

1991

Optimal Capital Income Taxation and Long Run Debt with Borrowing Constraints

S Rao Aiyagari

Dan Peled

Follow this and additional works at: <https://ir.lib.uwo.ca/economicsresrpt>

 Part of the [Economics Commons](#)

Citation of this paper:

Aiyagari, S Rao, Dan Peled. "Optimal Capital Income Taxation and Long Run Debt with Borrowing Constraints." Department of Economics Research Reports, 9113. London, ON: Department of Economics, University of Western Ontario (1991).

29878

ISSN: 0318-725X
ISBN: 0-7714-1340-8

RESEARCH REPORT 9113
OPTIMAL CAPITAL INCOME TAXATION
AND LONG RUN DEBT
WITH BORROWING CONSTRAINTS

by ECONOMICS REFERENCE CENTRE

S. Rao Aiyagari
and
Dan Peled

MAR - 6 1997

UNIVERSITY OF WESTERN ONTARIO

October 1991

Department of Economics

Social Science Centre

University of Western Ontario

London, Ontario, Canada

N6A 5C2

**OPTIMAL CAPITAL INCOME TAXATION AND LONG RUN DEBT
WITH BORROWING CONSTRAINTS**

S. Rao Aiyagari and Dan Peled*

(Revised) October 1991

ABSTRACT

For a wide class of dynamic models, Chamley [1986] has shown that the optimal capital income tax rate is zero in the long run. Lucas [1990] has argued that for the U.S. economy there is a significant welfare gain from switching to this policy. We argue that the presence of binding borrowing constraints always leads to a positive optimal tax rate on capital income even in the long run. In addition, such models are capable of determining the optimal long run level of government debt independently of its initial level. For an important class of models pioneered by Bewley [1980], borrowing constraints will always bind on some of the people in each period. Our analysis is also applicable to overlapping generations models without bequest motives, or those with bequest motives but binding bequest constraints. An important problem for the future is to determine if a realistically parameterized Bewley type model with binding borrowing constraints can support the observed level of the capital income tax rate as optimal in the long run.

*Aiyagari-Federal Reserve Bank of Minneapolis, Minneapolis, MN 55480, U.S.A.; Peled-The Technion-Israel Institute of Technology, Haifa, 32000, Israel and The University of Western Ontario, London, Ontario, Canada.

The views expressed herein are those of the authors and not necessarily those of the Federal Reserve Bank of Minneapolis or the Federal Reserve system.

I. Introduction

For a wide class of dynamic models, Chamley [1986] has shown that the optimal capital income tax rate is zero in the long run. Lucas [1990] has argued that for the U.S. economy there is a significant welfare gain from switching to this policy. We argue that the presence of binding borrowing constraints always leads to a positive optimal tax rate on capital income even in the long run¹. For an important class of models pioneered by Bewley [1980], borrowing constraints will always bind on some of the people in each period. Our analysis is also applicable to overlapping generations models (hereafter, OLG) without bequest motives, or those with bequest motives but binding bequest constraints. An important future problem is to determine if a realistically parameterized Bewley type model (with the U.S. economy in mind) with binding borrowing constraints can support the observed level of the capital income tax rate as optimal in the long run.

Another important implication of models with borrowing constraints is that it is possible to determine the optimal long run level of government debt independently of initial conditions. In the class of models considered by Chamley [1985,1986], there is no theory of the long run level of government debt independent of the initial level of the debt. Roughly speaking, if the initial level of debt is higher, the labor income tax will be higher by just enough to pay for the higher interest charges and the debt is simply rolled over forever. On the other hand, in models with borrowing constraints government debt serves a liquidity role and it is possible to determine the optimal long run level independently of the initial level.

¹Jones, Manuelli and Rossi [1990] show that if government expenditures are endogenous and productive, then the capital income tax rate can be positive in the long run.

Note that second best taxation also plays a role here. With lump sum taxation, all that can be said is that the debt level should be high enough to make the borrowing constraints not bind. Further increases in the level of debt are neutral. Obviously, under second best taxation this will not be true since higher levels of debt will be associated with greater distortions which will lower welfare.

The intuition behind why borrowing constraints lead to a positive tax rate on capital income may be explained as follows. The possibility of being borrowing constrained in some future periods leads to a precautionary saving motive for agents which increases their saving and hence capital accumulation, and thereby lowers the return on capital below the rate of time preference². That is, borrowing constraints lead to excess (i.e., greater than the optimal level of) capital³. Therefore, a positive tax rate on capital income will be needed to reduce capital accumulation and bring the pre-tax return on capital to equality with the rate of time preference.

The outline of the paper is as follows. In Section II we present Chamley's [1986] result for a model with infinitely lived agents and some heterogeneity. Clearly, some heterogeneity among agents is needed to permit borrowing and lending, and thereby to permit borrowing constraints to bind. The purpose of this exposition is to highlight the importance of nonbinding borrowing constraints for Chamley's result.

A special case of this model is considered in Section III—the Bewley-Townsend model with two types of agents, periodically varying

²See, Bewley [no date]. That borrowing constraints coupled with uninsured idiosyncratic income risks lead to a precautionary demand for wealth was also noted by Scheinkman and Weiss [1986].

³This should be distinguished from the standard notion of capital overaccumulation which refers to an inefficiently high level of capital.

individual labor productivities, and borrowing constraints. We show that for some parameter values the borrowing constraints will bind and, hence, that the optimal capital income tax rate must be positive even in the long run.

In Section IV, we extend this result to a Bewley model with a continuum of agents, stochastic and idiosyncratic shocks to labor productivities, and borrowing constraints. We also comment on the difficulties in computing solutions to these types of models. Section V concludes.

In the Appendix we show that the Bewley-Townsend model of Section III has an equivalent representation as an OLG model with bequest motives and nonnegativity constraints on bequests. Whenever the borrowing constraints bind in the Bewley-Townsend model, the bequest constraints will bind in the equivalent OLG representation. Hence, the Bewley-Townsend model with binding borrowing constraints is equivalent to an OLG model with bequest motives but binding bequest constraints, or equivalently, to an OLG model without bequest motives⁴. Therefore, the results of Section III carry over to the equivalent OLG model and may be interpreted as follows. Even in OLG models in which all agents and the social planner share a common rate of time preference, the optimal capital income tax rate in the long run will be positive if the bequest constraints bind.

II. The Chamley Result

Consider an economy with I infinitely lived agents, indexed by i . Periods are indexed by $t \in \{0, 1, 2, \dots\}$. Let, c_{it} , n_{it} be agent i 's consumption of the single good in the economy and his labor supply in period

⁴Woodford [1990] has made this point.

t. Each agent has one unit of time available for work and leisure so that $(1 - n_{it})$ is agent i's consumption of leisure in period t. We assume that agents differ in their labor productivities. Let θ_{it} be agent i's labor productivity in period t, i.e., if agent i supplies n units of labor in period t, then the effective units of labor supplied by him in period t are $\theta_{it}n$. Each agent has preferences over infinite streams of consumption and leisure given by the functional: $\sum_{t=0}^{\infty} \beta^t U(c_{it}, 1-n_{it})$, where $0 < \beta < 1$ is a common utility discount factor, and $U(\cdot)$ is a period utility function assumed to be strictly increasing, strictly concave, bounded and continuously differentiable.

Let, $n_{ot} = \sum_i \theta_{it} n_{it}$ be the total effective units of labor supplied in period t. The aggregate production technology is represented by the production function $f(k_t, n_{ot})$ which gives the total output produced in period t and k_t is the capital available in period t. We assume that $f(\cdot)$ is homogeneous of degree one and continuously differentiable with positive and diminishing returns to each factor. We assume that capital depreciates at the constant rate δ . Let $r_t = f_1(k_t, n_{ot}) - \delta$, and $w_t = f_2(k_t, n_{ot})$ be the pre-tax return on capital and the pre-tax wage rate, respectively. We let, $r_{xt} = (1 - \tau_{kt}) r_t$, $w_{xt} = (1 - \tau_{nt}) w_t$ be the after-tax return on capital and the after-tax wage, respectively, where τ_{kt} , τ_{nt} are the tax rates on capital and labor income, respectively.

Let a_{it} be the assets held by agent i at the beginning of period t. Let $p_t = \prod_{j=1}^t (1 + r_{xj})^{-1}$ be the price of date t consumption in terms of date zero consumption. Then, each agent maximizes $\sum_{t=0}^{\infty} \beta^t U(c_{it}, 1-n_{it})$ over time paths for c_{it} and n_{it} subject to:

$$(2.1a) \quad c_{it} + a_{i,t+1} = \theta_{it} w_{xt} n_{it} + (1 + r_{xt}) a_{it}$$

$$(2.1b) \quad \lim_{t \rightarrow \infty} p_t a_{1,t+1} = 0.$$

Let b_t be the face value of government debt outstanding at the beginning of period t . Let g_t be government consumption in period t . The equilibrium conditions in the asset and goods markets are as follows.

$$(2.2a) \quad \sum_i a_{it} = k_t + b_t$$

$$(2.2b) \quad \sum_i c_{it} + g_t + k_{t+1} - (1-\delta)k_t = f(k_t, n_{ot}).$$

By summing the individual budget constraints (2.1) over i , and using the market clearing conditions (2.2), we can derive the following form of the government budget constraint.

$$(2.3) \quad g_t + r_{xt} b_t = b_{t+1} - b_t - w_{xt} n_{ot} - r_{xt} k_t + f(k_t, n_{ot}) - \delta k_t$$

Note that if we substitute for the after-tax returns r_{xt} , w_{xt} and use the homogeneity of the production function $f(\cdot)$, the above form of the government budget constraint is equivalent to the following more traditional form.

$$(2.4) \quad g_t + r_t b_t = b_{t+1} - b_t + \sum_i \tau_{nt} w_{it} n_{it} + \sum_i \tau_{kt} r_{it} a_{it}.$$

The government's optimal tax problem may be stated as follows.

Maximize:

$$(2.5) \quad \begin{aligned} \mathcal{L} = & \sum_{t=0}^{\infty} \beta^t \sum_i U(c_{it}, 1-n_{it}) \\ & + \sum_{t=0}^{\infty} \beta^t \sum_i \xi_{it} [\theta_{it} w_{xt} n_{it} + (1+r_{xt}) a_{it} - c_{it} - a_{1,t+1}] \\ & + \sum_{t=0}^{\infty} \beta^t \lambda_t [f(k_t, \sum_i \theta_{it} n_{it}) - \sum_i c_{it} - g_t - k_{t+1} + (1-\delta)k_t] \end{aligned}$$

$$\begin{aligned}
& + \sum_{t=0}^{\infty} \beta^t \sum_{i=1}^n \nu_{it} [U_2(c_{it}, 1-n_{it}) - \theta_{it} w_{xt} U_1(c_{it}, 1-n_{it})] \\
& + \sum_{t=0}^{\infty} \beta^t \sum_{i=1}^n \alpha_{it} [U_1(c_{it}, 1-n_{it}) - \beta(1+r_{x,t+1}) U_1(c_{i,t+1}, 1-n_{i,t+1})].
\end{aligned}$$

The government is maximizing social welfare given by the sum of individual discounted sums of utilities subject to the individual budget constraints, the resource constraint, and individual optimality conditions. Note that with government debt defined as aggregate individual asset holdings minus capital (equation (2.2a)), the government budget constraint (equation (2.3)) is implied by the individual budget constraints (equations (2.1a)) and the resource constraint (equation (2.2b)). Therefore, the government budget constraint need not be entered as an additional constraint in the government's optimization problem. The government's choice variables are taken to be time paths for c_{it} , n_{it} , a_{it} , k_t , r_{xt} , and w_{xt} .

We will assume that the a_{it} 's are unrestricted other than by (2.1), i.e., there are no borrowing constraints. Further, c_{it} is assumed to be strictly positive always, so that the individual intertemporal optimality condition holds as an equality. Under these conditions, it is very easy to exhibit the Chamley result. Simply differentiate (2.5) w.r.t. k_{t+1} and $a_{i,t+1}$ to obtain the following FONCs

$$(2.6a) \quad -\lambda_t + \beta \lambda_{t+1} (1+r_{t+1}) = 0$$

$$(2.6b) \quad -\xi_{it} + \beta \xi_{i,t+1} (1+r_{x,t+1}) = 0.$$

A steady state is defined as an equilibrium with time invariant per capita output, capital, effective labor and consumption, and Lagrange multipliers that are bounded and bounded away from zero. Of course, among the necessary conditions for such a steady state are time invariant per

capita government consumption and labor productivity. An individual's labor productivity, however, can vary over time, and, hence, so can his labor supply, consumption and asset holdings. What matters for our purposes is that the agent smooths his consumption profile given his fluctuating labor income by accumulating and decumulating assets, unrestricted by non-negativity constraints, thus satisfying his first order conditions with equality.

In such a steady state equations (2.6) imply that $r = r_x = 1/\beta - 1$. This follows from the fact that unity is the only value for both $\beta(1+r)$ and $\beta(1+r_x)$ that is consistent with Lagrange multipliers that are bounded and bounded away from zero. Hence, the optimal tax rate on capital income in the long run is zero.

This exposition also serves to illustrate the potential importance of borrowing constraints for this result. Suppose we have the additional restrictions that the a_{it} must be nonnegative. Then (2.6b) is replaced by

$$(2.6b') \quad -\xi_{it} + \beta \xi_{i,t+1} (1+r_{x,t+1}) \leq 0, \text{ with } = \text{ if } a_{i,t+1} > 0.$$

In any steady state, so long as the nonnegativity constraints on the a_{it} 's are binding on some individuals in each period, $r_x < 1/\beta - 1 = r$, implying a positive tax rate on capital income even in the long run. For example, one possible type of steady state in a Bewley-Townsend type model is where half the people are borrowing constrained in one period and the other half are borrowing constrained in the following period. In the steady state of such a model the ξ_{it} 's will be varying periodically over time and the constraints (2.6b') will bind on half the people in one period and on the other half of the people in the next period. We consider this

possibility in the next section. We also show how this type of model bears a very close resemblance to an OLG model, so that our analysis is directly applicable to a certain class of OLG models.

Finally, notice the difference between the Ramsey optimal tax problem as posed here and the "comparative steady state" problem. In the latter, the government is choosing time invariant tax rates *assuming* a steady state with optimizing agents. If, for instance, the government cannot tax labor income, then positive government consumption necessarily leads to a positive tax rate on capital income. In contrast, the Ramsey optimal policy will be to levy a high tax on capital income initially, thereby generating surpluses and accumulating claims against the private sector. In the long run, the tax rate on capital income is reduced to zero and government consumption is financed out of the interest income generated by the claims against the private sector. Obviously, the difference stems from recognising the disincentive effect of capital income taxes on capital accumulation, which is ignored in the "comparative steady state" analysis.

III. A Bewley-Townsend Model

The model is a special case of the Section II model and is obtained by assuming that there are only two types of agents ($i=1,2$) and that their labor productivities fluctuate periodically. In addition, borrowing constraints are imposed so that individual asset holdings are required to be nonnegative. The labor productivities are specified as follows.

$$(3.1a) \quad \{\theta_{1t}\} = \{\theta_1, \theta_2, \theta_1, \theta_2, \dots\}, \theta_1 > \theta_2,$$

$$(3.1b) \quad \{\theta_{2t}\} = \{\theta_2, \theta_1, \theta_2, \theta_1, \dots\}.$$

It will be convenient to switch notation slightly and index agents by the productivity shock received. That is, we will use c_{1t} , n_{1t} , and $a_{1,t+1}$ to denote the consumption, labor supply and assets held (at the end of the period) in period t of the individual who received the productivity shock θ_1 . With this change, the government's programming problem can be written as follows.

$$\begin{aligned}
 (3.2) \quad \mathcal{L} = & \sum_{t=0}^{\infty} \beta^t \sum_1 U(c_{1t}, 1-n_{1t}) \\
 & + \sum_{t=0}^{\infty} \beta^t \xi_{1t} [\theta_1 w_{xt} n_{1t} + (1+r_{xt}) a_{2t} - c_{1t} - a_{1,t+1}] \\
 & + \sum_{t=0}^{\infty} \beta^t \xi_{2t} [\theta_2 w_{xt} n_{2t} + (1+r_{xt}) a_{1t} - c_{2t} - a_{2,t+1}] \\
 & + \sum_{t=0}^{\infty} \beta^t \lambda_t [f(k_t, \sum_1 \theta_i n_{it}) - \sum_1 c_{it} - g_t - k_{t+1} + (1-\delta)k_t] \\
 & + \sum_{t=0}^{\infty} \beta^t \sum_1 \nu_{1t} [U_2(c_{1t}, 1-n_{1t}) - \theta_1 w_{xt} U_1(c_{1t}, 1-n_{1t})] \\
 & + \sum_{t=0}^{\infty} \beta^t \alpha_{1t} [U_1(c_{1t}, 1-n_{1t}) - \beta(1+r_{x,t+1}) U_1(c_{2,t+1}, 1-n_{2,t+1})] \\
 & + \sum_{t=0}^{\infty} \beta^t \alpha_{2t} [U_1(c_{2t}, 1-n_{2t}) - \beta(1+r_{x,t+1}) U_1(c_{1,t+1}, 1-n_{1,t+1})].
 \end{aligned}$$

We now give an example with particular functional forms and parameter values to show the possibility of binding borrowing constraints and a positive optimal capital income tax rate even in the long run.

Let,

$$(3.3a) \quad f(k, n) = k^a n^{1-a}, \quad 0 < a < 1,$$

$$(3.3b) \quad U(c, 1-n) = \ln(c) + \psi \ln(1-n), \quad \psi > 0,$$

$$(3.3c) \quad g = \gamma f(k, n), \quad 0 < \gamma < 1.$$

Our method for constructing the example is the following. We start with the provisional assumption that the borrowing constraints never bind and calculate the steady state. Then, the Chamley result applies and the

return on capital (pre- and post tax) equals $1/\beta - 1$. The tax rate on wage income is determined by government consumption and interest on government debt. The individuals' choice problems together with the resource constraint can be used to solve for a_1 and a_2 . If the solution for a_2 violates the borrowing constraint, then there cannot be a steady state with nonbinding borrowing constraints. Consequently, the capital income tax rate must be positive even in the long run.

If $r = r_x = 1/\beta - 1$, then $c_1 = c_2 = c$, say. Using the individual budget constraints and the optimality conditions, it is easy to calculate that,

$$(3.4a) \quad a_1 = \beta[(1+\psi)c - w_x(\theta_2 + \beta\theta_1)/(1+\beta)]/(1-\beta)$$

$$(3.4b) \quad a_2 = \beta[(1+\psi)c - w_x(\theta_1 + \beta\theta_2)/(1+\beta)]/(1-\beta)$$

$$(3.4c) \quad n_1 = 1 - \psi c / (\theta_1 w_x)$$

$$(3.4d) \quad n_2 = 1 - \psi c / (\theta_2 w_x)$$

$$(3.4d) \quad n_o = \theta_1 n_1 + \theta_2 n_2 = \theta_1 + \theta_2 - 2\psi c / w_x$$

We assume that interest on government debt is a fraction γ' of gross output. The government budget constraint simplifies in the steady state to the following.

$$(3.5) \quad (w - w_x)n_o = g + rb \\ = (\gamma + \gamma')f(k, n_o).$$

Using the Cobb-Douglas form of the production function we have,

$$(3.6) \quad w_x = [wn_o - (\gamma + \gamma')f(k, n_o)]/n_o \\ = (1 - a - \gamma - \gamma')f(k, n_o)/n_o.$$

Note that $f(k, n_0)/n_0$ can be expressed entirely in terms of the parameters of the example using the fact that $f_1(k, n_0) = r + \delta$. Therefore, w_x can be expressed entirely in terms of the parameters of the example.

Next, using the resource constraint (at steady state) together with (3.4d) we can solve for c as follows.

$$\begin{aligned}
 (3.7) \quad c + c &= f(k, n_0) - \delta k - g \\
 &= n_0 [(1-\gamma)f(k, n_0) - \delta k] / n_0 \\
 &= (\theta_1 + \theta_2 - 2\psi c / w_x) [(1-\gamma)f(k, n_0) - \delta k] / n_0.
 \end{aligned}$$

The solution for c is given by the following.

$$(3.8a) \quad c = (1/2)(\theta_1 + \theta_2)H / [1 + \psi H / w_x], \text{ where}$$

$$(3.8b) \quad H = [(1-\gamma)f(k, n_0) - \delta k] / n_0.$$

We can now substitute (3.8) in (3.4b) to come up with the following condition for violating the borrowing constraint.

$$(3.9a) \quad (1+\psi)(1/2)(\theta_1 + \theta_2) / [(w_x/H) + \psi] < (\theta_1 + \beta\theta_2) / (1+\beta) \text{ iff } a_2 < 0.$$

$$(3.9b) \quad (1+\psi)(1/2)(\theta_1 + \theta_2) / [(w_x/H) + \psi] > (\theta_2 + \beta\theta_1) / (1+\beta) \text{ iff } a_1 > 0.$$

In addition, we need to ensure that $n_1 > 0$. For this it is necessary and sufficient to require that $n_2 > 0$ which leads to the following.

$$(3.10) \quad \theta_2 > \psi(1/2)(\theta_1 + \theta_2) / [(w_x/H) + \psi] \text{ iff } n_2 > 0.$$

Using (3.6) and (3.8b) we can write,

$$\begin{aligned}
 (3.11) \quad w_x/H &= (1-a-\gamma-\gamma')f(k, n_0)/[(1-\gamma)f(k, n_0)-\delta k] \\
 &= (1-a-\gamma-\gamma')/[1-\gamma-\delta k/f(k, n_0)] \\
 &= (1-a-\gamma-\gamma')/[1-\gamma-\delta a/f_1(k, n_0)] \\
 &= (1-a-\gamma-\gamma')/[1-\gamma-\delta a/(r+\delta)] \\
 &< 1 \text{ provided } \gamma' \geq 0.
 \end{aligned}$$

This shows that so long as γ' is nonnegative (which we will assume is the case) condition (3.9b) is always satisfied, i.e., a_1 is always positive. Therefore, to get a binding borrowing constraint and positive labor supplies we need the following condition.

$$(3.12) \quad \psi(1/2)(\theta_1 + \theta_2)/[(w_x/H) + \psi] < \min\{\theta_2, (\psi/(1+\psi))(\theta_1 + \beta\theta_2)/(1+\beta)\}.$$

There exist parameter values satisfying the above conditions, and for such values the optimal solution (in the long run) can be found as follows⁵. The following description also shows how this model of optimal taxation with borrowing constraints also determines the optimal long run level of government debt independently of initial conditions.

First, we set a_{2t} to zero and throw out the last constraint in the programming problem (3.2). This leads to the following formulation.

$$(3.13) \quad \mathcal{L} = \sum_{t=0}^{\infty} \beta^t \sum_1 U(c_{1t}, 1-n_{1t})$$

⁵As an example, it can be verified that the following parameter values satisfy condition (3.12): $\psi = 1$, $\beta = 2/3$, $a = 0.06$, $\delta = 0.1$, $\gamma = 0.09$, $\gamma' = 0$, $\theta_1 = 1$, and $\theta_2 = 0.5$.

$$\begin{aligned}
& + \sum_{t=0}^{\infty} \beta^t \xi_{1t} [\theta_1 w_{1xt} n_{1t} - c_{1t} - a_{1,t+1}] \\
& + \sum_{t=0}^{\infty} \beta^t \xi_{2t} [\theta_2 w_{2xt} n_{2t} + (1+r_{xt}) a_{1t} - c_{2t}] \\
& + \sum_{t=0}^{\infty} \beta^t \lambda_t [f(k_t, \sum \theta_i n_{it}) - \sum c_{it} - g_t - k_{t+1} + (1-\delta)k_t] \\
& + \sum_{t=0}^{\infty} \beta^t \sum \nu_{it} [U_2(c_{it}, 1-n_{it}) - \theta_i w_{ixt} U_1(c_{it}, 1-n_{it})] \\
& + \sum_{t=0}^{\infty} \beta^t \alpha_{1t} [U_1(c_{1t}, 1-n_{1t}) - \beta(1+r_{x,t+1}) U_1(c_{2,t+1}, 1-n_{2,t+1})]
\end{aligned}$$

The steady state versions of the FONCs w.r.t. k_{t+1} , $a_{1,t+1}$, c_{1t} , c_{2t} , n_{1t} , n_{2t} , w_{xt} , $(1+r_{x,t+1})$ are as follows. As before, we use n_0 to denote the effective labor supply $(\theta_1 n_1 + \theta_2 n_2)$.

$$(3.14a) \quad -\lambda + \beta \lambda [f_1(k, n_0) + 1 - \delta] = 0$$

$$(3.14b) \quad -\xi_1 + \beta \xi_2 (1+r_x) = 0$$

$$(3.14c) \quad U_1(c_1, 1-n_1) - \xi_1 - \lambda + \nu_1 [U_{21}(c_1, 1-n_1) - \theta_1 w_{1x} U_{11}(c_1, 1-n_1)] + \alpha_1 U_{11}(c_1, 1-n_1) = 0.$$

$$(3.14d) \quad U_1(c_2, 1-n_2) - \xi_2 - \lambda + \nu_2 [U_{21}(c_2, 1-n_2) - \theta_2 w_{2x} U_{11}(c_2, 1-n_2)] - \alpha_1 (1+r_x) U_{11}(c_2, 1-n_2) = 0.$$

$$(3.14e) \quad -U_2(c_1, 1-n_1) + \xi_1 \theta_1 w_{1x} + \lambda \theta_1 f_2(k, n_0) + \nu_1 [-U_{22}(c_1, 1-n_1) + \theta_1 w_{1x} U_{12}(c_1, 1-n_1)] - \alpha_1 U_{12}(c_1, 1-n_1) = 0.$$

$$(3.14f) \quad -U_2(c_2, 1-n_2) + \xi_2 \theta_2 w_{2x} + \lambda \theta_2 f_2(k, n_0) + \nu_2 [-U_{22}(c_2, 1-n_2) + \theta_2 w_{2x} U_{12}(c_2, 1-n_2)] + \alpha_1 (1+r_x) U_{12}(c_2, 1-n_2) = 0.$$

$$(3.14g) \quad \xi_1 \theta_1 n_1 + \xi_2 \theta_2 n_2 - \nu_1 \theta_1 U_1(c_1, 1-n_1) - \nu_2 \theta_2 U_1(c_2, 1-n_2) = 0$$

$$(3.14h) \quad \xi_2 a_{1t} - \alpha_1 U_1(c_2, 1-n_2) = 0.$$

One way to think about solving this system of equations is as follows. First, note that condition (3.14a) immediately determines the capital/effective labor ratio k/n_0 , and hence also $f_2(k, n_0)$. For a given value of r_x the consumers decision problem can be solved to find c_1 , c_2 , n_1 ,

n_2 , and a_1 as functions of w_x . These can be substituted in the resource constraint to solve for w_x (Note that net output $[f(k, n_0) - \delta k]$ is proportional to n_0 which is also a function of w_x). These solutions can now be substituted in the six equations (3.14b)-(3.14g) to solve for the six multipliers ξ_1 , ξ_2 , λ , ν_1 , ν_2 , and α_1 . Note that the equations (3.14b)-(3.14g) are linear in the multipliers. The last equation (3.14h) can now be used to pin down the value of r_x .

The above description verifies that this model of optimal taxation with borrowing constraints also pins down the long run level of government debt independently of initial conditions. Once r_x and w_x are known, a_1 and k are known and hence the level of government debt $b = (a_1 - k)$ is determined.

IV. The Bewley Model

In this section we consider the problem of optimal capital income taxation in a Bewley type model (see, for eg. Bewley 1980) with a continuum of agents receiving stochastic, idiosyncratic shocks to labor productivities which are uninsured. While the model of the previous section was conceptually useful to explain the basic idea, it seems far removed from reality to be a useful guide to the magnitudes of the optimal tax rates or the level of government debt. It may be possible to parameterize a model of the Bewley [no date] type and compute the optimal capital income tax rate in the long run to get an idea of how far from zero the tax rate could be due to the presence of realistic borrowing constraints. It would also be interesting to determine the optimal long run level of debt and the wage tax. Recently, models of this type have been used to address quantitatively

a variety of questions⁶.

We assume that there is a continuum of agents indexed by $i \in [0,1]$. The labor productivity of agent i in period t is denoted θ_{it} . We assume that θ_{it} follows a finite stationary Markov chain with no transient states, and with a unique ergodic set with no cyclically moving subsets. We assume that the Markov process for θ_{it} is independent across i so that there is no aggregate uncertainty.

An agent starts with some assets a_{i0} and a realized productivity shock θ_{i0} in period 0, and maximizes $E_0\{\sum \beta^t U(c_{it}, 1-n_{it})\}$ subject to the sequence of budget constraints and borrowing constraints given by

$$(4.1a) \quad c_{it} + a_{i,t+1} = \theta_{it} w_{xt} n_{it} + (1+r_{xt})a_{it},$$

$$(4.1b) \quad a_{it} \geq 0.$$

It is convenient to adopt a different description of this problem. Essentially, we index an agent by the pair (a, θ) , the assets and the realized productivity shock that he begins a period with. We let $J_t(a, \theta)$ be the joint cross-section density of agents according to asset holdings and productivity shocks, and take $J_0(a, \theta)$ as a given initial condition. The evolution of $J_t(\cdot)$ over time will have to be determined as part of the solution. Let,

$$(4.2a) \quad w^{xt} = \{w_{xt}, w_{x,t+1}, w_{x,t+2}, \dots\},$$

$$(4.2b) \quad r^{xt} = \{r_{xt}, r_{x,t+1}, r_{x,t+2}, \dots\},$$

⁶See, for example, Imrohoroglu [1988,1989] on the costs of inflation and the costs of the business cycle, Diaz-Gimenez and Prescott [1989] on monetary policy and asset returns, Huggett [1989] on the risk free interest rate, and Aiyagari and Gertler [1991] on asset returns with transactions costs.

An agent's decision problem can be expressed in terms of the following Bellman's equation.

$$(4.3) \quad V(a_t, \theta_t, w^{xt}, r^{xt}) = \text{Max}\{U(c_t, 1-n_t) + \beta E_t V(a_{t+1}, \theta_{t+1}, w^{x,t+1}, r^{x,t+1})\},$$

subject to: $c_t + a_{t+1} = w_{xt} \theta_t n_t + (1+r_{xt}) a_t$.

The solution to the above problem will consist of the following decision rules.

$$(4.4a) \quad c_t = c(a_t, \theta_t, w^{xt}, r^{xt})$$

$$(4.4b) \quad n_t = n(a_t, \theta_t, w^{xt}, r^{xt})$$

$$(4.4c) \quad a_{t+1} = a(a_t, \theta_t, w^{xt}, r^{xt}).$$

Using (4.4c) and the transition probability matrix describing the Markov chain for θ , we can update the given initial joint density $J_0(a, \theta)$ to obtain $J_t(\cdot)$ for all t . Note that these joint densities for $t \geq 1$, will depend on the sequence of after-tax prices. To make this dependence explicit, we will denote them by $J_t(a, \theta, w^{x0}, r^{x0})$.

The government's optimal tax problem may now be described as follows.

Maximize

$$(4.5) \quad \mathcal{L} = \int V(a, \theta, w^{x0}, r^{x0}) J_0(a, \theta) da d\theta +$$

$$\Sigma \beta^t \lambda_t [f(k_t, \int \theta n(a, \theta, w^{xt}, r^{xt}) J_t(a, \theta, w^{x0}, r^{x0}) dad\theta) + (1-\delta)k_t - k_{t+1}$$

$$- \int c(a, \theta, w^{xt}, r^{xt}) J_t(a, \theta, w^{x0}, r^{x0}) dad\theta - g_t].$$

Note that in the above problem, the only constraint is the resource constraint in which we have already substituted for the effective labor supply in period t the expression $\int \theta n(a, \theta, w^{xt}, r^{xt}) J_t(a, \theta, w^{x0}, r^{x0}) dad\theta$, and for per capita consumption in period t the expression $\int c(a, \theta, w^{xt}, r^{xt}) J_t(a, \theta, w^{x0}, r^{x0}) dad\theta$. The government budget constraint need

not be included since the individual decision rules automatically satisfy individual budget constraints, which together with the resource constraint implies the government budget constraint.

The result that the optimal capital income tax rate must be positive even in the long run can be shown as follows. The FONC w.r.t. k_{t+1} in the above planning problem is (for $k_{t+1} > 0$),

$$(4.6) \quad -\lambda_t + \beta \lambda_{t+1} [f_1(k_{t+1}, n_{e,t+1}) + 1 - \delta] = 0.$$

The intertemporal FONC for a typical agent is given by

$$(4.7) \quad U_1(c_t, 1-n_t) \geq \beta(1+r_{x,t+1})E_t\{U_1(c_{t+1}, 1-n_{t+1})\},$$

with equality if $a_{t+1} > 0$, and $E_t\{\cdot\}$ being the expectation conditional on information available at t .

A steady state equilibrium in this economy consists of: (i) time invariant pre- and post-tax factor prices, r , r_x , w , w_x ; (ii) a time invariant cross-section distribution of agents' asset holdings and productivities, $J(a, \theta)$, and (iii) Lagrange multipliers that are bounded and bounded away from zero. A typical agent's decisions in period t , written in general form in equations (4.4) simplify to functions of (a_t, θ_t) alone, by virtue of the constancy of post-tax factor prices w_{xt} and r_{xt} . It follows that per capita capital k_t and effective labor n_{et} will also be time invariant.

From equations (4.6), since $\{\lambda_t\}$ must be bounded and bounded away from zero, it follows that,

$$(4.8) \quad r = \lim_{t \rightarrow \infty} r_{t+1} = \lim_{t \rightarrow \infty} [f_1(k_{t+1}, n_{e,t+1}) - \delta] = 1/\beta - 1.$$

In contrast the individual FONC (4.7) implies that $r_x < 1/\beta - 1$. To see the latter, let $\psi(a_t, \theta_t) \equiv U_1(c(a_t, \theta_t), 1 - n(a_t, \theta_t))$, denote the marginal utility of consumption of an agent in period t making optimal decisions with beginning of period assets a_t and productivity θ_t , where $(a_t, \theta_t) \in \Omega$. Our assumptions about the stochastic processes $\{\theta_{1t}\}$ and the time invariance of the individual decision rules imply that Ω is some compact subset of R^2 . Since $\psi(\cdot)$ is continuous, it attains a minimum on Ω , say $\psi^* \equiv \min \{\psi(a, \theta) : (a, \theta) \in \Omega\} = \psi(a^*, \theta^*)$ for some $(a^*, \theta^*) \in \Omega$. Now assume by way of contradiction that $\beta(1+r_x) \geq 1$ in the steady state version of (4.7). Then we have:

$$(4.9) \quad \psi(a, \theta) \geq E\{\psi(a', \theta') : (a, \theta)\}, \quad \forall (a, \theta) \in \Omega.$$

Obviously (4.9) cannot hold for $(a, \theta) = (a^*, \theta^*)$, unless the marginal utility of consumption $\psi(a, \theta)$ is constant on Ω . But that possibility is ruled out by $U(\cdot)$ being strictly concave, implying a value function $V(\cdot)$ in (4.3) which is strictly concave in beginning of period assets, and the fact that $\psi(\cdot)$ must be the partial derivative of $V(\cdot)$ with respect to beginning of period assets. It follows that $\beta(1+r_x) < 1$, so that $r_x < 1/\beta - 1 = r$, implying a positive tax rate on capital income even in the long run.

We end this section with a few comments on the computational difficulties involved in solving such a model. First of all, the dynamic programming approach outlined in equation (4.4) is not computationally feasible since it involves an infinite dimensional state vector. While it is true that in a steady state the tax rates and the distributions $J_t(\cdot)$

will be constant, this cannot be imposed at the outset. Choosing the optimal tax rates involves computing how the values of the functions $V(\cdot)$, $n(\cdot)$, $c(\cdot)$, and $J_t(\cdot)$ respond to changes in the tax rates and this seems to require knowledge of these functions outside of the steady state. In brief, the computational difficulties appear to be quite formidable.

V. Conclusion

We have shown that in an otherwise standard neoclassical model of capital accumulation, Ramsey taxation with binding borrowing constraints implies a strictly positive tax rate on capital income even in the long run. It would be interesting to pursue this question quantitatively to see whether a realistically specified model can explain the observed level of the capital income tax rate as optimal in the long run. The Bewley type model described in Section IV seems to be an appropriate framework for such an analysis. However, the computational difficulties involved appear to be quite formidable and this remains a task for the future.

APPENDIX

Here we exhibit the equivalence between the Bewley-Townsend model of Section III and a class of OLG models with two period lived agents and bequest motives. To see this, interpret (c_{1t}, n_{1t}) and (c_{2t}, n_{2t}) as the consumption and labor supply of the "young" in period t (the generation that is born in period t), and (c_{2t}, n_{2t}) as the consumption and labor supply of the "old" in period t (the generation that was born in the previous period and will die in period t). The bequest motive is captured by "dynastic" preferences which are represented by the following alternative form of the social planner's preferences.

$$(A.1) \quad \sum_{t=0}^{\infty} \beta^t \sum_1 U(c_{1t}, 1-n_{1t}) = U(c_{20}, 1-n_{20}) + \sum_{t=0}^{\infty} \beta^t [U(c_{1t}, 1-n_{1t}) + \beta U(c_{2,t+1}, 1-n_{2,t+1})]$$

In (A.1), we interpret $U(c_{20}, 1-n_{20})$ as the utility of the initial old generation in period 0, and $[U(c_{1t}, 1-n_{1t}) + \beta U(c_{2,t+1}, 1-n_{2,t+1})]$ as the lifetime utility of the generation born in period $t \geq 0$. With this interpretation, the social planner has the same preferences as the dynastic head (the initial old in period 0).

The budget constraints of the agents may be interpreted as follows. Let q_t be the bequest made by the old at t to the young at t , and let s_t be the saving of the young in period t . We interpret these as follows.

$$(A.2a) \quad q_t = a_{2,t+1} + (1+r_{xt})a_{2t}$$

$$(A.2b) \quad s_t = a_{1,t+1} + a_{2,t+1}$$

With the above interpretation, the budget constraints of the agents

(the first two constraints in the programming problem (3.2)) can be rewritten as follows.

$$(A.3a) \quad \theta_1 w_{xt} n_{1t} + q_t = c_{1t} + s_t$$

$$(A.3b) \quad \theta_2 w_{xt} n_{2t} + (1+r_{xt})s_{t-1} = c_{2t} + q_t.$$

The resource constraint is straightforward. The individual optimality conditions are also straightforward, except the last one. To see it in more familiar light, note that the optimality condition for an old agent at t who is choosing the bequest level q_t is given by

$$(A.4) \quad U_1(c_{2t}, 1-n_{2t}) \geq U_1(c_{1t}, 1-n_{1t}) \text{ with } = \text{ if } q_t > 0.$$

By virtue of the intertemporal optimality condition for a young agent at t (the fifth constraint in the programming problem (3.2)) the above condition can be seen to be equivalent to the last constraint in the programming problem. This completes the description of the equivalence between the Bewley-Townsend model and a class of OLG models with two period lives and bequest motives.

At least as far as steady states are concerned, the borrowing constraints in the Bewley-Townsend model are exactly analogous to the nonnegativity constraints on bequests in the OLG interpretation. Typically, in a steady state, the borrowing constraint may bind on the agent receiving the low productivity shock θ_2 but will not bind on the agent receiving the high productivity shock θ_1 . It is obvious from the definition of q_t that in a steady state, the borrowing constraint will bind if and only if the nonnegativity constraint on bequests binds. When this happens, the last

constraint in the programming problem (3.2) will hold with strict inequality. We may, therefore, drop this constraint, and set a_2 to zero. Now the model looks exactly like an OLG model with two period lived agents and no bequest motives. The social planner's and the individual agents' preferences have the special property that all agents and the social planner have the same rate of time preference. By the argument of section II, if the optimal tax policy results in a steady state with binding borrowing (equivalently, bequest) constraints, then the capital income tax rate must be positive in such a steady state. The special property of the agents' and the social planner's preferences alluded to above shows that this result arises not because of any discrepancies between time perspectives (a point emphasized by Chamley) but due to binding borrowing (or bequest) constraints. Incidentally, the above equivalence also shows that analysis of optimal taxation in the Bewley-Townsend model with binding borrowing constraints is also applicable to a particular class of OLG models with two period lived selfish agents.

REFERENCES

- Aiyagari, S. Rao and Mark Gertler. 1991. "Asset returns with transactions costs and uninsured individual risk," *Journal of Monetary Economics*, 27, 311-331.
- Bewley, Truman. 1980. "The optimum quantity of money," in John Kareken and Neil Wallace (eds.), *Models of Monetary Economies*, Federal Reserve Bank of Minneapolis, 169-210.
- _____ (no date). "Interest bearing money and the equilibrium stock of capital," mimeo.
- _____ and R. Radner. 1980. "Stationary monetary equilibrium with a continuum of independently fluctuating consumers," mimeo.
- Chamley, Christophe. 1985. "Efficient taxation in a stylized model of intertemporal general equilibrium," *International Economic Review*, 26, 451-68.
- _____. 1986. "Optimal taxation of capital income in general equilibrium with infinite lives," *Econometrica*, 54:3, 607-22.
- Diaz-Gimenez, Javier and Edward C. Prescott. 1989. "Asset returns in computable general equilibrium heterogeneous agent economies," Research Department, Federal Reserve Bank of Minneapolis.
- Huggett, Mark. 1989. "The risk-free rate in heterogeneous agent economies," Department of Economics, University of Minnesota.
- Imrohorglu, A. 1988. "The welfare cost of inflation under imperfect insurance," Department of Finance and Business Economics, University of Southern California.
- _____. 1989. "Costs of business cycles with indivisibilities and liquidity constraints," *Journal of Political Economy*, 97(6), 1364-83.
- Jones, L. E., Manuelli, R. E., and Rossi, P. E. 1990. "Optimal taxation in

convex models of equilibrium growth," Working Paper, MEDS/KGSM, Northwestern University.

Lucas, Robert E. Jr. 1990. "Supply side economics: An analytical overview," *Oxford Economic Papers*, 42(2), April, 293-316.

Scheinkman, Jose A. and Laurence Weiss. 1986. "Borrowing constraints and aggregate economic activity," *Econometrica*, 54:1, 23-45.

Townsend, Robert. 1980. "Models of money with spatially separated agents," in John Kareken and Neil Wallace (eds.), *Models of Monetary Economies*, Federal Reserve Bank of Minneapolis, 265-303.

Woodford, Michael. 1990. "Public debt as private liquidity," *AER Papers and Proceedings*, May, 80(2), 382-88.