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by

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1. INTRODUCTION

The pursuit of consistency in social choices has generated a large, and predominantly nihilistic, literature (e.g., [1], [2], [3], [4], [6], [7], [8], [9], [15], [16]). All of the references quoted have produced mainly negative conclusions on the possibility of achieving consistency in the presence of "reasonable" other conditions on the social choice process.

Consistency in social choices has been interpreted in a number of related ways. In Arrow's seminal work [1], it was interpreted in terms of social rationality in the form of the requirement that the social choice rule generates binary social preferences, of the form xRy [1], such as to yield a social ordering of the set X of social alternatives. In addition to the properties of completeness[2] and reflexivity,[3] this requires in particular the requirement of

(a) transitivity: $[\forall x, y, z \in X : (xRy \;\&\; yRz) \rightarrow (xRz)]$.

The social choice set C(S) is then definable in terms of the R-relations by

$$C(S) = \{x \in S : xRz \quad \text{for all} \quad z \in S\}$$

Impossibility theorems involving property (a) have been proved by Arrow [1], Blau [2], Kemp and Ng [7], Inada [4], Wilson [16] and others.

Weaker forms of consistency than full transitivity are the following:

(b) quasi-transitivity: $[\forall x, y, z \in X \quad (xPy \& yPz) \rightarrow (xPz)]$

(c) path independence: $[C(S_1 \cup S_2) = C(C(S_1) \cup C(S_2)) \quad \text{for all} \quad S_1, S_2 \subseteq X]$

(d) property α: $[x \in S_1 \subset S_2 \rightarrow [x \in C(S_2) \rightarrow x \in C(S_1)] \quad \text{for all} \quad x \in X]$

(e) property β: $[[x_1, x_2 \in C(S_1) \;\&\; S_1 \subset S_2] \rightarrow [x \in C(S_2) \leftrightarrow y \in C(S_2)] \text{ for all } x_1, x_2 \in X]$

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[1] where R has the familiar interpretation "considered at least as good as," P has the interpretation "strictly preferred to" and I "considered indifferent to."

[2] i.e., $[\forall x, z \in X : (x \neq z) \rightarrow (xRz \;\underline{\vee}\; zRx)]$.

[3] i.e., $[\forall x \in X, xRx]$.

(f) Strong Path Independence: $[C(S_1) \cap S_2 \neq \emptyset] \rightarrow [C([S_2 - S_1] \cup [C(S_1) \cap S_2]) = C(S_2)]$

Impossibility theorems involving the achievement of properties (b) and (d) are proved by Fishburn [3] and Mas-Colell and Sonnenschein [9], and similar strong impossibility theorems involving property (c) by Suzumura [15]. Moreover Plott [12] shows that imposition of property (f) is equivalent to the transitivity requirement (a). Similarly Sen [13] shows that any social choice rule which generates non-empty C(S) for all $S \subseteq X$ and which satisfies properties α and β also requires the generation of a social ordering over X, to which Arrow's theorem then applies. Hence consistency in social choices still appears difficult to achieve.

Despite the apparent difficulty of achieving consistency, the desire for some form of consistency, involving at least some of the above properties, very naturally persists. In particular, without, for instance, path independence, the social choice over what may well be important alternatives, such as Presidential candidates, different economic policies, or job appointees, becomes dependent upon the arbitrary feature of simply the sequence in which the alternatives are considered.

A central question in social choice theory therefore remains that of how one can find a social choice rule with some or all of the above consistency properties. Previous work has approached the problem by imposing other conditions, in addition to consistency, and then has demonstrated nonexistence of any such social choice rule. These conditions have, in particular, included Arrow's Independence of Irrelevant Alternatives (IIA) condition:

IIA: Let $R_1,\ldots,R_n$ and $R_1',\ldots,R_n'$ be two sets of individual orderings over X, with corresponding social choice sets C(S) and $C'(S)$. Then we require for all $S \subseteq X$: $[\forall i: \; x_1 R_i x_2 \text{ iff } x_1 R_i' x_2, \text{ for all } x_1, x_2 \in S] \rightarrow C(S) = C'(S)$,

i.e., the social choice over S must be independent of individual preferences outside S.

Our own approach, however, will be a constructive one, of imposing all of Arrow's conditions except the IIA condition, and then examining the degree of independence from "outside" preferences which is still possible given our desired goal of achieving consistency in our social choices.[1] These other conditions include:

(i) Non-dictatorship (ND): $[\nexists i: \; x_1 P_i x_2 \rightarrow x_1 P x_2 \;$ for $\; \forall x_1, x_2 \in X]$.

(ii) Unrestricted Domain (UD): the domain of the social choice rule $R = f(R_1, \ldots, R_n)$ shall include all logically possible combinations of individual orderings.

(iii) The Strong Pareto Principle (SP): $[\{x_1 R_i x_2 \;$ for all $\; i\} \rightarrow x_1 R x_2]$ & $[\{x_1 R_i x_2 \;$ for all $i \;$ & $\exists j: \; x_1 P_j x_2\} \rightarrow x_1 P x_2]$.

Since the achievement of a SWF, and hence a social ordering, itself guarantees all of the consistency properties (a) - (f), we shall maintain as our goal the achievement of a SWF over X. In order to examine the constructive possibilities for the achievement of such consistency in social choices, we will first examine the following topic.

2. INDEPENDENT FORMULATIONS OF SOCIAL CHOICES

It is clear that consistency in social choices itself implies a definite restriction upon the content of social choice sets C(S) for at least some $S \subset X$, once we have already determined the content of $C(S')$ for some other $S' \subset X$. In particular once we have determined the social preferences $x P y$ and $y P z$, we have no remaining degree of freedom under conditions (a) and (b) in avoiding the social preference $x P z$. Hence $C(\{x, z\})$ cannot then be formulated independently of our social choices over $\{x, y\}$ and $\{y, z\}$ if consistency is to be guaranteed.

This suggests that one way to ensure consistency in social choices is simply to restrict the number of independently formulated social preferences

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[1] in doing so we are continuing an approach initiated in Mayston [10], [11].

we make to some subset of all the $n(n-1)/2$ possible binary combinations from within X, for $|X| = n$, and then make use of the transitive closure of this "basic" set of social preferences, in order to generate a complete social ordering over X. Thus we may simply independently formulate, as part of the basic set of social preferences, $x P y$ and $y P z$, with $C(\{x, z\})$ then determined by the transitive closure of this basic set.

In section 3 we shall examine ways in which we may select such a "basic" set of social preferences. First, however, we seek to establish conditions under which the social choice rule f is free to independently formulate a social preference between a given pair $\{x_1, x_2\}$, rather than having $C(\{x_1, x_2\})$ restricted in content by the social preference already chosen for some other $C(\{x_3, x_4\})$ sets, given our maintained desire for consistency. We have now the following definition being relevant:

<u>Definition 1</u>: A binary social choice set $C(\{x, z\})$ will be said to be <u>formulated independently</u> of $C(\{v, y\})$ in the construction of a SWF over X (written $(x, z)\,FI\,(v, y)$) iff the content of $C(\{x, z\})$ is unrestricted by the substitution of any one of the alternative social choices $C^*(\{v, y\}) = v \vee y \vee \{v, y\}$, for any initial $C(\{v, y\})$, under any given set of individual preferences over X within the desired domain, where we have $v, x, y, z \in X$ and $\{x, z\} \neq \{v, y\}$.

Let us suppose that we are presented with a set X of social alternatives, with accompanying individual preference orderings over these alternatives. We may then ask the question, what position must the alternatives $\{v, x, y, z\}$ occupy in the individual preference orderings in order for us to have $(x, z)\,FI\,(v, y)$? In other words, what restrictions upon the individuals' preferences over $\{v, y, x, z\}$ are necessary if $(x, z)\,FI\,(v, y)$ is to be consistent with the achievement of a SWF over X? We will make use here of the following definition of necessity due to Sen [13, p. 183] and Inada [5].

<u>Definition 2</u>: A condition on the set of individual preferences is <u>necessary</u> if every violation of the condition yields a list of individual orderings such that some assignment of these orderings over some number of individuals will make the individual preference pattern lie outside the domain of the social choice rule f acting as a SWF.

<u>Definition 3</u>: An ordered triple (x, y, z) will be said to satisfy

$$ER(x, y, z) \quad \text{iff} \quad (\exists i: \; x P_i y P_i z) \rightarrow (\forall j: \; z P_j x \rightarrow z P_j y P_j x).$$

We may now state:

<u>Proposition 1</u>: A necessary condition for (x, z) FI (v, y) to hold in the achievement of a SWF[1] over X satisfying condition SP, for all $v \in X$, is that ER(x, y, z) holds for any given $y \in X$.

<u>Proof</u>: Suppose that ER(x, y, z) does not hold for some $y \in X$. Such a breach of ER(x, y, z) occurs iff $x P_i y P_i z$ for some i, and either (a) $z P_j x R_j y$ or (b) $y R_j z P_j x$ for some individual j. The definition of necessity permits an assignment such that only i and j are concerned[2] over the triple {x, y, z}. Condition SP requires in case (a), $x P y$. Substitution of the social choice $y \in C(\{v, y\})$ for $v = z$ implies $y R z$, requiring under transitivity $x P z$, and hence restricting $C(\{x, z\})$ to equal {x}. In case (b), SP implies $y P z$, which with $C(\{v, y\}) = x$ for $v = x$ implies $x P y$, and hence given transitivity, $x P z$, again restricting

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[1] The same condition, by a directly parallel proof, may be shown to apply also to the achievement of simply quasi-transitivity, or acyclicity.

[2] i.e., other than indifferent.

C({x, z}). Thus ER(x, y, z) must hold for any $y \in X$, if C({x, z}) is to be unrestricted, and hence if (x, z) FI (v,y) is to hold for all $v \in X$.

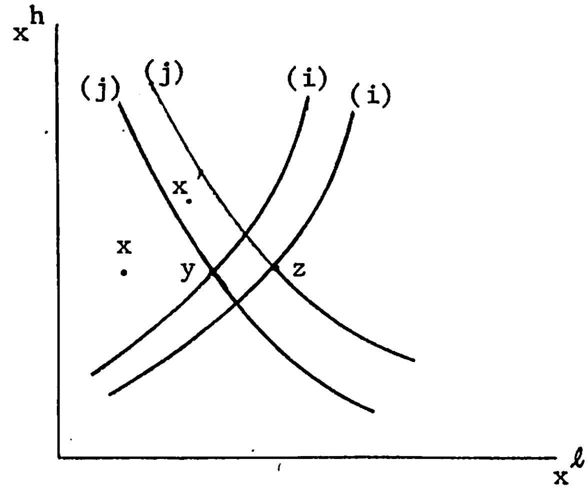

Figure 1

If we examine the illustrative Figure 1, where we have the two concerned individuals i and j having crossing indifference surfaces, we may note that ER(x, y, z) does hold, although ER(x′, y, z) does not. Hence through applying Proposition 1, we cannot have (x′, z) FI (v, y) for all $v \in X$, although Proposition 1 does not rule out the possibility of (x, z) FI (v, y) in the process of achieving a SWF over X.

Given our maintained desire for the achievement of a SWF, our policy must then be to abandon any attempt at having (x′, z) FI (v, y) for all $v \in X$, whenever x′, y, z are in inappropriate relative positions in the individuals' preference orderings. Rather, if we are to attempt independent formulations of social preferences, we must concentrate upon pairs such as {x, z } and {v, y} where ER(x, y, z) does hold and hence where (x, z) FI (v, y) is not excluded by Proposition 1.

We may contrast our own approach here with those of previous treatments of social choice theory, such as those of Sen and Pattanaik [14], Inada [5], and others, which have sought to restrict the domain of individual preferences in order to permit majority rule to act as a SWF. Majority rule insists upon taking the majority preference as representing the social preference over every pair of alternatives within X, independently of whether or not this yields intransitivities with other binary majority rule preferences already formulated. We then have the following strong theorem applying:

Proposition 2:[1] The necessary and sufficient condition for a set of individual orderings to be in the domain of the method of majority rule acting as a SWF is that every triple of alternatives must satisfy extremal restrictedness, as defined by:

Definition 4: A triple {x, y, z} satisfies extremal restrictedness iff ER(x, y, z) holds for every ordered triple attainable from the triple {x, y, z}.

Kramer [8] has shown the extremal restrictedness condition to be a very strong restriction upon individual preferences in the context of a multidimensional policy space with convex individual preferences. In particular he shows that it will be broken whenever at any point x in the space there exist two individuals with differing marginal rates of substitution in the space of characteristics, and hence whenever any two individuals have crossing indifference surfaces. There will then exist points such as x′, y, z in Figure 1 which are in breach of the ER(x′, y, z) condition for at least one such triple (x′, y, z) in the space.

Thus to require (x, z) FI (v, y) for all $v \in X$, and hence from Proposition 1 ER(x, y, z), for all ordered triples (x, y, z) is to risk the requirement of severe restrictions upon individual preferences, which are clearly incompatible with our maintained condition UD.

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[1] See Sen [13, p. 185].

The approach which we will follow in this paper, therefore, does not try to impose $(x, z)\, FI\, (v, y)$ for all $v \in X$ and for all triples $\{x, y, z\}$ independently of the relative position of x, y and z in the individual preference orderings. Rather, as stressed above, it asks what position must x and z occupy in the individual preference orderings relative to y if we are to have admissible the possibility of $(x, z)\, FI\, (v, y)$ for all $v \in X$? If they are in an inappropriate position, we must simply abandon our attempt at an independent formulation of $C(\{x, z\})$.

3. DETERMINATION OF A BASIC SET OF SOCIAL PREFERENCES

We mentioned earlier the search for a "basic" set of social preferences which could be used to generate a complete SWF over X. Let us now introduce the following concept of a basis.

<u>Definition 5</u>: A set of binary social choice sets $C(\{x, z\})$ will be said to form a <u>basis</u> for the construction of a SWF over X satisfying condition SP, denoted $BA(X)$, iff for all $C(\{x, z\}) \in BA(X)$:

(i) $\sim (\forall i: \; x R_i z) \;\&\; \sim (\forall i: \; z R_i x)$ and

(ii) $(x, z)\, FI\, (v, y)$ for all $v, y \in X : \{v, y\} \neq \{x, z\}$ and

(iii) $[(u, w)\, FI\, (v, y)$ for all $v, y \in X : \; \{v, y\} \neq \{u, w\}]$ does not hold for $C(\{u, w\}) \notin BA(X)$.

Thus a basis under the above definition contains only those binary social choice sets which involve alternatives between which at least some individuals disagree, and consists of all and only those binary social choice sets which can be formulated independently of the social choice sets over all other distinct pairs within X. Once we have selected such a basic set of social preferences, we may then proceed to fill the gaps in our complete set of binary social preferences through taking the transitive closure of social preferences in our basic set, combined with appropriate use of the Pareto condition SP.

Let us now suppose that we attempt to construct such a basis by making a whole series of independent social preferences of the form $C(\{y_1, y_k\})$ $k=2,\ldots,H$, or of the form $C(\{y_k, y_{k+1}\})$ $k=1,\ldots,H-1$. We can then ask the question, under what conditions will an independent formulation of all of these binary social choice sets be consistent with the achievement of a SWF over X, i.e., when will all of these choice sets be candidates to enter the basis BA(X)?

Again we will focus upon the relative position which the alternatives in this sequence must occupy in the individuals' preference ordering if such an independent formulation is to be possible.

Definition 6: A sequence $(y_1,\ldots,y_H)$ of alternatives will be said to lie in successive individual indifference classes within X iff for all individuals i who are concerned over the sequence:

$[y_1 P_i y_2 P_i \ldots, y_{H-1} P_i y_H] \vee [y_H P_i y_{H-1} \ldots y_2 P_i y_1]$ for $y_k \in X$, $k=1,\ldots,H$.

Proposition 3: A necessary and sufficient condition for each $C(\{y_1, y_k\})$ to belong to BA(X) for $k=2,\ldots,H$ is that for some labelling of the alternatives within $\hat{Y} = \{y_1,\ldots,y_H\}$, the ordered sequence $Y = (y_1,\ldots y_H)$ lies in successive individual indifference classes. In addition for any such set of sequences $(Y_1,\ldots,Y_K)$ whose respective $C(\{y_1, y_k\})$ sets enter the basis, we require that there exists no series of the form

$$y_K^{(1)} \tilde{R}\, y_k^{(2)},\; y_\rho^{(2)} \tilde{R}\, y_j^{(3)},\; y_q^{(3)} \tilde{R}\, y_r^{(4)},\ldots, y_s^{(q)} \tilde{R}\, y_t^{(1)} \qquad (1)$$

where $x \tilde{R} y$ denotes the (weak) Pareto dominance relation $[x R_i y, \forall i]$ and $Y_j = (y_1^{(j)},\ldots,y_H^{(j)})$ for all $j=1,\ldots,K$.

Proof: From Definition 5, it is necessary that $\exists\, i, j$: $y_k P_i y_1$ & $y_1 P_j y_k$ for $k=2,\ldots,H$. Definition 2 permits us to specify just two concerned individuals over Y, viz. i and j. Completeness of individual preferences requires $y_k R_j y_{k'}$ for some labelling of each pair within $Y = \{y_1,...,y_H\}$. $y_k I_i y_{k'}$ implies $y_k \tilde{R} y_{k'}$, which with conditions SP and $y_{k'} P y_1$ would imply $y_k P y_1$, breaching $(y_k, y_1) FI(y_{k'}, y_1)$. Hence $y_k I_i y_{k'}$ must not hold for any pair within Y, implying a labelling such that $y_H P_i y_{H-1} P_i \ldots P_i y_1$, implying for all $H-1 > k > 1$: $y_{k+1} P_i y_k P_i y_1$. From Proposition 1, it is necessary for condition (ii) in Definition 5 to hold, that $y_1 P_j y_k P_j y_{k+1}$ for $H-1 > k > 1$, implying $y_1 P_j y_2 P_j \ldots P_j y_H$.

Since for no sequence of $C(\{y_1^{(j)}, y_k^{(j)}\})$ $k=2,...,H_j$, $j=1,...,K$ do we have three pairs in the same triple of alternatives involved, the only way in which transitivity can imply a restriction upon any such $C(\{y_1^{(j)}, y_k^{(j)}\})$ is if there are intermediate (weak) Pareto dominance relations holding across different sequences, which (1) prevents. Hence the SCR is free to specify the content of each $C(\{y_1^{(j)}, y_k^{(j)}\})$ independently of all other $C(\{y_1, y_k\})$, causing the conditions of Proposition 3 to be both necessary and sufficient for each $C(\{y_1^{(j)}, y_k^{(j)}\})$ to enter the basis.

Proposition 4: Necessary and sufficient conditions for a series of binary social choice sets $C(\{y_k, y_{k+1}\})$ to belong to a basic set BA(X) for $k=1,...,H-1$ such that $y_k P_i y_{k+1}$ $(k=1,...,H-1)$ for some individual i, are that

(a) the sequence $(y_1,...,y_H)$ lies in successive individual indifference classes.

(b) for any such set of sequences $(Y_1,...,Y_K)$ whose respective $C(\{y_k, y_{k+1}\})$ enter the basis, we require again that there exists no series of the form (1).

<u>Proof</u>: From Definition 5, we require $y_{k+1} P_j y_k$ given $y_k P_i y_{k+1}$. From Proposition 1 with two concerned individuals, $y_k P_i y_{k+1} P_i y_{k+2}$ requires then $y_{k+2} P_j y_{k+1} P_j y_k$ for $k=1,\ldots,H-2$, requiring the sequence $(y_1,\ldots,y_H)$ to lie in successive individual indifference classes. The remainder of the proof is parallel to that of Proposition 3.

Let us now examine our above results in the context of Arrow's Independence of Irrelevant Alternatives condition. Propositions 3 and 4 emphasize the necessity of our sequence of alternatives lying in successive individual indifference classes if the accompanying series of binary social choice sets are to be capable of an independent formulation, and hence qualify for membership of the basis BA(X). If we were to impose IIA, we would require each successive social choice set $C(\{y_k, y_{k+1}\})$[1] to be determined <u>irrespective</u> of the position of the sequence $(y_1,\ldots,y_H)$ in the individuals' preference ordering, for any given set of individual preferences over simply the immediate pair $\{y_k, y_{k+1}\}$ being considered.

Thus imposition of IIA would prevent us from achieving any guarantee we might otherwise have that the requirements of Propositions 3 and 4 would be fulfilled, once we permit individual preferences to vary according to our UD condition, and hence permit changes in individual preferences outside $\{y_k, y_{k+1}\}$ in such a way as to breach the requirements of Propositions 3 and 4.

If we therefore relax insistence upon IIA, the question still remains as to whether some other form of independence is desirable or possible in

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[1]or $C(\{y_1, y_k\})$.

the formulation of a binary social choice set of the form $C(\{x_1, x_2\})$. Arrow's IIA condition regards the social choice over $\{x_1, x_2\}$ as involving a move between two alternatives, with the only relevant individual preference information being the individuals' directions of preferences over these immediate two alternatives. In contrast, a different perspective upon the same social choice would be to regard the move from x_1 to x_2 as involving moves across successive individual indifference classes for each individual, as in Figure 2. In this context we introduce the following concept.

Definition 7: An individual's Between (x_1, x_2) set, written $B_i(x_1, x_2)$, is defined as follows:

$$B_i(x_1, x_2) = \{x \in X : x_1 R x \;\&\; x R_i x_2\}$$

corresponding therefore to the intersection of individual i's no-better-than-x_1 set and his no-worse-than-x_2 set.

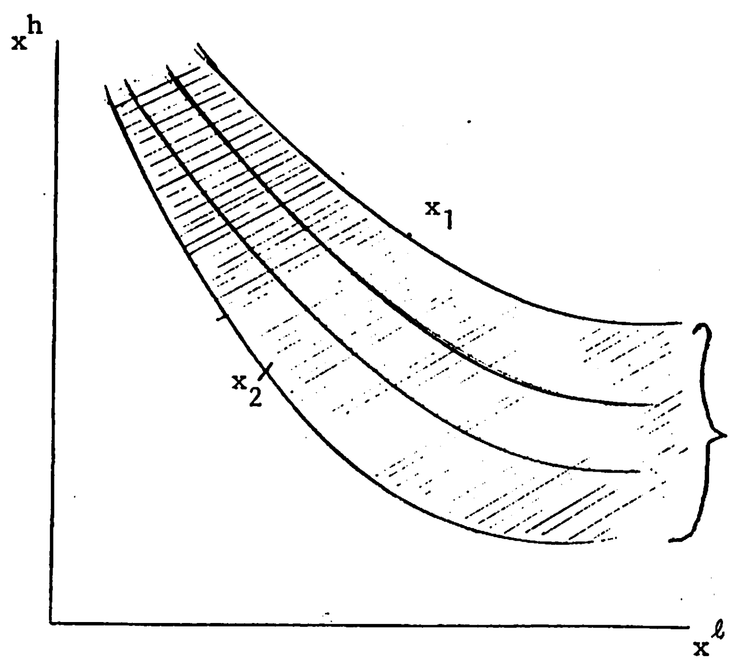

Figure 2

Since $B_i(x_1, x_2) = \emptyset$ for $x_2 P_i x_1$, we find it useful to define also the non-directional set:

$$\hat{B}_i(x_1, x_2) \equiv [B_i(x_1, x_2)] \cup [B_i(x_2, x_1)]$$

Suppose that we hold constant each individual's preferences involving all successive individual indifference classes for the individual, so that the move from x_1 to x_2 involves crossing the same successive individual indifference classes as previously. We may then require constancy of the corresponding social choice set $C(\{x_1, x_2\})$, as in the following Weak Independence of Irrelevant Preferences (WIIP) condition:

Condition WIIP: Let $(R_1, \ldots, R_n)$ and $(R_1', \ldots, R_n')$ be two sets of individual orderings such that the corresponding individual $\hat{B}$etween (x_1, x_2) sets are equal, i.e., $\hat{B}_i(x_1, x_2) = \hat{B}_i'(x_1, x_2)$ for $i=1,\ldots,n$, and where R and R$'$ designate the corresponding social preferences. Then we require for all $x_1, x_2 \in X$: $[\forall i: (\forall x, y \in \hat{B}_i(x_1, x_2) : x R_i y \leftrightarrow x R_i' y)] \rightarrow [x_1 R x_2 \leftrightarrow x_1 R' x_2]$

4. A BASIS FOR THE CONSTRUCTION OF A SWF

In order to investigate whether there exist any SWF's satisfying WIIP, or indeed even stronger forms of independence from outside preferences, let us return to our concept of a basis. Again this corresponds to a complete set of binary social choice sets whose contents the social choice rule is free to choose in an unrestricted way, without being restricted by the contents chosen for other such binary choice sets. Let us now provide an example of the use of such a basis in the construction of a SWF, with its desired consistency properties. In doing so we will make use of the following assumption:

A1. There exists at least one characteristic of the social state for each individual i (e.g., i's money income), which we may write y^{in}, consumed without externality, such that for each social state $x \in X$, there exists a social state $y^i(x)$ differing from some fixed reference social state $\bar{y}$ simply by its level of y^{in}, such that $x I_i y^i(x)$, i.e., the individual is indifferent between x and $y^i(x)$. In addition we assume that each individual i always prefers to have more rather than less of y^{in}, for given levels of the other characteristics of the social state.

Assumption A1 implies that the individual i can always be compensated by some variation in y^{in} about $\bar{y}$ for any initial movement away from any social state $x \in X$, as in Figure 3.

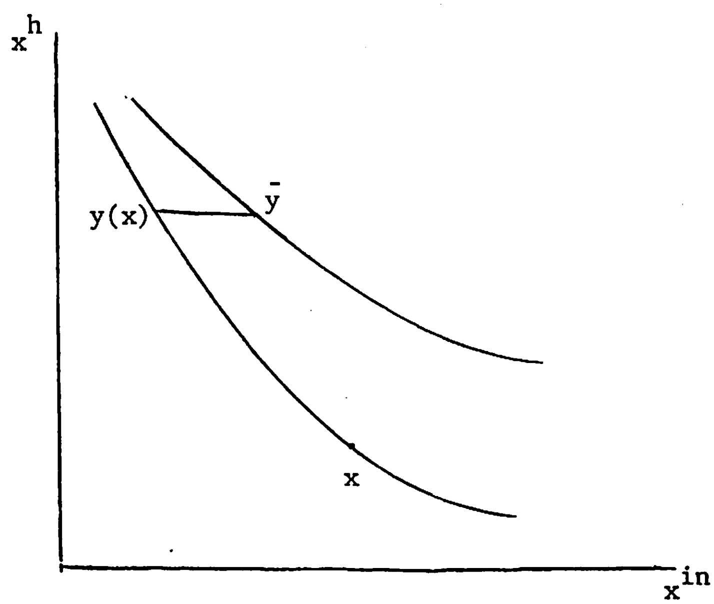

Figure 3

Let us now begin our process of providing a basis for the construction of a SWF over X. We will start at the initial reference social state $\bar{y}$ referred to in Assumption A1. From Proposition 4, we know that if we want the binary social choice sets $C(\{y_k, y_{k+1}\})$ k=1,...,H-1 to be in the basis, the associated sequence of alternatives $(y_1,\ldots,y_H)$ must be such as to lie in successive individual indifference classes for all the individuals concerned, whenever they are for one individual. One such sequence here is provided by the following procedure:

(i) We start at $\bar{y}$ and then make a series of equal and opposite small incremental unit changes in the level of the characteristics y^{in} and y^{mn} away from the social state $\bar{y}$, increasing y^{in} and reducing y^{mn}, for the value i=1. Under assumption A1, each y^{in} is consumed without externality and such a sequence of points lies in successive individual indifference classes for individuals i and m, with all other individuals unconcerned over points in the sequence.

As a further step in the construction of our basis, we will also include within BA(X) the successive binary social choice sets $C(\{y'_k, y'_{k+1}\})$ associated with the additional sequences of points $(y'_1,\ldots,y'_H)$:

(ii) the reverse sequence to (i) involving starting at $\bar{y}$ and then making a series of equal and opposite small incremental unit changes in the level of the characteristics y^{in} and y^{mn} away from the social state $\bar{y}$, for the value i=1, increasing y^{mn} and reducing y^{in}.

We moreover extend the changes involved in the sequences of points in (i) and (ii) such that for any $x \in X$, there is contained within one or other of these sequences a point which individual i considers indifferent to x, and similarly a point which individual m considers indifferent to x. Such points must exist by virtue of assumption A1.

(iii) For each level of y^{1n} in steps (i) and (ii), we repeat the process for i=2. Then for each level of $(y^{1n}, y^{2n},\ldots,y^{jn})$, we repeat the steps (i) and (ii) for $i=j+1$, for $j=2,\ldots,m-2$.

(iv) We consider next a new reference social state $\hat{y}(\Delta y^{mn})$ differing from $\bar{y}$ simply by an amount Δy^{mn} of individual m's consumption of y^{mn}, and repeat sequences (i) - (iii) for all distinct values of Δy^{mn}, with $\hat{y}(o) = \bar{y}$.

Each of the sequences in steps (i) - (iv) involves making equal and opposite variations in the level of the characteristics y^{in} and y^{mn} which, under assumption A1, place the individuals concerned in successive individual indifference classes, thereby satisfying the first condition (a) of Proposition 4 for independent formulations of social preferences between each pair in these sequences.

Moreover the only way in which (weak) Pareto dominance can hold between social states in different sequences in (i) - (iv) is if two individuals have (weakly) greater levels of y^{in} and y^{mn}, since all other characteristics, other than y^{jn} for j=1,...,m, of the social state are held constant in moving between the different sequences. In order for one sequence, say (s), to have (weakly) greater levels of y^{in} and y^{mn} than another, say (t), requires $(\Delta y^{mn})^{(s)} \geq (\Delta y^{mn})^{(t)}$ where these are the respective levels of Δy^{mn} in the two sequences in step (iv), including the possibility $\Delta y^{mn} = 0$. Hence for (1) to hold we require

$$(\Delta y^{mn})^{(1)} \geq (\Delta y^{mn})^{(2)} \geq \ldots \geq (\Delta y^{mn})^{(q)} \geq (\Delta y^{mn})^{(1)}$$

which is clearly possible only if all $(\Delta y^{mn^{(s)}})$ are equal. Such occurs only for those sequences involved in steps (i) - (iii) above for a common value of Δy^{mn} referred to in step (iv). It is clear that a Pareto dominance relation of the form (1) cannot hold for these sequences, since the sum $\sum_i y^{in}$ must remain constant for all members of these sequences. The necessary and sufficient conditions (a) and (b) of Proposition 4 in order for each $C(\{y_k, y_{k+1}\})$ involved in the sequences (i) - (iv) to enter the basis are therefore satisfied.

Hence the SCR has freedom to independently specify the content of each such $C(\{y_k, y_{k+1}\})$ in the construction of a SWF over X. In order to provide concreteness to our analysis, we will make the choice:

$$C(\{y_k, y_{k+1}\}) = \{y_k, y_{k+1}\} \quad \text{implying } y_k \, I \, y_{k+1} \qquad (2)$$

for each of the sequences involved in steps (i) - (iv), as would be implied by a majority rule decision in each case, since only individual i and m disagree over any given pair for the appropriate choice of i.

We may now show that the set of binary social choice sets $C(\{y_k, y_{k+1}\})$ generated by steps (i) - (iv) and (2) are also here sufficient to generate a SWF over X, since through use of our transitivity requirement and condition SP, we may generate a unique value to $C(\{x, z\})$ for all $x, z \in X$, such that the consistency properties of a SWF over X hold.

Consider any pair of points $x, z \in X$. Then from assumption A1, there must exist points $y(x)$ and $y(z)$ such that $x \, I_i \, y(x)$ for $i=1,\ldots,m$ where $y(x)$ differs from $\bar{y}$ simply by the level of the y^{in} characteristics for $i=1,\ldots,m$, and similarly for $y(z)$. Hence under condition SP, we have $x \, I \, y(x)$, $z \, I \, y(z)$. By the transitive closure of $y_1 \, I \, y_2$, $y_2 \, I \, y_3$ etc. we may readily demonstrate the implication $y_h \, I \, y_k$ for all h, k defined by steps (i) - (iii), including the case $y_k = \hat{y}(\Delta y^{mn})$ for the appropriate value of Δy^{mn}. Moreover under conditions SP and A1, we must have $\hat{y}(\Delta y^{mn}) P \hat{y} (\widetilde{\Delta} y^{mn})$ whenever $\Delta y^{mn} > \Delta \tilde{y}^{mn}$, with such a series of social preferences being transitive for all values of such Δy^{mn}. Hence all points in the sequences (i) - (iii) for the same value of Δy^{mn} are in the same social indifference classes, and all those with greater values of Δy^{mn} are in higher social indifference classes, thereby yielding transitivity across sequences. Moreover the definition of steps (i) - (iv) covers all possible combinations of $(y^{1n}, y^{2n}, \ldots, y^{mn})$ for the given other characteristics of the social state $\bar{y}$. Hence these points include $y(x)$ and $y(z)$, with a well-defined social preference relation then defined between them, and hence between x and z, given $x \, I \, y(x)$ and $z \, I \, y(z)$ under condition SP, together with our requirement of transitivity. We have then the same social preference relation between x and z as between y(x) and y(z). Moreover we know from above that since y(x) and y(z) lie in sequences described by steps (i) - (iv) the relationship between y(x), y(z) and any other y(w) will be transitive. Hence so also will be the relationship between any $x, z, w \in X$. Reflexivity and completeness follow immediately.

Such an independent formulation of the sequences of binary social choice sets of the form $C(\{y_k, y_{k+1}\})$ described by steps (i) - (iv) above therefore provides a basis for the generation of a transitive, complete and reflexive set of social preference over X, and hence for the construction of a SWF over X.

Our above procedure may be shown to be equivalent to the criterion:

$$zRx \quad \text{as} \quad \sum_i [y^{in}(z) - y^{in}(x)] \geq 0 \quad \text{for all } x, z \in X \tag{3}$$ [1]

and hence to satisfy the following independence condition, involving a strengthening of the degree of independence from outside individual preferences (i.e., other than $zR_i x$ or $xR_i z$ for each i) beyond that involved in WIIP.

Independence of Irrelevant Preferences (IIP): Let $R_1, \ldots, R_n$ and $R_1', \ldots, R_n'$ be any two sets of individual preference orderings over X such that for each individual i, i's preferences are the same in both cases along some common single ordered sequence of points $(\theta_1^i, \ldots, \theta_t^i)$ across[2] each respective $\hat{B}_i(x, z)$ set. Then we require the respective social choice sets $C(\{x, z\})$ and $C'(\{x, z\})$ to be equal.

The relevant sequence of points for individual i involved in the criterion (3), and the above basis, is that involved in varying each y^{in}

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[1] where from A1, $y^{in}(x)$ is the level of characteristic y^{in} in a social state considered by i indifferent to x and differing only from $\bar{y}$ by the level of y^{jn} $(j=1, \ldots, m)$.

[2] I.e., such that there exists one and only one member of the sequence in each individual indifference class contained within $\hat{B}_i(x, z)$.

about the point $\bar{y}$ between the values $y^{in}(z)$ and $y^{in}(x)$, with then (3) directly satisfying condition IIP, and hence also WIIP. In addition, (3) can readily be shown to generate a social ordering over X satisfying also conditions SP, ND and UD, subject to assumption A1.

Thus we have:

Proposition 5: There exists a SWF satisfying conditions SP, ND, UD, IIP and WIIP under assumption A1.

5. CONCLUSION

Our concern in this paper has been with an investigation of how we may accomplish the achievement of consistency in social choices. Previous approaches to social choice have often viewed the problem as one of simply the formation of social choices directly over alternatives. In particular Arrow's Independence of Irrelevant Alternatives condition has regarded individual preferences over the immediate alternatives to be the only relevant individual preference information to the formation of social choices. As is evident from the numerous impossibility theorems which surround use of the IIA condition in conjuction with reasonable other conditions, there would seem to be a basic conflict between the achievement of consistency in social choices and the imposition of IIA in such a context.

Our own approach in contrast has been to regard the social choice over alternative social states, such as x and z, as involving the intermediate factor of the crossing of successive individual indifference classes between those in which x and z lie for each individual. We thereby regard the process of consistent aggregation[1] in social choice as a two-stage

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[1]Note that social choice theory may be regarded as part of a more general theory of aggregation, as in Wilson [17].

process. The first involves the formation of individual preference orderings by each individual and hence the assignment by him of alternatives to successive individual indifference classes. The second stage involves the formation of social preferences in such a way as to respect the structure imposed upon the alternatives by each individual in determining his preferences.

In Propositions 3 and 4, we saw that if we were to formulate a sequence of binary social choice sets, of the form $C(\{y_1, y_k\})$ or $C(\{y_k, y_{k+1}\})$ independently, in the sense that the content of these choice sets was unrestricted by the contents chosen for any other choice sets, then the associated sequence $(y_1,\ldots,y_H)$ of alternatives had to lie in successive individual indifference classes for each concerned individual, if consistency was to be guaranteed. Similar remarks applied to the conditions under which a binary social choice set could enter a basis, i.e., a basic set of binary social preferences each of which could be formulated independently as either P, I or contra P. We also provided an example of such a basis, and showed how this basic set of binary social preferences could be used to generate a complete social ordering over X through the application of the Pareto condition SP and the requirement of transitivity.

Whilst relaxing the Independence of Irrelevant Alternatives condition, the SWF generated by our basis nonetheless did satisfy an Independence of Irrelevant Preferences condition involving the requirement of the same social choice set whenever we maintained constant individual preferences not simply over the immediate alternatives, say $\{x, z\}$, but rather over some sequence of alternatives across successive individual indifference classes between those in which x and z lie, thereby preserving important parts of the way in which individuals themselves structure the alternatives.

Our conclusions therefore suggest that consistency is indeed possible in social choice under reasonable conditions, provided that we make the appropriate shift of emphasis away from simply aggregation of individual preferences across the immediate alternatives being considered. Rather our conclusions suggest the need for a reorientation of social choice theory towards treating social choices as involving the crossing of successive individual indifference classes for each concerned individual between the social states under consideration.

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