literature, we assume that the firm's objective is to maximize its stock market value.

For mathematical convenience we will use a continuous-time model. The mathematics used is stochastic control theory, which was introduced into the economics literature by Merton [16,17], who has demonstrated its wide applicability to problems involving intertemporal uncertainty. In particular, Merton has developed a continuous time model of the stock market [18], which as he points out, is actually a model of the demand side of a general equilibrium securities market.⁴ As we will see, our model is a simple model of the supply side of this general equilibrium model. The very important but extremely difficult task of putting these two sides together is not attempted here.

II. The Model

We will assume the firm produces a single homogeneous output using homogeneous labour and homogeneous capital. These variables are related by the production function

$$Q(t) = F(K(t), L(t))$$

which is assumed to be concave. The prices of output, labour, and capital goods will be denoted $p(t)$, $w(t)$, $q(t)$, respectively. We will express these

jointly by the vector $P(t) = (p(t), w(t), q(t))$. We assume that the firm is a perfect competitor in its input and output markets, taking $P(t)$ as given. Following the neoclassical investment literature [4,12,13,26], we will assume that investment incurs adjustment costs,⁵ and for simplicity we will assume the adjustment costs are a function of gross investment. Thus the adjustment cost function will be written $v(I(t), P(t))$, where $I(t)$ is the level of gross investment at time t ($v(I(t), P(t))$ includes qI , the value of capital goods).⁶ We assume that $v_I > 0$ for $I > 0$ and $v_{II} > 0$ (i.e., that there are increasing marginal adjustment costs). For analytical convenience it is assumed that there are no adjustment costs for varying labour.

At each instant of time the firm must choose the amount of labour and the level of gross investment. We assume that the firm knows (with certainty) all current prices, but is uncertain about future prices. It is also assumed that the rate of depreciation of capital is also uncertain, so that although gross investment is controlled by the firm, net investment is uncertain. Thus the net cash flow of the firm at time t is

$$p(t)F(K(t), L(t)) - w(t)L(t) - v(I(t), P(t)) \quad (1)$$

Assuming that the level of production does not affect the amount of depreciation (a perhaps unreasonable, but common assumption), and since $P(t)$ is known at time t , under any plausible optimizing behavior, the firm will choose $L(t)$ so as to maximize (1) with respect to $L(t)$. The maximized value of (1) can be written

$$\Pi(p(t), w(t), K(t)) - v(I(t), P(t)) \quad (2)$$

where $\Pi(\)$ is the short-run profit function.⁷ As is well known, Π is a concave function of K and a convex function of (p, w) . Now, given the investment policy $I(t)$, we have a complete description of the process generating the net cash flow of the firm.

We will now turn our attention to the financial structure of the firm.

Define

$S(t)$ = price per share (ex-dividend) at time t

$d(t+h)$ = accumulated dividends per share paid during the period $(t, t+h)$

$X(t+h)$ = accumulated net cash flow over the period $(t, t+h)$

$V(t)$ = stock market value at time t (ex-dividend)

$N(t)$ = number of shares at time t

$D(t+h)$ = accumulated total dividends paid during the period $(t, t+h)$

We assume the firm's objective (at time t) is to maximize $V(t)$, i.e., to maximize the wealth of its current shareholders. We assume that the firm is a perfect competitor in the capital markets. There would seem to be some disagreement in the literature as to what this assumption implies. In papers by Jensen and Long [7] and Stiglitz [25], it was claimed that in the context of the CAPM, "competitive" value maximizing firms would choose non-(Pareto) optimal investment policies. However, Merton and Subrahmanyam [19] later showed that this result was in fact due to non-competitive behavior of firms in the capital markets. The essence of the Jensen and Long, and Stiglitz results was that their firms assumed they could affect the aggregate amount of investment in a given project, and thus they faced a non-constant "cost-of-capital". This is analogous to a "perfect competitor" facing a downward sloping demand curve. Merton and Subrahmanyam argued therefore that a correct interpretation of perfect competition in the capital markets is that a firm's investment projects can be undertaken by several other existing and potential firms. Thus a perfect competitor in the capital markets must be a firm whose "size" (given by the amount of investment in its projects) is "small" relative to the total investment on similar projects, and therefore, it must take its cost of capital (or "required rate of return") as given. We will take this then to be our definition of perfect competition.

It remains now to discuss how a firm determines its cost of capital.

In a CAPM world, the firm's cost of capital would be given by the CAPM equilibrium:

$$R(t;h)-1 = i + \left[\frac{\bar{r}_m(t;h) - i}{\sigma_m^2} \right] \sigma \quad (3)$$

where $R(t;h)$ the expected gross rate of return (cost of capital) on the firm between time t and $t+h$, i is the riskless rate of return, $\bar{r}_m(t,h)$ is the expected rate of return on the market, σ_m^2 is the variance of the market return, and σ is the covariance between the firm and the market. By our assumption of perfect competition, σ_m^2 , \bar{r}_m , and σ are not affected by the firm's decisions, so that $R(t;h)$ is exogenous to the firm. Notice that

$$R(t;h) = E_t \left\{ \frac{\tilde{S}(t+h) + \tilde{d}(t+h)}{S(t)} \right\} \quad (4)$$

Let $k(t)$ be the instantaneous cost of capital, i.e., $R(t;h) = 1 + k(t) \cdot h + o(h)$.

Merton [18] has shown that the CAPM equation (3) will not generally hold in a dynamic model of the stock market. However, $R(t;h)$ will be determined in such a model, but not by the simple form of (3).

Conceptually then we see a model in which a firm determines its current $R(t;h)$ from the stock market's capitalization of firms in its "industry group". Therefore we assume that $R(t;h)$ is known (or estimated) by the firm at time t , but $R(t';h)$ for $t' > t$ is uncertain. Allowing $R(t';h)$ to be uncertain is very important, since the assumption that $R(t;h)$ is determined from market evidence would be conceptually inconsistent with $R(t';h)$ being deterministic. Thus we are able to capture more of the flavor of reality in our simple model.

We will now derive the optimality conditions for the value maximizing firm. From the usual identities

$$\begin{aligned} V(t+h) &= N(t+h)S(t+h) = N(t)S(t+h) + [N(t+h)-N(t)]S(t+h) \\ V(t) &= N(t)S(t) \end{aligned} \quad (5)$$

we have

$$V(t+h) = V(t) \frac{S(t+h)}{S(t)} + \Delta N(t)S(t+h) \quad (6)$$

$V(t+h)$, $S(t+h)$, and $\Delta N(t)$ are uncertain at time t , so we can write

$$E_t \{ \tilde{V}(t+h) \} = V(t)E_t \left\{ \frac{\tilde{S}(t+h)}{S(t)} \right\} + E_t \{ \Delta \tilde{N}(t) \tilde{S}(t+h) \} \quad (7)$$

where E_t is the expectation at time t , and random variables are denoted with tildes.

From (4), we can rewrite (7) as

$$E_t \{ \tilde{V}(t+h) \} = V(t)R(t;h) + E_t \{ \Delta \tilde{N}(t) \tilde{S}(t+h) - \frac{V(t)\tilde{d}(t+h)}{S(t)} \} \quad (8)$$

where, by assumption $R(t;h)$ is given at time t .

Since $\frac{V(t)}{S(t)} = N(t)$ and $N(t)\tilde{d}(t+h) = \tilde{D}(t+h)$, (8) can be written

$$E_t \{ \tilde{V}(t+h) \} = V(t)R(t;h) + E_t \{ \Delta \tilde{N}(t) \tilde{S}(t+h) - \tilde{D}(t+h) \} \quad (9)$$

We must now consider the firm's financial policy. It could, of course, raise funds by either debt or equity financing. We will assume the Modigliani-Miller theorems hold, so that the firm's method of financing is irrelevant.⁸ Therefore, for simplicity we assume that external financing always takes the form of equity issues. Thus we have the accounting identity

$$\Delta \tilde{N}(t) \tilde{S}(t+h) + \tilde{X}(t+h) \equiv \tilde{D}(t+h) \quad (10)$$

which allows us to write (10)

$$E_t \{ \tilde{V}(t+h) \} = V(t)R(t;h) - E_t \{ \tilde{X}(t+h) \} \quad (11)$$

Finally, solving for $V(t)$, we have

$$V(t) = [R(t;h)]^{-1} E_t \{ \tilde{V}(t+h) + \tilde{X}(t+h) \} \quad (12)$$

Notice that $V(t)$ is the value of shares of the current shareholders at time t , and that the "dilution" effects of new equity issues have been captured in (5) - (7).

Now we must make some assumptions about how uncertainty in the model is generated. Initially we will assume that the uncertain prices, rate of return, and depreciation are generated by $I\delta$ processes, which are continuous-time analogues of Markov processes. (Later we will show that our basic results do not depend on the Markov assumption.)

Using the usual notation,⁹ we will write these processes:

$$di = \alpha_i(P, k; t) dt + \sqrt{\sigma_{ii}(P, k; t)} dz_i \quad i=p, w, q, k \quad (13)$$

$$dK = (I - \delta K) dt + \sqrt{\theta(K, t)} dz_K$$

The dz 's are standard normal variates, and the covariance between the processes will be denoted $\sigma_{ij}(P, k; t)$ $i, j = p, w, q, k$.

Notice that so far we have not made any behavioral assumptions about the firm, i.e., (12) is independent of the behavior of the firm. To proceed any further we must impose our assumption of stock market value maximization, i.e., we assume the firm chooses $I(t)$ so as to maximize $V(t)$. Let

$$J(K, P, k, t) = \max_{\{I(t)\}} V(t) \quad (14)$$

(given $K(t) = K, P(t) = P, k(t) = k$, and (13)).

Then by the Principle of Optimality we can write (12):

$$J(K, P, k, t) = \max_{\{I\}} [R(t; h)]^{-1} E_t \{ \tilde{J}(K(t+h), P(t+h), k(t+h), t+h) + \tilde{X}(t+h) \} \quad (15)$$

Expanding $R(t; h)$ in a Taylor's Series,

$$R(t; h) = 1 + k(t)h + o(h) \quad (16)$$

so that

$$[R(t; h)]^{-1} = 1 - k(t)h + o(h) \quad (17)$$

Also,

$$\tilde{X}(t+h) = X(t)h + \tilde{o}(h) \quad (18)$$

Substituting (17) and (18) in (15), rearranging terms and dividing by h ,

$$0 = \max_{\{I\}} E_t \left[\frac{(1-k(t)h+o(h))\tilde{J}(t+h)-J(t)}{h} + \frac{(1-k(t)h+d(h))(X(t)h+o(h))}{h} \right] \quad (19)$$

Following Merton's notation, let

$$\begin{aligned} \mathcal{L}[J] = & J_K(I-\delta K) + \sum_{i=p,w,q,k} J_i \alpha_i + J_t + J_{KK} \theta(K) \\ & + \frac{1}{2} \sum_i \sum_j J_{ij} \sigma_{ij} \end{aligned} \quad (20)$$

From now on we will write $J(K,P,k,t)$ as $J(t)$.

By the differentiability of the processes¹⁰

$$\lim_{h \rightarrow 0} E_t \left\{ \frac{\tilde{J}(t+h) - J(t)}{h} \right\} = \mathcal{L}[J] \quad (21)$$

Taking the limit of (19) as $h \rightarrow 0$, using (21), we have the Bellman equation

$$0 = \max_{\{I\}} [-k(t)J(t) + \mathcal{L}[J] + X(t)] \quad (22)$$

Finally, since $X(t) = \Pi(p,w,K) - v(I;P)$, we can write (22) as

$$0 = \max_{\{I\}} [-k(t)J(t) + \mathcal{L}[J] + \Pi(p,w,K) - v(I;P)] \quad (23)$$

This is the fundamental equation, in implicit form, which generates the maximized stock market value, $J(t)$, and the optimal investment policy $I(t)$.

The power of the assumptions of value maximization and perfect competition in the capital markets is remarkable, since (23) was derived only from these assumptions and manipulation of accounting identities.

Notice that the first-order conditions for the required maximization with respect to I are

$$J_K - V_I(I^*, P) = 0 \quad (24)$$

which has an obvious interpretation. Solving (24) for $I^*(J_K, P)$ and substituting into (23) gives us a deterministic partial differential equation generating $J(K, P, k, t)$.

III. Relationship to the Neoclassical Literature

In this section we will establish the relationship between the model of section II and the neoclassical literature. Consistent with this literature we will consider a model of an expected discounted net-cash-flow maximizing firm. In an important departure from this literature we will assume that the discount rate, $r(t)$, used by the firm is random. This would be consistent, for example, with the discount rate being an (uncertain) interest rate.

The processes generating P and K are as assumed in (13). The process generating $r(t)$ will be written

$$dr = \alpha_r(r, t)dt + \sqrt{\sigma_{rr}(r, t)} dz_r \quad (25)$$

The firm's maximization problem can be written

$$\max_{\{I(t)\}} E_0 \left\{ \int_0^{\infty} e^{-\int_0^t r(s)ds} [\Pi(p, w, K) - v(I; P)] dt \right\} \quad (26)$$

Assuming a solution exists, define

$$S(K, P, r, t) = \max_{\{I(s)\}} E_t \left\{ \int_t^{\infty} e^{-\int_t^s r(x)dx} [\Pi(p, w, K) - v(I, P)] ds \right\} \quad (27)$$

i.e., $S(K, P, r, t)$ is the maximum expected-discounted profits of the firm starting at time t with $K(t) = K$, etc.

Now consider a small time interval, h .

$$S(K, P, r, t) = \max_{\{I(s)\}} E_t \left\{ \int_t^{t+h} e^{-\int_t^s r(x) dx} [\Pi(p, w, K) - v(I, P)] ds \right. \\ \left. + \int_{t+h}^{\infty} e^{-\int_t^s r(d) dx} [\Pi(p, w, K) - v(I, P)] ds \right\} \quad (28)$$

Now

$$E_t \left\{ \int_{t+h}^{\infty} e^{-\int_t^s r(x) dx} [\Pi(p, w, K) - v(I, P)] ds \right\} =$$

$$E_t \left\{ e^{-\int_t^{t+h} r(y) dy} E_{t+h} \left\{ \int_{t+h}^{\infty} e^{-\int_{t+h}^s r(s) ds} [\Pi - v] ds \right\} \right\} \quad (29)$$

$$\text{Let } E_t \left\{ e^{-\int_t^{t+h} r(x) dx} \right\} = [\mathfrak{M}(t, h)]^{-1}$$

Then, by the Principle of Optimality, using (29) we can write (28)

$$S(t) = \max_{\{I\}} E_t \left\{ \int_t^{t+h} e^{-\int_t^s r(x) dx} [\Pi - v] ds + [\mathfrak{M}(t, h)]^{-1} S(t+h) ds \right\} \quad (30)$$

Proceeding in exactly the same manner as our analysis in equations (15-23), we can write the final Bellman equation

$$0 = \max_{\{I\}} \{-r(t)S(t) + \mathfrak{L}[S] + \Pi(p, w, K) - v(I; P)\} \quad (31)$$

Notice that if $k(t) = r(t)$, then equations (25) and (31) are identical. Thus maximized stock market value is equivalent to maximized expected discounted net-cash-flow (with a discount rate equal to the "required return," $k(t)$), and so

the optimal investment policies are identical. This would also be true of course, even for a more general version of our value maximizing model, in which the process generating $k(t)$ is affected by the firm's actions. The neoclassical "dual" of that model would be a firm which had monopoly power in its capital markets. The importance of this demonstrated duality between value maximizing and expected value maximizing models should not be underestimated. We have shown that the results derived from models which make the seemingly unsatisfactory ad hoc assumption of expected value maximization are of more general interest because they can be rationalized within a model of value maximization. However our task is not finished at this point since the neoclassical literature has generally assumed that the discount (interest) rate used to discount the expected value is assumed to be an exogenous deterministic function of time. This makes the results derived in the neoclassical literature of less interest, since assuming the cost of capital is deterministic is quite unreasonable. Therefore we will proceed to analyze the properties of the optimal investment policy. As we will see, the elegance and simplicity of continuous time techniques allows us to considerably expand on the results derived in the neoclassical literature.

IV. Properties of the Optimal Investment Policy

We can analyze the properties of the optimal investment policy by analyzing (23). The first order conditions for the required maximization with respect to I in (23) are:

$$0 = J_{K} - v_{I}(I^*;P) \quad (32)$$

By assumption, $v_{II} > 0$, so that the second order conditions for the instantaneous maximization with respect to I hold. Since $v_{II} > 0$, (32) can be solved for I, giving us

$$I^* = f(J_{K};P) \quad (33)$$

Substituting (33) into (23), we have the partial differential equation in explicit form

$$0 = -k(t)J(t) + \dot{J}[J] + \Pi(p,w,K) - v(f(J_{K};P),P) \quad (34)$$

The rest of the second order conditions for the problem require:

$$a) J_{KK} \leq 0$$

$$b) \lim_{t \rightarrow \infty} E_t \left\{ e^{-\int_0^t r(s) ds} J(t) \right\} = 0 \quad (35)$$

which is the transversality condition.

Condition (a) follows directly from the assumption of concavity of the production function and convexity of the adjustment costs function. Condition (b) requires conditions on the stochastic processes (13). We will discuss this in more detail as it applies to some examples we will present later. In the following we will assume a solution exists.

The first obvious property of the optimal investment policy is

$$\partial I^* / \partial K \leq 0 \quad (34)$$

which follows directly from (32) and (35)a). Likewise the effects of changes in prices on the current level of investment will be determined by the signs of J_{Ki} , $i=p, w, q, k$. Notice that it will not generally be true, for example, that $J_{Kp} > 0$ (or even $J_p > 0$) so that the effect on current investment of an increase in output price is not predictable without more assumptions on the stochastic processes (13). We will return to this point when we consider the special case of constant-returns-to-scale technology. We also cannot say anything in general about the effects of increasing uncertainty, however defined, on investment.

However, consider the following comparative dynamics experiment. Suppose the instantaneous expected change in p changes at each time t from $\alpha_p(P, k; t)$ to $[\alpha_p(P, k; t) + \epsilon]$. What would have to be the change in the instantaneous variance of the process generating p (σ_{pp}) in order for the value of the firm, J , to remain invariant? (If J is invariant, then by (33) the optimal investment policy is invariant also.) From (34) (and (20)) we see that if α_p is changed to $\alpha_p + \epsilon$, in order for the solution of (34) to remain invariant, σ_{pp} must change to $\sigma_{pp} + \delta_{pp}(P, K, r, t)$, where

$$\delta_{pp}(P, K, r, t) = - 2 \frac{J_p}{J_{pp}} \epsilon \quad (37)$$

Thus, we can write the mean-variance trade-off which leaves S invariant as:

$$\frac{\epsilon}{\delta_{pp}} = -\frac{1}{2} \frac{J_{pp}}{J_p} \quad (38)$$

Notice that this result is completely analogous to the Arrow-Pratt interpretation of the coefficient of absolute risk aversion $(-U''/U')$ for a static expected-utility maximizer.

V. Constant>Returns-to-Scale Technology

In this section we will assume that $F(K,L)$ is homogeneous of degree one (and concave). The result of this assumption is that the short-run profit function, $\Pi(p,w,K)$ can be written

$$\Pi(p,w)K \quad (39)$$

where $\Pi(p,w)$ is convex and homogeneous of degree one in (p,w) .

Examination of (34) for this case shows that a solution of (34) is of the form

$$J(K,P,r,t) = A(P,k,t)K + B(P,k,t) \quad (40)$$

where the functions A,B are solutions of the following set of differential equations:

$$\begin{aligned} \text{a) } \Pi(p,w) - (\delta+k)A(P,k,t) + \dot{A}[A] &= 0 \\ \text{b) } v(f(A),P) + Af(A;P) + \dot{A}[B] &= 0 \end{aligned} \quad (41)$$

Notice that these equations are not fully simultaneous, since (a) can be solved independently of (b). Notice that $B(P,k;t)$ is the "value" of a firm with zero capital stock. Notice also that $J_K (=A)$ is independent of the form of v (although, by (32), $I^*(t)$ is not).

The first important property of (40) is that by (33), $I^* = f(A; P, k, t)$ and so investment is independent of the level of the capital stock. Notice also that since A and B are not functions of K, $\theta(K)$ (the instantaneous variance of the capital accumulation process) does not appear in (41)a) or b), so that the value of the firm and the investment policy are independent of the variance of the capital accumulation process. This is in spite of the fact that randomness in depreciation implies that net investment is necessarily random also.

Because of the form of (40) we could consider a comparative dynamics experiment of the kind described earlier (section IV), but which in this case would leave the investment policy, but not the value of the firm invariant--i.e., which would leave $A(P, k, t)$ invariant. Using the same argument as for (37) and (38), we have

$$\frac{\epsilon}{\delta_{pp}} = - \frac{1}{2} \frac{A_{pp}}{A_p} \quad (42)$$

If the processes (13) are stationary, i.e., the instantaneous expected rate of change and variance of each process are not functions of time, then examination of (40) shows that $J(K, P, k, t) = J(K, P, k)$. As mentioned earlier, it will not in general be true that $J_{Kp} > 0$ (which would imply $\partial I^*/\partial p > 0$). However, if the price process is stationary, and independent of the other processes, then by the Markov property of the processes it must be the case that $J_p > 0$. Therefore, by the form of the solution to the constant-returns case (40), $B_p > 0$, since otherwise, for K sufficiently small, $J_p \leq 0$. Likewise, we must have $A_p \geq 0$, otherwise for K sufficiently large, $J_p \leq 0$. Therefore in this special case $J_{Kp} \geq 0$, so that $\partial I^*/\partial p \geq 0$.

Although $\Pi(p,w)$ is homogeneous of degree one in (p,w) , $J_K(K,P,k,t)$ will not generally have this property. However, if

$\alpha_i(P,k,t)$ is homogeneous of degree one in P

$\sigma_{ij}(P,k,t)$ is homogeneous of degree two in P

(43)

$$i, j = p, w, q, k$$

then it is easy to show that there is a solution of (41)a) such that

$A(P,k,t)$ is homogeneous of degree one in P .

(44)

This result follows from the fact that if $A(P,k,t)$ is homogeneous of degree one in P , then, given (42), every term of $\int[A]$ is also. Therefore if (42) holds and $v(I;P)$ is homogeneous of degree one in P , by (32), the optimal investment policy is homogeneous of degree zero in P .

As we see from (41)a) and b), A and B will generally be functions of all prices. However, if we assume that

$\alpha_i(P,k;t)$ is not a function of q , for $i \neq q$

$\sigma_{iq}(P,k;t) = 0$, for $i \neq q$

(45)

then (40) is of the special form

$$J(K,P,r,t) = A(K,p,w,k,t)K + B(K,P,k,t)$$

(46)

Thus, in this case A (and therefore the optimal investment policy), is independent of the stochastic process generating q . Of course, by (32), the optimal investment policy is not independent of the current value of q .

VI. Non-Markovian Processes

Although the Markov assumption may be reasonable when applied to movements of stock prices in a stationary economy, it is not a particularly appealing assumption when applied to price movements in goods markets. As Merton [17] has shown, the stochastic optimal control technique can be applied to quite general non-Markovian processes. As an example, consider the moving-average learning model discussed by Merton, applied to the $p(t)$ -process. In this process, the expected rate-of-change of p is estimated by a moving average of previous realized changes. Thus the dynamics, for a simple case, can be written¹¹

$$dp/p = \alpha_p dt + \sqrt{\sigma_{pp}} dz_p \quad (47)$$

where α_p is estimated by

$$\hat{\alpha}_p(t) = \frac{1}{t+\tau} \int_{-\tau}^t dp/p \quad (48)$$

The joint dynamics of the α and p processes can then be written

$$dp/p = \hat{\alpha}_p dt + \sqrt{\sigma_{pp}} dz_p \quad (49)$$

$$d\hat{\alpha} = \frac{\sqrt{\sigma_{pp}}}{t+\tau} dz_p .$$

The effect of this change in the model is that now J is also a function of $\hat{\alpha}$, so equation (23) is changed by the addition of the terms $J_{\hat{\alpha}}$ and $J_{\hat{\alpha}\hat{\alpha}}$ in $\mathcal{L}[J]$. Notice that this sort of change in (23) will not affect the results derived in sections IV or V. Thus, our results are not dependent on our initial Markovian assumption.

VII. Some Examples

In this section we present two examples. The purpose of the examples is to show the differential effects of uncertainty on the investment policies of two firms with constant-returns-to-scale technologies exhibiting different degrees of factor substitutability. In both cases, for simplicity, we will assume that k is fixed, and that p and w are generated by stationary "geometric Brownian motion" processes. This is summarized:

$$\begin{aligned}\alpha_i(P,t) &= \alpha_i^i, & i=p,w \\ \sigma_{ij}(P,t) &= \sigma_{ij}^i j; & i,j=p,w\end{aligned}\tag{50}$$

Since the processes (50) satisfy (43), the solution of (41) a) will be independent of the process generating q .¹²

a. Fixed Coefficients Technology

First we will assume the technology can be written

$$Q = \min(K, L/\beta)\tag{51}$$

In this case the short-run profit function can be written

$$\Pi(p,w,K) = \max\{(p-\beta w)K, 0\}K\tag{52}$$

The solution of (41a)

$$A(P,k) = \max\left\{\left[\frac{P}{\delta+k-\alpha_p} - \frac{\beta w}{\delta+k-\alpha_w}\right], 0\right\}\tag{53}$$

and we see that in this case the optimal investment policy is independent of the variances of the processes generating p and w . In fact, the optimal investment policy is identical to the one generated by assuming that p and w are non-random exponential functions of time, with growth rates α_p and α_w , respectively. Notice that this result depends not only on the assumption of fixed-coefficients technology, but also on the fact that the α_i 's were linear functions of their arguments.

The transversality condition (35), b) requires that the expression in brackets be bounded.

b. Cobb-Douglas Technology

If the technology is Cobb-Douglas, the short-run profit function is of the form

$$\Pi(p,w,K) = w^m p^{1-m} K, \quad m < 0 \quad (54)$$

The solution of ()a) is

$$A(P,k) = c w^m p^{1-m} \quad (55)$$

where

$$c = \left[\delta + r + \alpha_p (m-1) + \alpha_w (-m) + \frac{1}{2} \sigma_{pp} (m-m^2) + \frac{1}{2} \sigma_{ww} (m-m^2) + \sigma_{wp} (-m+m^2) \right]^{-1}$$

Notice, by (32) that

$$\partial I^* / \partial \sigma_{pp}, \quad \partial I^* / \partial \sigma_{ww} > 0 \quad (56)$$

The transversality condition (35), b) requires that c be bounded.

Notice that the increased substitutability of the Cobb-Douglas technology over the fixed coefficients technology resulted in $A_{\sigma_{ii}} > 0$, i.e., increasing variance results in an increase in the "value" of the firm and of investment.

We can also easily derive (42) for this case, and we see that

$$\epsilon / \delta_{pp} = - \frac{1}{2} \frac{(-m(1-m))}{1-m} = \frac{1}{2} \frac{m}{1-m} < 0 \quad (57)$$

It can be shown that increasing σ_{ii} in (50) results in a "mean-preserving spread" in the i -process, since process (50) results in a log-normal distribution of $P(t)$.

VII. Summary and Conclusions

In this paper we have developed a simple model of optimal investment for a perfectly competitive stock market value maximizing firm. Using only the assumptions of perfect competition and value maximization we were able to derive the conditions for optimality simply from a manipulation of accounting identities. We have extended the results derived in the neoclassical literature, and in particular, showed that our general results do not require Markovian processes.

This paper could serve as a basis for further research in at least two other areas. First we have developed a simple model which can be looked on as the supply side of a general equilibrium securities market. Merton [18] has developed a continuous time capital asset pricing model which is a theory of the pricing of securities, for given behavior of firms. Our model is a theory of the behavior of a competitive firm, given market valuation. A truly general equilibrium model would put these two sides together, with firm behavior and market valuation being determined simultaneously.

The second area of future research which this paper suggests is in the specification of empirical investment demand functions. Since we have derived the fundamental partial differential equation which generates the optimal investment policy, this equation can likely be approximately solved for realistic specifications, generating suitable functional forms for empirical analysis. This would hopefully allow the effects of uncertainty to be captured more adequately in empirical investment functions.

This paper has dealt only with a perfectly competitive firm which faces only price uncertainty. We are devoting further research to models of an imperfectly competitive firm (i.e., a firm which can affect its cost of capital), and to models of a firm facing technological uncertainty.

Footnotes

*Much of this paper has benefited immeasurably from conversations with Robert Merton. Unfortunately, all remaining inadequacies are mine.

¹The main references are Lorie and Smith [11], Hartman [5,6], and Norstrom [20].

²For the standard arguments see Van Horne [27].

³See Merton and Subrahmanyam [19]. We will discuss this point later in this paper when we define competitive conditions.

⁴See Merton [18], p. 886.

⁵Although there has been some criticism in the literature about the ad hoc nature of the adjustment costs assumption, in the context of a continuous time model there must be adjustment costs. Otherwise the model would imply that it is possible to invest an arbitrarily large amount at constant marginal cost in an arbitrarily small amount of time, which is clearly implausible.

⁶Thus the production side of our model is formally similar to that of Gould [4].

⁷See McFadden [14].

⁸Merton has shown that in a continuous time model, the MM theorem holds under quite general conditions.

⁹See Kushner [8,9].

¹⁰See Kushner [8,9].

¹¹See Merton [17].

¹²See section V. Since the processes (13) are stationary, A will also not be a function of time.

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