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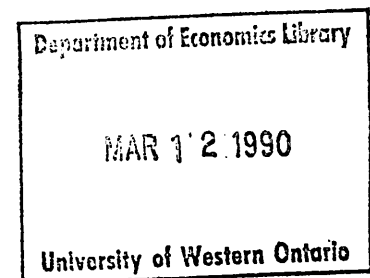
WICKSELL'S CUMULATIVE PROCESS

AS NONCONVERGENCE TO RATIONAL EXPECTATIONS

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1.   Introduction

In his (1968) AEA Presidential address, Milton Friedman argued, among other things, that controlling interest rates tightly was not a feasible monetary policy. His argument was a variation on Wicksell's cumulative process. Start in full employment with no actual or expected inflation. Let the monetary authority peg the nominal interest rate below the natural rate. This will require monetary expansion, which will eventually cause inflation. When expected inflation rises in response to actual inflation, the Fisher effect will put upward pressure on the interest rate. More monetary expansion will be required to maintain the peg. This will make inflation accelerate until the policy is abandoned. Likewise, if the interest rate is pegged above the natural rate deflation will accelerate until the policy is abandoned. Since no one knows the natural rate the policy is doomed one way or another.

This argument, which was once quite uncontroversial, at least among monetarists, has lost its currency. One obvious reason is that the argument invokes adaptive expectations, and there appears to be no way of reformulating it under rational expectations. McCallum (1986) shows that in conventional rational–expectations models monetary policy can peg the nominal rate, even in the face of random shifts in the natural rate, without producing runaway inflation or deflation. This can be accomplished by systematically reducing monetary growth, and hence reducing expected inflation, whenever the natural rate is expected to rise. Furthermore, as McCallum points out, pegging the nominal rate at a lower value will produce a lower average rate of inflation, not the ever–higher inflation predicted by Friedman.

There is a rational–expectations argument, due to Sargent and Wallace (1975), to the effect that interest–pegging will make the price level indeterminate. However, as McCallum and others have pointed out, the indeterminacy in the Sargent–Wallace argument arises from a failure fully to specify monetary policy. Furthermore, as Laidler (1983) has emphasized, indeterminacy is not the same as the dynamic instability of Wicksell's cumulative process.

Thus the rational–expectations revolution has almost driven the cumulative process from the literature. Modern textbooks treat it as a relic of pre–rational–expectations

thought. For example, Sargent's (1987, p. 99) brief reference characterizes Wicksell's stability analysis as merely "important in the history of economic thought." The only other reference in the same text (p. 461) identifies Wicksell's "observation" as indeterminacy of the price level. Blanchard and Fischer (1989, pp. 577–80) present a model of the cumulative process but warn the reader that it is "hard to believe that individuals continue to form expectations of inflation adaptively if the inflation rate is ever–accelerating." They go on to examine interest–pegging under rational expectations and conclude that the danger of instability is "not inherent to such a policy." McCallum (1989) does not even refer to the cumulative process.

With the disappearance of Friedman's argument has come a revival in belief that interest–pegging is feasible, desirable, and even conducive to stability. Barro (1989) argues that interest–pegging is "reasonable" (p. 4), and presents evidence to the effect that the Federal Reserve Board has in fact pegged interest rates, although randomly, since World War II. Woodford (1988) argues that whereas monetary control may yield rational–expectations equilibria with the price level affected by extrinsic (sunspot) uncertainty, interest–pegging does not.

The purpose of this paper is to argue that, contrary to these rational–expectations arguments, Wicksell's cumulative process is not only possible, but inevitable, not just in a conventional macro model but also a flexible–price, micro–based finance–constraint model, whenever the interest rate is pegged. The paper follows Laidler (1983) and Cottrell (1989) in arguing that the essence of the cumulative process lies not in an economy's rational–expectations equilibria but in the disequilibrium adjustment process by which people try to acquire rational expectations. It argues that in a conventional macro model the process cannot converge under interest–pegging if people use any reasonable rule for formulating their beliefs, and that the cumulative process is a manifestation of the nonconvergence.

Thus the results obtained under adaptive expectations should be regarded not as a relic, but as an example drawn from a much more general analysis of real–time belief–formation of the sort studied in the literature on convergence (or nonconvergence) to rational expectations equilibrium (for example, Frydman and Phelps, 1983). Several recent authors (for example, Evans, 1985, Marcet and Sargent, 1986, and Woodford, 1986) have

suggested using the approach of this literature to address the question of which equilibrium will actually be reached in a world of multiple equilibria. The present paper uses the approach instead to address the question of what kind of monetary policy allows any rational–expectations equilibrium at all to be reached in a world where multiplicity is not at issue.

Section 2 below lays out a conventional macro model exhibiting the cumulative process under adaptive, but not rational expectations, when the nominal interest rate is pegged. Section 3, which contains the main result of the paper, shows that the same process will appear under any learning rule that satisfies a minimal assumption, the violation of which would arguably imply that people are refusing to learn from experience. Section 4 extends the result to a finance–constraint model. Section 5 considers further generalizations. Section 6 shows that the cumulative process will arise under less rigid interest control, and characterizes the extent to which the monetary authority can control the rate of interest without generating the process. Section 7 contains concluding remarks and suggestions for further research.

2.   A Conventional Model of the Cumulative Process

Friedman's argument is formalized by the following equation system:

(1) $y_t = y(R E_t(1/\pi_{t+1}))$; $R > 1$, $y' < 0$

(2) $\pi_{t+1} E_t(1/\pi_{t+1}) = f(y_t)$; $f, f' > 0$, $f(y^*) = 1$

(3) $E_t(1/\pi_{t+1}) - E_{t-1}(1/\pi_t) = \gamma[(1/\pi_t) - E_{t-1}(1/\pi_t)]$; $0 < \gamma < 1$

where y_t is demand for output, $\pi_{t+1} \equiv P_{t+1}/P_t$ the inflation factor and R the (pegged) nominal interest factor. The IS curve is (1); (2) is an expectations–augmented Phillips curve in which the left side is the average price–setter's expected relative price $P_{t+1} E_t(1/P_{t+1})$ and y^* is the natural level of output; and (3) is a convenient form of adaptive expectations. Assuming R pegged makes the LM curve redundant. The natural interest factor (one plus the natural interest rate) is r^*, with:

(4) $y(r^*) = y^*$, $r^* > 0$

Define:

(5) $\quad x_t \equiv 1/\pi_t,$

(6) $\quad \hat{x}_t \equiv E_{t-1}x_t,$

and

(7) $\quad x^* \equiv r^*/R > 0,$

where x_t (resp. $\hat{x}_t$) is the actual (resp. expected) "return to hoarding money." Define the function:

(8) $\quad h(x) \equiv [f(y(Rx))]^{-1}$

The *IS* and Phillips curves collapse to:

(9) $\quad x_{t+1} = \hat{x}_{t+1}h(\hat{x}_{t+1}); \quad h, \ h' > 0, \ h(x^*) = 1.$

The source of the cumulative process is the result in (9) that $h' > 0$. Because of this, any departure of the expectation $\hat{x}_{t+1}$ from its equilibrium value x^* will cause on overreaction of the actual value x_{t+1}, which in turn will generate a false signal. Specifically, if people have been overly pessimistic, in the sense that $\hat{x}_{t+1} < x^*$, they will be led to believe instead that they have been overly optimistic, becuase they will observe $x_{t+1} < \hat{x}_{t+1}$.

To see the connection with Wicksell more clearly, recall that, by (7), the equilibrium expectation x^* depends upon the level at which the nominal interest rate has been pegged. Thus whether or not people are overly pessimistic to begin with depends entirely upon whether or not the (real) market rate of interest $R\hat{x} - 1$ has been set below the natural rate $Rx^* - 1$.

Formally, adaptive expectations can be written as:

(10) $\quad \hat{x}_{t+1} - \hat{x}_t = \gamma(x_t - \hat{x}_t); \quad 0 < \gamma < 1.$

From (9) and (10):

(11) $\quad \hat{x}_{t+1} - \hat{x}_t = \gamma \hat{x}_t [h(\hat{x}_t) - 1]$

Equation (11) describes the evolution of $\hat{x}_t$ given any initial guess $\hat{x}_0$. It has two rest points: 0 and x^*. The former is a degenerate equilibrium with infinite inflation. The latter is the regular rational-expectations equilibrium, in which the market and natural rates are equal. The former rest point is stable, the latter unstable, as shown in Figure 1.
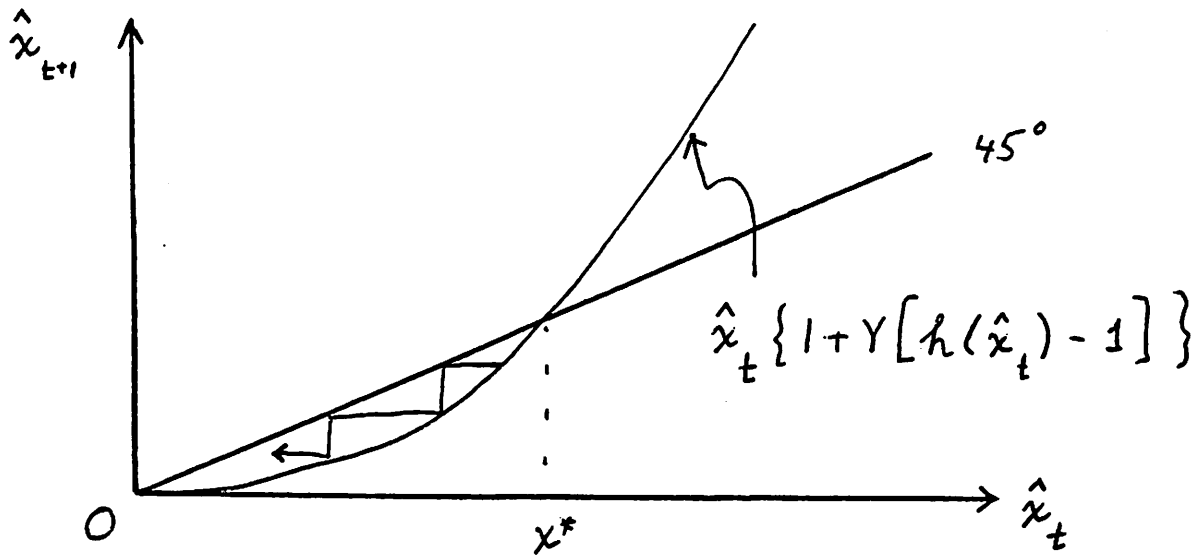
Figure 1. The cumulative process under adaptive expectations. The expected return to hoarding money falls to zero and inflation explodes when the market rate is less than the natural rate ($\hat{x}_t < x^*$).

As in Friedman's exposition, if the market rate has been set below the natural rate, the return to money will be less than expected, the expected return will fall even further below x^*, and inflation will increase forever.

3.   The Cumulative Process with General Learning

Adaptive expectations can be thought of as a particular rule by which people attempt to form rational expectations. The cumulative process implies that the rule doesn't work. Therefore it is unreasonable to suppose, as the above does, that the rule will be used forever. We must allow people to make fuller use of their adaptive capabilities.

On the other hand it would not be reasonable to assume people capable of forming rational expectations ab ovo. Even if nature endows us with innate capabilities of various sorts, it seems highly unlikely that the capability of rational macroeconomic forecasting is one of them. Rational expectations can be acquired only through experience, if at all.

These considerations suggest that $\hat{x}_t$ should depend upon experience, through a learning rule that can be specified independently of the macroeconomic relationship (9)

generating the variable being forecast, and that allows maximal flexibility. Suppose, accordingly, that:

(12) $\hat{x}_{t+1} = J_t(\Omega_t); \; t \geq 0$

where $\Omega_t \equiv \{x_\tau\}_0^t$ and $\{J_t\}$ is a sequence of functions. The initial guess $\hat{x}_0$ is arbitrary.

One example would be the least–squares learning rule of Bray–Savin (1986) and Marcet–Sargent (1986), which in this case amounts to using the sample mean:

(13) $\hat{x}_{t+1} = (t+1)^{-1} \sum_0^t x_\tau$

The rule (13) could also be rationalized as the way a nonparametric econometrician, or a Bayesian with diffuse priors, would forecast x_{t+1} under the maintained hypothesis that $\{x_\tau\}_0^\infty$ is a sequence of independent draws from an unchanging distribution. If the economy were actually in a rational expectations equilibrium, that maintained hypothesis would be correct (with $x_t = x^* \; \forall t$).

However, a rule as simple as (13) would be no more likely than adaptive expectations to survive if forecasts did not converge to rational expectations. So we must be more general. Accordingly, assume merely that the J_t's pay attention to experience, in the minimal sense that if x_t has always been underestimated (resp. overestimated) in the past, and has always risen (resp. fallen) in the past, then this period's forecast will be higher (resp. lower) than last period's. More formally:

> Assumption 1: *For any $t \geq 1$; if for all $\tau = 0,...,t-1$:*
>
> $x_\tau >$ *(resp. $<$) $\hat{x}_\tau$ and (if $\tau \geq 1$) $x_\tau >$ (resp. $<$) $x_{\tau-1}$, then:*
>
> $\hat{x}_t >$ *(resp. $<$) $\hat{x}_{t-1}$.*

If people violated Assumption 1 we would have to say that they are failing to learn from experience.

Even though Assumption 1 does not rule out any learning behaviour that genuinely attempts to learn from experience, the cumulative process will arise under interest–pegging. Again, the key to the process is the fact that $h' > 0$, which makes the system overreact to a

shortfall of the expected return below x^*. The false signal of an actual return less than expected will lead people to reduce their expectation even further below x^*. More formally:

Proposition 1. Suppose $\{x_t, \hat{x}_t\}_0^\infty$ are generated by (9) and (12), with $\hat{x}_0 > 0$ given. Let $\{J_t\}_0^\infty$ satisfy Assumption 1. If $\hat{x}_0 >$ (resp. $<$) x^*, then $\{\hat{x}_t\}_0^\infty$ is a strictly increasing (resp. decreasing) sequence.

Proof: Take the case where $\hat{x}_0 > x^*$. Suppose, contrary to the Proposition, that $\{\hat{x}_t\}$ is not strictly increasing. Let t be the first date at which $\hat{x}_t \leq \hat{x}_{t-1}$. By construction:

$$\hat{x}_{t-1} > \hat{x}_{t-2} > ... > \hat{x}_0 > x^*.$$

From this and (9):

$$h(\hat{x}_{t-1}) > h(\hat{x}_{t-2}) > ... > h(\hat{x}_0) > 1.$$

From these inequalities and (9):

$$x_\tau = \hat{x}_\tau h(\hat{x}_\tau) > \hat{x}_\tau \quad \forall \tau = 0,1,...,t-1,$$

and:

$$x_\tau = \hat{x}_\tau h(\hat{x}_\tau) > \hat{x}_{\tau-1} h(\hat{x}_{\tau-1}) = x_{\tau-1} \quad \forall \tau = 1,2,...,t-1.$$

This establishes the premise of Assumption 1. Therefore $\hat{x}_t > \hat{x}_{t-1}$, a contradiction. The proof when $\hat{x}_0 < x^*$ is analogous.‖

4.   A Finance–Constraint Model of the Cumulative Process

The present section shows that the price stickiness and lack of microfoundations of the above model are inessential to the cumulative process, by constructing a finance–constraint model with an almost identical dynamic structure. Consider a world such as that described by Lucas (1980) where agents are endowed with a flow of perishable commodities that appear identical to the outside observer. All agents have an absolute aversion to consuming their own endowments, and trade using fiat money, with a payment lag that induces a finance constraint (Clower, 1967; Kohn, 1981) on their decisions.

Time is discrete. Each period, trade proceeds as follows. First, each agent receives the money for his sale of goods in the previous period. Then a bond market opens, in which promises to pay one unit of money with certainty next period are traded for money, and the bonds issued last period are redeemed. There is no lag in receipt of payment for bond sales. Finally the goods market convenes, in which money available this period is exchanged for goods available this period.

Assume instantaneous price flexibility. An alternative source of nonneutrality is needed in order for the monetary authority to have even a momentary influence on the nominal interest rate when peoples' expectations are predetermined. To this end, the model has heterogeneous agents, and monetary policy produces a systematic distribution effect.[1] Specifically, there are two types of representative agents. The first, called shortsighted, have a two—period horizon. The second, called farsighted, have an infinite horizon.

All quantities are measured per farsighted agent. Shortsighted agents comprise overlapping generations with a constant endowment of e when young, and derive utility only from consumption when old. Thus each period the young sell e and the old spend eP_{t-1}, the proceeds from last period's sale.

Farsighted agents have a constant endowment y. They receive lump—sum transfers from the monetary authority at the beginning of each period. They face the finance constraints:

$$(14) \qquad P_t c_t \leq M_{t-1} + T_t + B_{t-1} - B_t/R_{t+1},$$

and the budget constraints:

$$(15) \qquad M_t - M_{t-1} = P_t y + T_t + B_{t-1} - P_t c_t - B_t/R_{t+1},$$

where c_t is consumption, M_{t-1} is money held just before receiving the transfer T_t, and B_t is the number of bonds demanded.

---

[1]This channel of monetary policy may seem unfamiliar to some readers. However, it was frequently invoked by classical quantity theorists to explain the short—run nonneutrality of money (Patinkin, 1972). It was also used more recently by Grossman and Weiss (1983) and Rotemberg (1984) in whose models monetary injections reduce the rate of interest by redistributing real cash balances toward those individuals who are in the market this period and have a relatively low propensity to spend out of real balances, and away from those who are out, who spend all their cash. Distribution effects also play an important role in overlapping generation models.

Subject to (14) and (15) for all t, and a No–Ponzi–Game constraint, the farsighted

agent maximizes $E_t \sum\limits_{j=0}^{\infty} \beta^j u(c_{t+j})$ where $\beta = 1/(1 + \delta)$, $\delta > 0$, and u is a smooth function on R_+

with $u' > 0$, $u'' < 0$ and $\lim\limits_{c \to 0} u'(c) = \infty$. If $R_{t+1} > 1$, necessary first–order conditions are:

(16) $\qquad u'(c_t) = R_{t+1} E_t[\beta u'(c_{t+1})/\pi_{t+1}]$

and

(17) $\qquad M_t = P_t y$,

Condition (16) is the usual condition for life–cycle consumption choices. Condition (17)

asserts that the finance constraint must be binding.

Each period, the price level and interest rate clear the goods and bond market. Total

consumption will equal total supply: $e + y$, of which the shortsighted consume $eP_{t-1}/P_t = e/\pi_t$.

Therefore:

(18) $\qquad c_t = y + e(1 - 1/\pi_t)$.

Inflation imposes a tax on the money holdings of the shortsighted, and hence raises c_t in (18).

This tax is the forced savings that earlier writers recognized as a channel for real effects of

monetary policy.[2]

If $R_t > 1$, then (17) implies the total money held in the economy at the start of period

t will be $M^s_{t-1} = P_{t-1}(y + e)$. Define $\mu_t \equiv M^s_t/M^s_{t-1}$. It follows that if $R_t, R_{t+1} > 1$ then:

(19) $\qquad \pi_t = \mu_t$.

Assume:

(20) $\qquad R > \max\{1, e/\beta(e + y)\}$.

Then it is straightforward to verify that there exists a unique stationary perfect–foresight

equilibrium with the constant interest factor R, and that it has the constant inflation factor:

(21) $\qquad \pi^* \equiv R\beta$

Thus the natural rate of interest $(R/\pi^* - 1)$ is the rate of time preference δ.

When the economy is not in a rational–expectations equilibrium, the beliefs

underlying the expectations operator E_t in (16) are complicated to describe in full. But

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[2]See Hayek (1932) for references to this notion among early classical writers.

according to (16) there is only one parameter of these beliefs that matters for the choice of c_t, namely the term multiplying R_{t+1} on the right side of (16). This term plays the same role as $\hat{x}_{t+1}$ in the previous section's model, and it has the same interpretation; the expected return to hoarding money. For purposes of the present model, redefine x_t as:

(22) $\quad x_t \equiv \beta\, u'[y + e(1 - 1/\pi_t)]/\pi_t.$

It follows that in order for the monetary authority to keep $R_{t+1} = R$, the monetary expansion factor must equal π_t, which in turn must be a solution to:

(23) $\quad u'[y + e(1 - 1/\pi_t)] = R\hat{x}_{t+1}.$

A (positive) solution exists, and is unique, if and only if:

(24) $\quad \hat{x}_{t+1} > u'(y + e)/R,$

and can be expressed as:

(25) $\quad \pi_t = \pi(\hat{x}_{t+1}) > 0,$

with:

(26) $\quad \pi' = R\pi^2/eu'' < 0.$

To interpret (26) note that $R\hat{x}_{t+1}$ is the opportunity cost of consumption to the farsighted. If $\hat{x}_{t+1}$ increases, the increased cost will reduce farsighted consumption. To clear the goods market at an unchanged nominal rate of interest the monetary authority must reduce forced saving, allowing the shortsighted to consume more.

When (24) is not satisfied the interest peg is no longer feasible. The interpretation is that with such a low opportunity cost the farsighted will want to consume more than the economy's GNP, so no amount of forced saving will clear the goods market. The rate of interest must rise.

Redefine the function h as:

(27) $\quad h(x) \equiv \beta R/\pi(x).$

It follows that:[3]

(28) $\quad x_t = \hat{x}_{t+1} h(\hat{x}_{t+1}); \; h, h' > 0, h(x^*) = 1,$

---

[3]From (23), $u'[y + e(1 - 1/\pi(x^*))] = Rx^*$. From this and (29), $\pi(x^*) = R\beta$. Substituting this into (27) yields $h(x^*) = 1$.

where

(29) $x^* \equiv u'[y + e(1 - 1/R\beta)]/R.$

Note that x^* is the value of x_t in the perfect foresight equilibrium, is well defined, and satisfies the existence condition (24).[4]

Equation (28) plays the role of (9) in the previous model, which contained the key to the cumulative process. The only difference is one of timing; x_t appears on the left side of (28) in place of x_{t+1}, reflecting the assumption of price–flexibility. Because $h' > 0$, the economy will overreact to any shortfall of the expected future return to money below x^*, by reducing the current realized return even further below x^*.

The economic reason for overreaction in this case closely follows Friedman's argument. To make people initially too pessimistic, start in perfect–foresight equilibrium with $x_t = \hat{x}_{t+1} = x^*$, and then reduce the nominal rate of interest. According to (29) this will raise x^*, without changing $\hat{x}_{t+1}$. The fall in the market rate will induce the farsighted to consume more, and this increase in demand will drive up inflation. Both reactions will reduce $x_t (= \beta u'(c_t)/\pi_t)$ below $\hat{x}_{t+1}$, even though $\hat{x}_{t+1}$ has been made lower than x^*.

Instability of x^* under learning is what one would expect from the results of Grandmont and Laroque (1986). By substituting x_{t+1} for $\hat{x}_{t+1}$ in (28) we see that x^* is stable under perfect foresight dynamics, or "indeterminate". Grandmont and Laroque show that indeterminacy in a one–dimensional system like (28) implies local instability under a restricted class of learning rules.[5] The following argument demonstrates instability under a broader class, using an argument almost identical to that of Proposition 1.

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[4] x^* is well defined if $y + e(1 - 1/R\beta) > 0$, which follows from (20); it satisfies (24) because $u'' < 0$ and, by (20), $1/R\beta > 0$.

[5] That is, the class of rules: $\hat{x}_{t+1} = J(x_{t-1}, x_{t-2}, ..., x_{t-T})$ for a fixed function J and a fixed memory–length T.

Out of perfect–foresight equilibrium, $\hat{x}_{t+1}$ derives from peoples' beliefs concerning the distribution of future inflation, interest rates and transfers, and their beliefs concerning how their beliefs will change in the future. All of these things it is reasonable to suppose the individual takes as given. But there is one determinant of an individual's return on money which he chooses, namely his own current consumption. An increase in c_t will result in less wealth at $t+1$ and hence a different (generally higher) value of $u'(c_{t+1})$. To simplify the analysis assume that the individual ignores the effect of his own current consumption when predicting the return to money. (This assumption is relaxed at no substantive cost in the Appendix.) He treats the problem of determining $\hat{x}_{t+1}$ as one of forecasting an exogenous random variable.

To avoid simultaneous interactions between x_t and $\hat{x}_{t+1}$, assume that the information available at t does not include x_t. That is: $\Omega_t \equiv \{x_\tau\}_0^{t-1}$. As before:

(30) $\qquad \hat{x}_{t+1} = J_t(\Omega_t); \ t = 1,2,...,$

and the initial guess $\hat{x}_1$ is arbitrary.

Assumption 1 must now be supplemented to include the initial revision at date 1 (when $\hat{x}_2$ is formed). At that date people have not yet observed a forecast error. The amended assumption is:

Assumption 1′:

(a)   *If $x_0 >$ (resp. $<$) $\hat{x}_1$, then $\hat{x}_2 \geq$ (resp. $\leq$) $\hat{x}_1$.*

(b)   *For any $t \geq 2$; if for all $\tau = 1,2,...,t-1$:*

$\qquad x_\tau >$ *(resp. $<$) $\hat{x}_\tau$ and $x_\tau \geq$ (resp. $\leq$) $x_{\tau-1}$, then:*

$\qquad \hat{x}_{t+1} >$ *(resp. $<$) $\hat{x}_t$.*

The new part (a) simply asserts that if a forecast is revised before any forecast error is observed it will be upward (resp. downward) if the first observation (for date 0) is greater (resp. less) than the first forecast (for date 1). The weakening of an inequality in (b) is done to allow the possibility that no revision is made at date 1. Although slightly stronger than

Assumption 1, this amended assumption does not appear to rule out any sensible learning rules.

Given Assumption $1'$, the cumulative process will again be inevitable. As before, if $\hat{x} < x^*$ the economy will overreact; subsequent forecast revisions will all be downward, and will never catch up with the fall in x_t. During this process the market rate: $R/\pi(\hat{x}_{t+1}) - 1$ will be less than the natural rate: $R/\pi(x^*) - 1 = \beta^{-1} - 1 = \delta$, and, by (26), inflation will be rising. More formally:

Proposition 1': Suppose $\{x_t, \hat{x}_{t+1}\}_0^\infty$ is generated by (28) and (30) with $\hat{x}_1 > 0$ given. Let $\{J_t\}_1^\infty$ satisfy Assumption $1'$. If $\hat{x}_1 > $ (resp. $<$) x^*; then $\hat{x}_2 \geq$ (resp. $\leq$) $\hat{x}_1$ and $\{\hat{x}_t\}_2^\infty$ is a strictly increasing (resp. decreasing) sequence.

Proof: Take the case where $\hat{x}_1 > x^*$. By (28) $x_0 = \hat{x}_1 h(\hat{x}_1) > \hat{x}_1 h(x^*) = \hat{x}_1$. So, by (a) of Assumption $1'$:

(31) $\hat{x}_2 \geq \hat{x}_1$.

So it remains to show that $\{\hat{x}_t\}_2^\infty$ is strictly increasing. Suppose the contrary. Let t be the first date ≥ 2 such that $\hat{x}_{t+1} \leq \hat{x}_t$. Then by (31) and the definition of t:

$$\hat{x}_t > ... > \hat{x}_2 \geq \hat{x}_1 > x^*.$$

From this and (28):

$$h(\hat{x}_t) > ... > h(\hat{x}_2) \geq h(\hat{x}_1) > 1.$$

From these inequalities and (28):

$$x_\tau = \hat{x}_{\tau+1} h(\hat{x}_{\tau+1}) > \hat{x}_{\tau+1} \geq \hat{x}_\tau; \ \forall \tau = 1,...,t-1,$$

and:

$$x_\tau = \hat{x}_{\tau+1} h(\hat{x}_{\tau+1}) \geq \hat{x}_\tau h(\hat{x}_\tau) = x_{\tau-1}; \ \forall \tau = 1,...,t-1.$$

This establishes the premise of Assumption $1'$ (b). Therefore $\hat{x}_{t+1} > \hat{x}_t$, a contradiction. The proof when $\hat{x}_1 < x^*$ is analogous.$\|$

5. Generalizing the Analysis

The analysis can be generalized in several directions without losing the cumulative process. For example, heterogeneity of preferences[6] and beliefs among farsighted agents can be accomodated. The cumulative process does not depend on any uniformity across agents. Let there be n farsighted agents, with utility functions U_i, learning rules J_{it}, and initial guesses $\hat{x}_{i1}$; $i = 1,...,n$. Then:

(22') $\quad x_{it} \equiv \beta\, U_i'(c_{it})/\pi_t$; $i = 1,...,n$,

(23') $\quad U_i'(c_{it}) = R\hat{x}_{it+1}$; $i = 1,...,n$,

and

(18') $\quad \sum_{i=1}^{n} c_{it} = y + e(1 - 1/\pi_t)$

If a temporary equilibrium exists at all with the interest factor pegged at R, then π_t will be the solution to (23') and (18'):

(25') $\quad \pi_t = \pi(\hat{x}_{1t+1},...,\hat{x}_{nt+1}) > 0$

with

(26') $\quad \pi$ decreasing in $\hat{x}_{it+1}$; $i = 1,...,n$.

Redefine h as:

(27') $\quad h(x_1,...,x_n) \equiv \beta R/\pi(x_1,...,x_n)$

then:

(28') $\quad \begin{cases} x_{it} = \hat{x}_{it+1}\ h(\hat{x}_{1t+1},...,\hat{x}_{nt+1}); & i=1,...,n;\ t=0,1,... \\ h > 0,\ h\ increasing\ in\ each\ argument. \end{cases}$

In a rational expectations equilibrium, $h(x_1,...,x_n) = 1$. Suppose:

(30') $\quad \hat{x}_{it+1} = J_{it}(\Omega_{it})$; $i = 1,...,n$; $t = 1,2,...$

where $\Omega_{it} \equiv \{x_{i\tau}\}_0^{t-1}$. Interpret Assumption 1' as applying to each of the rules $\{J_{it}\}$. Define $\hat{x}_t \equiv (\hat{x}_{1t},...,\hat{x}_{nt})$. Then:

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[6]Except for the rate of time-preference δ which we assume the same for all agents in order to avoid well-known degeneracy of long-run equilibrium with additive intertemporal preferences and heterogeneous rates of time preference.

Proposition 1″: Suppose $\{x_{it},...,x_{nt}\,;\,\hat{x}_{it+1},...,\hat{x}_{nt+1}\}_0^\infty$ is generated by (28′) and (30′) with $\hat{x}_1$ > 0 given. Let $\{J_{it}\}_1^\infty$ satisfy Assumption 1′ for all i. If $h(\hat{x}_1)$ > (resp. <) 1; then $h(\hat{x}_2)$ ≥ (resp. ≤) $h(\hat{x}_1)$ and $\{h(\hat{x}_t)\}_2^\infty$ is a strictly increasing (resp. decreasing) sequence.

Proof: Take the case where $h(\hat{x}_1)$ > 1. By (28′) $x_{i0} = \hat{x}_{i1}\,h(\hat{x}_1) > \hat{x}_{i1}\,\forall_i$. So, by (a) of Assumption 1′:

(31′) $\hat{x}_{i2} \geq \hat{x}_{i1}$ ∀i.

By (28′) and (31′), $h(\hat{x}_2) \geq h(\hat{x}_1)$. So it remains to show that $\{h(\hat{x}_t)\}_2^\infty$ is strictly increasing. By (28′) it suffices to show that $\{\hat{x}_{it}\}_2^\infty$ is strictly increasing ∀i. Suppose the contrary. Let $t \geq 2$ be the first date such that for some j, $\hat{x}_{jt+1} \leq \hat{x}_{jt}$. By (31′) and the definition of t:

$\hat{x}_{it} > ... > \hat{x}_{i2} \geq \hat{x}_{i1}$ ∀i.

From these and (28′):

$h(\hat{x}_t) > ... > h(\hat{x}_2) \geq h(\hat{x}_1) > 1$.

From these inequalities and (28′):

$x_{i\tau} = \hat{x}_{i\tau+1}\,h(\hat{x}_{\tau+1}) > \hat{x}_{i\tau+1} \geq \hat{x}_{i\tau}\,\forall i,\,\forall \tau=1,...,t-1$

and:

$x_{i\tau} = \hat{x}_{i\tau+1}\,h(\hat{x}_{\tau+1}) \geq \hat{x}_{i\tau}\,h(\hat{x}_\tau) = x_{i\tau-1}\,\forall i,\,\forall \tau=1,...,t-1.$

This establishes the premises of Assumption 1′(b) for each i. Therefore $\hat{x}_{it+1} > \hat{x}_{it}$ ∀i, a contradiction. The proof when $h(\hat{x}_1)$ < 1 is analogous.∥

Likewise, the analysis can be generalized to allow intrinsic uncertainty. This generalization is important not because intrinsic uncertainty is needed in order to have a meaningful distinction between interest–pegging and monetary control (variations in expectations during the learning process are sufficient for this purpose) but because (a) the cumulative process is robust to adding random variables on which people could condition their beliefs, and (b) the above instability proofs rely upon a strict monotonicity of $\{x_t\}$ which might not occur with intrinsic randomness.

The model of section 3 can easily incorporate random shocks to the natural rate. Replace the IS curve (1) with:

(1′) $y_t = y(R\hat{x}_{t+1}, \theta_t), y_1 > 0$

where θ_t is iid on the finite set $\{\theta^1,...,\theta^n\}$. There is a separate natural rate $r^{*i} > 0$ for each θ^i, satisfying $y(r^{*i}, \theta^i) = y^*$. Define:

(7') $\quad x^{*i} \equiv r^{*i}/R > 0 \quad \forall i,$

and

(8') $\quad h(x, \theta) \equiv [f(y(Rx, \theta))]^{-1}.$

The IS and Phillips curve yield:

(9') $\quad x_{t+1} = \hat{x}_{t+1} \, h(\hat{x}_{t+1}, \theta_t) \, ; \, h, h_1 > 0, \, h(x^{*i}, \theta^i) = 1 \quad \forall i.$

People can observe the realization of θ_t when forecasting x_{t+1}. Suppose they follow the practice of nonparametric econometricians with enough data, and base their forecast exclusively on the experience that has been associated with the same realization. Define:

$$S_t^i \equiv \{\tau \mid \tau \leq t \text{ and } \theta_{\tau-1} = \theta^i\}$$

Whenever $\theta_t = \theta^i$ they use a learning rule $J_t^i (\Omega_t^i)$, where:

$$\Omega_t^i \equiv \{x_\tau \mid \tau \in S_t^i\}.$$

Then the evolution of the economy over all dates $t+1$ with $\theta_t = \theta^i$ will proceed independently of what happens in any other date, and will be given by (9') and:

(12') $\quad \hat{x}_{t+1} = J_t^i (\Omega_t^i)$

Assumption 1 must be modified to restrict all inequalities to $\tau \in S_t^i$. The analysis is then formally equivalent to that of section 3, and the cumulative process again emerges.

In general, however, it is not possible for the adjustment processes associated with different states to proceed independently of one another. When they do not, the analysis of dynamics becomes complicated, and results under very general learning rules are hard to obtain. However, the cumulative process will still be exhibited under more special rules.

An extreme example of interdependence arises when people can observe x_t only with observational error. They cannot condition expectations on this random error because it is unobservable. Suppose, for example, in the finance–constraint model, that the agent observes $x_t + \theta_t$ instead of the true return, where θ_t is iid and $E(\theta_t) = 0$. Suppose people use the sample mean of these noisy observations as their estimator:

$$\hat{x}_{t+1} = (1/t) \sum_{0}^{t-1} (x_\tau + \theta_\tau).$$

Equivalently:

$$\hat{x}_{t+1} - \hat{x}_t = (1/t) \left[x_{t-1} - \hat{x}_t + \theta_{t-1} \right].$$

From this and (28):

(32) $\qquad \hat{x}_{t+1} - \hat{x}_t = (1/t) \left[\hat{x}_t \left(h(\hat{x}_t) - 1 \right) + \theta_{t-1} \right].$

The analysis of Ljung (1977) can now be applied. Consider the differential equation whose right hand side is the expected value of the term multiplying ($1/t$) on the right hand side of (32):

(33) $\qquad \dfrac{d}{dt} \hat{x} = \hat{x} \left(h(R\hat{x}) - 1 \right).$

The two stationary points of (33) are 0 and x^*. By Theorem 2 of Ljung, $\hat{x}_t$ cannot converge to x^* with positive probability if the derivative of the RHS of (33) is positive at x^*. That derivative is $x^* h'(x^*) > 0$. Therefore $\hat{x}_t$ cannot converge with positive probability to x^*.

Of course the cumulative process is not perfectly general; it is possible to construct models in which learning converges under interest–pegging. However, such generalizations tend to carry with them peculiar implications. In the case of the conventional macro model in section 3 it is clear that an upward–sloping IS curve ($y' > 0$) or a Phillips' curve that showed price setters reducing prices in the face of excess demand ($f' < 0$) could avoid the cumulative process. For then $h' < 0$ in (9). However, either of these features would appear highly implausible from a conventional Keynesian or monetarist perspective.

Likewise, the finance–constraint model of section 4 could be generalized to allow stability for some specifications. Suppose, for example, that all of the transfers went to the old shortsighted agents. Then a simple accounting excercise shows that as long as $R_t > 1$ the consumption of the farsighted would equal $c_t = y/\pi_t$, so that

(22″) $\qquad x_t = \beta \, U'(y/\pi_t)/\pi_t,$

and:

(23″) $\qquad U'(y/\pi_t) = R_{t+1}\hat{x}_{t+1}$

In this case, setting $R_{t+1} = R$ and solving (23″) would yield $\pi(\hat{x}_{t+1})$ with $\pi' > 0$. An increase in the expected return to holding money would raise the price level, holding the

interest rate pegged. This is the analogue to an upward–sloping IS curve. Instability would not necessarily arise under an interest peg because in this case $h' < 0$ in (28).

Not only would there be an analogue to an upward–sloping IS curve, but the impact effect of monetary policy on interest rates would also be perverse. If the authority kept μ_t on a preset course and allowed R_{t+1} to be determined by (23″), then, as before, $\mu_t = \pi_t$ if $R_t > 1$. So, from (23″), holding the expectation $\hat{x}_{t+1}$ fixed, an increase in the rate of monetary expansion $\mu_t(= \pi_t)$ would raise the nominal (and real) interest factor.

$(\partial R_{t+1}/\partial \mu_t = -U''y/\hat{x}_{t+1}\pi_t^2 > 0)$. In this case, by giving transfers to the shortsighted the monetary authority would be causing forced dissaving.

Thus at least within the class of models considered above, generalizations that would not yield the cumulative process would also yield striking subsidiary implications at odds with standard macroeconomic analysis. At the least this should create a presumption that Wicksellian instability will be the consequence of interest–pegging.

6.   Alternative Monetary Policies

The cumulative process is inevitable under a pegged interest rate, but not under a pegged rate of monetary expansion. In the finance–constraint model if μ_t were held constant at μ then as long as R_{t+1} remained greater than unity π_t would equal μ, by (19). Thus x_t would equal the constant:

$$\beta u' \ [y + e(1-1/\mu)] \ / \ \mu.$$

Under any reasonable learning rule $\hat{x}_{t+1}$ would converge to this constant.

This result relies heavily on the fixed velocity of circulation in the finance–constraint model, which rules out Cagan's (1956) stability problem.[7] The point remains that instability is inevitable under a pegged interest rate but not under a pegged rate of monetary expansion.

This section examines the related question of whether a less rigid form of interest control might avoid the cumulative process. Specifically, consider the finance–constraint

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[7]Sargent and Marcet (1956) examine the Cagan problem and show almost sure convergence for certain parameter values in a linear model with least–squares learning.

model, and suppose the monetary authority allows the interest rate to adjust to inflation, according to:

(34) $R_{t+1} = a + b\pi_t$; $b > 0$.

The pure peg studied above is the limiting case with $a > 0$ and $b = 0$. When $a > 0$ the authority allows the "backward–looking real rate" $R_{t+1} / \pi_t - 1$ to fall when inflation rises, but not by as much as under a pure peg. When $a < 0$ the authority makes the nominal rate "overreact" to inflation by raising the backward–looking real rate.

The case of $a = 0$, where the authority attempts to keep the real rate pegged, is the dividing line between stability and instability. If the nominal rate does not overreact to inflation then the cumulative process will arise. Otherwise beliefs can converge.

In perfect foresight equilibrium the inflation and interest factors must satisfy $\pi^* = R^*\beta$ as before. So, from (34):

(35) $\pi^* = a\beta / (1 - b\beta)$, $R^* = a / (1 - b\beta)$.

In order for this equilibrium to be well–defined we need to assume the equivalent of (20):

(36) $a / (1 - b\beta) > max \{1, e / \beta(e + y)\}$.

From (23) and (34):

(37) $u' [y + e (1 - 1/\pi_t)] = (a + b\pi_t) \hat{x}_{t+1}$

From (36) and the implicit function theorem a unique (positive) solution $\pi(\hat{x}_{t+1})$ exists to (37), with $a + b\pi(\hat{x}_{t+1}) > 1$, in a neighbourhood of $x^* \equiv \beta u'[y + e(1 - 1/\pi^*)] / \pi^* > 0$, with:[8]

(38) $\pi(x^*) \equiv \pi^*$,

and

(39) $\pi'(x) < 0$.

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[8]From (36), x^* is well–defined. Let $f(\pi, x) \equiv u' [y + e(1 - 1/\pi)] - (a + b\pi) x$. Then $f(\pi^*, x^*) = x^* \pi^* / \beta - (a + b\pi^*)x^*$, by the definition of x^*; this and the definition of π^* imply $f(\pi^*, x^*) = 0$. Note that $f_1(\pi, x) < 0$ when $f(\pi, x)$ is well defined and $(\pi, x) > 0$. So π^* is the unique solution to (37) when $x = x^*$.

Also, by (36) $a + b\pi^* > 1$. The extension of $\pi(x)$ in a neighbourhood of x^* follows from the implicit function theorem, with $\pi'(x) = (a + b\pi)/f_1 < 0$.

So for $\hat{x}_{t+1}$ in this neighbourhood

(40) $\pi_t = \pi(\hat{x}_{t+1}) > 0$.

Redefine h once more as:

(41) $h(x) \equiv \beta a/\pi\ (x) + \beta b$.

Then in a neighbourhood of x^*:

(42) $h > 0$ and $h' \gtrless 0$ as $a \gtrless 0$

and:

(43) $h\ (x^*) = 1$.

From (22), (37), (40), and (41);

(44) $x_t = \hat{x}_{t+1}\ h(\hat{x}_{t+1})$

In the case where $a > 0$, then (42) ~ (44) imply (28), so under Assumption $1'$, by

Proposition $1'$ the cumulative process will exist. However, when $a < 0$ convergence can

occur. Consider, for example, the least–square learning rule:

(45) $\hat{x}_{t+1} = (1/t) \sum_{0}^{t-1} x_t$.

Then:

(46) $\hat{x}_{t+1} - \hat{x}_t = (1/t)\ \hat{x}_t [h(\hat{x}_t) - 1]$.

Since $h' < 0$ the system (46) is easily seen to converge to x^* if it remains in the

neighbourhood where h is well–defined. From (46), $0 < \partial\hat{x}_{t+1}/\partial\hat{x}_t < 1$ for large t. So for large

t, $\hat{x}_t$ will not leave the neighbourhood $(\underline{x}, \bar{x})$ in which h is well defined and $a + b\pi > 1$.
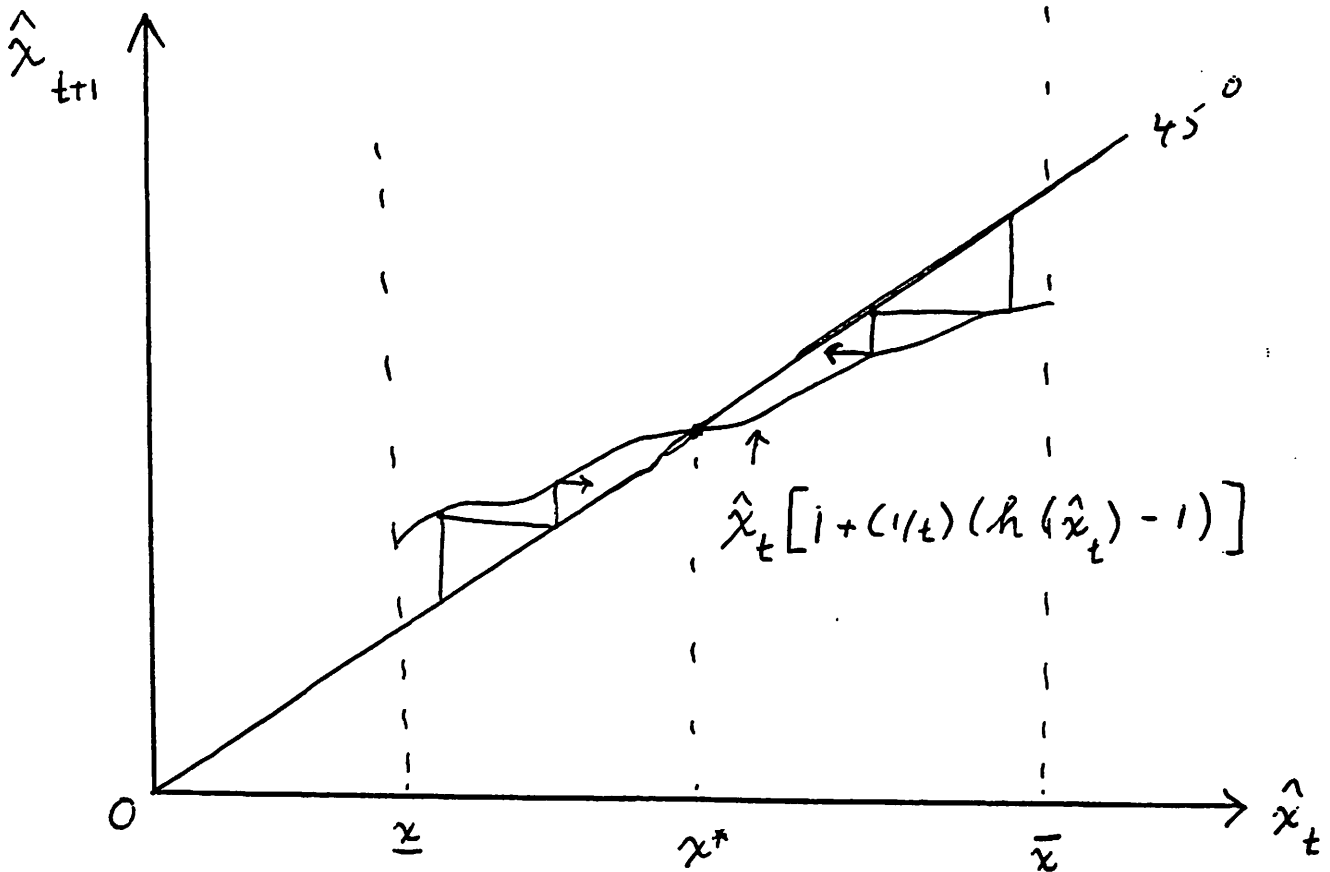
Figure 2. When the real rate of interest is made an increasing function of the rate of inflation, convergence can occur.

7. Conclusion

In summary, the above analysis shows a critical difference between pegging the rate of monetary expansion and pegging the rate of interest; stability is possible under the former but not under the latter. Although well–behaved rational–expectations equilibria exist, they are not stable when the monetary authority attempts to dampen the effect of changing expectations on the nominal rate of interest. The instability that follows from interest–stabilization is a manifestation of Wicksell's cumulative process. This process is an inevitable consequence of interest–pegging in both a conventional Keynesian macro–model and a flexible–price finance–constraint model, and under minimal assumptions on learning behaviour. Under the same assumptions on learning it is also an inevitable consequence of a

more flexible interest–control policy, as long as the monetary authority tries to prevent the rate of interest from adjusting as much as point–for–point with the rate of inflation.

It should be emphasized that the dynamic structure of the above model is extremely simple. Wicksellian instability might conceivably be offset by the interaction between learning and other forms of adjustment in a more complicated multidimensional system. For example, a low–interest policy might offset the Wicksellian process by stimulating investment and hence driving the natural rate down to the market rate.[9] It would be interesting to explore such issues in a more general setting.

It should also be emphasized that the analysis assumes all information Ω_t available for use in conditioning the forecasts $\hat{x}_{t+1}$ consists of observations generated under the regime of interest–pegging. This raises the question of whether experience under earlier, different regimes might be employed to make expectations converge.

The long–run nature of our results should also be stressed. The analysis of Poole (1970) suggests that interest–stabilization might constitute a sensible short–run policy if there is enough randomness in the demand–for–money function. The present analysis is best taken as a warning against extending that kind of policy too far into the future.

Perhaps the most important lesson of the analysis is that the assumption of rational expectations can be misleading, even when used to analyze the consequences of a fixed monetary regime. If the regime is not conducive to expectational stability, then the consequences can be quite different from those predicted under rational expectations. An economist who focussed only on rational–expectations equilibria would conclude from the above models that a policy of interest control would produce stable inflation in the long run, whereas in fact, if our analysis is correct, it would lead to accelerating inflation or deflation, and possibly to the collapse of the regime.

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[9]Wicksell himself (1905, pp. 198–199) explicitly recognized that the forced saving associated with low interest rates might have this effect.

This suggests that in general any rational—expectations analysis of monetary policy should be supplemented with a stability analysis of the sort conducted in this paper, to determine whether or not the rational expectations equilibrium could ever be observed. It also suggests expectational stability as a criterion for evaluating alternative monetary policies.

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APPENDIX

This appendix generalizes the analysis of section 4 to allow $\hat{x}_{t+1}$ to depend upon c_t.

Define:

$$\Omega'_t \equiv \{x_0,...,x_{t-1};\, c_0,...,c_t\} \equiv (\Omega''_t, c_t);\ t = 1, 2,...$$

and let:

$$(30') \quad \hat{x}_{t+1} = J_t (\Omega'_t);\ t = 1, 2,...$$

where the functions J_t satisfy:

Assumption A.1′

(a) If $c_1 \geq$ (resp. $\leq$) c_0 and $x_0 >$ (resp. $<$) $\hat{x}_1$, then $\hat{x}_2 \geq$ (resp. $\leq$) $\hat{x}_1$.

(b) For any $t \geq 2$ if:

(i) $c_t \geq$ (resp. $\leq$) c_{t-1}, and

(ii) for all $\tau = 1,2,...,\ t-1$:

$$x_\tau > (resp. <)\ \hat{x}_\tau\ and\ x_\tau > (resp. <)\ x_{\tau-1};$$

then $\hat{x}_{t+1} >$ (resp. $<$) $\hat{x}_t$.

(c) J_t is non–decreasing in c_t, holding Ω''_t fixed; for all $t \geq 1$.

This modification of Assumption 1′ allows the individual to anticipate that higher consumption today may reduce wealth and thus increase the marginal utility of consumption next period. It also allows him to increase the forecast $\hat{x}_{t+1}$ despite a history of overestimated and falling x_τ, if he is also planning to increase c_t. Nevertheless Wicksell's process will still arise. From (18) and (25):

$$(47) \quad c_t = c(\hat{x}_{t+1}) \equiv y + e(1-1/\pi\ (\hat{x}_{t+1})),\ c' < 0;\ t = 0,\ 1,...$$

Thus:

Proposition A.1′. Suppose $\{c_t, x_t, \hat{x}_{t+1}\}_0^\infty$ is generated by (28), (30′) and (47) with $\hat{x}_1 > 0$ given. Let $\{J_t\}_1^\infty$ satisfy Assumption A.1′. If $\hat{x}_1 >$ (resp. $<$) x^*; then $\hat{x}_2 \geq$ (resp. $\leq$) $\hat{x}_1$ and $\{\hat{x}_t\}_2^\infty$ is a strictly increasing (resp. decreasing) sequence.

<u>Proof</u>. Take the case where $\hat{x}_1 > x^*$. By (28) $x_0 = \hat{x}_1 h(\hat{x}_1) > \hat{x}_1 h(x^*) = \hat{x}_1$. By this and Assumption A.1′(a), $J_1(\Omega_1'',c_0) \geq \hat{x}_1$. Suppose, contrary to the Proposition, that $\hat{x}_2 < \hat{x}_1$. Then by (47), $c_1 > c_0$. By this and Assumption A.1′(c):

$$\hat{x}_2 = J_1(\Omega_1'',c_1) \geq J_1(\Omega_1'',c_0).$$

Therefore $\hat{x}_2 \geq \hat{x}_1$, a contradiction. This establishes that:

(31″) $\hat{x}_2 \geq \hat{x}_1$.

So it remains to show that $\{\hat{x}_t\}_2^\infty$ is strictly increasing. Suppose the contrary. Let t be the first date ≥ 2 such that $\hat{x}_{t+1} \leq \hat{x}_t$. Note that, by (31″) and the definition of t:

$$\hat{x}_t > \dots > \hat{x}_2 \geq \hat{x}_1 > x^*.$$

From this and (28):

$$h(\hat{x}_t) > \dots > h(\hat{x}_2) \geq h(\hat{x}_1) > 1.$$

From these inequalities:

$$x_\tau = \hat{x}_{\tau+1} h(\hat{x}_{\tau+1}) > \hat{x}_{\tau+1} \geq \hat{x}_\tau \; ; \; \tau=1,\dots,t-1,$$

and:

$$x_\tau = \hat{x}_{\tau+1} h(\hat{x}_{\tau+1}) \geq \hat{x}_\tau h(\hat{x}_\tau) = x_{\tau-1}; \; t=1,\dots,t-1.$$

It follows from Assumption A.1′ (b) that

$$J_t(\Omega_t'',c_{t-1}) > \hat{x}_t.$$

By (47):

$$c_t = c(\hat{x}_{t+1}) \geq c(\hat{x}_t) = c_{t-1}$$

So, by Assumption A.1′(c):

$$\hat{x}_{t+1} = J_t(\Omega_t'',c_t) \geq J_t(\Omega_t'',c_{t-1})$$

Therefore $\hat{x}_{t+1} > \hat{x}_t$, a contradiction. The proof when $\hat{x}_1 < x^*$ is analogous. ‖