1999

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53955001

RESEARCH REPORT 9915

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November 1999

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SOLUTIONS FOR SOME DYNAMIC PROBLEMS WITH UNCERTAINTY AVERSION*

by

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November 17, 1999

Abstract

In a discounted expected-utility problem, tomorrow's utilities are aggregated across tomorrow's states by the expectation operator. In our problems, this aggregation is accomplished by a Choquet integral of the form $\int u \, dP^\alpha$, where α specifies uncertainty aversion. We solve all finite-state problems by either a closed form or a finite-dimensional iteration, and show that uncertainty aversion reduces the perceived return on investment, thereby decreasing the saving rate given elastic preferences and increasing the saving rate given inelastic preferences.

1. Introduction

A useful distinction can be made between risk, where the information about like-lihoods can be represented by a probability measure, and uncertainty, where the information about likelihoods is less precise. Non-expected utility functions can specify both risk aversion and uncertainty aversion. The literature on uncertainty aversion in a static setting that is most relevant to our work includes Schmeidler (1989, whose concept of uncertainty aversion resembles ours), Epstein (1999, who proposes an interesting alternative), Quiggin (1982), Yaari (1987), Gilboa (1987), Gilboa and Schmeidler (1989), and Chateauneuf (1991).

One particularly simple non-expected utility function is specified by the Choquet integral $\int u \, dP^\alpha$ satisfying

$$\int u \, dP^\alpha = \min\left\{ \int u \, dQ \,\Big|\, (\forall A \subseteq S)\, Q(A) \geq P^\alpha(A) \right\},$$

where $S = \{1, 2, \ldots, \#S\}$ is a finite set; $u \in \Re_-^S \cup \Re_+^S = (-\infty, 0]^{\#S} \cup [0, +\infty)^{\#S}$ is the uncertain prospect to be valued; $P : \mathcal{P}(S) \to [0, 1]$ is a probability measure [$\mathcal{P}(S)$ is the set of all subsets of S]; $\alpha \in [1, +\infty)$; and the choice variable in the minimization problem is the probability measure Q. If $\alpha = 1$, the only imaginable probability measure is P itself. Hence uncertainty vanishes and $\int u \, dP^\alpha$ reduces to expected utility. But as α increases from 1, every $P^\alpha(A)$ decreases, and the set of imaginable probability measures expands. Given this uncertainty, the objective $\int u \, dP^\alpha$ values u under the most pessimistic of these imaginable probability measures. For this reason, objectives with $\alpha \in (1, +\infty)$ are said to exhibit uncertainty aversion. Symmetrically, uncertainty appeal is specified by

$$\int u \, dP^\alpha = \max\left\{ \int u \, dQ \,\Big|\, (\forall A \subseteq S)\, Q(A) \leq P^\alpha(A) \right\},$$

where $\alpha \in (0, 1)$.

We will call $\alpha \in (0, +\infty)$ uncertainty aversion. This is imprecise for two reasons. First, the concept is defined only within our very narrow parametric class of preferences. Second, α controls not only uncertainty aversion (by specifying the max or min operator), but also uncertainty itself (there is less uncertainty when α is near 1).

We employ this specification of uncertainty aversion in dynamic problems by considering utility functions which satisfy the recurrence relation

$$u = c^{1-\rho}/(1-\rho) + \beta \int v \, dP^\alpha,$$

where the scalar u is today's utility, the scalar c is today's consumption, the vector v lists tomorrow's utility in each of tomorrow's states, $\rho \in (0,1) \cup (1,+\infty)$ is dynamic inelasticity, and $\beta \in (0,+\infty)$ is the discount factor.

Epstein and Wang (1994, 1995) consider similar problems which are more general in that they admit infinite state spaces and preferences from outside our narrow parametric class. However, we are able to solve all problems with either closed forms or finite-dimensional iterations and to derive the effect of uncertainty aversion on the saving rate. In addition, our assumption that impatience exceeds some measure of "average" growth is less restrictive than their assumption that impatience exceeds growth along the most fortuitous sequence of events. (Ozaki and Streufert (1999) contains preliminary results of a similar nature for continuum state spaces.)

We study three cases. In the first (Section 3.1), the transition probabilities P_s is independent of today's state s. Here we derive a closed-form solution to the dynamic problem and show that increases in α decrease the perceived return to saving and thereby decrease the saving rate when preferences are elastic (that is, when $\rho \in (0,1)$) and increase the saving rate when preferences are inelastic (that is, when $\rho \in (1,+\infty)$).

In the second case (Section 3.2), we assume that higher states are not only associated with higher returns, but also with more favourable transition probabilities. In this case, we find a particularly tractable finite-dimensional iteration which derives the parameters of the solution, and again show that uncertainty aversion leads to an decrease in saving in elastic problems and an increase in saving in inelastic problems.

The third case (Section 3.3) imposes no restriction on the transition probabilities. Here we find a finite-dimensional but somewhat less tractable iteration which derives the parameters of the solution.

All lemmas, proofs, and numbered definitions are collected in the Appendix. The

proofs are based upon the dynamic programming techniques in Ozaki and Streufert (1996), which is hereafter referred to as OS.

2. Framework

2.1. Dynamic Problems with Uncertainty Aversion

Let $S = \{1, 2, \ldots, \#S\}$ be a finite state space, and suppose that the state $s \in S$ evolves exogenously over time according to the transition matrix $P \gg o$.[1] The matrix element $p_{s_+|s}$ in the s_+-th row and s-th column is the probability that tomorrow's state is s_+ given that today's state is s. Accordingly, the s-th column $p_{\cdot|s}$ gives the probability vector over tomorrow's states given that today's state is s.

We have three preference parameters: dynamic inelasticity $\rho \in (0, 1) \cup (1, +\infty)$, the discount factor $\beta > 0$, and uncertainty aversion $\alpha > 0$.

When $\rho \in (0, 1)$, define the utility function $U : \prod_{t \geq 0} \Re_+^{S^t} \to \bar{\Re}_+$ by[2][3]

$$U_s(_0c) = \lim_{T \to \infty} c_0^{1-\rho}/(1-\rho) + \beta \int c_1^{1-\rho}/(1-\rho) + \cdots \beta \int c_T^{1-\rho}/(1-\rho)dP^\alpha \cdots dP_s^\alpha, \quad (1)$$

where P_s is the probability measure derived from the s-th column of the transition matrix P, and an arbitrary consumption process $_0c = (c_0, c_1, c_2, \ldots)$ consists of $c_0 \in \Re_+$, $c_1 \in \Re_+^S$, $c_2 \in \Re_+^{S^2}, \ldots$. U is well-defined because the above sequence of nonnegative numbers is weakly increasing. For example, if $\alpha = 1$,

$$U_s(_0c)$$
$$= \lim_{T \to \infty} c_0^{1-\rho}/(1-\rho) + \beta \int c_1^{1-\rho}/(1-\rho) + \cdots \beta \int c_T^{1-\rho}/(1-\rho)dP \cdots dP_s$$
$$= \lim_{T \to \infty} c_0^{1-\rho}/(1-\rho) + \beta \int c_1^{1-\rho}/(1-\rho)dP_s + \cdots \beta^T \int \int \cdots \int c_T^{1-\rho}/(1-\rho)dP \cdots dP_s$$

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[1]Weakening $P \gg o$ to $(\exists t)\, P^t \gg o$ is like weakening $\beta > 0$ to $\beta \geq 0$: its only consequence is to admit the possibility of optimal processes which do not obey the true policy function K^*.

[2]In order to understand the text's use of the Choquet integral, it suffices to understand the first two equalities in the introduction. Meanwhile, the proofs in the Appendix employ the definition of the Choquet integral and a number of additional concepts which are explained in Definition 1. The introduction's equalities hold by Lemmas 3 and 5.

[3]This utility function satisfies the introduction's recurrence relation over the relevant commodity space. This fact is established as U4 in OS Theorems D and E, and each of this paper's theorems is proven by applying one of those two theorems.

$$= \lim_{T \to \infty} \sum_{t=0}^{T} \beta^t E_s c_t^{1-\rho}/(1-\rho)$$

$$= \sum_{t=0}^{\infty} \beta^t E_s c_t^{1-\rho}/(1-\rho)$$

where the first, third, and fourth equalities are definitions and the second holds because $\alpha = 1$.

When $\rho \in (1, +\infty)$, define the utility function $U : \prod_{t \geq 0} \Re_+^{S^t} \to \bar{\Re}_-$ by (1). In this instance, U is defined by a weakly decreasing sequence of nonpositive numbers. Again, $U_s({}_0c)$ reduces $\sum_{t=0}^{\infty} \beta^t E_s c_t^{1-\rho}/(1-\rho)$ when $\alpha = 1$.

Finally, let $R \in (0, +\infty)^S$ give the gross return to saving as a function of the state. We assume without loss of generality that R is weakly increasing in the sense that $R_1 \leq R_2 \leq \cdots \leq R_{\#S}$. Accordingly, tomorrow's income $y_+ \in \Re_+$ is derived from tomorrow's state $s_+ \in S$ and today's saving $x \in \Re_+$ by $y_+ = R_{s_+} x$. A consumption process ${}_0c$ is *feasible* from y if there exists a corresponding saving process ${}_0x \in \prod_{t \geq 0} \Re_+^{S^t}$ such that

$$c_0 = y - x_0 \quad \text{and} \quad (\forall t \geq 1) \quad c_t = R x_{t-1} - x_t.$$

In summary, we have the parameters S, P, ρ, β, α, and R. The problem is to maximize the utility function U, given some initial state s, over the collection of consumption processes which are feasible from some initial income y. It is of particular interest to see the way in which the problem's solution is affected by the parameter α.

2.2. Solutions

We solve such a problem by constructing its *true value function* $J^* : S \times \Re_+ \to \bar{\Re}$ defined by

$$J_s^*(y) = \max\{ U_s({}_0c) \mid {}_0c \text{ is feasible from } y \};$$

by constructing its *true policy function* $K^* : S \times \Re_+ \twoheadrightarrow \Re_+$ defined by

$$K_s^*(y) = \arg\max\left\{ (y-x)^{1-\rho}/(1-\rho) + \beta \int J_{s_+}^*(R_{s_+} x) P_s^\alpha(ds_+) \,\middle|\, x \in [0, y] \right\};$$

and by showing that K^* *characterizes optimality* in the sense that[4]

$$\arg\max\{\, U_s(_0c)\mid {}_0c \text{ is feasible from } y\,\}$$

$$= \{\, _0c\mid (\exists _0x)\ x_0 \in K_s^*(y) \text{ and } c_0 = y - x_0;$$

$$(\forall t \geq 1)\ x_t \in K^*(Rx_{t-1}) \text{ and } c_t = Rx_{t-1} - x_t\,\}.$$

2.3. Examples

Suppose that $S = \{1,2\}$, that $(\forall s, s_+)\ p_{s_+\mid s} = .5$, and that $R_1 < R_2$ (a weak inequality is always assumed). If $\alpha = 1$, tomorrow's utilities are aggregated by the expectation operator $\int u\, dP = .5u_1 + .5u_2$, and Theorems A$^+$ and A$^-$ establish that optimality is characterized by

$$K_s^*(y) = \{\, k^*y\,\}, \quad \text{where} \quad k^* = [\beta(.5R_1^{1-\rho} + .5R_2^{1-\rho})]^{1/\rho},$$

provided that $k^* < 1$. On the other hand, if $\alpha = 2$, tomorrow's utilities are aggregated with uncertainty aversion by $\int u\, dP^2 = .75\min\{u_1, u_2\} + .25\max\{u_1, u_2\}$, and the same propositions establish that optimality is characterized by

$$K_s^*(y) = \{\, k^*y\,\}, \quad \text{where} \quad k^* = [\beta(.75R_1^{1-\rho} + .25R_2^{1-\rho})]^{1/\rho},$$

provided that $\beta(.25R_1^{1-\rho} + .75R_2^{1-\rho}) < 1$ when $\rho \in (0,1)$ and that $k^* < 1$ when $\rho \in (1, +\infty)$ [the first inequality implies $k^* < 1$ when $\rho \in (0,1)$].

This k^* can be interpreted as the saving rate. As might be anticipated from prior knowledge of discounted expected utility models (that is, the case $\alpha = 1$), the saving rate increases in β, is ambiguously affected by ρ, increases with R when $\rho \in (0,1)$ and decreases with R when $\rho \in (1, +\infty)$. It is of particular interest here to see how k^* changes when α changes from 1 to 2. Since $R_1 < R_2$, we see that k^* decreases when $\rho \in (0,1)$ and increases when $\rho \in (1, +\infty)$. Intuitively, uncertainty aversion causes the consumer to place less weight on the higher state, and thus, to perceive less return on her investment.

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[4]The proofs use further dynamic programming concepts such as Bellman's equation. These are given in Definition 9.

This causes her to save less if her preferences are dynamically elastic and to save more if her preferences are dynamically inelastic.

3. Theorems

3.1. Transition Matrices with Identical Columns

Given a probability vector $p \in \Re_+^S$ and an exponent $\alpha \in (0, +\infty)$, define

$$\check{p}^\alpha = \langle (\sum_{s \geq \sigma} p_s)^\alpha - (\sum_{s > \sigma} p_s)^\alpha \rangle_\sigma$$

where $\sum_{s > \#S}$ is set equal to zero. For example, if $S = \{1, 2\}$ and $p = (.5, .5)$, then $\check{p}^\alpha = (1 - (.5)^\alpha, (.5)^\alpha)$, and in particular, $\check{p}^2 = (.75, .25)$. Thus it seems that increases in α push more probability weight down from higher states and into lower states. This intuition corresponds to our use of the symbol $(\check{\cdot})$.

To be precise, one probability vector $p \in \Re_+^S$ is *stochastically lower* than another probability vector $q \in \Re_+^S$ if $(\forall \sigma > 1)$ $\sum_{s \geq \sigma} p_s \leq \sum_{s \geq \sigma} q_s$. Note that $\check{p}^{\alpha'}$ is stochastically lower than $\check{p}^\alpha$ whenever $\alpha' \geq \alpha$. Also note that if p is stochastically lower than q, and if $x \in \Re_+^S$ is weakly increasing (resp. decreasing), then $\sum_s x_s p_s \leq$ (resp. $\geq$) $\sum_s x_s q_s$.

One assumption in Theorem A^+ requires a symmetric concept. Given a probability vector $p \in \Re_+^S$ and an exponent $\alpha \in (0, +\infty)$, define

$$\hat{p}^\alpha = \langle (\sum_{s \leq \sigma} p_s)^\alpha - (\sum_{s < \sigma} p_s)^\alpha \rangle_\sigma$$

where $\sum_{s < 1}$ is set equal to zero. For example, if $S = \{1, 2\}$ and $p = (.5, .5)$, then $\hat{p}^\alpha = ((.5)^\alpha, 1 - (.5)^\alpha)$, and in particular, $\hat{p}^2 = (.25, .75)$. This operation pushes probability weights up from lower states and into higher states in the sense that $\hat{p}^{\alpha'}$ is stochastically higher than $\hat{p}^\alpha$ whenever $\alpha' \geq \alpha$. Note both $\check{p}^1$ and $\hat{p}^1$ equal p.

Theorem A^+: *Assume that $\rho \in (0, 1)$ and that all the columns of P are identical to some probability vector $p \in \Re_+^S$. Let $k^* = (\beta \sum_{s+} R_{s+}^{1-\rho} \check{p}_{s+}^\alpha)^{1/\rho}$, and assume*

$$\text{either} \quad \alpha \in (0, 1] \quad \text{and} \quad k^* < 1$$

$$\text{or} \quad \alpha \in [1, +\infty) \quad \text{and} \quad \sum_{s+} R_{s+}^{1-\rho} \hat{p}_{s+}^\alpha < \beta^{-1}.$$

Then (by Theorem B^+),

$$J_s^*(y) = (1 - k^*)^{-\rho} y^{1-\rho} / (1 - \rho) \quad and$$

$$K_s^*(y) = \{ k^* y \} \quad characterizes\ optimality.$$

(Proof 16.)

Theorem A^-: *Assume that $\rho \in (1, +\infty)$ and that the columns of P are identical. Let $k^* = (\beta \sum_{s_+} R_{s_+}^{1-\rho} \breve{p}_{s_+}^\alpha)^{1/\rho}$. Then (by Theorem B^-), if $k^* < 1$,*

$$J_s^*(y) = (1 - k^*)^{-\rho} y^{1-\rho} / (1 - \rho) \quad and$$

$$K_s^*(y) = \{ k^* y \} \quad characterizes\ optimality;$$

and, if $k^* \geq 1$, $J^*(y) = -\infty$ and any feasible process is optimal. (Proof 19.)

Once again, it is of particular interest to see how the saving rate

$$k^* = (\beta \sum_{s_+} R_{s_+}^{1-\rho} \breve{p}_{s_+}^\alpha)^{1/\rho}$$

varies with the exponent α. If $\alpha' \geq \alpha$, then $\breve{p}^{\alpha'}$ is stochastically lower than $\breve{p}^\alpha$. Consequently, $\sum_{s_+} R_{s_+}^{1-\rho} \breve{p}_{s_+}^{\alpha'} \leq \sum_{s_+} R_{s_+}^{1-\rho} \breve{p}_{s_+}^\alpha$ when $\rho \in (0, 1)$ and the opposite when $\rho \in (1, +\infty)$. This is because $R^{1-\rho}$ is weakly increasing when $\rho \in (0, 1)$ and is weakly decreasing when $\rho \in (1, +\infty)$. Hence, the saving rate weakly decreases with α when $\rho \in (0, 1)$ and weakly increases with α when $\rho \in (1, +\infty)$. Once again, uncertainty aversion causes the consumer to place less weight on higher states, and thus, to perceive less return on her investment. This causes her to save less if her preferences are dynamically elastic and to save more if her preferences are dynamically inelastic.

As might be anticipated, the above solutions are contingent upon assumptions requiring that impatience exceed some measure of growth. In most instances, this is handled by $k^* < 1$, which is algebraically equivalent to $\sum_{s_+} R_{s_+}^{1-\rho} \breve{p}_{s_+}^\alpha < \beta^{-1}$. However, in the instance $\rho \in (0, 1)$ and $\alpha \in [1, +\infty)$, the theorem requires $\sum_{s_+} R_{s_+}^{1-\rho} \hat{p}_{s_+}^\alpha < \beta^{-1}$, which is at least as strong as the former inequality because $R^{1-\rho}$ is weakly increasing when $\rho \in (0, 1)$ and because $\hat{p}^\alpha$ is stochastically higher than $\breve{p}^\alpha$ when $\alpha \in [1, +\infty)$.

3.2. Transition Matrices with Stochastically Ordered Columns

The columns of P are said to be *stochastically ordered* if $(\forall s < \#S)$ $p_{\cdot|s}$ is stochastically lower than $p_{\cdot|s+1}$. Because R was assumed to be weakly increasing, lower states are already associated with lower returns. If P is stochastically ordered, lower states are also associated with less fortunate transition probabilities.

Unfortunately, the following theorems do not derive closed-form solutions. Rather, they compute coefficients for the true value function by means of an iteration in $\Re_+^S$. This iteration is defined by means of the function $\breve{b} : \Re_+^S \to \Re_+^S$ specified by

$$(\breve{b}j)_s = (1 + (\beta \sum_{s_+} j_{s_+} R_{s_+}^{1-\rho} \breve{p}_{s_+|s}^\alpha)^{1/\rho})^\rho .$$

[$\breve{b}j$ is weakly increasing in j, weakly increasing in R when $\rho \in (0,1)$, and weakly decreasing in R when $\rho \in (1, +\infty)$.]

Like the preceding theorems, all results are contingent upon assumptions requiring that impatience exceed some measure of growth. When $\rho \in (0,1)$ and $\alpha \in (0,1]$, this is done by $\lambda(\Gamma^{1-\rho} \breve{P}^\alpha) < \beta^{-1}$, where $\Gamma^{1-\rho}$ is the diagonal matrix whose s_+-th element is $R_{s_+}^{1-\rho}$, $\breve{P}^\alpha$ is the matrix whose s-th column is $\breve{p}_{\cdot|s}^\alpha$, and $\lambda(\cdot)$ yields the dominant eigenvalue (Luenberger, 1979, p. 191) of the enclosed matrix. When $\rho \in (0,1)$ and $\alpha \in [1, +\infty)$, this is done by $\lambda(\Gamma^{1-\rho} \hat{P}^\alpha) < \beta^{-1}$ where $\hat{P}^\alpha$ is the matrix whose s-th column is $\hat{p}_{\cdot|s}^\alpha$. When $\rho \in (1, +\infty)$, this is done by requiring that the limit of the iterative process is finite (or equivalently, that the saving rate in every state is less than unity).

Theorem B⁺: *Assume that $\rho \in (0,1)$, that the columns of P are stochastically ordered, and that*

$$\text{either} \quad \alpha \in (0,1] \quad \text{and} \quad \lambda(\Gamma^{1-\rho} \breve{P}^\alpha) < \beta^{-1}$$

$$\text{or} \quad \alpha \in [1, +\infty) \quad \text{and} \quad \lambda(\Gamma^{1-\rho} \hat{P}^\alpha) < \beta^{-1}.$$

Then $j^* \equiv \lim_{n \to \infty} \breve{b}^n o \in \Re_+^S$ and $j^+ \equiv \mathbf{1}'[I - \beta \Gamma^{1-\rho} \breve{P}^\alpha]^{-1} \in \Re_+^S$ are well-defined (and hence finite), $j^* = \lim_{n \to \infty} \breve{b}^n j^+$, j^* is the only weakly increasing solution to $j = \breve{b}j$ which satisfies $o \le j \le j^+$,

$$J_s^*(y) = j_s^* y^{1-\rho}/(1 - \rho), \quad \text{and}$$

$$K_s^*(y) = \{ [((j_s^*)^{1/\rho} - 1)/(j_s^*)^{1/\rho}]y \} \text{ characterizes optimality}.$$

(Proof 15.)

Theorem B⁻: *Assume that $\rho \in (1, +\infty)$ and that the columns of P are stochastically ordered. Then (by Theorem C⁻) $j^* \equiv \lim_{n \to \infty} \check{b}^n \mathbf{o} \in \bar{\Re}_+^S$ is well-defined; if $j^* \in \Re_+^S$, then*

$$J_s^*(y) = j_s^* y^{1-\rho}/(1 - \rho) \text{ and}$$

$$K_s^*(y) = \{ [((j_s^*)^{1/\rho} - 1)/(j_s^*)^{1/\rho}]y \} \text{ characterizes optimality};$$

and, if $(\exists s)$ $j_s^* = +\infty$, then $(\forall s)$ $J_s^*(y) = -\infty$ and any feasible process is optimal. (Proof 18.)

Once again, increases in α decrease the saving rate when $\rho \in (0, 1)$ and increase the saving rate when $\rho \in (1, +\infty)$. When $\rho \in (0, 1)$, both $R^{1-\rho}$ and every $\check{b}^n j$ are weakly increasing. Thus, the definition of $\check{b}$ suggests that an increase in α forces probability weight down to lower states and thereby decreases every $\check{b}^n j$ (full details are in Lemma 13). Hence, an increase in α decreases j^* and thereby decreases the saving rate. On the other hand, when $\rho \in (1, +\infty)$, both $R^{1-\rho}$ and every $\check{b}^n j$ are weakly *decreasing*. As a result, an increase in α has the reverse effect.

3.3. The General Case

Like the preceding pair of theorems, the following pair of theorems compute coefficients for the true value function by means of an iteration in $\Re_+^S$. In this case, the iteration is defined by means of the function $b : \Re_+^S \to \Re_+^S$ specified by

$$(bj)_s = \left(1 + \left(\beta \int j_{s_+} R_{s_+}^{1-\rho} P_s^\alpha(ds_+)\right)^{1/\rho}\right)^\rho \quad \text{when} \quad \rho \in (0, 1)$$

and by

$$(bj)_s = \left(1 + \left(\beta \int j_{s_+} R_{s_+}^{1-\rho} P_s^{\alpha'}(ds_+)\right)^{1/\rho}\right)^\rho \quad \text{when} \quad \rho \in (1, +\infty)$$

Theorem C⁺: *Suppose that $\rho \in (0,1)$ and that*

either $\alpha \in (0,1]$ and $(\exists z \in (0,\alpha))$ $(\alpha/(\alpha-z))\lambda(\Gamma^{(1-\rho)/z}P)^z < \beta^{-1}$

or $\alpha \in [1,+\infty)$ and $\alpha\lambda(\Gamma^{1-\rho}P) < \beta^{-1}$.

Then $j^* \equiv \lim_{n\to\infty} b^n o \in \Re_+^S$ is well-defined (and hence finite),

$$J_s^*(y) = j_s^* y^{1-\rho}/(1-\rho), \quad and$$

$$K_s^*(y) = \{ [((j_s^*)^{1/\rho} - 1)/(j_s^*)^{1/\rho}]y \} \ characterizes \ optimality.$$

(Proof 14.)

Theorem C⁻: *Assume that $\rho \in (1,+\infty)$. Then $j^* \equiv \lim_{n\to\infty} b^n o \in \bar{\Re}_+^S$ is well-defined; if $j^* \in \Re_+^S$,*

$$J_s^*(y) = j_s^* y^{1-\rho}/(1-\rho) \quad and$$

$$K_s^*(y) = \{ [((j_s^*)^{1/\rho} - 1)/(j_s^*)^{1/\rho}]y \} \ characterizes \ optimality;$$

and, if $(\exists s)$ $j_s^* = +\infty$, then $(\forall s)$ $J_s^*(y) = -\infty$ and every feasible process is optimal. (Proof 17.)

APPENDIX

No definition, lemma, or proof appears after it has been used (how boring).

Definition 1: Let $\mathcal{P}(S)$ denote the set of all subsets of a finite set S. A *probability capacity* is a function $\theta : \mathcal{P}(S) \to [0,1]$ which satisfies $\theta(\phi) = 0$, $\theta(S) = 1$, and $(\forall A, B)$ $A \subseteq B \Rightarrow \theta(A) \leq \theta(B)$. The *Choquet integral* of a nonnegative vector $u \in \Re_+^S$ with respect to a probability capacity θ is defined by

$$\int u_s \theta(ds) = \int_0^{+\infty} \theta(\{s|u_s \geq y\})dy,$$

where the right-hand-side is an improper Riemann integral which must be well-defined (but not necessarily finite) because its integrand weakly decreases with y. The *conjugate*

of a probability capacity θ is the function $\theta' : \mathcal{P}(S) \to [0,1]$ defined by $(\forall A)$ $\theta'(A) = 1 - \theta(\sim A)$, where $\sim A$ denotes the complement of A in S. The *Choquet integral* of a nonpositive vector $u \in \bar{\Re}^S_-$ with respect to θ is $\int u_s \theta(ds) = -\int -u_s \theta'(ds)$. A probability capacity is *concave* (resp. *convex*) if

$$(\forall A, B \subseteq S) \quad \theta(A \cup B) + \theta(A \cap B) \le (\text{resp. } \ge) \; \theta(A) + \theta(B).$$

(A probability measure is a probability capacity which is additive, that is, both concave and convex.) The next three lemmas follow immediately from these definitions.

Lemma 2: *Suppose θ is a probability capacity. Then*

$$(\forall u \in \Re^S_+)(\forall a, b \in \Re_+) \quad \int (a + bu) d\theta = a + b \int u \, d\theta.$$

Lemma 3: *Suppose P is a probability measure. Then P^α is concave when $\alpha \in (0,1]$ and P^α is convex when $\alpha \in [1, +\infty)$.*

Lemma 4: *Suppose θ is a probability capacity. Then θ is concave (resp. convex) if and only if θ' is convex (resp. concave).*

Lemma 5: *If θ is a concave probability capacity, then*

$$(\forall u \in \Re^S_- \cup \Re^S_+) \quad \int u \, d\theta = \max \left\{ \int u \, dQ \, \middle| \, (\forall A \subseteq S) \, Q(A) \le \theta(A) \right\}.$$

If θ is a convex probability capacity, then

$$(\forall u \in \Re^S_- \cup \Re^S_+) \quad \int u \, d\theta = \min \left\{ \int u \, dQ \, \middle| \, (\forall A \subseteq S) \, Q(A) \ge \theta(A) \right\}.$$

Proof: Schmeidler (1986, Proposition 3) derives both results when $u \in \Re^S_+$. Now suppose $u \in \Re^S_-$. If θ is concave, then

$$\int u \, d\theta$$
$$= -\int -u \, d\theta'$$
$$= -\min \left\{ \int -u \, dQ \, \middle| \, (\forall A \subset S) \, Q(A) \ge \theta'(A) \right\}$$
$$= \max \left\{ \int u \, dQ \, \middle| \, (\forall A \subset S) \, Q(A) \ge \theta'(A) \right\}$$

$$= \max\left\{ \int u\, dQ \,\middle|\, (\forall A \subset S)\; 1 - Q(\sim A) \geq 1 - \theta(\sim A) \right\}$$

$$= \max\left\{ \int u\, dQ \,\middle|\, (\forall A \subset S)\; Q(\sim A) \leq \theta(\sim A) \right\}$$

$$= \max\left\{ \int u\, dQ \,\middle|\, (\forall A \subset S)\; Q(A) \leq \theta(A) \right\},$$

where the first equality holds by Definition 1, the second holds by Schmeidler's result for $-u \in \Re_+^S$ since θ' is convex (Lemma 4), and the fourth holds by Definition 1 and the fact that Q is a probability measure. A symmetric argument can be made when θ is convex. $\square$

Lemma 6 (Schmeidler, 1986, Proposition 3): *If θ is a concave probability capacity, then*

$$(\forall u, v \in \Re_+^S) \quad \int (u + v) d\theta \leq \int u\, d\theta + \int v\, d\theta.$$

If θ is a convex probability capacity, then

$$(\forall u, v \in \Re_+^S) \quad \int (u + v) d\theta \geq \int u\, d\theta + \int v\, d\theta.$$

Lemma 7: *Suppose θ is a convex probability capacity. Then*

$$(\forall u, v \in \Re_+^S) \quad \int (u + v) d\theta - \int u\, d\theta \leq \int v\, d\theta'.$$

Proof: We argue

$$\int (u + v) d\theta - \int u\, d\theta$$

$$= \min\left\{ \int (u + v) dQ \,\middle|\, (\forall A)\; Q(A) \geq \theta(A) \right\} - \min\left\{ \int u\, dQ \,\middle|\, (\forall A)\; Q(A) \geq \theta(A) \right\}$$

$$\leq \int (u + v) dQ^* - \int u\, dQ^*$$

$$= \int v\, dQ^*$$

$$\leq \max\left\{ \int v\, dQ \,\middle|\, (\forall A)\; Q(A) \geq \theta(A) \right\}$$

$$= \max\left\{ \int v\, dQ \,\middle|\, (\forall A)\; Q(\sim A) \geq \theta(\sim A) \right\}$$

$$= \max\left\{ \int v\, dQ \,\middle|\, (\forall A)\; 1 - Q(\sim A) \leq 1 - \theta(\sim A) \right\}$$

$$= \max\left\{ \int v\, dQ \,\middle|\, (\forall A)\; Q(A) \leq \theta'(A) \right\}$$

$$= \int v\, d\theta'.$$

The first equality holds by Lemma 5 and the convexity of θ, and the first inequality holds by defining Q^* as the minimizer in the problem for $\int u\, d\theta$. The second equality holds by the linearity of an integral defined with respect to a probability measure, and the second inequality holds because Q^* is in the feasible set (since it's the same feasible set as earlier). The penultimate equality holds because Q is a measure and because of the definition of the conjugate θ'. The final equality holds by Lemma 5 and the concavity of θ' (which follows from Lemma 4 and the convexity of θ). $\square$

Lemma 8: *Let $A \in (0, +\infty)^{S^2}$ be a positive matrix and $\lambda(A)$ its dominant eigenvalue. Then $(\forall y, x \in \Re_+^S)(\forall m > \lambda(A))(\exists k \in \Re_+)(\forall t \geq 1)\ y'A^t x < km^t$.*

Proof: Take any $y, x \in \Re_+^S$ and any m such that $\lambda(A) < m$. By the genericity of distinct eigenvalues and the continuity of $\lambda(\cdot)$, there exists a matrix B with distinct eigenvalues such that $A \leq B$ and $\lambda(B) < m$. Further, since B has distinct eigenvalues, there exist an eigenvector matrix M and a diagonal matrix Λ of eigenvalues such that $B = M\Lambda M^{-1}$. Consequently,

$$
\begin{aligned}
& y'A^t x \\
\leq\ & y'B^t x \\
=\ & y'M\Lambda^t M^{-1}x \\
=\ & \sum_s (y'M)_s \lambda_s^t (M^{-1}x)_s \\
\leq\ & \sum_s |\lambda_s^t| |(y'M)_s (M^{-1}x)_s| \\
\leq\ & \lambda(B)^t \sum_s |(y'M)_s (M^{-1}x)_s| \\
<\ & m^t k,
\end{aligned}
$$

where $(y'M)_s$ is the s-th element of the row vector $y'M$, $(M^{-1}x)_s$ is the s-th element of the column vector $M^{-1}x$, and $k \equiv \sum_s |(y'M)_s (M^{-1}x)_s|$. $\square$

Definition 9: A function $J : S \times \Re_+ \to \bar{\Re}$ is an *admissible value function* if it is upper semicontinuous, weakly increasing in its second argument, and satisfies $J^- \leq J \leq J^+$, where $J_s^-(y) \equiv U_s(\mathbf{o})$ is an extremely pessimistic value function and $J_s^+(y) \equiv$

$U_s(y, Ry, R^2y, \ldots)$ is an extremely optimistic value function. Define *Bellman's operator*
B from the set of admissible value functions into itself by

$$(\forall s)(\forall y) \quad BJ_s(y) = \max\left\{ (y-x)^{1-\rho}/(1-\rho) + \beta \int J_{s_+}(R_{s_+}x)P_s^\alpha(ds_+) \,\Big|\, x \in [0,y] \right\}.$$

Finally, J solves *Bellman's equation* if $J = BJ$.

Lemma 10: *Let J and $j \in \Re_+^S$ be such that $J_{s_+}(y_+) = j_{s_+}y_+^{1-\rho}/(1-\rho)$. Then*

$$\max\left\{ (y-x)^{1-\rho}/(1-\rho) + \beta \int J_{s_+}(R_{s_+}x)P_s^\alpha(ds_+) \,\Big|\, x \in [0,y] \right\}$$
$$= (bj)_s y^{1-\rho}/(1-\rho), \quad and$$
$$\operatorname{arg\,max}\left\{ (y-x)^{1-\rho}/(1-\rho) + \beta \int J_{s_+}(R_{s_+}x)P_s^\alpha(ds_+) \,\Big|\, x \in [0,y] \right\}$$
$$= \left\{ [((bj)_s^{1/\rho} - 1)/(bj)_s^{1/\rho}]y \right\}.$$

The first equality is equivalent to $(BJ)_s(y) = (bj)_s y^{1-\rho}/(1-\rho)$ when J is admissible (see Definition 9).

Proof: First suppose $\rho \in (0,1)$. For notational ease, fix s and set

$$w = \left(\beta \int j_{s_+} R_{s_+}^{1-\rho} P_s^\alpha(ds_+) \right)^{1/\rho}.$$

Then

$$\max\left\{ (y-x)^{1-\rho}/(1-\rho) + \beta \int J_{s_+}(R_{s_+}x)P_s^\alpha(ds_+) \,\Big|\, x \in [0,y] \right\}$$
$$= \max\left\{ (y-x)^{1-\rho}/(1-\rho) + \beta \int j_{s_+}(R_{s_+}x)^{1-\rho}/(1-\rho)P_s^\alpha(ds_+) \,\Big|\, x \in [0,y] \right\}$$
$$= \max\left\{ (y-x)^{1-\rho}/(1-\rho) + \beta \int j_{s_+}R_{s_+}^{1-\rho}P_s^\alpha(ds_+)x^{1-\rho}/(1-\rho) \,\Big|\, x \in [0,y] \right\}$$
$$= \max\left\{ (y-x)^{1-\rho}/(1-\rho) + w^\rho x^{1-\rho}/(1-\rho) \,|\, x \in [0,y] \right\}$$
$$= (y - wy/(1+w))^{1-\rho}/(1-\rho) + w^\rho(wy/(1+w))^{1-\rho}/(1-\rho)$$
$$= (y/(1+w))^{1-\rho}/(1-\rho) + w^\rho(wy/(1+w))^{1-\rho}/(1-\rho)$$
$$= [1 + w^\rho(w)^{1-\rho}](1/(1+w))^{1-\rho}y^{1-\rho}/(1-\rho)$$
$$= [1 + w](1/(1+w))^{1-\rho}y^{1-\rho}/(1-\rho)$$
$$= (1+w)^\rho y^{1-\rho}/(1-\rho)$$
$$= (bj)_s y^{1-\rho}/(1-\rho),$$

where the second equality holds by Lemma 2 and the nonnegativity of $x^{1-\rho}/(1-\rho)$, the maximization operator is resolved by the first-order condition

$$(y-x)^{-\rho}(-1) + w^{\rho}x^{-\rho} = 0$$

$$w^{\rho}x^{-\rho} = (y-x)^{-\rho}$$

$$w^{\rho}(y-x)^{\rho} = x^{\rho}$$

$$wy - wx = x$$

$$wy/(1+w) = x$$

and the last equality follows from the identity $(bj)_s = (1+w)^{\rho}$. The same first-order condition and identity determines the argmax as

$$\{wy/(1+w)\}$$
$$= \{[((1+w)-1)/(1+w)]y\}$$
$$= \{[((bj)_s^{1/\rho}-1)/(bj)_s^{1/\rho}]y\}.$$

Second suppose $\rho \in (1, +\infty)$. For notational ease, fix s and set

$$w = \left(\beta \int j_{s+} R_{s+}^{1-\rho} P_s^{\alpha\prime}(ds_+)\right)^{1/\rho}.$$

Then

$$\max\left\{(y-x)^{1-\rho}/(1-\rho) + \beta \int J_{s+}(R_{s+}x)P_s^{\alpha}(ds_+) \,\middle|\, x \in [0,y]\right\}$$
$$= \max\left\{(y-x)^{1-\rho}/(1-\rho) + \beta \int j_{s+}(R_{s+}x)^{1-\rho}/(1-\rho)P_s^{\alpha}(ds_+) \,\middle|\, x \in [0,y]\right\}$$
$$= \max\left\{(y-x)^{1-\rho}/(1-\rho) - \beta \int -j_{s+}(R_{s+}x)^{1-\rho}/(1-\rho)P_s^{\alpha\prime}(ds_+) \,\middle|\, x \in [0,y]\right\}$$
$$= \max\left\{(y-x)^{1-\rho}/(1-\rho) - \beta \int j_{s+}R_{s+}^{1-\rho}P_s^{\alpha\prime}(ds_+)[-x^{1-\rho}/(1-\rho)] \,\middle|\, x \in [0,y]\right\}$$
$$= \max\left\{(y-x)^{1-\rho}/(1-\rho) + \beta \int j_{s+}R_{s+}^{1-\rho}P_s^{\alpha\prime}(ds_+)x^{1-\rho}/(1-\rho) \,\middle|\, x \in [0,y]\right\}$$
$$= \max\{(y-x)^{1-\rho}/(1-\rho) + w^{\rho}x^{1-\rho}/(1-\rho) \,|\, x \in [0,y]\},$$

where the second equality follows from the definition of the Choquet integral over non-positive integrands and the third equality holds by Lemma 2 and the nonnegativity of

$[-x^{1-\rho}/(1-\rho)]$. The remainder of this case is proven by following the remainder of the preceding paragraph verbatim (the only difference is that w has been defined differently here). $\square$

Lemma 11: *Suppose that P is the probability measure corresponding to the probability vector p, and that $\alpha > 0$. Then, for any $u \in \Re_+^S$,*

if u is weakly increasing, $\quad \int u_s P^\alpha(ds) = \sum_s u_s \breve{p}_s^\alpha \quad and \quad \int u_s P^{\alpha\prime}(ds) = \sum_s u_s \hat{p}_s^\alpha$;

if u is weakly decreasing, $\quad \int u_s P^\alpha(ds) = \sum_s u_s \hat{p}_s^\alpha \quad and \quad \int u_s P^{\alpha\prime}(ds) = \sum_s u_s \breve{p}_s^\alpha$.

Proof: Suppose u is weakly increasing. Then by the definition of the Choquet integral,

$$
\int u_s P^\alpha(ds)
$$
$$
= \int_0^{+\infty} P^\alpha(\{i \mid u_i \geq y\})dy
$$
$$
= \int_0^{u_1} P^\alpha(\{i \mid u_i \geq y\})dy + \sum_{s>1} \int_{u_{s-1}}^{u_s} P^\alpha(\{i \mid u_i \geq y\})dy + \int_{u_{\#S}}^{+\infty} P^\alpha(\{i \mid u_i \geq y\})dy
$$
$$
= \int_0^{u_1} P^\alpha(\{1,2,\ldots,\#S\})dy + \sum_{s>1} \int_{u_{s-1}}^{u_s} P^\alpha(\{s,s+1,\ldots,\#S\})dy + \int_{u_{\#S}}^{+\infty} P^\alpha(\phi)dy
$$
$$
= u_1(\sum_i p_i)^\alpha + \sum_{s>1}(u_s - u_{s-1})(\sum_{i\geq s} p_i)^\alpha + 0
$$
$$
= \sum_s u_s(\sum_{i\geq s} p_i)^\alpha - \sum_{s>1} u_{s-1}(\sum_{i\geq s} p_i)^\alpha
$$
$$
= u_{\#S}(p_{\#S})^\alpha + \sum_{s<\#S} u_s(\sum_{i\geq s} p_i)^\alpha - \sum_{s<\#S} u_s(\sum_{i>s} p_i)^\alpha
$$
$$
= u_{\#S}(p_{\#S})^\alpha + \sum_{s<\#S} u_s[(\sum_{i\geq s} p_i)^\alpha - (\sum_{i>s} p_i)^\alpha]
$$
$$
= \sum_s u_s \breve{p}_s^\alpha,
$$

and by the definition of the conjugate $(P^\alpha)'$,

$$
\int u_s (P^\alpha)'(ds)
$$
$$
= \int_0^{+\infty} 1 - P^\alpha(\{i \mid u_i < y\})dy
$$
$$
= \int_0^{u_1} 1 - P^\alpha(\{i \mid u_i < y\})dy + \sum_{s>1} \int_{u_{s-1}}^{u_s} 1 - P^\alpha(\{i \mid u_i < y\})dy
$$
$$
+ \int_{u_{\#S}}^{+\infty} 1 - P^\alpha(\{i \mid u_i < y\})dy
$$

$$= \int_0^{u_1} 1 - P^\alpha(\phi)dy + \sum_{s>1} \int_{u_{s-1}}^{u_s} 1 - P^\alpha(\{1, 2, \ldots, s-1\})dy$$

$$+ \int_{u_{\#S}}^{+\infty} 1 - P^\alpha(\{1, 2, \ldots, \#S\})dy$$

$$= u_1 + \sum_{s>1} [u_s - u_{s-1}][1 - (\sum_{i<s} p_i)^\alpha] + 0$$

$$= u_1 + \sum_{s>1} u_s[1 - (\sum_{i<s} p_i)^\alpha] - \sum_{s>1} u_{s-1}[1 - (\sum_{i<s} p_i)^\alpha]$$

$$= \sum_s u_s[1 - (\sum_{i<s} p_i)^\alpha] - \sum_{s<\#S} u_s[1 - (\sum_{i\le s} p_i)^\alpha]$$

$$= u_{\#S}[1 - (\sum_{i<\#S} p_i)^\alpha] + \sum_{s<\#S} u_s[(\sum_{i\le s} p_i)^\alpha - (\sum_{i<s} p_i)^\alpha]$$

$$= \sum_s u_s \hat{p}_s^\alpha .$$

A symmetric argument holds when u is weakly decreasing. $\square$

Lemma 12: *Suppose that the columns of P are identical, and let*

$$k^* = (\beta \sum_{s_+} R_{s_+}^{1-\rho} \check{p}_{s_+}^\alpha)^{1/\rho} .$$

Then,

$$\text{if } k^* < 1, \quad (\forall s) \lim_{n\to\infty} (\check{b}^n \mathbf{o})_s = (1 - k^*)^{-\rho} < +\infty ,$$

$$\text{and if } k^* \ge 1, \quad (\forall s) \lim_{n\to\infty} (\check{b}^n \mathbf{o})_s = +\infty .$$

Proof: Define $b : \Re_+ \to \Re_+$ by $b(j) = (1 + k^* j^{1/\rho})^\rho$. This paragraph shows that $(\forall n \ge 1)(\forall s)\ (\check{b}^n \mathbf{o})_s = b^n(0)$ by induction on $n \ge 1$. At $n = 1$, we have $(\forall s)\ (\check{b}\mathbf{o})_s = b(0)$ since both equal 1. Then, at any $n \ge 2$,

$$(\forall s) \quad (\check{b}^n \mathbf{o})_s$$

$$= (1 + (\beta \sum_{s_+} (\check{b}^{n-1}\mathbf{o})_{s_+} R_{s_+}^{1-\rho} \check{p}_{s_+|s}^\alpha)^{1/\rho})^\rho$$

$$= (1 + (\beta \sum_{s_+} b^{n-1}(0) R_{s_+}^{1-\rho} \check{p}_{s_+|s}^\alpha)^{1/\rho})^\rho$$

$$= (1 + (b^{n-1}(0))^{1/\rho}(\beta \sum_{s_+} R_{s_+}^{1-\rho} \check{p}_{s_+|s}^\alpha)^{1/\rho})^\rho$$

$$= (1 + (b^{n-1}(0))^{1/\rho} k^*)^\rho$$

$$= b(b^{n-1}(0))$$

$$= b^n(0),$$

where the first five equalities follow from the definition of $\breve{b}$, the inductive hypothesis, algebra, the definition of k^*, and the definition of b.

Since $(\forall n)$ $b^n(0) = (1 + k^* b^{n-1}(0)^{1/\rho})^\rho$ by the definition of b,

$$(\forall n) \quad b^n(0)^{1/\rho} = 1 + k^* b^{n-1}(0)^{1/\rho}.$$

Hence $\langle b^n(0)^{1/\rho} \rangle_n$ obeys a linear difference equation.

If $k^* < 1$, Luenberger (1979, Theorem 1, p. 157) shows that $\lim_{n \to \infty} b^n(0)^{1/\rho} = z^*$, where $z^* \in \Re_+$ is the unique solution to $z^* = 1 + k^* z^*$. Hence $z^* = (1 - k^*)^{-1}$ and $\lim_{n \to \infty} b^n(0) = (1 - k^*)^{-\rho}$. Therefore, by the proof's first paragraph, $(\forall s)$ $\lim_{n \to \infty} (\breve{b}^n \mathbf{o})_s = (1 - k^*)^{-\rho}$.

On the other hand, if $k^* \geq 1$, the difference equation shows that $\lim_{n \to \infty} b^n(0)^{1/\rho} = +\infty$, and hence, that $\lim_{n \to \infty} b^n(0) = +\infty$. Therefore, by the proof's first paragraph, $(\forall s)$ $\lim_{n \to \infty} (\breve{b}^n \mathbf{o})_s = +\infty$. □

Lemma 13: *Suppose that the columns of P are stochastically ordered. Define $\breve{b}_\alpha$ as $\breve{b}$ with $\alpha > 0$ explicit rather than implicit. Suppose $\alpha' \geq \alpha$. Then $(\forall n)$ $\breve{b}_{\alpha'}^n \mathbf{o} \leq \breve{b}_\alpha^n \mathbf{o}$ when $\rho \in (0,1)$, and $(\forall n)$ $\breve{b}_{\alpha'}^n \mathbf{o} \geq \breve{b}_\alpha^n \mathbf{o}$ when $\rho \in (1, +\infty)$.*

Proof: Suppose $\rho \in (0,1)$. We prove by induction on $n \geq 1$ that (1) $(\forall n)$ $\breve{b}_{\alpha'}^n \mathbf{o} \leq \breve{b}_\alpha^n \mathbf{o}$ and (2) $(\forall n)$ $\breve{b}_{\alpha'}^n \mathbf{o}$ is weakly increasing. Both statements hold at $n = 1$ since $\breve{b}_{\alpha'}^1 \mathbf{o} = \breve{b}_\alpha^1 \mathbf{o} = 1$. Then, at any $n \geq 2$,

$$
\begin{aligned}
(\breve{b}_{\alpha'}^n \mathbf{o})_s \\
&= (1 + (\beta \sum_{s+} (\breve{b}_{\alpha'}^{n-1} \mathbf{o})_{s+} R_{s+}^{1-\rho} \breve{p}_{s+|s}^{\alpha'})^{1/\rho})^\rho \\
&\leq (1 + (\beta \sum_{s+} (\breve{b}_{\alpha'}^{n-1} \mathbf{o})_{s+} R_{s+}^{1-\rho} \breve{p}_{s+|s}^{\alpha})^{1/\rho})^\rho \\
&\leq (1 + (\beta \sum_{s+} (\breve{b}_{\alpha}^{n-1} \mathbf{o})_{s+} R_{s+}^{1-\rho} \breve{p}_{s+|s}^{\alpha})^{1/\rho})^\rho \\
&= (\breve{b}_\alpha^n \mathbf{o})_s
\end{aligned}
$$

by the definition of $\breve{b}_{\alpha'}$, by inductive hypothesis (2) and the fact that $R^{1-\rho}$ is weakly increasing, by inductive hypothesis (1), and by the definition of $\breve{b}_\alpha$. Finally, at any $n \geq 2$,

$$(\breve{b}_{\alpha'}^n \mathbf{o})_s = (1 + (\beta \sum_{s+} (\breve{b}_{\alpha'}^{n-1} \mathbf{o})_{s+} R_{s+}^{1-\rho} \breve{p}_{s+|s}^{\alpha'})^{1/\rho})^\rho$$

Lemma 8 and because $\lambda(\boldsymbol{\Gamma}^{(1-\rho)/z}\boldsymbol{P}) < (w\beta)^{-1/z}$ is equivalent to the assumption $(\alpha/(\alpha - z))\lambda(\boldsymbol{\Gamma}^{(1-\rho)/z}\boldsymbol{P})^z < \beta^{-1}$; the second strict inequality holds because $(\forall s)$ $m_s < (w\beta)^{-1/z}$; and the last equality holds because δ was set equal to β.

As promised, the above bound on g_s^t holds because

$$
\begin{aligned}
&g_s^t \\
&= N_s^t(\prod_{q=1}^{t}\gamma_{s_q}^\varepsilon) \\
&= \int \cdots \int\int \prod_{q=1}^{t} R_{s_q}^{1-\rho} P_{s_{t-1}}^\alpha(ds_t) P_{s_{t-2}}^\alpha(ds_{t-1}) \cdots P_s^\alpha(ds_1) \\
&\leq w\left[\int \cdots w\left[\int w\left[\int \prod_{q=1}^{t}(R_{s_q}^{1-\rho})^{1/z} P_{s_{t-1}}(ds_t)\right]^{z/z} P_{s_{t-2}}(ds_{t-1})\right]^z \cdots^{1/z} P_s(ds_1)\right]^z \\
&= w\left[w^{t-1}\int \cdots \int\int \prod_{q=1}^{t} R_{s_q}^{(1-\rho)/z} P_{s_{t-1}}(ds_t) P_{s_{t-2}}(ds_{t-1}) \cdots P_s(ds_1)\right]^z \\
&\leq w^t\left[\int \cdots \int\int \prod_{q=1}^{t} R_{s_q}^{(1-\rho)/z} P_{s_{t-1}}(ds_t) P_{s_{t-2}}(ds_{t-1}) \cdots P_s(ds_1)\right]^z \\
&= w^t[\sum_{s_1}\cdots\sum_{s_{t-1}}\sum_{s_t}\prod_{q=1}^{t} R_{s_q}^{(1-\rho)/z} p_{s_t|s_{t-1}}p_{s_{t-1}|s_{t-2}}\cdots p_{s_1|s}]^z \\
&= w^t[\mathbf{1}'(\boldsymbol{\Gamma}^{(1-\rho)/z}\boldsymbol{P})^t\mathbf{1}_s]^z .
\end{aligned}
$$

The first equality is the definition of g_s^t (see OS Assumption N8), and the second follows from the definitions of N, γ, and ε. The first inequality is derived in the following paragraph, and the second inequality holds because $w > 1$ and $z < 1$. The algebra behind the last equality is similar to that of (5) below.

As promised, the first weak inequality above holds because

$$
\begin{aligned}
&\int u\,dP^\alpha \\
&= \int_0^{+\infty} [P(\{s\,|\,u_s \geq y\})]^\alpha dy \\
&\leq b + \int_b^{+\infty} [P(\{s\,|\,u_s \geq y\})]^\alpha dy \\
&\leq b + \int_b^{+\infty} \left[\int u^{1/z}dP/y^{1/z}\right]^\alpha dy \\
&= b + \int_b^{+\infty} [b^{1/z}/y^{1/z}]^\alpha dy
\end{aligned}
$$

$$= b + b^{\alpha/z} \int_b^{+\infty} y^{-\alpha/z} dy$$

$$= b + b^{\alpha/z}[-b^{1-\alpha/z}/(1 - \alpha/z)]$$

$$= [\alpha/(\alpha - z)]b$$

$$= w \left[\int u^{1/z} dP \right]^z ,$$

where $b \equiv (\int u^{1/z} dP)^z$ is employed for notational ease. The second inequality follows from the observation that $y^{1/z} P(\{s \mid u_s \geq y\}) \leq \int_{\{s \mid u_s \geq y\}} u^{1/z} dP \leq \int u^{1/z} dP$ (a version of the Chebyshev inequality). The integral is evaluated in the fourth equality using the assumption $z < \alpha$.

Second consider $\alpha \in [1, +\infty)$. Define N by $N_s u = \int u \, dP_s^{\alpha\prime}$. All the OS assumptions are immediate except for N4 which follows from Lemma 7 because P_s^α is convex by Lemma 3; N5 which follows from Lemma 6 because $(P_s^\alpha)'$ is concave by Lemmas 3 and 4; and N9 which follows from

$$\sup_s \lim_{t \to \infty} (g_s^t)^{1/t}$$

$$\leq \sup_s \lim_{t \to \infty} (\alpha^t \mathbf{1}'(\Gamma^{1-\rho} P)^t \mathbf{1}_s)^{1/t}$$

$$\leq \sup_s \lim_{t \to \infty} (\alpha^t k_s m_s^t)^{1/t}$$

$$= \sup_s \alpha m_s$$

$$< \beta^{-1} = \delta^{-1} ,$$

where the weak inequality bounding g_s^t is derived below; $(\forall s)$ k_s and

$$m_s \in (\lambda(\Gamma^{1-\rho} P), (\alpha\beta)^{-1})$$

exist to satisfy the first strict inequality because of Lemma 8 and the assumption $\alpha\lambda(\Gamma^{1-\rho} P) < \beta^{-1}$; the second strict inequality holds because $(\forall s)$ $m_s < (\alpha\beta)^{-1}$; and the last equality holds because δ was set equal to β.

As promised, the above bound on g_s^t holds because

$$g_s^t$$

$$= N_s^t (\prod_{q=1}^t \gamma_{s_q}^\varepsilon) \tag{5}$$

$$= \int \cdots \int \int \prod_{q=1}^{t} R_{s_q}^{1-\rho} P_{s_{t-1}}^{\alpha\prime}(ds_t) P_{s_{t-2}}^{\alpha\prime}(ds_{t-1}) \cdots P_s^{\alpha\prime}(ds_1)$$

$$\leq \alpha \int \cdots \alpha \int \alpha \int \prod_{q=1}^{t} R_{s_q}^{1-\rho} P_{s_{t-1}}(ds_t) P_{s_{t-2}}(ds_{t-1}) \cdots P_s(ds_1)$$

$$= \alpha^t \sum_{s_1} \cdots \sum_{s_{t-1}} \sum_{s_t} \prod_{q=1}^{t} R_{s_q}^{1-\rho} p_{s_t|s_{t-1}} p_{s_{t-1}|s_{t-2}} \cdots p_{s_1|s}$$

$$= \alpha^t \sum_{s_1} \cdots \sum_{s_{t-1}} \sum_{s_t} R_{s_t}^{1-\rho} p_{s_t|s_{t-1}} R_{s_{t-1}}^{1-\rho} p_{s_{t-1}|s_{t-2}} \cdots R_{s_1}^{1-\rho} p_{s_1|s}$$

$$= \alpha^t \sum_{s_1} \cdots \sum_{s_{t-1}} (\mathbf{1}'(\Gamma^{1-\rho}P))_{s_{t-1}} R_{s_{t-1}}^{1-\rho} p_{s_{t-1}|s_{t-2}} \cdots R_{s_1}^{1-\rho} p_{s_1|s}$$

$$= \alpha^t \sum_{s_1} \cdots (\mathbf{1}'(\Gamma^{1-\rho}P)^2)_{s_{t-2}} \cdots R_{s_1}^{1-\rho} p_{s_1|s}$$

$$= \alpha^t (\mathbf{1}'(\Gamma^{1-\rho}P)^t)_s$$

$$= \alpha^t \mathbf{1}'(\Gamma^{1-\rho}P)^t \mathbf{1}_s .$$

The first equality is the definition of g_s^t (see OS assumption N8), and the second follows from the definitions of N, γ, and ε. The weak inequality follows from $\alpha \in [1, +\infty)$ because $(\forall A \subseteq S)\ P^{\alpha\prime}(A) = 1 - P^\alpha(\sim A) = 1 - (1 - P(A))^\alpha \leq \alpha P(A)$ (the first equality is the definition of the conjugate capacity and the inequality holds for any number such as $P(A) \in [0,1]$ because $\alpha \in [1, +\infty)$). The remainder is algebra.

We now derive the theorem's conclusions. By (3), $J^- = 0$, Lemma 10, and the theorem's definition of j^*,

$$J_s^*(y) \tag{6}$$

$$= [\lim_{n \to \infty} B^n J^-]_s(y)$$

$$= [\lim_{n \to \infty} B^n 0]_s(y)$$

$$= [\lim_{n \to \infty} b^n \mathrm{o}]_s y^{1-\rho}/(1-\rho)$$

$$= j_s^* y^{1-\rho}/(1-\rho).$$

Since the utility function U is finite-valued by OS Theorem D, J^* is finite-valued, and thus, the above equality implies that j^* is finite. Finally,

$$K_s^*(y)$$

$$= \arg\max \left\{ (y-x)^{1-\rho}/(1-\rho) + \beta \int J_{s_+}^*(R_{s_+}x) P_s^\alpha(ds_+) \ \middle| \ x \in [0, y] \right\}$$

$$= \{[((bj^*)_s^{1/\rho} - 1)/(bj^*)_s^{1/\rho}]y\}$$

$$= \{[((j_s^*)^{1/\rho} - 1)/(j_s^*)^{1/\rho}]y\}.$$

by the definition of K^*, by (6) and Lemma 10, and by the fact that $(bj^*)_s = j_s^*$ (this holds because $(bj^*)_s y^{1-\rho}/(1-\rho) = (BJ^*)_s(y) = J_s^*(y) = j_s^* y^{1-\rho}/(1-\rho)$ by (6) and Lemma 10, (2), and (6) again). The converse of (4) holds because $\beta > 0$ and $P \gg$ o. This and (4) itself imply that K^* characterizes optimality. $\square$

Proof 15 (for Theorem B$^+$): First we establish that

$$J^* \text{ exists and is the unique admissible solution to Bellman's equation,} \tag{7}$$

$$J^* = \lim_{n \to \infty} B^n J^+, \tag{8}$$

$$J^* = \lim_{n \to \infty} B^n J^-, \quad \text{and} \tag{9}$$

$$\arg\max\{U_s(_0c) \,|\, _0c \text{ is feasible from } y\} \tag{10}$$

$$\supset \{_0c \,|\, (\exists_0 x)\, x_0 \in K_s^*(y) \text{ and } c_0 = y - x_0;$$

$$(\forall t \geq 1)\ x_t \in K^*(Rx_{t-1}) \text{ and } c_t = Rx_{t-1} - x_t\}$$

(see Definition 9 and note that all limits are pointwise). We do this by applying OS Theorems B and D: set $\bar{\delta}$, δ, ε, ζ_s, and γ_s there equal to β, β, $1 - \rho$, 0, and R_s here, and define N as follows in the theorem's two cases.

First consider $\alpha \in (0, 1]$. Define N by $N_s u = \int u\, dP^\alpha$. All the OS assumptions are immediate except for N4 and N5 which follow from Lemma 6 because P_s^α is concave (Definition 1) by Lemma 3 (note $N = M$ in OS), and N9 which follows from

$$\sup_s \lim_{t \to \infty} (g_s^t)^{1/t}$$

$$= \sup_s \lim_{t \to \infty} (1'(\Gamma^{1-\rho}\check{P}^\alpha)^t 1_s)^{1/t}$$

$$\leq \sup_s \lim_{t \to \infty} (k_s m_s^t)^{1/t}$$

$$= \sup_s m_s$$

$$< \beta^{-1} = \delta^{-1},$$

where the first equality is derived below (1 is a vector of ones and 1_s is a unit vector with a 1 in the s-th position); $(\forall s)$ k_s and $m_s \in (\lambda(\Gamma^{1-\rho}\check{P}^\alpha), \beta^{-1})$ exist to satisfy the first

strict inequality because of Lemma 8 and the assumption $\lambda(\Gamma^{1-\rho}\check{P}^{\alpha}) < \beta^{-1}$; the second strict inequality holds because $(\forall s)$ $m_s < \beta^{-1}$; and the last equality holds because δ was set equal to β. As promised, the above equality for g_s^t holds because

$$
\begin{aligned}
g_s^t \\
&= N_s^t(\prod_{q=1}^{t} \gamma_{s_q}^\varepsilon) \\
&= \int \cdots \int \int \prod_{q=1}^{t} R_{s_q}^{1-\rho} P_{s_{t-1}}^\alpha(ds_t) P_{s_{t-2}}^\alpha(ds_{t-1}) \cdots P_s^\alpha(ds_1) \\
&= \int \cdots \int \int R_{s_t}^{1-\rho} P_{s_{t-1}}^\alpha(ds_t) R_{s_{t-1}}^{1-\rho} P_{s_{t-2}}^\alpha(ds_{t-1}) \cdots R_{s_1}^{1-\rho} P_s^\alpha(ds_1) \\
&= \int \cdots \int \sum_{s_t} R_{s_t}^{1-\rho} \check{p}_{s_t|s_{t-1}}^\alpha R_{s_{t-1}}^{1-\rho} P_{s_{t-2}}^\alpha(ds_{t-1}) \cdots R_{s_1}^{1-\rho} P_s^\alpha(ds_1) \\
&= \int \cdots \int (\mathbf{1}'\Gamma^{1-\rho}\check{P}^\alpha)_{s_{t-1}} R_{s_{t-1}}^{1-\rho} P_{s_{t-2}}^\alpha(ds_{t-1}) \cdots R_{s_1}^{1-\rho} P_s^\alpha(ds_1) \\
&= \int \cdots \sum_{s_{t-1}} (\mathbf{1}'\Gamma^{1-\rho}\check{P}^\alpha)_{s_{t-1}} R_{s_{t-1}}^{1-\rho} \check{p}_{s_{t-1}|s_{t-2}}^\alpha \cdots R_{s_1}^{1-\rho} P_s^\alpha(ds_1) \\
&= \int \cdots (\mathbf{1}'(\Gamma^{1-\rho}\check{P}^\alpha)^2)_{s_{t-2}} \cdots R_{s_1}^{1-\rho} P_s^\alpha(ds_1) \\
&\cdots \\
&= (\mathbf{1}'(\Gamma^{1-\rho}\check{P}^\alpha)^t)_s \\
&= \mathbf{1}'(\Gamma^{1-\rho}\check{P}^\alpha)^t \mathbf{1}_s .
\end{aligned}
$$

The first equality is the definition of g_s^t (see OS assumption N8), the second follows from the definitions of N, γ, and ε, and the third follows from Lemma 2. Since $R_{s_t}^{1-\rho}$ weakly increases in s_t, Lemma 11 derives the sum over s_t and stochastic ordering implies that it weakly increases with s_{t-1}. This sum is written more easily as $(\mathbf{1}'\Gamma^{1-\rho}\check{P}^\alpha)_{s_{t-1}}$. Since both $(\mathbf{1}'\Gamma^{1-\rho}\check{P}^\alpha)_{s_{t-1}}$ and $R_{s_{t-1}}^{1-\rho}$ weakly increase in s_{t-1}, Lemma 11 derives the sum over s_{t-1} and stochastic ordering implies that it weakly increases with s_{t-2}. This sum is written more easily as $(\mathbf{1}'(\Gamma^{1-\rho}\check{P}^\alpha)^2)_{s_{t-2}}$. Repeat this process to sum over $s_{t-2}, s_{t-3}, \ldots, s_1$.

Second consider $\alpha \in [1, +\infty)$. Define N by $N_s u = \int u\, dP_s^{\alpha'}$. All the OS assumptions are immediate except for N4 which follows from Lemma 7 because P_s^α is convex by Lemma 3; N5 which follows from Lemma 6 because $P_s^{\alpha'}$ is concave by Lemmas 3 and 4;

and N9 which follows from

$$\sup_s \lim_{t \to \infty} (g_s^t)^{1/t}$$

$$= \sup_s \lim_{t \to \infty} (\mathbf{1}'(\Gamma^{1-\rho}\hat{P}^\alpha)^t \mathbf{1}_s)^{1/t}$$

$$\leq \sup_s \lim_{t \to \infty} (k_s m_s^t)^{1/t}$$

$$= \sup_s m_s$$

$$< \beta^{-1} = \delta^{-1},$$

where the first equality is derived below; $(\forall s)$ k_s and $m_s \in (\lambda(\Gamma^{1-\rho}\hat{P}^\alpha), \beta^{-1})$ exist to satisfy the first strict inequality because of Lemma 8 and the assumption $\lambda(\Gamma^{1-\rho}\hat{P}^\alpha) < \beta^{-1}$; the second strict inequality holds by $(\forall s)$ $m_s < \beta^{-1}$; and the last equality holds because δ was set equal to β. As promised, the above equation for g_s^t holds because

$$g_s^t$$

$$= N_s^t (\prod_{q=1}^t \gamma_{s_q}^\varepsilon)$$

$$= \int \cdots \int \int \prod_{q=1}^t R_{s_q}^{1-\rho} P_{s_{t-1}}^{\alpha\prime}(ds_t) P_{s_{t-2}}^{\alpha\prime}(ds_{t-1}) \cdots P_s^{\alpha\prime}(ds_1)$$

$$= \int \cdots \int \int R_{s_t}^{1-\rho} P_{s_{t-1}}^{\alpha\prime}(ds_t) R_{s_{t-1}}^{1-\rho} P_{s_{t-2}}^{\alpha\prime}(ds_{t-1}) \cdots R_{s_1}^{1-\rho} P_s^{\alpha\prime}(ds_1)$$

$$= \int \cdots \int \sum_{s_t} R_{s_t}^{1-\rho} \hat{p}_{s_t|s_{t-1}}^\alpha R_{s_{t-1}}^{1-\rho} P_{s_{t-2}}^{\alpha\prime}(ds_{t-1}) \cdots R_{s_1}^{1-\rho} P_s^{\alpha\prime}(ds_1)$$

$$= \int \cdots \int (\mathbf{1}'\Gamma^{1-\rho}\hat{P}^\alpha)_{s_{t-1}} R_{s_{t-1}}^{1-\rho} P_{s_{t-2}}^{\alpha\prime}(ds_{t-1}) \cdots R_{s_1}^{1-\rho} P_s^{\alpha\prime}(ds_1)$$

$$= \int \cdots \sum_{s_{t-1}} (\mathbf{1}'\Gamma^{1-\rho}\hat{P}^\alpha)_{s_{t-1}} R_{s_{t-1}}^{1-\rho} \hat{p}_{s_{t-1}|s_{t-2}}^\alpha \cdots R_{s_1}^{1-\rho} P_s^{\alpha\prime}(ds_1)$$

$$= \int \cdots (\mathbf{1}'(\Gamma^{1-\rho}\hat{P}^\alpha)^2)_{s_{t-2}} \cdots R_{s_1}^{1-\rho} P_s^{\alpha\prime}(ds_1)$$

$$\cdots$$

$$= (\mathbf{1}'(\Gamma^{1-\rho}\hat{P}^\alpha)^t)_s$$

$$= \mathbf{1}'(\Gamma^{1-\rho}\hat{P}^\alpha)^t \mathbf{1}_s.$$

The first equality is the definition of g_s^t (OS assumption N8), the second follows from the definitions of N, γ, and ε, and the third follows from Lemma 2. Since $R_{s_t}^{1-\rho}$ weakly increases in s_t, Lemma 11 derives the sum over s_t and stochastic ordering implies that

it weakly increases with s_{t-1}. This sum is written more easily as $(1'\Gamma^{1-\rho}\hat{P}^{\alpha})_{s_{t-1}}$. Since both $(1'\Gamma^{1-\rho}\hat{P}^{\alpha})_{s_{t-1}}$ and $R^{1-\rho}_{s_{t-1}}$ weakly increase in s_{t-1}, Lemma 11 derives the sum over s_{t-1} and stochastic ordering implies that it increases with s_{t-2}. This sum is written more conveniently as $(1'(\Gamma^{1-\rho}\hat{P}^{\alpha})^2)_{s_{t-2}}$. Repeat this process to sum over $s_{t-2}, s_{t-3}, \ldots, s_1$.

Before deriving the theorem's conclusions, we show by induction on $n \geq 0$, that, if j is weakly increasing, then $(\forall n)$ $b^n j$ is weakly increasing and

$$(\forall n) \quad b^n j = \breve{b}^n j. \tag{11}$$

Both statements hold at $n = 0$ (b^0 and $\breve{b}^0$ denote the identity map). Next consider any $n \geq 1$. Note that

$$(b^{n-1} j)_{s_+} R^{1-\rho}_{s_+}$$

is weakly increasing in s_+ because $(b^{n-1} j)_{s_+}$ weakly increases in s_+ by the inductive hypothesis and because $R^{1-\rho}_{s_+}$ also weakly increases in s_+. So, the definition of b, Lemma 11 and the weak increase of $(b^{n-1} j)_{s_+} R^{1-\rho}_{s_+}$, the inductive hypothesis, and the definition of $\breve{b}$ imply

$$
\begin{aligned}
(b^n j)_s & \\
= & \left(1 + \left(\beta \int (b^{n-1} j)_{s_+} R^{1-\rho}_{s_+} P^{\alpha}_s(ds_+)\right)^{1/\rho}\right)^{\rho} \\
= & \left(1 + \left(\beta \sum_{s_+} (b^{n-1} j)_{s_+} R^{1-\rho}_{s_+} \breve{p}^{\alpha}_{s_+|s}\right)^{1/\rho}\right)^{\rho} \\
= & \left(1 + \left(\beta \sum_{s_+} (\breve{b}^{n-1} j)_{s_+} R^{1-\rho}_{s_+} \breve{p}^{\alpha}_{s_+|s}\right)^{1/\rho}\right)^{\rho} \\
= & (\breve{b}^n j)_s .
\end{aligned}
$$

Furthermore, the weak increase of $(b^{n-1} j)_{s_+} R^{1-\rho}_{s_+}$ in s_+ and stochastic ordering together imply that $(b^n j)_s$ is weakly increasing in s.

We will first prove the equality concerning J^* and then go through the remaining conclusions in the order in which they are stated. By (9), $J^- = 0$, Lemma 10, (11), and the theorem's definition of j^*,

$$J^*_s(y) \tag{12}$$

$$= \left[\lim_{n \to \infty} B^n J^- \right]_s (y)$$

$$= \left[\lim_{n \to \infty} B^n 0 \right]_s (y)$$

$$= \left[\lim_{n \to \infty} b^n \mathbf{o} \right]_s y^{1-\rho} / (1 - \rho)$$

$$= \left[\lim_{n \to \infty} \breve{b}^n \mathbf{o} \right]_s y^{1-\rho} / (1 - \rho)$$

$$= j_s^* y^{1-\rho} / (1 - \rho) .$$

Since the utility function is finite-valued by OS Theorem D, J^* is finite-valued, and thus, the above equality also justifies the definition of $j^* \in \Re_+^S$ as a vector of nonnegative real numbers.

Before demonstrating the well-definition of j^+, we establish in this paragraph that

$$\sum_{q=0}^{\infty} \beta^q (\mathbf{1}'(\Gamma^{1-\rho} \breve{P}^\alpha)^q)_s < +\infty . \tag{13}$$

In the case $\alpha \in (0, 1]$,

$$\sum_{q=0}^{\infty} \beta^q (\mathbf{1}'(\Gamma^{1-\rho} \breve{P}^\alpha)^q)_s$$

$$= (\mathbf{1}' \sum_{q=0}^{\infty} \beta^q (\Gamma^{1-\rho} \breve{P}^\alpha)^q)_s$$

$$< +\infty ,$$

where the inequality follows from the assumption $\lambda(\Gamma^{1-\rho} \breve{P}^\alpha) < \beta^{-1}$ and Luenberger (1979, p. 198). In the case $\alpha \in [1, +\infty)$,

$$\sum_{q=0}^{\infty} \beta^q (\mathbf{1}'(\Gamma^{1-\rho} \breve{P}^\alpha)^q)_s$$

$$\leq \sum_{q=0}^{\infty} \beta^q (\mathbf{1}'(\Gamma^{1-\rho} \hat{P}^\alpha)^q)_s$$

$$= (\mathbf{1}' \sum_{q=0}^{\infty} \beta^q (\Gamma^{1-\rho} \hat{P}^\alpha)^q)_s$$

$$< +\infty ,$$

where the first inequality follows from the statement (b) proven in the remainder of this paragraph, and the second inequality follows from the assumption $\lambda(\Gamma^{1-\rho} \hat{P}^\alpha) < \beta^{-1}$ and Luenberger (1979, p. 198). The remainder of this paragraph proves by induction on

$q \geq 0$ that (a) the row vector $\mathbf{1}'(\varGamma^{1-\rho}\check{P}^{\alpha})^q$ is weakly increasing and (b) $\mathbf{1}'(\varGamma^{1-\rho}\check{P}^{\alpha})^q \leq \mathbf{1}'(\varGamma^{1-\rho}\hat{P}^{\alpha})^q$. Both are obviously true at $q = 0$ (where $A^0 \equiv I$ for any matrix A). Now consider any $q \geq 1$. Note that the row vector

$$[\mathbf{1}'(\varGamma^{1-\rho}\check{P}^{\alpha})^{q-1}\varGamma^{1-\rho}] \quad \text{is weakly increasing} \tag{14}$$

since $\mathbf{1}'(\varGamma^{1-\rho}\check{P}^{\alpha})^{q-1}$ is a weakly increasing row vector by the inductive hypothesis (a), and since $\varGamma^{1-\rho}$ is a weakly increasing diagonal matrix. Statement (a) follows from

$$\mathbf{1}'(\varGamma^{1-\rho}\check{P}^{\alpha})^q$$
$$= [\mathbf{1}'(\varGamma^{1-\rho}\check{P}^{\alpha})^{q-1}\varGamma^{1-\rho}]\check{P}^{\alpha},$$

(14), and stochastic ordering. Statement (b) follows from

$$\mathbf{1}'(\varGamma^{1-\rho}\check{P}^{\alpha})^q$$
$$= [\mathbf{1}'(\varGamma^{1-\rho}\check{P}^{\alpha})^{q-1}\varGamma^{1-\rho}]\check{P}^{\alpha}$$
$$\leq [\mathbf{1}'(\varGamma^{1-\rho}\check{P}^{\alpha})^{q-1}\varGamma^{1-\rho}]\hat{P}^{\alpha}$$
$$\leq [\mathbf{1}'(\varGamma^{1-\rho}\hat{P}^{\alpha})^{q-1}\varGamma^{1-\rho}]\hat{P}^{\alpha}$$
$$= \mathbf{1}'(\varGamma^{1-\rho}\hat{P}^{\alpha})^q.$$

Here, the first inequality holds by (14) and the fact that each column of $\check{P}^{\alpha}$ is stochastically lower than the corresponding column of $\hat{P}^{\alpha}$ because $\alpha \in [1, +\infty)$; and the second inequality holds by inductive hypothesis (b).

(13) implies that $\sum_{q=0}^{\infty}(\beta\varGamma^{1-\rho}\check{P}^{\alpha})^q$ is finite. Consequently, Luenberger (1979, first paragraph of lemma's proof) establishes that

$$\sum_{q=0}^{\infty}(\beta\varGamma^{1-\rho}\check{P}^{\alpha})^q = [I - \beta\varGamma^{1-\rho}\check{P}^{\alpha}]^{-1}. \tag{15}$$

These two observations justify the definition of j^+ as $\mathbf{1}'[I - \beta\varGamma^{1-\rho}\check{P}^{\alpha}]^{-1} \in \mathfrak{R}_+^S$.

Before establishing that $j^* = \lim_{n\to\infty}\check{b}^n j^+$, note that

$$J_s^+(y) \tag{16}$$
$$= U_s(y, Ry, R^2y, \ldots)$$

$$= \lim_{t\to\infty} y^{1-\rho}/(1-\rho) + \beta\int (R_{s_1}y)^{1-\rho}/(1-\rho) + \cdots \beta\int(\prod_{q=1}^{t-1} R_{s_q}y)^{1-\rho}/(1-\rho)$$

$$+\beta\int(\prod_{q=1}^{t} R_{s_q}y)^{1-\rho}/(1-\rho)P^\alpha_{s_{t-1}}(ds_t)P^\alpha_{s_{t-2}}(ds_{t-1})\cdots P^\alpha_s(ds_1)$$

$$= \lim_{t\to\infty} y^{1-\rho}/(1-\rho)\cdot$$
$$\left\{1+\beta\int\left(1+\cdots\beta\int\left(1+\beta\int R^{1-\rho}_{s_t}P^\alpha_{s_{t-1}}(ds_t)\right)R^{1-\rho}_{s_{t-1}}P^\alpha_{s_{t-2}}(ds_{t-1})\cdots\right)R^{1-\rho}_{s_1}P^\alpha_s(ds_1)\right\}$$

$$= \lim_{t\to\infty} y^{1-\rho}/(1-\rho)\cdot$$
$$\left\{1+\beta\int\left(1+\cdots\beta\int\left(1+\beta(\mathbf{1}'\Gamma^{1-\rho}\check{P}^\alpha)_{s_{t-1}}\right)R^{1-\rho}_{s_{t-1}}P^\alpha_{s_{t-2}}(ds_{t-1})\cdots\right)R^{1-\rho}_{s_1}P^\alpha_s(ds_1)\right\}$$

$$= \lim_{t\to\infty} y^{1-\rho}/(1-\rho)\cdot$$
$$\left\{1+\beta\int\left(1+\cdots\beta(\mathbf{1}'\Gamma^{1-\rho}\check{P}^\alpha)_{s_{t-2}}+\beta^2(\mathbf{1}'(\Gamma^{1-\rho}\check{P}^\alpha)^2)_{s_{t-2}}\cdots\right)R^{1-\rho}_{s_1}P^\alpha_s(ds_1)\right\}$$

$$= \lim_{t\to\infty} y^{1-\rho}/(1-\rho)\cdot$$
$$\{1+\beta(\mathbf{1}'\Gamma^{1-\rho}\check{P}^\alpha)_s+\cdots+\beta^{t-1}(\mathbf{1}'(\Gamma^{1-\rho}\check{P}^\alpha)^{t-1})_s+\beta^t(\mathbf{1}'(\Gamma^{1-\rho}\check{P}^\alpha)^t)_s\}$$

$$= y^{1-\rho}/(1-\rho)\sum_{q=0}^{\infty}\beta^q(\mathbf{1}'(\Gamma^{1-\rho}\check{P}^\alpha)^q)_s$$

$$= y^{1-\rho}/(1-\rho)(\mathbf{1}'[I-\beta\Gamma^{1-\rho}\check{P}^\alpha]^{-1})_s$$

$$= j^+_s y^{1-\rho}/(1-\rho).$$

The first three equalities hold by the definition of J^+, the definition of U, and Lemma 2. Since $R^{1-\rho}_{s_t}$ weakly increases in s_t, Lemma 11 derives the fourth equality and stochastic ordering implies that $(\mathbf{1}'\Gamma^{1-\rho}\check{P}^\alpha)_{s_{t-1}}$ weakly increases in s_{t-1}. Since both $R^{1-\rho}_{s_{t-1}}$ and $(\mathbf{1}'\Gamma^{1-\rho}\check{P}^\alpha)_{s_{t-1}}$ weakly increase in s_{t-1}, Lemma 11 derives the fifth equality and stochastic ordering implies that both $(\mathbf{1}'(\Gamma^{1-\rho}\check{P}^\alpha)^2)_{s_{t-2}}$ and $(\mathbf{1}'\Gamma^{1-\rho}\check{P}^\alpha)_{s_{t-2}}$ are weakly increasing in s_{t-2}. The sixth equality, and the fact that every term of the form $(\mathbf{1}'(\Gamma^{1-\rho}\check{P}^\alpha)^q)_s$ weakly increases in s, are obtained by repeating this process at $t-2, t-3, \ldots, 1$. The final two equalities hold by (15) and the theorem's definition of j^+. The final three equalities and the monotonicity in s derived two sentences ago demonstrate that j^+ is weakly increasing.

By (12), by (8), by (16) and Lemma 10, and by (11),

$$j^*_s y^{1-\rho}/(1-\rho)$$
$$= J^*_s(y)$$

$$= [\lim_{n\to\infty} B^n J^+]_s(y)$$

$$= [\lim_{n\to\infty} b^n j^+]_s y^{1-\rho}/(1-\rho)$$

$$= [\lim_{n\to\infty} \breve{b}^n j^+]_s y^{1-\rho}/(1-\rho).$$

Hence $j^* = \lim_{n\to\infty} \breve{b}^n j^+$.

j^* is weakly increasing because $j^* = \lim_{n\to\infty} \breve{b}^n o$ by definition and $\lim_{n\to\infty} \breve{b}^n o$ is weakly increasing by the sentence containing (11). j^* satisfies $o \le j^* \le j^+$ by the admissibility of J^* derived in (7), by $J^- = 0$, by (12), and by (16). j^* satisfies

$$j^* = \breve{b} j^* \tag{17}$$

because $j_s^* y^{1-\rho}/(1-\rho) = J_s^*(y) = (BJ^*)_s(y) = (bj^*)_s y^{1-\rho}/(1-\rho) = (\breve{b}j^*)_s y^{1-\rho}/(1-\rho)$ by (12), (7), Lemma 10 and (12), and (11).

On the other hand, suppose j' is weakly increasing and satisfies both $o \le j' \le j^+$ and $j' = \breve{b}j'$. Define $J_s'(y) = j_s' y^{1-\rho}/(1-\rho)$. The first supposition about j' implies J' is admissible because of $J^- = 0$ and (16). The second implies $(BJ')_s(y) = (bj')_s y^{1-\rho}/(1-\rho) = (\breve{b}j')_s y^{1-\rho}/(1-\rho) = j_s' y^{1-\rho}/(1-\rho) = J_s'(y)$ by Lemma 10 and (11). Thus J' is an admissible solution to Bellman's equation, and hence, $J' = J^*$ since J^* is the only admissible solution to Bellman's equation by (7). Consequently, $j' = j^*$ by the definition of J' and (12).

Finally,

$$K_s^*(y)$$
$$= \arg\max\left\{ (y-x)^{1-\rho}/(1-\rho) + \beta \int J_{s+}^*(R_{s+}x)P_s^\alpha(ds_+) \,\middle|\, x \in [0, y] \right\}$$
$$= \{ [((bj^*)_s^{1/\rho} - 1)/(bj^*)_s^{1/\rho}]y \}$$
$$= \{ [((j_s^*)^{1/\rho} - 1)/(j_s^*)^{1/\rho}]y \}$$

by the definition of K^*, by Lemma 10 and (12), and by (17). The converse of (10) holds because $\beta > 0$ and $P \gg o$. This and (10) itself imply that K^* characterizes optimality. $\square$

Proof 16 (for Theorem A^+) : This paragraph establishes Theorem B^+'s assumptions. If $\alpha \in (0,1]$, $\lambda(\Gamma^{1-\rho}\breve{P}^\alpha) = \sum_{s+} R_{s+}^{1-\rho}\breve{p}_{s+}^\alpha < \beta^{-1}$, where the equality holds by

Luenberger (1979, last sentence of p. 194) and the assumption that the columns of P are identical, and the inequality is algebraically equivalent to the assumption $k^* < 1$. If $\alpha \in [1, +\infty)$, $\lambda(\Gamma^{1-\rho}\hat{P}^{\alpha}) = \sum_{s+} R_{s+}^{1-\rho}\hat{p}_{s+}^{\alpha} < \beta^{-1}$, where the equality again holds by Luenberger (1979, p. 194) and the inequality has been assumed directly.

When $\alpha \in (0,1]$, $k^* < 1$ is assumed. When $\alpha \in [1, +\infty)$, the assumption $\sum_{s+} R_{s+}^{1-\rho}\hat{p}_{s+}^{\alpha} < \beta^{-1}$ implies $k^* < 1$ because $\sum_{s+} R_{s+}^{1-\rho}\check{p}_{s+}^{\alpha} \leq \sum_{s+} R_{s+}^{1-\rho}\hat{p}_{s+}^{\alpha}$ since $\check{p}^{\alpha}$ is stochastically lower than $\hat{p}^{\alpha}$ when $\alpha \in [1, +\infty)$. By Theorem B$^+$'s definition of j^* and Lemma 12, $(\forall s)$ $j_s^* = (\lim_{n\to\infty} \check{b}^n\mathbf{o})_s = (1 - k^*)^{-\rho}$. Therefore Theorem B$^+$ allows us to conclude that $J_s^*(y) = j_s^* y^{1-\rho}/(1-\rho) = (1 - k^*)^{-\rho} y^{1-\rho}/(1-\rho)$ and that optimality is characterized by $K_s^*(y) = \{ [((j_s^*)^{1/\rho} - 1)/(j_s^*)^{1/\rho}]y \} = \{ k^* y \}$. $\square$

Proof 17 (for Theorem C$^-$) : OS Theorems A and E immediately imply that

J^* exists and is the greatest admissible solution to Bellman's equation, (18)

$$J^* = \lim_{n\to\infty} B^n J^+ \text{ , and} \tag{19}$$

$$\arg\max\{ U_s(_0c) \,|\, _0c \text{ is feasible from } y \} \tag{20}$$

$$\supset \{ _0c \,|\, (\exists_0 x)\ x_0 \in K_s^*(y) \text{ and } c_0 = y - x_0;$$

$$(\forall t \geq 1)\ x_t \in K^*(Rx_{t-1}) \text{ and } c_t = Rx_{t-1} - x_t \}$$

(see Definition 9 and note that all limits are pointwise).

Since $b\mathbf{o} \geq \mathbf{o}$, and since b is weakly increasing, it must be that $(\forall n \geq 1)$ $b^{n+1}\mathbf{o} \geq b^n\mathbf{o}$. Thus $\langle b^n\mathbf{o}\rangle_{n=1}^{\infty}$ is a weakly increasing sequence, and consequently, $j^* \equiv \lim_{n\to\infty} b^n\mathbf{o}$ must exist in $\bar{\mathfrak{R}}_+^S$. Hence j^* is well-defined.

In Definition 9, B is defined on the space of admissible value functions. In this proof, extend the domain of B to include inadmissible value functions of the form $J_s(y) = j_s y^{1-\rho}/(1-\rho)$ for some $j \in \bar{\mathfrak{R}}_+^S$ (the well-definition of the extension follows from Lemma 10). Note that by Lemma 10 and the definition of j^*,

$$[\lim_{n\to\infty} B^n\mathbf{0}]_s(y) = [\lim_{n\to\infty} b^n\mathbf{o}]_s y^{1-\rho}/(1-\rho) = j_s^* y^{1-\rho}/(1-\rho). \tag{21}$$

First, suppose that $(\exists s)$ $j_s^* = +\infty$. This implies that $(\forall s)$ $j_s^* = +\infty$ since $P \gg \mathbf{o}$

by assumption. Hence, by (19), by $J^+ \leq 0$, and by (21),

$$J_s^*(y)$$
$$= \ [\lim_{n \to \infty} B^n J^+]_s(y)$$
$$\leq \ [\lim_{n \to \infty} B^n 0]_s(y)$$
$$= \ j_s^* y^{1-\rho}/(1-\rho)$$
$$= \ -\infty.$$

Consequently, any feasible consumption process is optimal.

Second, suppose that $j^* \in \Re_+^S$. By the definition of j^*, the continuity of b, and the definition of j^* again,

$$bj^* = b(\lim_{n \to \infty} b^n o) = \lim_{n \to \infty} b(b^n o) = j^*. \tag{22}$$

Also note that by (21) and Lemma 10, by (22), and by (21) again,

$$[B(\lim_{n \to \infty} B^n 0)]_s(y) \tag{23}$$
$$= \ (bj^*)_s y^{1-\rho}/(1-\rho)$$
$$= \ j_s^* y^{1-\rho}/(1-\rho)$$
$$= \ [\lim_{n \to \infty} B^n 0]_s(y).$$

Define $A : \{J : S \times \Re_+ \to \bar{\Re}_-\} \to \{J : S \times \Re_+ \to \bar{\Re}_-\}$ by

$$(AJ)_s(y) = y^{1-\rho}/(1-\rho) + \beta \int J_{s_+}(R_{s_+} y) P_s^\alpha(ds_+),$$

and note that

$$\lim_{n \to \infty} (A^n 0)_s(y)$$
$$= \ \lim_{n \to \infty} (A(A^{n-1} 0))_s(y)$$
$$= \ \lim_{n \to \infty} y^{1-\rho}/(1-\rho) + \beta \int (A^{n-1} 0)_{s_1}(R_{s_1} y) P_s^\alpha(ds_1)$$
$$= \ \lim_{n \to \infty} y^{1-\rho}/(1-\rho) + \beta \int (A(A^{n-2} 0))_{s_1}(R_{s_1} y) P_s^\alpha(ds_1)$$
$$= \ \lim_{n \to \infty} y^{1-\rho}/(1-\rho) + \beta \int (R_{s_1} y)^{1-\rho}/(1-\rho)$$

$$+\beta \int (A^{n-2}0)_{s_2}(R_{s_2}R_{s_1}y)P^{\alpha}_{s_1}(ds_2)P^{\alpha}_s(ds_1)$$

$$= \lim_{n\to\infty} y^{1-\rho}/(1-\rho) + \beta \int (R_{s_1}y)^{1-\rho}/(1-\rho) + \beta \int (R_{s_2}R_{s_1}y)^{1-\rho}/(1-\rho) + \cdots$$

$$\beta \int (A0)_{s_{n-1}}(\prod_{q=1}^{n-1} R_{s_q}y)P^{\alpha}_{s_{n-2}}(ds_{n-1})\cdots P^{\alpha}_{s_1}(ds_2)P^{\alpha}_s(ds_1)$$

$$= \lim_{n\to\infty} y^{1-\rho}/(1-\rho) + \beta \int (R_{s_1}y)^{1-\rho}/(1-\rho) + \beta \int (R_{s_2}R_{s_1}y)^{1-\rho}/(1-\rho) + \cdots$$

$$\beta \int \{(\prod_{q=1}^{n-1} R_{s_q}y)^{1-\rho}/(1-\rho) + 0\}P^{\alpha}_{s_{n-2}}(ds_{n-1})\cdots P^{\alpha}_{s_1}(ds_2)P^{\alpha}_s(ds_1)$$

$$= J^+_s(y).$$

By (19), by $J^+ \leq 0$ and the monotonicity of B, by $B(J) \leq A(J)$ for all J in the domain of B, and by the preceding equation,

$$J^*$$
$$= \lim_{n\to\infty} B^n J^+$$
$$\leq \lim_{n\to\infty} B^n 0$$
$$\leq \lim_{n\to\infty} A^n 0$$
$$= J^+.$$

Thus $J^* \leq \lim_{n\to\infty} B^n 0 \leq J^+$. By applying B repeatedly to all three of these value functions, and by encapsulating the two inequalities with equalities from (18) and (19), we obtain

$$J^*$$
$$= \lim_{m\to\infty} B^m J^*$$
$$\leq \lim_{m\to\infty} B^m(\lim_{n\to\infty} B^n 0)$$
$$\leq \lim_{m\to\infty} B^m J^+$$
$$= J^*.$$

Thus $J^* = \lim_{m\to\infty} B^m(\lim_{n\to\infty} B^n 0)$. This, (23), and (21) imply

$$J^*_s(y)$$

$$
= [\lim_{m\to\infty} B^m(\lim_{n\to\infty} B^n 0)]_s(y)
$$

$$
= [\lim_{n\to\infty} B^n 0]_s(y)
$$

$$
= j_s^* y^{1-\rho}/(1-\rho).
$$

Finally, by the definition of K^*, by the preceding equality and Lemma 10, and by (22),

$$
K_s^*(y)
$$
$$
= \arg\max\left\{ (y-x)^{1-\rho}/(1-\rho) + \beta\int J_{s_+}^*(R_{s_+}x)P_s^\alpha(ds_+) \,\Big|\, x\in[0,y] \right\}
$$
$$
= \{ [((bj^*)_s^{1/\rho}-1)/(bj^*)_s^{1/\rho}]y \}
$$
$$
= \{ [((j_s^*)^{1/\rho}-1)/(j_s^*)^{1/\rho}]y \}.
$$

The converse of (20) holds because $\beta>0$ and $P\gg 0$. This and (20) itself imply that K^* characterizes optimality. $\square$

Proof 18 (for Theorem B$^-$) : To apply Theorem C$^-$, we only need that $(\forall n)$ $b^n 0 = \check{b}^n 0$. We will use induction on n to show both this equality and the fact that $(\forall n)$ $(b^n 0)_s$ is weakly decreasing in s. Both hold at $n=1$ because $b0 = \check{b}0 = 1$. Next consider any $n\geq 2$. Note that

$$
(b^{n-1} 0)_{s_+} R_{s_+}^{1-\rho}
$$

is weakly decreasing in s_+ because $(b^{n-1} 0)_{s_+}$ is decreasing in s_+ by the inductive hypothesis and because $R_{s_+}^{1-\rho}$ is also decreasing in s_+ because $\rho\in(1,+\infty)$. Hence Lemma 11 and the inductive hypothesis imply

$$
(b^n 0)_s
$$
$$
= \left(1 + \left(\beta\int(b^{n-1} 0)_{s_+} R_{s_+}^{1-\rho} P_s^{\alpha\prime}(ds_+)\right)^{1/\rho}\right)^\rho
$$
$$
= (1 + (\beta\sum_{s_+}(b^{n-1} 0)_{s_+} R_{s_+}^{1-\rho}\check{p}_{s_+|s}^\alpha)^{1/\rho})^\rho
$$
$$
= (1 + (\beta\sum_{s_+}(\check{b}^{n-1} 0)_{s_+} R_{s_+}^{1-\rho}\check{p}_{s_+|s}^\alpha)^{1/\rho})^\rho
$$
$$
= (\check{b}^n 0)_s.
$$

Furthermore, the weak decrease of $(b^{n-1} 0)_{s_+} R_{s_+}^{1-\rho}$ in s_+ and the stochastic ordering of the columns of P together imply that $(b^n 0)_s$ is weakly decreasing in s. $\square$

Proof 19 (for Theorem A^-) : Suppose $k^* < 1$. By Theorem B^-'s definition of j^* and Lemma 12, $(\forall s)\ j_s^* = (\lim_{n\to\infty} \breve{b}^n o)_s = (1 - k^*)^{-\rho} \in \Re_+$. Therefore, Theorem B^- implies that $J_s^*(y) = j_s^* y^{1-\rho}/(1 - \rho) = (1 - k^*)^{-\rho} y^{1-\rho}/(1 - \rho)$ and that optimality is characterized by $K_s^*(y) = \{ [((j_s^*)^{1/\rho} - 1)/(j_s^*)^{1/\rho}]y \} = \{ k^* y \}$.

On the other hand, suppose $k^* \geq 1$. By Theorem B^-'s definition of j^* and Lemma 12, $(\forall s)\ j_s^* = (\lim_{n\to\infty} \breve{b}^n o)_s = +\infty$. Therefore, Theorem B^- implies that $J_s^*(y) = -\infty$ and any feasible process is optimal. $\square$

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