1983

The Optimal Length of a Patent with Variable Output Elasticity and Returns to Scale in Research and Development

M. H. I. Dore

Joseph Kushner

I. Masse

ECONOMICS AND LAW WORKSHOP

83-~~10~~ 11

THE OPTIMAL LENGTH OF A PATENT WITH VARIABLE OUTPUT ELASTICITY AND RETURNS TO SCALE IN RESEARCH AND DEVELOPMENT

by

M.H.I. Dore, J. Kushner and I. Masse*

4032 SSC 4:00 p.m. October 6, 1983

* We would like to thank Lewis Soroka for helpful comments and assistance. Our intellectual debt to William Nordhaus is enormous, as will be obvious to anyone reading these pages.

Major funding for the Centre for Economic Analysis of Property Rights has been provided by the Academic Development Fund, The University of Western Ontario. Additional support has come from The Bureau of Policy Coordination, Consumer and Corporate Affairs. The views expressed by individuals associated with the Centre do not reflect official views of the Centre, The Bureau of Policy Coordination, or The University of Western Ontario.

THE OPTIMAL LENGTH OF A PATENT WITH VARIABLE OUTPUT ELASTICITY AND RETURNS TO SCALE IN RESEARCH AND DEVELOPMENT

INTRODUCTION

It is commonly held that an increase in the term of patent increases the reward to the inventor and thereby stimulates invention and innovation.[1] This benefit, however, is not without cost, for the term of monopoly power conferred by the patent is extended. From the social viewpoint, the optimal term of the patent is therefore the period in which the marginal social cost of conferring of monopoly power to the inventor equals the marginal social benefit arising from increased inventive and innovative activities.

---

[1] A dissenting view is held by Hirshleifer (1971) who conjectures that legal remedies to protect property rights such as patents are not necessary to stimulate invention because investment in R&D gives the firm superior information which has both direct and indirect benefits. Direct benefits would include new products and cost saving techniques whereas indirect benefits might arise through the use of the information in stock market and commodity investments.

Nordhaus (1969) was the first to develop a methodology to determine the optimal life of a patent. A welfare function for the industry to which the patent is applicable is specified and the function is then maximized with respect to the life of the patent. Recent studies have extended Nordhaus' methodology to show that competition for patents results in the complete exhaustion of all privately appropriable surpluses associated with the patent [for example, Kitti (1973), Dasgupta and Stiglitz (1980), and Berkowitz and Kotowitz (1979)]. Given the exhaustion of the surplus, the new inventions are not welfare improving and therefore the time optimality of patents is low. If on the other hand, the appropriable surplus is not dissipated, the optimal patent term is increased [see Stigler (1968) and McFetridge and Rafiquzzaman (1983)]. In another recent paper, Berkowitz and Kotowitz (1982) have even argued that it is not in the interest of a small open economy to have a patent system at all. Hence the optimal term of the patent depends on the market structure assumed in the model.

Almost all work cited here has relied heavily on Nordhaus' original specification of an invention possibility function, B(R), which is concave. This has meant that the numerical estimates of the optimal patent term have been based on a constant elasticity of cost reduction with respect to research.

In this paper we relax the assumption of constant elasticity by specifying a generalized invention possibility function (henceforth, IPF) which admits increasing as well as decreasing returns to scale.

Such an IPF is in agreement with empirical observations. Comanor (1965) found significant economies of scale in R & D in the pharmaceutical industry at low firm sizes (or low levels of R & D) and decreasing returns as firm size or the level of R & D increased beyond a certain level. Other evidence of increasing returns is given in Schmookler (1967) and Angilley (1973)[2].

The empirical evidence on the elasticity of cost reduction with respect to research (loosely called 'output elasticity' α) is also at variance with what is assumed in the literature. In his numerical estimates, Nordhaus (1969, p. 80) assumes the elasticity α to be 0.1. However, the estimated value of α is wide ranging. Excluding externalities in the production of knowledge, estimates of elasticity of output with respect to research and development range from 0.05 to 0.12 [for example see Minasian (1962), Mansfield (1965, 1968, 1977) and Terleckyj (1974)]. In a more recent study, Griliches (1980) applied Cobb-Douglas production functions to 900 firms and found the overall elasticity to be .07 ranging from an average of .1 for the more intensive R & D industries to .05 for the less intensive firms.

---

[2]If one applies E. A. G. Robsinson's (1953) classic analysis of scale effects to research and development, one would expect both increasing and decreasing returns. For a survey of the literature, see Morton I. Kamien and Nancy L. Schwartze (1975). It should also be noted that increasing returns to scale appear to be the motive in a large number of mergers [see Murray N. Friedman (1958)]. Based on the above, we therefore consider our generalization to be more realistic than the common assumption of only decreasing returns. F. M. Scherer (1972) in his geometric interpretation makes a similar modification.

The IPF used in this paper has a number of advantages. In particular it is shown that first, a socially optimal patent exists if and only if there are decreasing returns to research. However some comparative static results throw light on the increasing returns phase. Second, assuming a given social rate of discount, it is shown that the optimal patent depends only on the output elasticity (α) and demand elasticity. This fact considerably simplifies the numerical estimation of the patent term, which ranges from 40 years to just a few days, depending on the elasticities.

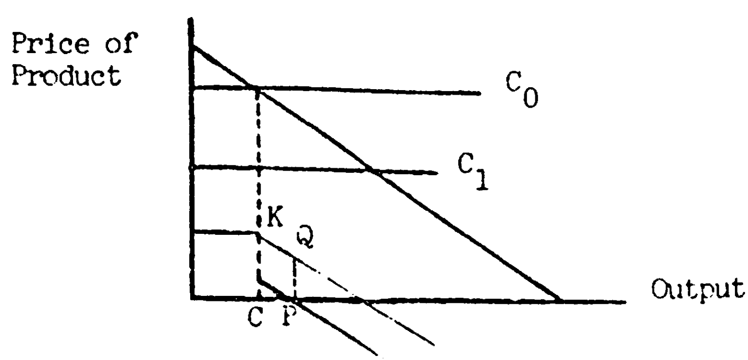
The plan of the paper is as follows: the first section begins with a graphical review of Nordhaus' model, followed by an analysis of the implications of a generalized IPF. Section two considers some comparative static results of the model. In the third section, the social cost of a patent is considered in terms of the deadweight loss, and the formula for the socially optimal patent is derived in terms of elasticities. Numerical simulations are also given and the results are compared with the constant elasticity case. In general, it is shown that the constant elasticity assumption over-estimates the optimal life of a patent.

§ THE MODEL

Consider a profit maximizing inventor who licences the patented productive technique to producers who in turn have the choice of using existing techniques or the patented technique. Following Nordhaus, we assume that (a) there is perfect knowledge regarding the existing technique, (b) the product is manufactured under constant cost conditions using either the existing technique or patented technique, and (c) the demand curve is downward sloping. Figure I shows the industry demand curve for the consumer product. The average cost using existing productive techniques and the patented technique is C_0 and C_1 respectively.

If the manufacturer produces to the left of E, the maximum per-unit royalty fee that can be charged is EG where $EG = C_0 - C_1$ and if he produces to the right of E, the maximum fee is the vertical distance between the demand curve and the new cost function. The derived demand curve for the patent is represented by JKL, where $CK = C_0 - C_1$, which is the vertical distance between the demand curve and the new cost function. The marginal revenue curve is given by JKMN. The optimal royalty will be where marginal revenue from royalties will be zero, as the marginal cost of a license is zero. Hence, the profit maximizing per unit royalty is CK where $CK = C_0 - C_1$. The product price and output remain unchanged. It should be noted that the royalty may be less than CK depending on the magnitude of the cost reduction.[3] Nordhaus

[3]According to Nordhaus, if the patent is for an exceptionally important technique, the result is a profit maximizing fee which is less than the average cost reduction. For example in the diagram below the profit maximizing fee is PQ where PQ < CK. In this case the price of the product is reduced and output is expanded.



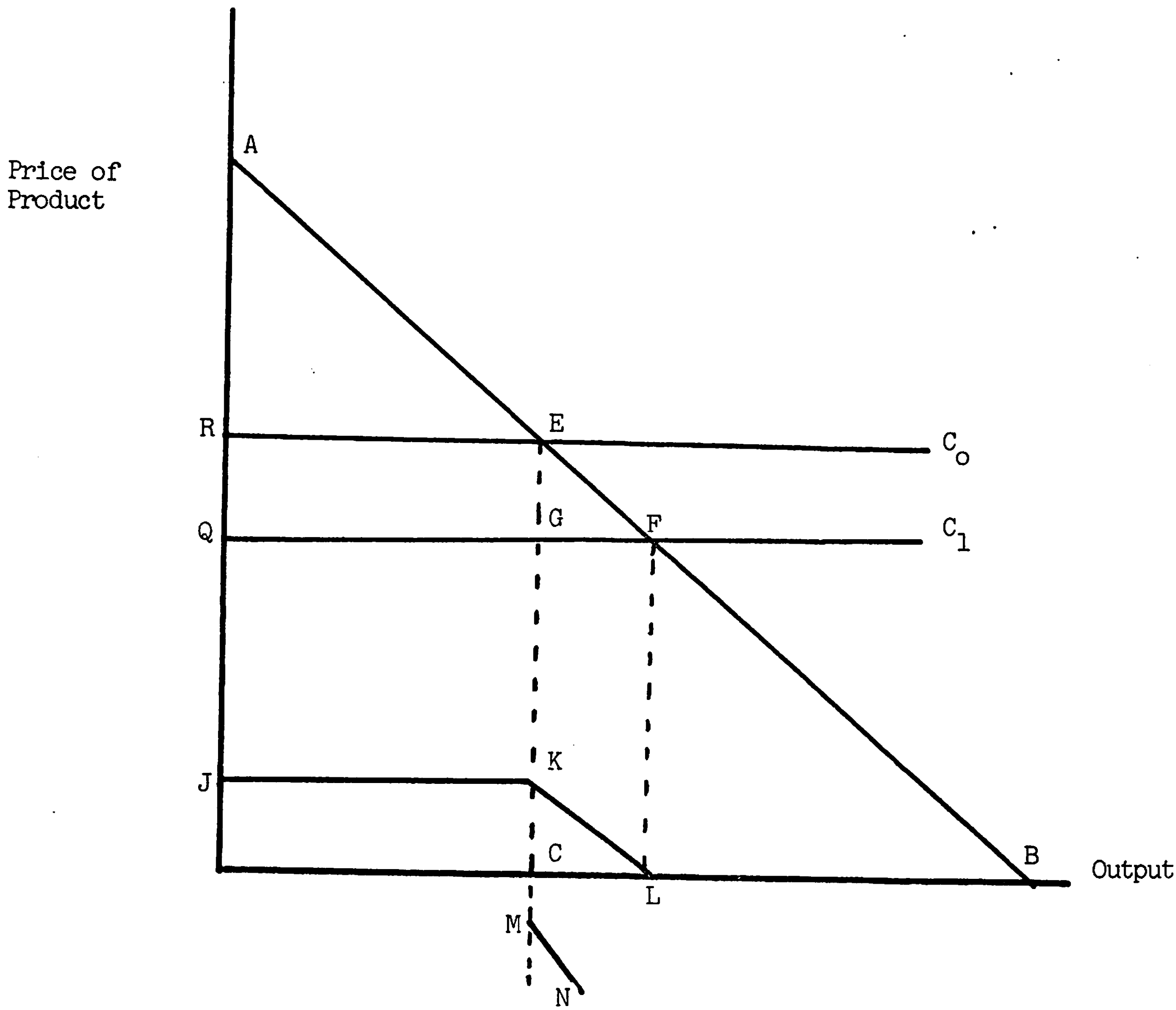


FIGURE 1

assigns the name "run-of-the-mill" to small inventions where the royalty rate equals the total cost reduction, and the name "drastic" to large inventions where the royalty rate is less than the cost reduction. Like Nordhaus we consider only run-of-the-mill inventions where production is carried out under perfectly competitive conditions.[4] Under these circumstances the optimal royalty fee, $y = C_0 - C_1$. (1)

Following Nordhaus, we introduce an IPF whereby the inventive output, measured by the percentage unit production cost reduction is positively related to the amount of research and development expenditure (R). Using Nordhaus' terminology,

$$\frac{C_0 - C_1}{C_0} = B(R) \quad (2)$$

Nordhaus assumed $B(R) = \beta R^{\alpha}$ where $\alpha < 1$ depicting diminishing returns to inventive activity. In this paper we adopt an IPF which initially depicts increasing returns and thereafter diminishing returns and incorporates a variable output elasticity.

---

[4]If we relax the assumption that the industry output is produced under conditions of perfect competition, the revenue to the investor is once again the per unit cost differential arising from the patented technique. The total revenue to the patentee is smaller however due to the monopolist's restricted output.

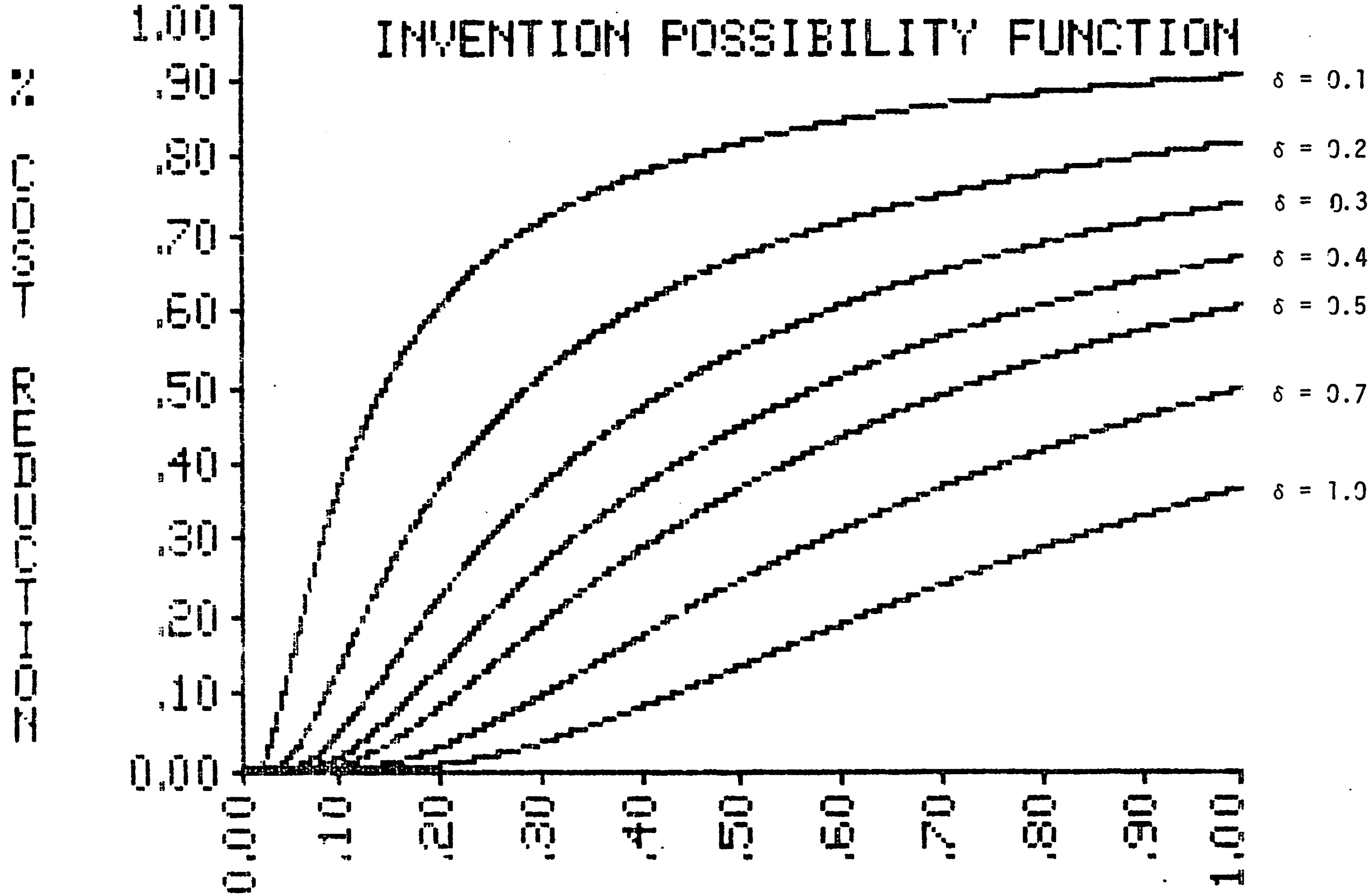


FIGURE 2

Note: R&D has been normalized to equal 1.

In order to reflect both variable returns to research as well as a variable output elasticity, let the IPF be represented by:

$$B(R) = e^{-\delta/R} \quad \delta > 0, \quad R > 0 \qquad (3)$$

where R stands for research and development expenditure (as before), and δ represents the 'ease of invention'. The smaller the value of δ the easier the invention and the larger the δ the more difficult the invention, since a larger δ leads to a smaller cost reduction.

Equation (3) is drawn in Figure 2 for different values of δ. Analysis of equation (3) in terms of the output elasticity makes the comparative statics (given in section two below) easier to follow. It can be shown that the output elasticity[5] α of B(R) is equal to δ/R, i.e. $\alpha = \delta/R$. Geometrically, the point where a ray from the origin is tangent to the curve (Figure 3), gives $\alpha = 1$. Let $R = R_0$, where $\alpha = 1$.

A further important property of equation (3) is that at the point of inflexion[6] in Figure 3, $\alpha = 2$. From this it is obvious that where $\alpha = 2$, $R = 1/2\ R_0$. That is, at the point of inflexion, where increasing returns end and diminishing returns begin, $R = \frac{1}{2} R_0$. Hence, α is a decreasing function of R, and the point where $\alpha = 2$ turns out to be useful in analyzing equation (11) below.

---

[5]As far as possible, we have retained the same notation as in Nordhaus (1969).

[6]At the point of inflexion $B''(R) = 0$, which implies that $\delta/R = 2$.

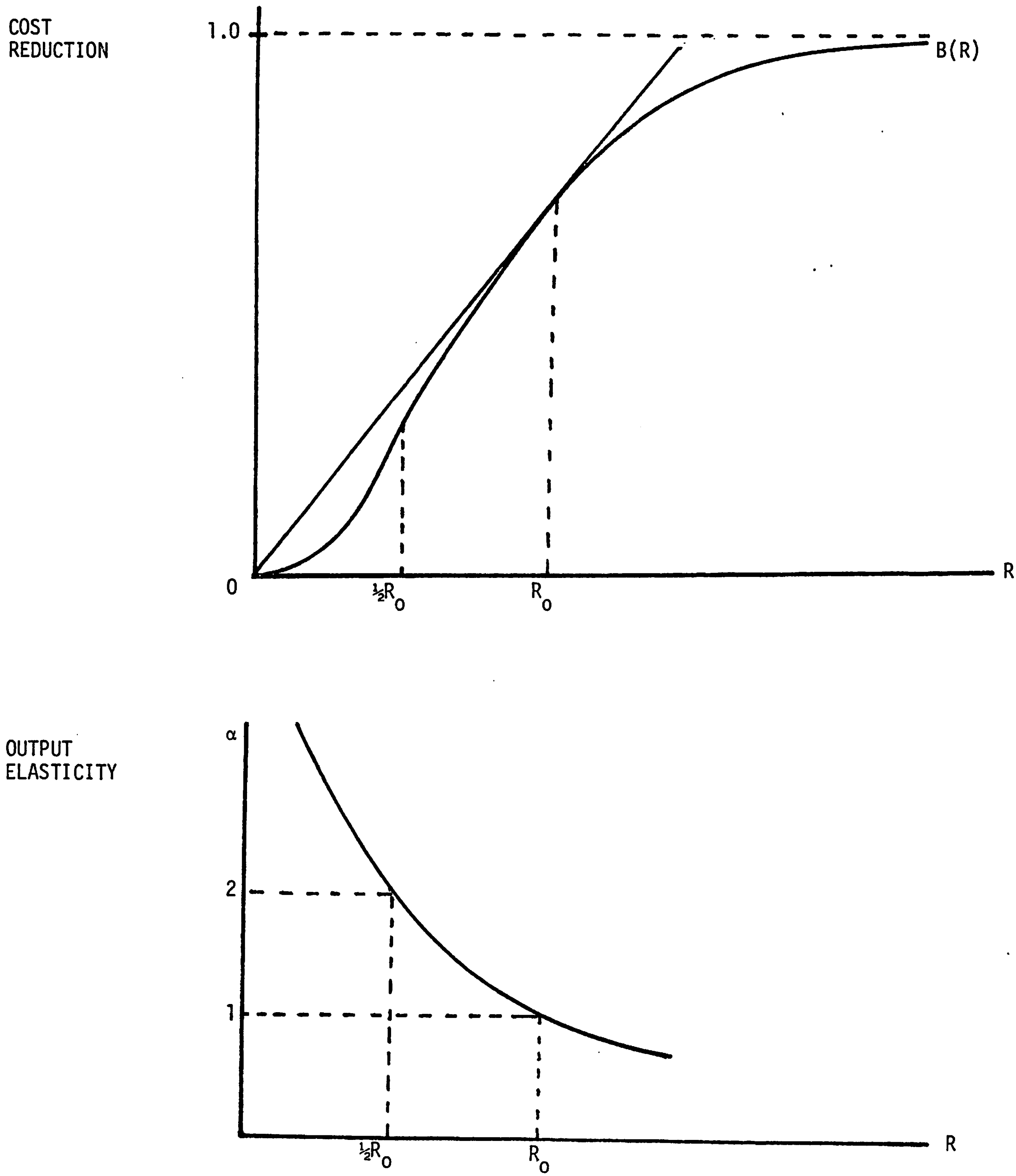


FIGURE 3

To return to the development of the model, in terms of Figure 1, the financial return to the inventor in any given year is determined by the per unit royalty fee and the total output less the inventor's costs. Using Nordhaus' terminology, the present value of the patent is given by:

$$V = \int_0^T yXe^{-rt}dt - sR \tag{4}$$

where V - discounted total profits
y = royalty rate per unit output
r = discount rate
s = cost of inventive inputs
R = level of research inputs
X = level of output

Assuming the inventor is a profit maximizing entrepreneur, we substitute equations (1), (2) and (3) into (4) and show that his profitability, as illustrated by equation (5), is determined by the success of the patent (measured by the per unit cost reduction, or royalty) and the length of the patent.

The resulting equation is:

$$V = \int_0^T B(R)X_0C_0e^{-rt}dt - sR \text{ where } X_0 \text{ is the initial level of output.} \tag{5}$$

Differentiating with respect to R (the level of research) we get

$$\frac{\partial V}{\partial R} = \int_0^T B'(R)X_0C_0e^{-rt}dt - s = 0$$

$$\text{i.e. } \int_0^T B'(R)X_0C_0e^{-rt}dt = s \tag{6}$$

In words, the optimal level of research occurs at the point where the discounted value of the increase in patent royalties (left hand side of equation (6)) is equal to the incremental cost of more research inputs.

For a maximum, the second order conditions imply that $B''(R) < 0$. For the function $B(R) = e^{-\delta/R}$,

$$B'(R) = \frac{\delta}{R^2} e^{-\delta/R}$$

and

$$B''(R) = \frac{\delta}{R^3} \left[\frac{\delta}{R} - 2 \right] e^{-\delta/R}$$

Clearly $B''(R) < 0$ if and only if $\frac{\delta}{R} < 2$, or $\alpha < 2$. In terms of economics, as long as the discounted value of the increase in patent royalties is greater than the marginal cost of research inputs, it pays to increase research. But the second order conditions imply that equation (6) must be stated as

$$\int_0^T B'(R) X_0 C_0 e^{-rt} dt = s, \ \alpha < 2 \tag{6'}$$

Without loss of generality we can let $C_0 = 1$. Then, integrating equation (6') with respect to time (t) we get

$$\int_0^T B'(R) X_0 e^{-rt} dt = \left[B'(R) X_0 \left(\frac{-1}{r} \right) e^{-rt} \right]_0^T$$

$$= B'(R) X_0 \left(\frac{-1}{r} \right) e^{-rT} - B'(R) X_0 \left(\frac{-1}{r} \right) e^{0}$$

$$= \frac{1}{r} B'(R) X_0 (1 - e^{-rT}) \tag{7}$$

Hence in equilibrium $\frac{1}{r} B'(R) X_0 (1 - e^{-rT}) = s$

$$\text{or} \quad B'(R) X_0 (1 - e^{-rT}) = rs, \quad \alpha < 2 \tag{8}$$

For the function $B(R) = e^{-\delta/R}$

the inventor's profit-maximizing equilibrium condition is:

$$e^{-\delta/R} \cdot \frac{\delta}{R^2} X_0 (1 - e^{-rT}) = rs, \quad \alpha < 2 \tag{9}$$

Rewriting in order to isolate R on the left-hand side of the equation (so as to facilitate derivation of the comparative static partial effects) we get

$$e^{-\delta/R}.R^{-2} = \frac{rs}{\delta X_0(1-e^{-rT})}$$

$$= \frac{rs}{\delta X_0 \psi} \quad \text{where } \psi = (1 - e^{-rT}) \tag{10}$$

Let $P = e^{-\delta/R}$, $Q = R^{-2}$ in equation (10). Then differentiating the left hand side of the equation with respect to R we get

$$(P'Q + Q'P) = \left(e^{-\delta/R}. \frac{\delta}{R^2}\right) R^{-2} + (-2R^{-3})(e^{-\delta/R})$$

$$= e^{-\delta/R} \left(\frac{\delta}{R^4} - \frac{2}{R^3}\right)$$

$$= \frac{e^{-\delta/R}}{R^3} \left(\frac{\delta}{R} - 2\right)$$

$$= \frac{e^{-\delta/R}}{R^3}\left[\alpha-2\right] \tag{11}$$

Clearly the sign of equation (11) depends on the value of α. Recall that when $\alpha > 2$, there are increasing returns to R. (As we have seen in the increasing returns phase, the inventor is not yet in equilibrium.) In this case the left hand side of equation (11) is positive. However when there are decreasing returns to R, $\alpha < 2$, and the left hand side of equation (11) is negative. As these results are a key to a great deal of subsequent analysis, it is worth summarizing them in Table 1.

Next it will be instructive to consider some comparative static results.

TABLE 1

SUMMARY ON VARIABLE RETURNS
AND OUTPUT ELASTICITY

EQUATION (11)		ELASTICITY		RETURNS TO SCALE
$P'Q + Q'P > 0$	IFF	$\alpha > 2$	IFF	increasing
$P'Q + Q'P < 0$	IFF	$\alpha < 2$	IFF	decreasing

§2. COMPARATIVE STATIC RESULTS

We have seen that as long as there are increasing returns to R, a profit-maximizing inventor's equilibrium does not exist. However, the comparative statics give some useful insight into the increasing returns phase. It should be mentioned that all the comparative static results that deal with the decreasing returns phase are consistent with those obtained by Nordhaus.

Four of the comparative static results are obtained in a straightforward manner by total differentiation of equation (10) and then using the results in Table 1:

$$(P'Q + Q'P)dR = \frac{rs}{\delta X_0 \psi} \tag{12}$$

The first of the four results show how research expenditures R vary with research costs s, i.e. we find $\frac{\partial R}{\partial s}$.

1. From equation (12), it is easily obtained.

$$\frac{\partial R}{\partial s} = \frac{r}{\delta X_0 \psi} \left[P'Q + Q'P\right]^{-1} \tag{13}$$

Since $r > 0$, $\delta > 0$, $X_0 > 0$, and $\psi > 0$, the sign of $\frac{\partial R}{\partial s}$ depends on $(P'Q + Q'P)$, which we shall call the <u>discriminant</u>. Hence

$$\frac{\partial R}{\partial s} > 0 \qquad \text{IFF } \alpha > 2$$

$$\frac{\partial R}{\partial s} < 0 \qquad \text{IFF } \alpha < 2$$

This shows that research expenditures (R) will increase even if research costs increase, as long as there are increasing returns to R, i.e. as long as $\alpha > 2$. When there are decreasing returns, there is an inverse relationship between R and s.

2. Next consider the effect of a change in the rate of interest r on the level of research R. From equation (12)

$$\frac{\partial R}{\partial r} = \frac{s}{\delta X_0 \psi} \cdot \left[1 - \frac{rTe^{-rT}}{1-e^{-rT}}\right] \left[P'Q + Q'P\right]^{-1}$$

Simplifying the second term in the product, we can write

$$\frac{\partial R}{\partial r} = \frac{s}{\delta X_0 \psi} \cdot \left[1 - \frac{rT}{e^{rT}-1}\right] \left[P'Q + Q'P\right]^{-1} \qquad (14)$$

The first term in equation (14) is obviously positive as $s > 0$, $\delta > 0$, $X_0 > 0$ and $\psi > 0$. The second term is positive also,[7] so that the sign of equation (14) depends on the discriminant:

$$\frac{\partial R}{\partial r} > 0 \qquad \text{IFF } \alpha > 2$$

$$\frac{\partial R}{\partial r} < 0 \qquad \text{IFF } \alpha < 2$$

In words, in the increasing returns range the level of research increases with the rate of interest but in the decreasing returns range an increase in the rate of interest reduces research.

---

[7] To prove this, suppose the contrary, i.e. suppose $\frac{rT}{e^{rT}-1} > 1$. This implies that $\log(1+rT) > rT$, which is impossible.

3. Consider now the effect of a change in the life of a patent (T) over which royalties are paid. That is, we wish to find the effect on the level of research due to a change in T. Again using equation (12),

$$\frac{\partial R}{\partial T} = - \frac{r^2 s e^{-rT}}{\delta X_0 \psi^2} \left[P'Q + Q'P\right]^{-1} \tag{15}$$

This time the first term is negative, since $r > 0$, $s > 0$, $\delta > 0$, $X_0 > 0$ and $T > 0$ so that $\psi = 1 - e^{-rT} > 0$. Again the sign of the equation depends on the discriminant.

$$\frac{\partial R}{\partial T} < 0 \qquad \text{IFF} \quad \alpha > 2$$

$$\frac{\partial R}{\partial T} > 0 \qquad \text{IFF} \quad \alpha < 2$$

This means that increasing the length of time over which royalties are received <u>reduces</u> research expenditures if there are increasing returns to research. However if there are decreasing returns, then increasing the life of a patent increases research expenditures.

4. Next consider the effect of a change in the initial level of output X_0 on research. Using equation (12),

$$\frac{\partial R}{\partial X_0} = - \frac{rs}{\delta X_0^2 \psi} \left[P'Q + Q'P\right]^{-1} \tag{16}$$

By the same argument as above, we can conclude that

$$\frac{\partial R}{\partial X_0} < 0 \qquad \text{IFF } \alpha > 2$$

$$\frac{\partial R}{\partial X_0} > 0 \qquad \text{IFF } \alpha < 2$$

This states that an increase in the level of output <u>reduces</u> R if there are increasing returns but increases research if there are decreasing returns.

5. Consider now the effect of a change in δ on the level of research R. It was stated before (see Figure 3), that as δ gets smaller the greater the cost reduction for any given level of R. Equivalently, the smaller the δ the smaller the amount of research necessary to achieve any given level of cost reduction. And the larger the δ the 'flatter' the B(R) curve, and the larger the amount of R needed to achieve any given level of cost reduction.

It is obvious that the value of the parameter δ relative to R determines the point of inflexion on the IPF, and hence the range of increasing returns; the smaller the δ the bigger the range of increasing returns (see Figure 2). Consonant with the earlier results we would expect R to <u>increase</u> as δ increases in the increasing returns range. In the decreasing returns range we would expect R to <u>decrease</u> as δ increases, i.e. as cost reduction becomes more difficult. These two results are easily proven.

Differentiate equation (10) totally and find $\frac{\partial R}{\partial \delta}$;

$$- \frac{e^{-\delta/R}}{R^3} + \frac{\delta e^{-\delta/R}}{R^4} \cdot \frac{\partial R}{\partial \delta} - \frac{2e^{-\delta/R}}{R^3} \cdot \frac{\partial R}{\partial \delta} = \frac{rS}{\delta^2 X_0 \psi} \qquad (17)$$

To simplify, substitute equation (10) again into equation (17):

$$\frac{rs}{\delta X_0 \psi} \left[- 1/R + \frac{1}{R} (\alpha - 2) \frac{\partial R}{\partial \delta} \right] = \frac{-rs}{\delta^2 X_0}$$

which yields (after some manipulation)

$$\frac{\partial R}{\partial \delta} = \frac{\alpha-1}{\alpha(\alpha-2)} \tag{18}$$

From equation (18) it is obvious that[8]

$$\frac{\partial R}{\partial \delta} > 0 \qquad \text{if } \alpha > 2$$

and $\frac{\partial R}{\partial \delta} < 0 \qquad \text{if } 1 < \alpha < 2$

6. In his work Nordhaus (1969, p. 75) also investigates the amount of cost reduction that will result from the inventor's profit maximizing behavior. This he calls <u>the size of the invention</u>, represented by B. Here we investigate how the size of the invention B varies with the life of the patent T.

Recall from equation (3) that the amount of cost reduction is a function of R, but that the amount of R that is undertaken depends on the length of the patent T. Thus,

$$\frac{\partial B}{\partial T} = \frac{\partial B}{\partial R} \cdot \frac{\partial R}{\partial T} \tag{19}$$

Now the size of the invention B can be obtained from equation (10) as follows:

$$e^{-\delta/R} \cdot R^{-2} = \frac{rs}{\delta X_0 \psi}$$

$$B = \frac{rsR^2}{\delta X_0 \psi}$$

and so $\frac{\partial B}{\partial R} = \frac{2rsR}{\partial X_0 \psi}$ (20)

That is, the <u>profit-maximizing cost-reduction</u> (B) is a rising function of research. To find how B varies with T,

---

[8] The purist will note that $\frac{\partial R}{0\delta} > 0$ again when $0 < \alpha < 1$. We are unable to give an economic interpretation for this change in sign.

substitute equations (15) and (20) into (19), and we have

$$\frac{\partial B}{\partial T} = \frac{2rsR}{\delta X_0 \psi} \cdot \left[\frac{-r^2 se^{-rT}}{\delta X_0 \psi^2}\right] \left[P'Q + Q'P\right]^{-1} \qquad (21)$$

Again the sign of the partial derivative depends on the sign of the discriminant. The first term in the product on the right hand side is positive and the second is negative. Hence

$$\frac{\partial B}{\partial T} < 0 \qquad \text{IFF } \alpha > 2$$

$$\frac{\partial B}{\partial T} > 0 \qquad \text{IFF } \alpha < 2$$

That is, the size of the invention decreases when the life of the patent is increased, if there are increasing returns to research. The size of the invention increases with the life of the patent when there are decreasing returns to research.

To summarize, when there are decreasing returns to research the profit-maximizing inventor will decrease R if either research costs or the rate of interest goes up, or if it becomes more difficult to achieve cost reductions (i.e. δ increases). The inventor will increase R if either the level of output or length of the patent increases. Alternatively stated, the size of the invention B increases as T increases.

When there are increasing returns all the relationships in the previous paragraph are reversed. Increasing returns imply that the profit-maximizing inventor has not yet reached his equilibrium. Intuitively it is clear that research should increase even if research costs or the rate of interest increase, as long as there are increasing returns. The same is true if δ increases. However, the size of the invention decreases if the life of the patent is increased. This suggests that in an industry that is going through an increasing returns phase, the granting of patents may reduce cost-reducing inventive activity, assuming all the other assumptions hold.

§3. THE SOCIALLY OPTIMAL PATENT

The above section shows that from the point of view of the profit maximizing inventor,the optimal life of a patent depends on the nature of returns to scale to research and development expenditure. However, from the point of view of society, a patent confers monopoly power and therefore the social cost of patents is the welfare loss due to this monopolistic resource misallocation. As illustrated in Figure I, with a patent the industry output is C whereas when the patent expires industry output is expanded to L. Defining welfare loss in the usual manner as consumer's and producer's surplus less resource cost, the area of the triangle EFG measures the welfare loss to society due to the misallocation of resources caused by the patent.

To determine the optimal length of a patent from society's viewpoint, the marginal net benefit in terms of increased inventive activity must be compared with the welfare loss due to monopolistic resource misallocation. The welfare function W for the inventive activity in the industry is the summation of the cost saving on the preinvention output (REGQ) plus the gain in consumer's surplus when the patent expires less the resource cost of inventive activity.[9]

$$W = \int_0^T \text{REGQ} + \int_T^\infty \text{REFQ} - \text{resource cost} \tag{22a}$$

Alternatively stated

$$W = \int_0^\infty \text{REGQ} + \int_T^\infty \text{EFG} - \text{resource cost} \tag{22b}$$

W is given by the expression

$$W = \int_0^\infty e^{-\delta/R} X_0 e^{-\rho t} dt + \int_T^\infty \frac{1}{2}(X_1 - X_0) e^{-\delta/R} e^{-\rho t} dt - sR \tag{23}$$

where ρ is the social rate of discount. Like Nordhaus, we assume that $\rho = r$. The first term reflects the gain in producer surplus over the period [0, T] and in consumer surplus for $[T, +\infty]$. The second term represents the gain in consumer surplus given by CDE from T to ∞ where X_0 is original output and X_1 is final output. The last term sR is the resource cost of research.

Let the demand function for the good be

$$X = E - \eta p.$$

Integrating equation (23) and replacing[10] $(X_1 - X_0)$ by $\eta e^{-\delta/R}$ and $(1 - e^{-\rho T})$ by ψ, we have

$$W = \frac{e^{-\delta/R} X_0}{\rho} + \frac{(e^{-\delta/R})^2 \, \eta(1-\psi)}{2\rho} - sR \tag{24}$$

---

[9] This is of course a partial equilibrium approach. For assumptions implied by it, see Nordhaus (1969, p. 76).

[10] Considering two points on the demand function and recalling that $C_0 = 1$, it can be shown that $p_0 - p_1 = \frac{C_0 - C_1}{C_0} \equiv e^{-\delta/R}$. Hence $X_1 - X_0 = \eta e^{-\delta/R}$

The objective of the policy maker is to maximize W, subject to the constraint relating the level of research to the life of the patent as given by equation (10). This can be achieved by expressing the objective function in the Lagrangean form:

$$L = \frac{e^{-\delta/R} X_0}{\rho} + \frac{n(1-\psi)e^{-2\delta/R}}{2\rho} - sR + \omega\left[e^{-\delta/R} \cdot \frac{\delta}{R^2} \psi X_0 - sr\right] \qquad (25)$$

where ω is the Lagrange multiplier.

Maximizing equation (22) with respect to ψ and R we have

$$\frac{\partial L}{\partial \psi} = \frac{-ne^{-2\delta/R}}{2\rho} + \omega e^{-\delta/R} \frac{\delta}{R^2} X_0 = 0 \qquad (26)$$

$$\frac{\partial L}{\partial R} = \frac{e^{-\delta/R} . \delta X_0}{\rho R^2} + \frac{n(1-\psi)\delta e^{-2\delta/R}}{\rho R^2} - s + \frac{\omega e^{-\delta/R} . \delta \psi X_0}{R^3} \left[\frac{\delta}{R} - 2\right] = 0 \qquad (27)$$

From equation (26), $\omega = \frac{R^2 ne^{-\delta/R}}{2\delta\rho X_0}$. Substitute this expression into equation (27), and recall that [11]

$$\rho s = e^{-\delta/R} . \frac{\delta}{R^2} \psi X_0 \quad \text{(because } \rho = r \text{, by assumption.)}$$

Finally substitute $B = e^{-\delta/R}$ (the size of the invention), and normalize X_0 to equal 1. It can be shown that

$$\psi = \frac{Bn+1}{Bn(\frac{1}{2} + \frac{1}{\alpha}) + 1} \qquad (28a)$$

[11] The substitution that follows is equivalent to $\frac{\partial L}{\partial \omega}$, i.e. respecting the constraint given by equation (10).

From the definition of ψ, it is clear that

$$T = -\frac{1}{\rho} \log (1 - \psi) \tag{29}$$

However, since B is uniquely determined by the output elasticity[12] α, it can be seen that the optimal life of the patent T depends only on two parameters: the elasticity of demand η and the output elasticity α, assuming a given social rate of discount. Hence we can also write equation (28a) as:

$$\psi = \frac{B\eta + 1}{B\eta(\frac{1}{2} - 1/\log B) + 1} \tag{28b}$$

Equations (28b) and (29) can then be used to compute the values of T.

Using these two equations we can obtain some numerical simulations for a range of demand and output elasticities (Table 2), where T is given in years. It will be seen that for any given output elasticity, T falls as the demand elasticity η increases.[13] For any given η, T falls as the output elasticity α decreases, i.e. as B increases. The numerical results show that the socially optimal patent ranges from about 40 years to just a few days.

Although the derivation of T is the same as that of Nordhaus[14] there are two important differences that make the numbers in Table 2 not comparable to Nordhaus'. First in Nordhaus, B is independent of the output elasticity; here it is not so. Second, Nordhaus makes the assumption that $\eta B<1$, in order to limit his computations to the

---

[12]Since $B = e^{-\alpha}$ (where $\alpha = \delta/R$), $\alpha = -\log B$

[13]We have used the same range of the demand elasticity as that in Nordhaus (1969, p. 81, Table 5.1).

[14]Equations (28a) and (29) are identical with Nordhaus' equation (5.13) and (5.14), (op. cit., p. 78).

TABLE 2

OPTIMAL PATENT LIFE IN YEARS SATISFYING EQUATIONS (25) AND (26)

Elasticity α	Corresponding β \ η	0.25	0.50	0.75	1.00	1.50	2.00	4.00	10.00
1.97	0.140	40.71	37.41	35.54	34.26	32.53	31.38	28.91	26.49
1.80	0.165	30.66	27.39	25.56	24.31	22.64	21.54	19.22	17.04
1.50	0.225	23.56	20.39	18.65	17.48	15.94	14.96	12.97	11.24
1.30	0.270	20.55	17.46	15.79	14.68	13.25	12.35	10.58	9.12
1.20	0.300	18.96	15.93	14.30	13.23	11.87	11.02	9.39	8.08
1.00	0.365	16.15	13.25	11.73	10.76	9.54	8.80	7.43	6.39
0.80	0.445	13.40	10.69	9.31	8.45	7.41	6.79	5.70	4.90
0.60	0.545	10.56	8.12	6.94	6.22	5.39	4.91	4.10	3.54
0.50	0.605	9.05	6.79	5.74	5.11	4.40	4.00	3.34	2.89
0.40	0.670	7.51	5.49	4.58	4.05	3.46	3.14	2.62	2.27
0.30	0.740	5.92	4.19	3.45	3.03	2.58	2.33	1.94	1.69
0.20	0.820	4.15	2.82	2.28	1.99	1.68	1.52	1.26	1.10
0.10	0.905	2.25	1.45	1.15	1.00	0.84	0.76	0.63	0.55
0.08	0.925	1.80	1.14	0.90	0.78	0.65	0.59	0.49	0.43
0.06	0.940	1.45	0.91	0.72	0.62	0.52	0.47	0.39	0.34
0.04	0.960	0.97	0.60	0.47	0.41	0.34	0.31	0.26	0.22
0.02	0.980	0.49	0.30	0.24	0.20	0.17	0.15	0.13	0.11
0.01	0.990	0.25	0.15	0.12	0.10	0.08	0.08	0.06	0.06
0.005	0.995	0.12	0.08	0.06	0.05	0.04	0.04	0.03	0.03

to the 'run-of-the-mill' inventions. In this paper no such assumption is necessary.[15]

The practical consequence of the assumption of the independence of B and α is to overestimate the optimal length of the patent. Table 3 below gives two examples that compare the optimal T value on the basis of the independence assumption as well as on the basis of this paper.

TABLE 3

OPTIMAL PATENT LIFE, A COMPARISON
OPTIMAL T

η	α and B independent B=0.605	This paper B=0.605	α and B independent B=0.405	This paper B=0.405
0.25	13.92	9.05	26.39	14.71
0.50	11.31	6.79	23.39	11.89
0.75	10.01	5.74	21.79	10.45
1.00	9.20	5.11	20.74	9.53
1.50	8.24	4.40	19.41	8.40
2.00	7.68	4.00	18.58	7.72
4.00	6.71	3.34	17.02	6.50
10.00	6.03	2.89	15.77	5.59

Notes: The columns marked 'α and B independent' are obtained by using Nordhaus' equations (5.13) and (5.14), Nordhaus, (op. cit., p. 78).

Before summarizing the main conclusions it would be useful to draw the main threads of the argument of section 2 and 3 together with the aid of a four quadrant diagram in Figure 4.

We begin in the second quadrant, where the IPF of equation (3) is drawn. It is a function of the output elasticity α, where $\alpha = \delta/R$.

---

[15] This is because our equation (3) is bounded by 1.

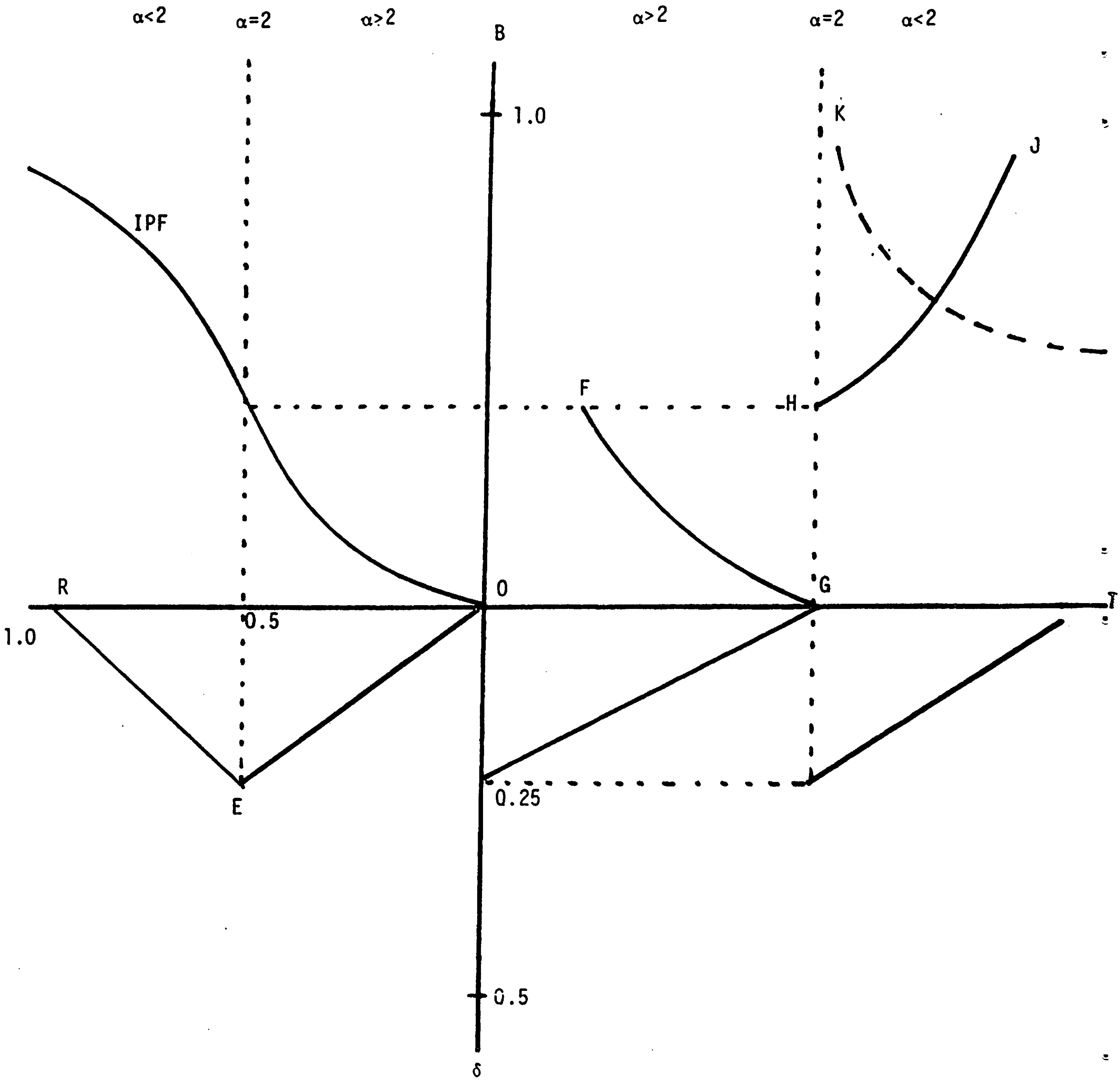


FIGURE 4

The parameter δ is the ease of invention: the smaller the δ the greater the cost reduction and the steeper the IPF. At the point of inflexion $\alpha=2$, and there are increasing returns to R in the region where $\alpha>2$, and decreasing returns to R where $\alpha<2$. For ease of exposition R has been normalized to be one.

In the third quadrant we introduce profit maximizing behavior on the part of the inventor. According to equation (18), $\frac{\partial R}{\partial \delta}$ is positive (line OE) when $\alpha>2$ and negative (line ER) when $\alpha<2$. Clearly if δ decreases the IPF gets steeper and OE is rotated anti-clockwise.

For the fourth quadrant, first note that

$$\frac{\partial R}{\partial \delta}\ \frac{\partial \delta}{\partial T} = \frac{\partial R}{\partial T}$$

so that

$$\frac{\partial \delta}{\partial T} = \frac{\partial R}{\partial T} / \frac{\partial R}{\partial \delta} \qquad (30)$$

i.e. we can use the ratio of equation (15) and equation (18). From these two equations we note that

$$\frac{\partial \delta}{\partial T} < 0$$

when $\alpha>2$, and is negative when $\alpha<2$, but equation (30) is undefined for $\alpha=2$, and hence the discontinuity.[16]

In the first quadrant we trace out the implications of the assumption of profit maximizing behavior on the part of the inventor. In the increasing returns phase we obtain the curve FG, which shows how the size of the invention B would vary with the patent T. FG is downward sloping when $\alpha>2$ as shown earlier in equation (21). The same equation shows that when there are decreasing returns to R,

---

[16]Alternatively equation (30) can be obtained directly by differentiating equation (10).

$\frac{\partial \dot{B}}{\partial T} > 0$, which is represented by the curve HJ, with a discontinuity at GH. It will be seen that the curvature of FG and HJ is determined by the respective curvatures of the portions of the IPF that generate FG and HJ.

Having considered the behavior of a profit maximizing inventor, we now introduce a welfare maximizing policy maker into the first quadrant with the line KL, which represents the locus of B and T that are socially optimal. KL is obtained from equation (28b) by finding $\frac{\partial B}{\partial T}\Big|_{dW=0}$. From equations (28) and (29) it is obvious that T is positive if and only if $\alpha < 2$. Given the constraint that T be positive, it is shown in the Appendix that KL is downward sloping if and only if $\alpha < 2 - \sqrt{B\eta(e^{\rho T}-1)}$, which reduces to $\alpha < 2$ if the social rate of discount were zero.

The socially optimal T can be represented by the intersection of HJ and KL, which is called T* in Figure 4. It is clear that T* exists if KL is downwardsloping. All the comparative statics of the socially optimal T obtained from equations (28b) and (29) can be derived by observing how T* varies in Figure 4. For example, if the demand elasticity η increases KL shifts down and T* falls. If α increases, then δ increases, IPF gets flatter and so HJ gets flatter. Consequently T* increases.

SUMMARY AND CONCLUSIONS

The object of this paper has been to recognize the empirical evidence on R&D which shows both variable returns to scale as well as a variable elasticity of cost reduction with respect to R&D. The empirical evidence calls for a generalized invention possibility function, such as the one used in this paper. Using such a function we show that the profit maximizing inventor has an equilibrium only when there are diminishing returns to R&D. However, the comparative statics of the inventor's equilibrium throws some useful insight into the increasing returns phase. For example, the paper suggests that an industry (such as the semiconductor industry) that is going through an increasing returns phase should not be granted patents, as this could reduce cost-reducing inventive activity.

The paper shows that the socially optimal life of a patent is a function of demand elasticity and output elasticity. In the existing literature the percentage of cost reduction has been treated as being independent of the output elasticity which leads to an overestimation of the life of the patent. We also show that for a given output elasticity, the optimal patent falls as the demand elasticity increases. For a given demand elasticity the patent falls as the output elasticity decreases. Finally, as inventive activity becomes more difficult, the optimal patent increases.

APPENDIX

PROPOSITION 1: $T > 0$ in equations (28) and (29) if and only if $\alpha < 2$.

PROOF. First note that in equation (29)

$$T > 0 \quad \text{IFF} \quad 0 < \psi < 1$$

and $0 < \psi < 1$ IFF

$$B\eta + 1 < B\eta(\tfrac{1}{2} + 1/\alpha) + 1$$

$$1 < \tfrac{1}{2} + 1/\alpha$$

$$\tfrac{1}{2} < 1/\alpha$$

$$\alpha < 2 \qquad \text{(A.1)}$$

PROPOSITION 2: $\left.\frac{\partial B}{\partial T}\right|_{dW=0} < 0$ IFF $\alpha < 2 - \sqrt{B\eta(e^{\rho T} - 1)}$

PROOF. Rewrite equation (28) and incorporate the inequality (A.1)

$$1 - e^{-\rho T} = \frac{B\eta + 1}{B\eta(\tfrac{1}{2} + 1/\alpha) + 1}$$

$$= \frac{1 + 1/B\eta}{\tfrac{1}{2} + 1/(2 - \alpha) + 1/B\eta} \qquad \text{(A.2)}$$

From equation (3), $\alpha = -\log B$, so that (A.2) becomes

$$\frac{+ 1/(2 + \log B) - \tfrac{1}{2}}{(\tfrac{1}{2} + 1/(2 + \log B) + 1/B\eta} - e^{-\rho T} = 0 \qquad \text{(A.3)}$$

$$e^{\rho T}[- \tfrac{1}{2} + 1/(2 + \log B)] = [\tfrac{1}{2} + 1/(2 + \log B) + 1/B\eta] \qquad \text{(A.4)}$$

Let (A.3) be $F(B, T) = 0$. Then by the implicit function theorem,

$$\left.\frac{\partial B}{\partial T}\right|_{dW=0} = - \frac{F_T}{F_B}$$

where F_i is the partial derivative with respect to i.

$$F_T = \rho e^{-\rho T}$$

$$F_B = \frac{-1}{B(2 + \log B)^2} [\tfrac{1}{2} + 1/(2 + \log B) + 1/B\eta]^{-1}$$

$$- [1/(2 + \log B) - \tfrac{1}{2}][\tfrac{1}{2} + 1/(2 + \log B) + 1/B\eta]^{-2} \left[\frac{-1}{B(2 + \log B)^2} - \frac{1}{B^2\eta}\right]$$

$$\left.\frac{\partial B}{\partial T}\right|_{dW=0} < 0 \quad \text{IFF} \quad F_B > 0$$

I.e IFF

$$\frac{-1}{B(2+\log B)^2}[\tfrac{1}{2}+1/(2+\log B)+1/B\eta]^{-1} > [1/(2+\log B)-\tfrac{1}{2}][\tfrac{1}{2}+1/(2+\log B)+1/B\eta]^{-2}\cdot\left[\frac{-B\eta-(2+\log B)^2}{B^2(2+\log B)^2\eta}\right]$$

Substitute (A.4) and note that $\log B = -\alpha$,

$$\frac{-1}{e^{\rho T}B(2-\alpha)^2}[-\tfrac{1}{2}+1/(2-\alpha)]^{-1} > \frac{1}{e^{2\rho T}}[1/(2-\alpha)-\tfrac{1}{2}][1/(2-\alpha)-\tfrac{1}{2}]^{-2}\left[\frac{-B\eta-(2-\alpha)^2}{B^2(2-\alpha)^2\eta}\right]$$

which simplifies to

$$-1 > e^{-\rho T}\left[\frac{-B\eta-(2-\alpha)^2}{B\eta}\right]$$

$$(2-\alpha)^2 > B\eta(e^{\rho T}-1)$$

$$2-\alpha > \sqrt{B\eta(e^{\rho T}-1)}$$

$$\therefore \alpha < 2-\sqrt{B\eta(e^{\rho T}-1)}$$

Q.E.D.

As a special case note that for $\rho = 0$

$$\left.\frac{\partial B}{\partial T}\right|_{dW=0} < 0 \qquad \text{IFF} \qquad \alpha < 2$$

REFERENCES

Anqilley, A. S., "Returns to Scale in Research in the Ethical Pharmaceutical Industry: Some Further Empirical Evidence", Journal of Industrial Economics, Dec. 1973, 22(2), pp. 81-93.

Bailey, M. N., "Research and Development Costs and Returns: The U.S. Pharmaceutical Industry", Journal of Political Economy, Jan./Feb. 1972, 80(1), pp. 70-85.

Berkowitz, M. K. and Y. Kotowitz, "Optimal Patent Design with Free Entry in the Invention Market", University of Toronto, Institute for Policy Analysis, 1979, Working Paper No. 7921.

_______________, "Patent Policy in an Open Economy", Canadian Journal of Economics, Feb. 1982, 15(1), pp. 1-17.

Comanor, W. S., "Research and Technical Change in the Pharmaceutical Industry", Review of Economic Statistics, May 1965, 47(2), pp. 182-190.

Dasgupta, P. and J. Stiglitz, "Industrial Structure and the Nature of Economic Activity", Economic Journal, June 1980, 90, pp. 263-93.

Friedman, Murray N., The Research and Development Factor in Mergers and Acquisitions, Study No. 16, U.S., Congress, Senate, Committee on the Judiciary, Subcommittee on Patents, Trademarks, and Copyrights, 85th Congress, 2nd Session, 1958.

Griliches, Z., "Returns to Research and Development Expenditures in the Private Sector",in J. W. Kendrick and B. Vaccara, New Developments in Productivity Measurement and Analysis, Chicago, University of Chicago Press, 1980.

Hirshleifer, J., "The Private and Social Value of Information and the Reward to Inventive Activity", American Economic Review, Sept. 1971, 61, pp. 561-574.

Kamien, Morton I. and Nancy L. Schwartze, "Market Structure and Innovation: A Survey", Journal of Economic Literature, March 1975, 13, pp. 1-37.

Kitti, Carole, "Patent Policy and the Optimal Timing of Innovations", (Unpublished Ph.D. dissertation, University of Chicago, 1973).

Mansfield, E., "Rates of Return from Industrial Research and Development", American Economic Review, May 1965, 55, pp. 310-322.

____________, Industrial Research and Technological Innovation, New York, W. W. Norton, 1968.

____________, "Social and Private Rates of Return from Industrial Innovations", Quarterly Journal of Economics, May 1977, 91, pp. 221-40.

McFetridge, Donald G. and R. Rafiquzzaman, "The Scope and Duration of the Patent Right and the Nature of Research Rivalry", paper presented to the Economics Analysis of Intellectual Property Rights Conference, University of Western Ontario, London, Ontario, April 1983.

Minasian, J. R., "The Economics of Research and Development" in R. Nelson, The Rate and Direction of Inventive Activity, Princeton: Princeton University Press, 1962, pp. 93-141.

____________, "Research and Development Production Functions and Rates of Return", American Economic Review, May 1969, 59, pp. 80-85.

Nordhaus, William D., Invention, Growth, and Welfare, Cambridge, Mass., M.I.T. Press, 1969.

Robinson, E. A. G., The Structure of Competitive Industry, London: Nisbet and Cambridge University Press, 1953.

Scherer, F. M., "Nordhaus' Theory of Optimal Patent Life: A Geometric Reinterpretation", American Economic Review, June 1972, 62, pp. 422-431.

Schmookler, Jacob, Invention and Economic Growth, Cambridge, Mass.: Harvard University Press, 1967.

Stigler, G. J., The Organization of Industry, Homewood: Irwin, 1968.

Terleckyj, N., Effects of Rand D on the Productivity Growth of Industries, An Exploratory Study, Washington, D.C.: Natural Planning Association, 1974.