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ABSTRACT

In this note we define and characterize two well–known topologies on spaces of preferences, namely the Kannai and closed–convergence topologies. We show that on spaces of economic preferences (e.g. continuous and monotonic preferences on R^{ℓ}_{+}), both topologies coincide and have a simple and appealing characterization. Namely, they agree with the questionnaire topology, which takes as a base the set of questionnaire sets

$$Q((x_1,y_1),\ldots,(x_n,y_n)) = \bigcap_{i=1}^{n} \{p|x_i\hat{p}y_i\}$$

where $\hat{p}$ denotes strict preference.

1. Introduction

In economic theory one often wishes to prove that a statement is true of all x in some set X, but because of some nonempty set of counterexample points $X_0 \subseteq X$ for which the statement is false, one must settle for proving that X_0 is small in some sense. One way to do this is to propose a "natural" measure on X, say μ, and then show that $\mu X_0 = 0$. More often, however, one demonstrates that X_0 is small by proposing a "natural" topology on X for which X_0 is nowhere dense; that is, for which $\overline{X - X_0} = X$. This is the same as saying that the complement of X_0 contains an open dense subset of X.

In this note we define and characterize two well–known topologies on spaces of preferences, namely the Kannai and closed–convergence topologies. A further, simpler characterization of these topologies is available when we restrict attention to spaces of economic preferences on certain types of sets, including the positive orthants R_+^ℓ of Euclidean spaces. By economic we mean that preferences are continuous in some preassigned topology on the set X of objects of preference, and that there is a natural partial order $\succ$ on X corresponding to "more than" with which preferences are consistent. Under suitable hypotheses on $\succ$ which are satisfied in the economic environments studied here, the Kannai and closed–convergence topologies coincide and are equal to a third topology with a much simpler characterization. This topology, dubbed the questionnaire topology, is built up from the collection of basic open sets

$$Q((x_1y_1),\ldots,(x_n,y_n)) = \bigcap_{i=1}^{n} \{p|x_i\hat{p}y_i\} \qquad (1)$$

where $\hat{p}$ denotes strict preference. The sets Q in (1) are called questionnaire sets. The rationale for the name is that if we ask an agent the n questions "Which do you prefer, x_i or y_i?" and the agent responds (truthfully) "x_i" to each question, then by posing this questionnaire we will have elicited precisely the information that the agent's preference lies in the set Q in

(1). It happens that with economic preferences of the type to be discussed here, answers of "indifference" to any question of the above form will occur only negligibly often, in a sense to be made precise shortly.

2. General Topology

Let X be any set; then a collection τ of subsets of X is called a topology on X if a) $\phi \in \tau$ and $X \in \tau$, b) if $0_1 \in \tau$ and $0_2 \in \tau$ then $0_1 \cap 0_2 \in \tau$, and c) if $0_\alpha \in \tau$ for all $\alpha \in A$, an index set, then $\bigcup_{\alpha \in A} 0_\alpha \in \tau$. The elements 0 of a topology τ on X are called open subsets of X, and $F \subseteq X$ is closed if X–F is open.

Two standard ways to define a topology are as follows. First, let $\mathscr{B}$ be any collection of subsets of X. Then there is a topology $\tau_{\mathscr{B}}$ on X which contains $\mathscr{B}$ and is minimal with respect to that property. In fact,

$$\tau_{\mathscr{B}} = \left\{ \bigcup_{\alpha \in A} 0_\alpha \middle| \text{ for each } \alpha \in A, \text{ an index set, } 0_\alpha = \bigcap_{i=1}^{n} B_i \text{ for some } B_1,\ldots,B_n \in B \right\} \cup \{\phi, X\}, \quad (2)$$

which is the set of all unions of finite intersections of elements of $\mathscr{B}$, together with the sets ϕ and X. Second, we can let Z denote the set of integers, X^Z the set of (countably) infinite sequences of elements of X, and then let C be any subset of $X^Z \times X$. A typical element of C may be denoted $(\{x_n | n \geq 1\}, x)$; and C has the interpretation that it defines a set of "convergent sequences" and their "limits". Any such definition of convergence induces a topology on X via

$$\tau_C = \{0 \subseteq X | \text{ if } x \in 0 \text{ and } (\{x_n | n \geq 1\}, x) \in C, \text{ then } x_n \in 0 \text{ for all sufficiently large } n\} \quad (3)$$

If $C = \phi$ (no sequence converges to any point) then $\tau_C = 2^X$, the discrete topology on X, while if $C = X^Z \times X$ (every sequence converges to every point in X) then $\tau_C = \{\phi, X\}$, the trivial topology on X. In general if $C_1 \subseteq C_2$ then $\tau_{C_2} \subseteq \tau_{C_1}$.

Now let X be a topological space with topology τ fixed for the remainder of this section; and let p be a preference on X — a complete, reflexive and transitive binary relation on X. We will let $\tilde{p}$ and $\hat{p}$ denote the symmetric and asymmetric factors of p — i.e. indifference and strict preference. We say that p is continuous if for every x the sets

$$\{z \in X | z \, \hat{p} \, x\} \qquad \{z \in X | x \hat{p} z\} \tag{4}$$

are open in τ. One can show that p is continuous if and only if the set

$$S(\hat{p}) = \{(x,y) \in X \times X | x \hat{p} y\} \tag{5}$$

is open in the product topology on X x X, denoted $\tau \times \tau$, which takes as a base the collection

$$\mathcal{B} = \{0_1 \times 0_2 | 0_1, 0_2 \in \tau\} \tag{6}$$

Since $\mathcal{B}$ is closed under finite intersections, the product topology is just the set of unions of elements of $\mathcal{B}$. Thus p is continuous if and only if $S(\hat{p})$ is a union of sets of the form $0_1 \times 0_2$, which is the same as saying that if $x\hat{p}y$ then $0_1 \hat{p} 0_2$ for some open product $0_1 \times 0_2$ containing the pair (x,y). (The notation $A\hat{P}B$ means that $x\hat{p}y$ whenever $x \in A$, $y \in B$, and $p \in P$. Also $A\hat{p}B$ means $A\{\hat{p}\}B$.)

3. The Kannai and Closed Convergence Topologies

Again, let a topology τ on X be fixed for the remainder of this section, and let $P_c(X)$ denote the set of continuous preferences on X. Then the Kannai topology on $P_c(X)$ is defined in Kannai (1970) to be the minimal topology τ_K, if it exists, such that the set

$$S = \{(x,y,p) \in X \times X \times P_c(X) | x\hat{p}y\} \tag{7}$$

is open in the product technology $\tau \times \tau \times \tau_K$ on $X \times X \times P_c(X)$. Often one shows that a collection of sets exists which is minimal with respect to a property by showing that at least one collection has the property and that any intersection of collections with the property also has the property; one then exhibits the minimal collection as the intersection of all collections with the property. But the second step in this line of reasoning does not seem to be elementary in the case of the Kannai topology, so the question of existence of τ_K will be answered here, as in Kannai (1970), by assuming some additional hypotheses on the topology on X.

The topological space (X,τ) is called normal if a) every singleton set $\{x\}$ is closed and b) if F_1 and F_2 are disjoint closed sets, then there are disjoint open sets 0_1 and 0_2 such that $F_1 \subseteq 0_1$ and $F_2 \subseteq 0_\alpha$. A set $E \subseteq X$ is called compact if for every family $\{0_\alpha | \alpha \in A\}$ of open sets such that $E \subseteq \bigcup_{\alpha \in A} 0_2$ there are a finite number of $\alpha_i \in A$ such that $E \subseteq \bigcup_1^n 0_{\alpha_i}$. In words, every open covering $\{0_\alpha | \alpha \in A\}$ has a finite subcovering $\{0_{\alpha_i} | 1 \leq i \leq n\}$. The closure of a set $E \subseteq X$, denoted E, is defined to be the minimal closed set containing E, or equivalently the intersection of all the closed sets containing E. The topology τ is called locally compact if every $x \in X$ is contained in some open 0 such that $\bar{0}$ is compact.

Lemma 1 (Kannai) Let (X,τ) be a normal, locally compact topological space. Then the Kannai topology on $P_c(X)$ exists and takes as a base the collection

$$\mathscr{B}_K = \{B(0_1,0_2) = \{p \in P_c(x) | \bar{0}_1 \hat{p} \bar{0}_2\} | 0_i \text{ open, } \bar{0}_i \text{ compact}\} \tag{8}$$

Proof (We include a proof here because Kannai's proof pertains to the case $X = R_+^\ell$ only, and is not self–contained. Moreover the hypotheses he states for the general case — local compactness and countability of a base for τ — do not seem to be sufficient).

We will show that the topology which takes $\mathscr{B}_K$ as a base actually makes S in (7) open in $\tau \times \tau \times \tau_K$, and then establish minimality by showing that if S is open in $\tau \times \tau \times \hat{\tau}$ for some topology $\hat{\tau}$ on $P_c(X)$, then each element of $\mathscr{B}_K$ must be open in $\hat{\tau}$.

To prove the first part, we assume that $x\hat{p}y$ and show that there are 0_x and 0_y open, with $\bar{0}_x$ and $\bar{0}_y$ compact, such that $x \in 0_x$, $y \in 0_y$, and $\bar{0}_x\hat{p}\bar{0}_y$; for then we will have

$$(x,y,p) \in 0_x \times 0_y \times B(0_x,0_y) \subseteq S \tag{9}$$

which shows that S is a union of the open product terms in (9) and hence open in $\tau \times \tau \times \tau_K$.

If $x\hat{p}y$ then $U_x\hat{p}U_y$ for some open U_x and U_y containing x and y. From normality there are closed sets F_x and F_y and open sets V_x and V_y, such that

$$x \in V_x \subseteq F_x \subseteq U_x \qquad y \in V_y \subseteq F_y \subseteq U_y \tag{10}$$

From local compactness there are 0_1 and 0_2 open such that $x \in 0_1$, $y \in 0_2$, and $\bar{0}_1$ and $\bar{0}_2$ are compact. Now let $0_x = 0_1 \cap V_x$ and $0_y = 0_2 \cap V_y$, to get all of the following inclusions:

$$\begin{aligned} &\bar{0}_x \subseteq \bar{0}_1 \quad && \bar{0}_x \subseteq F_x \subseteq U_x \\ &\bar{0}_y \subseteq \bar{0}_2 \quad && \bar{0}_y \subseteq F_y \subseteq U_y \end{aligned} \tag{11}$$

These together imply that $\bar{0}_x\hat{p}\bar{0}_y$ and that $\bar{0}_x$ and $\bar{0}_y$ are compact (a closed subset of a compact set is compact); also 0_x and 0_y are both open, all of which means that (9) holds as required.

For the second part, we begin with any $B(0_1,0_2) \in \mathscr{B}_K$ and show that $B(0_1,0_2) \in \tau$, by showing that if $p \in B(0_1,0_2)$ then $p \in 0 \subseteq B(0_1,0_2)$ for some $0 \in \hat{\tau}$. Since S is open in $\tau \times \tau \times \hat{\tau}$ we can write

$$S = \bigcup_{(x,y,p)\in S} 0_x \times 0_y \times O_p \tag{12}$$

with each $0_p \in \hat{\tau}$ and each 0_x and 0_y in τ. The collection of sets $\{0_x \times 0_y | (x,y) \in \bar{0}_1 \times \bar{0}_2, x\hat{p}y\}$ is an open covering of the product of compact sets $\bar{0}_1 \times \bar{0}_2$ which is therefore also compact; so we can write

$$\bar{0}_1 \times \bar{0}_2 \subseteq \bigcup_{i=1}^{n} 0_{x_i} \times 0_{y_i} \tag{13}$$

for a finite number n of <u>triples</u> $(x_i,y_i,p_i) \in S$. The set

$$0 = \bigcap_{i=1}^{n} 0_{p_i} \tag{14}$$

now satisfies $0 \in \hat{\tau}$ and $p \in 0$; and if $q \in 0$ then $0_{x_i} \hat{q} 0_{y_i}$ for $1 \leq i \leq n$. Given (13), this means that $\bar{0}_1 \hat{q} \bar{0}_2$, so $0 \subseteq B(0_1, 0_2)$. □

Another topology on spaces of continuous preferences was suggested by Debreu (1969), who used the topology of <u>closed convergence</u> on families of closed sets in the case where these sets correspond to elements of $P_c(X)$. If (Y,τ) is topological space and $\{F_n | n \geq 1\}$ is a sequence of closed subsets of Y then we define

$$\underline{\lim}_n F_n = \{y \in Y | \text{if } y \in 0 \in \tau, \text{ then } 0 \cap F_n \neq \phi \text{ for all sufficiently large } n\}$$

$$\overline{\lim}_n F_n = \{y \in Y | \text{if } y \in 0 \in \tau, \text{ then } 0 \cap F_n \neq \phi \text{ for infinitely many } n\} \tag{15}$$

Clearly $\underline{\lim}_n F_n \subseteq \overline{\lim}_n F_n$. We write $F_n \underset{c}{\rightarrow} F$ to denote that $\underline{\lim}_n F_n = F = \overline{\lim}_n F_n$, in which case we say that $\{F_n | n \geq 1\}$ closed converges to F. In practice all one needs to verify is that $\overline{\lim}_n F_n \subseteq F \subseteq \underline{\lim}_n F_n$.

Now we can let $Y = X \times X$ and define a notion of convergence of preferences by noting that if p is a continuous preference then the set

$$S(p) = \{(x,y,) \in X \times X | xpy\} \tag{16}$$

is a closed subset of $X \times X$. We write $p_n \underset{c}{\rightarrow} p$ to denote that $S(p_n)$ closed converges to $S(p)$ as defined above. In terms of elements of X, one can show that $p_n \underset{c}{\rightarrow} p$ if and only if

(1) If $x\hat{p}y$, then there are open 0_x and 0_y such that $x \in 0_x$, $y \in 0_y$, and $0_x \hat{p}_n 0_y$ for all sufficiently large n, and

(2) If xpy then for all open 0_x and 0_y containing x and y respectively, there is a sequence of pairs $\{(x_n,y_n)|n \geq 1\}$ such that $(x_n,y_n) \in 0_x \times 0_y$ and $x_n p_n y_n$ for all sufficiently large n. (17)

Lemma 2 Let (X,τ) be locally compact. Then the topology of closed convergence is at least as strong as the topology which takes $\mathscr{B}_K$(8) as a base. That is $\tau_{\mathscr{B}_K} \subseteq \tau_c = \{0 \subseteq P_c(x) | \text{if } p \in 0$ and $p_n \underset{c}{\rightarrow} p$, then $p_n \in 0$ for all sufficiently large n$\}$. (18)

Proof All we have to show is that the basic open sets $B(0_1,0_2)$ are open in τ_c. So let $p \in B(0_1,0_2)$ with $\bar{0}_1$ and $\bar{0}_2$ compact, and let $p_n \underset{c}{\rightarrow} p$; we have to show that $p_n \in B(0_1,0_2)$ for all sufficiently large n. If $(x,y) \in \bar{0}_1 \times \bar{0}_2$ then $x\hat{p}y$, so from (1) in (17) there are open 0_x and 0_y and n(x,y) such that $0_x \hat{p}_n 0_y$ for all $n \geq n(x,y)$. So

$$\bar{0}_1 \times \bar{0}_2 \subseteq \bigcup_{\substack{x \in \bar{0}_1 \\ x \in \bar{0}_2}} 0_x \times 0_y, \tag{19}$$

an open covering and hence

$$\bar{0}_1 \times \bar{0}_2 \subseteq \bigcup_1^m 0_{x_i} \times 0_{y_i} \tag{20}$$

for some finite subcovering. If $n \geq \max n(x_i,y_i)$ then $0_{x_i} \hat{p}_n 0_{y_i}$ for $1 \leq i \leq m$, which implies $p_n \in B(0_1,0_2)$. □

Conditions under which the reverse inclusion, and hence equality, holds in (18), are somewhat more specialized, but still occur often enough in economic theory so that it is probably not worth the effort to distinguish the two topologies.

First, notice that the definition of local compactness is equivalent to the statement that the topology can take as a base the set

$$\mathscr{B}_c = \{0 \in \tau | \bar{0} \text{ is compact}\} \tag{21}$$

A topology is defined to satisfy the <u>second axiom of countability</u> if it can take as a base a countable collection $\mathscr{B}_0$ of subsets of X; we also say in this case that τ is <u>second countable</u>. If τ is second countable and locally compact, then we can find a countable subset of $\mathscr{B}_c$ in (21) which also generates the topology. If $A \subseteq X$ then any topology τ on X induces a topology on A called the <u>relative topology on A</u>, which is denoted $\tau|_A$ and is defined by

$$\tau|_A = \{0 \cap A | 0 \in \tau\} \tag{22}$$

<u>Lemma 3</u> Let τ be normal, locally compact, and second countable, and let P be any subset of $P_c(X)$ such that for all $p \in P$ we have:

If xpy, then for every open 0_x and 0_y containing x and y respectively, there exist $\xi \in 0_x$ and $\eta \in 0_y$ such that $\xi \hat{p} \eta$. (23)

Then $\tau_K|_P = \tau_c|_P$.

<u>Proof</u> First, notice that given the condition (23) on preferences in P, 1) implies 2) in the definition of closed convergence for sequences $\{p_n | n \geq 1\} \subseteq P$. For if xpy and $(x,y) \in 0_x \times 0_y$ then we can let $(x_n, y_n) = (\xi, \eta)$ in (23) for all n, and from 1) we will have $\xi \hat{p} \eta$ for all sufficiently large n.

To prove the lemma we suppose that $E \subseteq P$ but $E \notin \tau_K|_P$, and try to show that $E \notin \tau_c|_P$ by constructing a sequence $\{p_n | n \geq 1\}$ that stays outside of E but closed converges to a preference in E.

If $E \notin \tau_K|_P$ then there is $p \in E$ such that every nonempty open set $0 \in \tau_K|_P$ containing p intersects $P - E$. In particular, if $p \in B(0_{1i}, 0_{2i})$ for $1 \leq i \leq n$, and

$$U_n = \bigcap_{i=1}^{n} B(0_{1i}, 0_{2i}), \tag{24}$$

then $U_n \cap (P-E) \neq \phi$. Since τ is normal and locally compact, and $S(\hat{p}) = \{(x,y)|x\hat{p}y\}$ is open, we know that every (x,y) in $S(\hat{p})$ is contained in some open product $0_x \times 0_y$ for which $\bar{0}_x\hat{p}\bar{0}_y$ and $\bar{0}_x$ and $\bar{0}_y$ are both compact. From second countablity, we can write

$$S(\hat{p}) \subseteq \bigcup_{i=1}^{\infty} 0_{x_i} \times 0_{y_i} \tag{25}$$

for some countable subset of those open products. Now, if we let $0_{1i} = 0_{x_i}$ and $0_{2i} = 0_{y_i}$ and define U_n according to (24), then we can pick a sequence of preferences $\{p_n|n \geq 1\}$ so that for every n,

$$p_n \in U_n \cap (P-E) \neq \phi. \tag{26}$$

So the proof is complete, once we show $p_n \xrightarrow{c} p$. If $x\hat{p}y$ then $(x,y) \in 0_{x_i} \times 0_{y_i}$ for one of the open products in (25); and for $n \geq i$ we will have $0_{x_i}\hat{p}_n 0_{y_i}$. This establishes (1) of closed convergence, and hence (2) according to the remarks above. □

It is worth remarking that the hypotheses of the lemma — specifically, normality and second compatibility — are sufficient to imply that the topology τ is metrizable. That is, there is a metric d on X such that τ can take as a base the set of spheroids

$$S = \{B_\varepsilon(x)|\varepsilon > 0, x \in X\} \tag{27}$$

where

$$B_\varepsilon(x) = \{z \in X|d(x,z) < \varepsilon\}. \tag{28}$$

This result is called the Urysohn Metrization Theorem.

Now recall the remarks on the questionnaire topology made at the end of the introductory section. It was asserted there that under suitable hypotheses, answers of "indifference" would occur negligibly often. This is true in the following sense:

Lemma 4 Let $p \in P_c(X)$ satisfy condition (23). Then the set of indifferent pairs

$$I(p) = \{(x,y) \in X \times X | x \tilde{p} y\} \tag{29}$$

is nowhere dense as a subset of $X \times X$ (in the product topology), and the indifference classes

$$I_x(p) = \{z \in Z | z \tilde{p} x\} \tag{30}$$

are all nowhere dense as subsets of X.

Proof Omitted.

Actually, in one of the examples to be discussed later on one can prove much more; if $X = R_+^\ell$ and p is monotonic then the indifference classes are all nowhere dense and have Lebesgue measure zero. Also, the set of indifferent pairs has product Lebesgue measure zero; and furthermore these results are true without any assumption of continuity.

4. Equivalence of τ_Q with τ_c and τ_K on Economic Preferences

The purpose of this paper is to show that on certain sets of economic preferences, the Kannai and closed convergence topologies both coincide with a third topology with a much simpler characterization, namely the questionnaire topology. So we turn now to a definition of what we mean by a domain of economic preferences. Loosely speaking, we need there to exist an ordering on X which is similar to "more than" on the real line. Formally, let $\succ$ be an asymmetric and transitive relation on X, and let $\succeq$ be the relation derived from $\succ$ via

$$x \succeq y \text{ if } x \in \overline{\{z \in X | z \succ y\}} \text{ or } y \in \overline{\{z \in X | x \succ z\}} \tag{31}$$

It is not hard to show that if p is a continuous preference which extends $\succ$ (i.e. for which $x \succ y$ implies $x\hat{p}y$), then $x \succeq y$ implies xpy. For what follows, we need there to exist a countable set $K \subseteq X$ which is dense (i.e. for which $\bar{K} = X$), and the following conditions on $\succ$:

(1) If x is not a minimal element according to $\succeq$, then for every open set $0'_x$ containing x there is an open set 0_x containing x, and $\xi \in 0'_x \cap K$, such that $0_x \succeq \xi$.

(2) If y is not a maximal element according to $\succeq$, then for every open set $0'_y$ containing y there is an open set 0_y containing y, and $\eta \in 0'_y \cap K$, such that $\eta \succeq 0_y$. (32)

(A maximal element according to $\succeq$ is an $x \in X$ such that $x \succeq z$ for all other $z \in X$, and a minimal element is one for which $z \succeq x$ for all other $z \in X$).

Proposition 1 Let X have a countable dense subset K, and let $P \subseteq P_c(X)$ be a set of continuous preferences on X which satisfy (23), and which extend an ordering $\succ$ satisfying (32). Then the restriction to P of the topology of closed convergence on $P_c(X)$ coincides with the questionnaire topology on P.

Proof It is easy to show that questionnaire sets are open in $\tau_{c|P}$; in fact this is true even if $P = P_c(x)$, the largest set on which τ_c makes sense, and whether or not X is separable.

For the reverse inclusion we will show that if $E \subseteq P$ is not questionnaire open then it is not open in $\tau_{c|P}$. That is, there are $p \in E$ and a sequence $\{p_n | n \geq 1\}$ which closed converges to p but stays outside of E.

If E is not questionnaire open then it is not true that every $p \in E$ is contained in some questionnaire $Q((x_1,y_1)\ldots(x_n,y_n)) \subseteq E$; so for some $p \in E$, every questionnaire containing p intersects P–E. Now consider the set

$$(K \times K) \cap S(\hat{p}) = \{(x,y) | x\hat{p}y,\ x \in K,\ y \in K\} = \{(x_1,y_1),\ldots,(x_n,y_n),\ldots\} \tag{33}$$

a countable set of pairs (x_i,y_i). If we let

$$Q_n = Q((x_1,y_1),\dots,(x_n,y_n)) = \{q | x_i \hat{q} y_i,\ 1 \le i \le n\}, \tag{34}$$

then from the above remarks $Q_n \cap (P-E) \neq \phi$; so we can pick a sequence $\{p_n | n \ge 1\}$ so that $p_n \in Q_n \cap (P-E)$ for all n. The proof will be complete once we show that $p_n \underset{c}{\rightarrow} p$; in view of (23), all we have to verify is (1) of closed convergence.

If $x\hat{p}y$ then $0'_x \hat{p} 0'_y$ for some open product $0'_x \times 0'_y$ containing (x,y). From (32) we can find $(\xi,\eta) \in (0'_x \times 0'_y) \cap (K \times K)$, and open sets 0_x and 0_y containing x and y respectively, such that $0_x \succeq \xi$ and $\eta \succeq 0_y$. Necessarily, $(\xi,\eta) = (x_i,y_i)$ for some i and (x_i,y_i) in (33), and for $n \ge i$ we will have $0_x p_n x_i \hat{p}_n y_i p_n 0_y$, which establishes $0_x \hat{p}_n 0_y$ for an open product $0_x \times 0_y$ containing (x,y). This establishes (1) of closed convergence and, hence concludes the proof of the proposition. □

<u>Proposition 2</u> Under the same hypotheses as Proposition 1, the Kannai topology on P exists and is equal to the questionnaire topology on P.

<u>Proof</u> We have to show that under these hypotheses, the set

$$S_P = \{(x,y,p) \in X \times X \times P | x\hat{p}y\} \tag{35}$$

is open in the product topology $\tau \times \tau \times \tau_Q|_P$, and that if S is also open in $\tau \times \tau \times \hat{\tau}|p$, then $\tau_Q|_P \subseteq \hat{\tau}|_P$. These steps will establish existence and equality simultaneously.

If $x\hat{p}y$ then the same 0_x, ξ, 0_y, and η of the last paragraph in the proof of Proposition 1 will suffice to yield

$$(x,y,p) \in 0_x \times 0_y \times Q(\xi,\eta) \cap P \subseteq S_P; \tag{36}$$

so S_P is therefore a union of the open products in (36) and hence open in $\tau \times \tau \times \tau_Q|_P$.

For minimality, suppose that S_P is open in $\tau \times \tau \times \hat{\tau}|_P$. If $x\hat{p}y$, then

$$(x,y,p) \in 0_x \times 0_y \times 0_p \subseteq S_p \tag{37}$$

for some open product with $0_p \in \hat{\tau}|_P$. In this case, $0_x \hat{0}_p 0_y$, so $p \in 0_p \subseteq Q(x,y)$. Thus Q(x,y) is a union of sets $O_p \in \hat{\tau}|_P$ and hence open in $\hat{\tau}|_P$; and the basic open sets $Q((x_1,y_1)...(x_n,y_n))$, being finite intersections of Q(x,y)'s, are also open in $\hat{\tau}|p$. So $\tau_Q|_P \subseteq \hat{\tau}|_P$, establishing minimality. □

Propositions 1 and 2 are both proven here with hypotheses that are more than sufficient. In particular, Proposition 1 does not seem to require condition (23) on preferences in P; and Proposition 2 does not require ξ and η to lie in any countable dense subset K, so X need not have such a subset for the result to hold. But these assumptions are not violated often enough in economics to make worthwhile the extra generality from relaxing them. As the next section shows, it is condition (32) on $\succ$ which is the hardest to satisfy.

5. Examples

In this section we present some economic examples of topological spaces (X,τ) and sets of preferences P on X which satisfy the hypotheses of Propositions 1 and 2. We also present some examples which do not satisfy all those hypotheses.

Example 1 Let $X_1 = R_+^\ell$ with the usual topology τ induced by the metric

$$d(x,y) = \max_{1 \leq i \leq \ell} |x_i - y_i| \tag{38}$$

Let $x \succ y$ denote that $x_i > y_i$ for all i, and let P be any subset of

$$P_{mc}(X_1) = \{p \in P_c(X_1) | \text{if } x_i > y_i \text{ for all i, then } x\hat{p}y\}, \tag{39}$$

the weakly monotonic preferences on X_1; and let K be the set of points in X_1 with rational coordinates. It is not hard to check that the triple (X_1,P,K) satisfies all the requirements of

Proposition 1.

Example 2 Let X_b be the set of bounded sequences of real numbers, with bound b;

$$X_b = \{\{x_n|n \geq 1\}|x_n \in R, \text{ and } |x_n| \leq b \text{ for all } n\} \quad (40)$$

Endow X_b with the product topology on R^Z (where Z is the set of integers); this topology takes as a base the set of products

$$\{\overset{\infty}{\underset{1}{X}} 0_i | 0_i \subseteq R \text{ is open for all i, and } 0_i = R \text{ for all sufficiently large i}\} \quad (41)$$

(where openness in (41) is defined according to the usual topology on R). Also, this topology is metrizable, by the metric

$$d(x,y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \min\{1, |x_i - y_i|\}. \quad (42)$$

Let $x \succ y$ denote that $x_i > y_i$ for all i, and let

$$K = \bigcup_{n=1}^{\infty} \bigcup_{\text{rational } r} \{x \in X_b | x_i \text{ rational}, i \leq n, x_i = r, i > n\}, \quad (43)$$

a countable dense subset of X_b. Finally let P be any subset of

$$P_{mc}(X_2) = \{p \in P_c(X_b)| \text{ if } x_i > y_i \text{ for all i, then } x\hat{p}y\}. \quad (44)$$

Then (X_b,P,K) satisfies all the assumptions of Propositions 1 and 2.

Example 3 Let $X_3 = R^Z$, the set of all sequences of real numbers. In the product topology on X_3 the set K in (43) is still dense; but $(X_3,P_{mc}(X_3), K)$ does not satisfy the conditions in (32), since no product–open set ever dominates or is dominated by any element of X_3.

Example 4 Let J = [a,b] and let $X_4 = D(J)$, the set of distribution functions on J;

$$X_4 = \{F: J \to [0,1] | F \text{ is left–continuous, non–decreasing, and } F(a) = 0\} \quad (45)$$

Endow X_4 with the topology of <u>weak convergence</u>, in which we define F_n to converge to F if

$$\int_J g(x)dF_n(x) \text{ converges to } \int_J g(x)dF(x) \tag{46}$$

for every bounded continuous function $g{:}J \to R$. This topology is metrizable, by the metric

$$d(F,G) = \inf\{\varepsilon | F(x-\varepsilon)-\varepsilon \leq G(x) \leq F(x+\varepsilon) + \varepsilon \;\; \forall x \in J\}, \tag{47}$$

the <u>Levy distance</u> between F and G. Define $F \succ G$ to mean that

$$\begin{aligned} &F(x) \leq G(x) \;\forall x \in J \\ &F(x) < G(x) \text{ for some } x \in J \end{aligned} \tag{48}$$

and let P be any subset of

$$P_{dc}(X_4) = \{p \in P_c(X_4) | \text{ if } F \succ G \text{ then } F\hat{p}G\}, \tag{49}$$

the continuous preferences on X_4 which agree with (first order) <u>stochastic dominance</u> (48). Let K be the countable set

$$K = \left\{\frac{1}{n} \sum_{i=1}^{n} \delta_{x_i} | x_i \text{ rational}, 1 \leq i \leq n < \infty\right\} \tag{50}$$

where

$$\delta_x(t) = \begin{cases} 0 & t \leq x \\ 1 & t > x, \end{cases} \tag{51}$$

a <u>point mass at x</u>. Then one can show that (X_4,P,K) satisfies all the hypotheses of Propositions 1 and 2.

<u>Example 5</u> Let J = {a,∞) or [−∞,b] or (−∞,∞), and proceed as in example 4, with $X_5 = D(J)$. The triple $(X_5,P_{dc}(X_5),K)$ does <u>not</u> satisfy the hypotheses of Propositions 1 and 2, since any spheroid $B_\varepsilon(F)$ either contains a sequence $\{F_n | n \geq 1\}$ which is undominated by any $G \in D(J)$ (if J = [a,∞)), or contains a sequence $\{F_n | n \geq 1\}$ which does not entirely dominate any $G \in D(J)$ (if J = [−∞,b]), or contains sequences of both types (if J = (−∞,∞)).

In examples 1, 2 and 4 the assertions are true for any subset P of the preferences described therein but this is not necessarily true when we consider proper subsets of the sets X involved. That is, it is possible that (X,P,K) satisfies the hypotheses of Proposition 1, but that (X',P',K') does not, where $X' \subseteq X$, $P' = P|_{X'}$, K' is a countable dense subset of X', and the topology is $\tau|_{X'}$. For example, if in example 1 we let

$$X' = \{x \in R_+^\ell | \Sigma x_i = 1\} \subseteq X_1 \tag{52}$$

and $\ell \geqq 2$, then no open subset of X' dominates any point in X' according to $\succsim$. Therefore (32) does not hold in this new environment.

6. Conclusion

On any set of continuous preferences $P \subseteq P_c(X)$ which satisfies the conditions of Propositions 1 and 2 there is a simple and appealing characterization of the Kannai and closed convergence topologies. Namely, under those conditions both topologies coincide with the questionnaire topology, or the topology of finite information on strict preference. Within such sets of preferences, a sequence $\{p_n | n \geq 1\}$ closed converges to p if and only if it questionnaire converges to p. That is, if for all x and y such that $x\hat{p}y$ we have $x\hat{p}_n y$ for all sufficiently large n (compare this simple condition to (1) and (2) of closed convergence (17)). The characterization holds also for any subset of such a family of preferences. For example in Example 1 one might wish to consider strictly monotonic preferences, or convex monotonic preferences, etc. Or in Example 4 one might want to consider strict subsets of $P_{dc}(X_4)$, such as expected utility preferences, or weighted utility preferences, etc.

The characterization is useful in proving that a certain type of subset of a preference set P is topologically small — i.e. nowhere dense in the relative topology on P. Using the closed convergence characterization one has to show that if P_0 is of the type in question, and $p \in \bar{P}_0$, then there exists a sequence $\{p_n | n \geq 1\}$ which closed converges to p and stays outside of $\bar{P}_0$.

Using the questionnaire characterization, one can assume that there is a nonempty questionnaire set $Q \subseteq \overline{P}_*$ and then show that P_* does not have the property defining the type of subset in question. The latter approach can make the proof substantially easier (see for example Redekop (1988)).

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