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First-Order Risk Aversion and Non-Differentiability*

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Summary First-order risk aversion happens when the risk premium π a decision maker is willing to pay to avoid the lottery $t \cdot \tilde{\varepsilon}$, $\mathrm{E}[\tilde{\varepsilon}] = 0$, is proportional, for small t, to t. Equivalently, $\partial\pi/\partial t|_{t=0^+} > 0$. We show that first-order risk aversion is equivalent to a certain non-differentiability of some of the local utility functions (Machina [7]).
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1 Introduction

Laboratory experiments have repeatedly shown that decision makers do not satisfy the expected utility hypothesis. The evidence did not discourage the use of expected utility, partly because it was a useful paradigm and partly due to the lack of convenient alternatives. Furthermore, it was not clear that the results obtained by expected utility theory do not hold for more general models. Indeed, Machina [7] argues that given a Fréchet differentiability condition, nonexpected utility functionals can be locally approximated by

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expected utility. Hence, many results of expected utility may be extended to nonexpected utility, especially comparative statics analysis, where local changes are examined (see Machina [8] and Chew and Nishimura [1]).

Expected utility analysis of decision making under risk strongly depends on the differentiability of the utility function. For example, consider a disturbance $\tilde{\varepsilon}$ whose expected value is zero. Multiply its outcomes by a positive number t and let t tend to zero. The risk premium $\pi(t)$ that a risk averse decision maker is willing to pay out of his present wealth level x^* to avoid the risk $t \cdot \tilde{\varepsilon}$ also declines to zero. For smooth expected utility, Pratt [10] showed that $\pi(t) = [-u''(x^*)/u'(x^*)]\sigma_{\tilde{\varepsilon}}^2 t^2 + o(t^2)$ which tends to zero at the rate t^2. However, if the utility function is not differentiable at x^*, then the risk premium is of the order of t. Of course, if the utility function is increasing, then it must be almost everywhere differentiable, and one may therefore convincingly argue that non-differentiability is of no importance in the expected utility model.

Suppose, however, that for every possible wealth level x^*, the expected utility approximation suggested by Machina [7] at the degenerate lottery $(x^*, 1)$ (i.e., x^* with probability 1) is non-differentiable at x^*. In that case, what is considered an exception within the expected utility model becomes the rule. Our work suggests that for a certain family of nonexpected utility theories, this may be the case. We prove this connection in Theorem 1 below by exploiting the concept of orders of risk aversion. In an earlier paper [11] we discussed situations where π tends to zero at rate t. We called the behavior of differentiable expected utility second-order risk aversion and the alternative first-order risk aversion (see also Montesano [9]). Second-order risk aversion implies that the derivative of the risk premium π with respect to t at $t = 0^+$ is zero, while under first-order risk aversion it is positive.

The concept of first-order risk aversion is not without economic meaning. Epstein [2] argues that first-order risk aversion permits more flexibility in functional forms. He shows that constant relative risk aversion under second-order attitude implies either enormous risk premiums for large gambles or almost zero premiums for moderate risks. By contrast, first-order risk aversion permits more realistic premiums. Epstein and Zin [4] show that models of first-order risk aversion explain stock and bond returns data much better than do models of second-order risk aversion. For other applications of this concept, see Epstein and Zin [3] and Segal and Spivak [11].

In the next section we discuss the concepts of orders of risk aversion and prove the equivalence between first-order risk aversion and the non-differentiability of the local utility functions.

2 Orders of Risk Aversion

Let (Ω, Σ, P) be a probability space. Let $\mathbf{x}(\mathbf{d})$ be the space of real random variables on Ω with values in the real interval $\mathbf{d} = [a, b]$. Elements of $\mathbf{x}$ are also called lotteries. The distribution function $F_{\tilde{x}}$ of a random variable $\tilde{x}$ is defined by $F_{\tilde{x}}(x) = P\{\omega \in \Omega : \tilde{x}(\omega) \leqslant x\}$. Assume that (Ω, Σ, P) is rich enough to generate all possible distribution functions on $\mathbf{d}$. For $\tilde{x} \in \mathbf{x}$, $x^* \in (a, b)$, and $t > 0$, let $x^* + t \cdot \tilde{x}$ be the lottery with the cumulative distribution function $F_{x^*+t\cdot\tilde{x}}(x^* + tx) = F_{\tilde{x}}(x)$. In other words, the lottery $x^* + t \cdot \tilde{x}$ is obtained from $\tilde{x}$ by multiplying its outcomes by t and adding x^*. Discrete random variables are denoted $(x_1, p_1; \ldots; x_n, p_n)$. Such a lottery yields x_i with probability p_i, $i = 1, \ldots, n$. The degenerate lottery $(x, 1)$ is denoted simply x and its cumulative distribution function F_x. Also, define $\| F - G \| = \int_{\mathbf{d}} | F(x) - G(x) | \, dx$.

Let $\succeq$ be a complete and transitive preference relation over $\mathbf{x}$ and define as usual $\succ$ and $\sim$ to be the strict preference and indifference relations obtained from $\succeq$, respectively. We assume that two random variables with the same distribution function are equally attractive. Consequently, the random variable $\tilde{x}$ stands for all random variables with the distribution function $F_{\tilde{x}}$. We assume throughout that the preference relation $\succeq$ is continuous in the topology of weak convergence. Also, assume that it is strictly monotonic with respect to first-order stochastic dominance. That is, $[\forall z,\ F_{\tilde{x}}(z) \leqslant F_{\tilde{y}}(z)$ and for some $z \in (a, b)$, $F_{\tilde{x}}(z) < F_{\tilde{y}}(z)]$ implies $\tilde{x} \succ \tilde{y}$. The functional $V : \mathbf{x} \to \mathbb{R}$ represents the preference relation $\succeq$ if for every $\tilde{x}, \tilde{y} \in \mathbf{x}$, $V(\tilde{x}) \geqslant V(\tilde{y})$ if and only if $\tilde{x} \succeq \tilde{y}$. The certainty equivalent of a lottery $\tilde{x}$ is defined as that number y such that $y \sim \tilde{x}$. The risk premium of the lottery $\tilde{x}$ is defined as the difference between the expected value of $\tilde{x}$ and its certainty equivalent.

Define a function $\pi(t; x^*, \tilde{x})$ implicitly by $x^* - \pi(t; x^*, \tilde{x}) \sim x^* + t \cdot \tilde{x}$. Since x^* is fixed for most of our discussion, we usually omit it from the expression π. Of course, $\pi(0; \tilde{x}) = 0$. If $\mathrm{E}[\tilde{x}] = 0$, then π is the risk premium of $x^* + t \cdot \tilde{x}$, and for a risk averse decision maker π is non-negative, it is increasing in t for $t > 0$ and decreasing in t for $t < 0$. (Risk aversion is defined with respect to mean preserving spreads). We assume that π is twice differentiable with respect to all its variables (with respect to t, for $t \neq 0$ only), and that at $t = 0$ both one sided derivatives with respect to t exist. The right-hand side derivative with respect to t at 0^+ is denoted by $D\pi(x^*, \tilde{x}) = \partial\pi(t; x^*, \tilde{x})/\partial t|_{t=0^+}$. Here too we often use the notation $D\pi(\tilde{x})$.

In [11] we defined a decision maker's attitude towards risk at x^* to be of order one if $D\pi(x^*, \tilde{x}) > 0$ for all $\tilde{x}$ such that $\mathrm{E}[\tilde{x}] = 0$. It was defined to be of order two if for all such $\tilde{x}$, $D\pi(x^*, \tilde{x}) = 0$ but $\partial^2\pi(t; x^*, \tilde{x})/\partial t^2|_{t=0^+} > 0$. If $\mathrm{E}[\tilde{x}] = 0$, then $\mathrm{E}[(-1) \cdot \tilde{x}] = 0$. It is therefore clear that first-order risk aversion requires non-differentiability of $\pi(t)$ at $t = 0$. Note that first- and second-orders of risk aversion do not imply risk aversion (see [11] for an example). This is the case because orders of risk aversion are local properties, while risk aversion is a global one.[1]

Suppose that the preference relation $\succeq$ with the representation functional V has the (set of) local utility functions $U(\cdot; F)$ (see Machina [7]). That is, for every $\tilde{x}$ and $\tilde{y}$,

$$V(\tilde{x}) - V(\tilde{y}) = \int U(x; F_{\tilde{y}}) d[F_{\tilde{x}}(x) - F_{\tilde{y}}(x)] + o(\| F_{\tilde{x}} - F_{\tilde{y}} \|) \tag{1}$$

Local utilities exist under the assumption that the functional V is Fréchet differentiable. Next we show that if such local utilities exist, then first-order risk aversion is equivalent to non-differentiability of the local utility $U(\cdot; F_{x^*})$ at x^*.

Theorem 1 *Suppose that the risk averse preference relation $\succeq$ with the representation functional V has local utilities as in Eq. (1). Then the following two conditions are equivalent.*

1. *The preference relation $\succeq$ satisfies first-order risk aversion at x^*.*

2. *The local utility $U(\cdot; F_{x^*})$ is not differentiable at x^*.*

Proof (1) $\Rightarrow$ (2): Let $\tilde{x} = (-1, \frac{1}{2}; 1, \frac{1}{2})$. Use Eq. (1) twice to obtain

$$V(x^* - \pi(t; \tilde{x})) = V(x^* + t \cdot \tilde{x}) \Rightarrow$$

$$U(x^* - \pi(t; \tilde{x}); F_{x^*}) = \frac{1}{2} U(x^* - t; F_{x^*}) + \frac{1}{2} U(x^* + t; F_{x^*}) + o(t) \tag{2}$$

[1]This is the place to mention that Lemma 1 in [11] is incorrect. Its correct version (which is also the one actually used in that paper) is that if $\mathrm{E}[\tilde{x}] = 0$ and the decision maker satisfies first-order risk aversion at x^*, then for a sufficiently small $s > 0$ and t, $x^* \succ x^* + t \cdot [\tilde{x} + s]$.

To see this, subtract $V(x^*)$ from both sides of the upper equation and $U(x^*; F_{x^*})$ from both sides of the lower equation. Observe that by first-order stochastic dominance, $\pi(t;\tilde{x}) \leqslant t$, therefore $\| F_{x^*-\pi(t;\tilde{x})} - F_{x^*} \| \leqslant t$. Also, $\| F_{x^*+t\cdot\tilde{x}} - F_{x^*} \| = t$. Denote the left and right derivatives of $U(\cdot; F_{x^*})$ by U_1^- and U_1^+, respectively. Differentiate both sides of Eq. (2) with respect to t to obtain

$$-U_1^-(x^*-\pi; F_{x^*})\frac{\partial \pi(t,\tilde{x})}{\partial t} = \frac{1}{2}U_1^-(x^*-t; F_{x^*}) + \frac{1}{2}U_1^+(x^*+t; F_{x^*}) + o'(t) \Rightarrow$$

$$D\pi(\tilde{x}) = \frac{U_1^-(x^*; F_{x^*}) - U_1^+(x^*; F_{x^*})}{2U_1^-(x^*; F_{x^*})}$$

Risk aversion implies concave local utility functions and monotomicity of preferences implies that the local utility functions are increasing (see Machina [7]). Hence $U_1^-(x^*, F_{x^*}) \neq 0$. It thus follows that if the above expression is positive, then $U(\cdot; F_{x^*})$ is not differentiable at x^*.

(2) $\Rightarrow$ (1): Let $\tilde{x} \not\equiv 0$ such that $\mathrm{E}[\tilde{x}] = 0$. Let $w_1 = \int_{x<0} dF_{\tilde{x}}(x)$ and $w_2 = \int_{x\geqslant 0} dF_{\tilde{x}}(x)$. Let y_1 and y_2 be the conditional averages of the negative and non-negative parts of $\tilde{x}$, respectively. That is, $y_1 = \int_{x<0} x dF_{\tilde{x}}(x)/w_1$ and $y_2 = \int_{x\geqslant 0} x dF_{\tilde{x}}(x)/w_2$. Then $\tilde{x}$ is a mean preserving spread of the lottery $\tilde{y} = (y_1, w_1; y_2, w_2)$. Since $\mathrm{E}[\tilde{x}] = \mathrm{E}[\tilde{y}] = 0$, it follows that $w_1 y_1 = -w_2 y_2$. Also, for every $t > 0$, $t \cdot \tilde{x}$ is a mean preserving spread of $t \cdot \tilde{y}$. Therefore, by risk aversion, $D\pi(\tilde{x}) \geqslant D\pi(\tilde{y})$. Similarly to Eq. (2), $U(x^* - \pi(t;\tilde{y})) = w_1 U(x^* + ty_1; F_{x^*}) + w_2 U(x^* + ty_2; F_{x^*}) + o(t)$. Hence $D\pi(\tilde{y}) = w_2 y_2 [U_1^-(x^*; F_{x^*}) - U_1^+(x^*; F_{x^*})]/U_1^-(x^*; F_{x^*}) > 0$. ∎

Theorem 1 states that, assuming Fréchet differentiability, first-order risk aversion depends on the non-differentiability of the local utility functions. Since expected utility theory is Fréchet differentiable and consistent with non-differentiable utility functions, economic results of first-order risk aversion must be consistent with non-differentiable expected utility. However, since the set of points where an increasing utility function can be non-differentiable is of measure zero, non-differentiable utility functions are of little consequence within the expected utility framework. On the other hand, for nonexpected utility functionals, the *local* utility function at the (degenerate) distribution F_{x^*} may be non-differentiable at the point $x = x^*$, for every x^*. (Formally, for every x^*, $U_1(x^*; F_{x^*})$ does not exist). In such cases non-differentiability becomes the norm, rather than the exception. Machina's [7] claim that non-expected utility behavior can be locally approximated by expected utility is formally correct, only that the approximating expected utility functional may be different from the standard one used in the literature, where the utility function is usually assumed to be differentiable. Therefore, first-order behavior may yield economic results that are qualitatively different from those of (smooth) expected utility.[2]

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[2]First-order risk aversion is not an empty concept. Gul's [5] disappointment aversion model is Fréchet differentiable, and has non-differentiable local utility functions as above. For a direct proof that Gul's model satisfies first-order risk aversion, see Loomes and Segal [6].

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