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[Abstract] This paper studies arbitrage-free conditions for multiperiod asset pricing in frictional financial markets with proportional transaction costs. We consider the Euclidean space for weakly arbitrage-free security markets and strongly arbitrage-free security markets, and establish the weakly arbitrage-free pricing theorem and the strongly arbitrage-free pricing theorem.

[Keywords] the first fundamental valuation theorems; frictional markets; weak arbitrage-freeness; strict arbitrage-freeness; arbitrage-free pricing theory

1. Introduction

Arbitrage-free asset pricing theory is of fundamental importance in neo-classical financial economics. Understanding arbitrage-free conditions is a crucial step in the study of general equilibrium theory with security markets. For incomplete asset markets, in particular, Duffie (1985, 1987, 1988, 1996) and Werner (1985, 1990) have laid down the foundation of general equilibrium theory and asset pricing theory. Here, the arbitrage-free pricing theory has been very powerful tool in the proof of the existence of general equilibrium for stochastic economies with incomplete financial markets.

Arbitrage-free conditions have also been an important step toward the general equilibrium theorem with security markets. Harrison & Kreps (1979) initiated the study of martingales and arbitrage in multi-period security markets. They first introduced a general theory of arbitrage in various economies with uncertainty. Kreps (1981) studied arbitrage and equilibrium in economies with infinitely many commodities and presented an abstract analysis of "arbitrage" in economies that have infinite dimensional commodity space. Dalang, Morton & Willinger (1990) studied equivalent martingale measures and no-arbitrage in stochastic securities market models. Back & Pliska (1991) studied the fundamental theorem of asset pricing with an infinite state space and showed some equivalent relations on arbitrage. Jacod & Sgiryaev (1998) studied local martingales and the fundamental asset pricing theorems in the discrete-time case. Dalang, Morton & Willinger (1990) and Jacod & Sgiryaev (1998) studied arbitrage-free model, weakly arbitrage-free model, and strongly arbitrage-free model, provided simple proofs of the two fundamental theorems of asset pricing theory. Jacod & Sgiryaev (1998) proved that these three concepts are equivalent to each another.

Friction in markets has attracted attention of several works in this field recently. Chen (1995) examined the incentives and economic roles of financial innovation and at the same time studied the effectiveness of the replication-based arbitrage valuation approach in frictional economies (the friction means holding constraints). Jouini & Kallal (1995) derived the implications of the absence of arbitrage in securities markets models where traded securities are subject to short-sales constraints and where the borrowing and lending rates differ, and showed that a securities price system is arbitrage free if and only if there exists a numeraire and an equivalent probability measure for which the normalized (by the numeraire) price processes of traded securities are supermartingales. Jouini & Kallal (1995) derived the implications from the absence of arbitrage in dynamic securities markets with bid-ask spreads. The absence of arbitrage is equivalent to the existence of at least an equivalent probability measure that transforms some process between the bid and the ask price processes of traded securities into a martingale. Pham & Touzi (1999) addressed the problem of characterization of no arbitrage (strongly arbitrage-free) in the presence of friction in a discrete-time financial model, and extended the fundamental theorem of asset pricing under a non-degeneracy assumption. The friction is described by the transaction cost rates for purchasing and selling the securities.

Farkas-Minkowski's Lemma and its strict version, Stiemke's Lemma, can be viewed as the mathematical counter part of the two-period asset pricing theory with arbitrage-free conditions. Stiemke's Lemma has been a very important tool in the study of the asset pricing theory with no-arbitrage conditions. In this work, we study the multi-period arbitrage-free security markets. We extend Farkas-Minkowski's Lemma and Stiemke's Lemma from two-period frictionless security markets to multi-period frictional security markets. Our main results are the first fundamental valuation theorems of asseet pricing in neo-classical financial economics.

Ross (1978) showed that the no-arbitrage condition is equivalent to the existence of a valuation or pricing operator with frictonless markets. Garman & Ohlson (1981) extended Ross (1978) to markets with propositional transaction costs, then proved that equilibrium prices in markets with propositional transaction costs equal prices in the "corresponding" markets with no frictions plus a "certain factor". Dermody & Prisman (1993) extended Garman & Ohlson (1981) to markets with increasing mmarginal transaction costs, and showed the precise relation of the "certain factor" to the structure of transaction costs. K.Ardalan (1999) emphasized that their result is applicable to financial markets with decreasing transaction costs.

Section 2 presents our model with frictionless security markets. Time, uncertainty and revealation of information is described by event tree over the finite time where the possible states of nature is finite. Security dividends, security prices and trading strategies are represented by processes in the event tree. Section 3 addresses the security markets with transaction costs. We examine proportional transaction costs, determine the total cost or gain process induced by (trading) a portfolio, then define the weakly arbitrage-free security markets and strongly arbitrage-free security markets, following Duffie (1988). In the next two sections, we establish the first fundamental valuation theorem of asset pricing (a necessary and sufficient condition for arbitrage-freeness) with weakly arbitrage-free security markets in Section 4 and strongly arbitrage-free security markets in Section 5. Section 6 concludes our article with some remarks on strongly arbitrage-free security markets.

2. Security Dividends, Security Prices and Trading Strategies

2.1 Event Tree

We use an event tree Ξ to describe time, uncertainty and revelation of information over a finite horizon. More precisely, let $\mathcal{T} = \{0, 1, \cdots, T\}$ denote the set of time periods and let Ω be a set of finite states of nature. The revelation of information is described by a sequence of partitions of Ω, $\mathcal{F} = (\mathcal{F}_0, \mathcal{F}_1, \cdots, \mathcal{F}_T)$, where the number of subsets in $\mathcal{F}_t$ is finite for $t = 0, 1, \cdots, T$ and $\mathcal{F}_{t+1}$ is finer than the partition $\mathcal{F}_t$ for all $t = 0, 1, \cdots, T-1$. At date $t = 0$ we assume that there exists no information so that $\mathcal{F}_0 = \{\emptyset, \Omega\}$. At date $t = T$ all information to be revealed is available by or at time T so that $\mathcal{F}_T = \mathcal{F}$. The information available at time $t \in \mathcal{T}$ is assumed to be the same for all agents in the economy (symmetric information) and is described by the subset of the partition $\mathcal{F}_t$ in which the state of nature lies. All agents in our economy are assumed to learn information according to an event tree Ξ. The set Ξ consisting of all vertices is called the event-tree induced by $\mathcal{F}$, which is a finite set of vertices.

The set of vertices which succeed a vertex $\xi \in \Xi$ is called the subtree $\Xi(\xi) = \{\xi' \in \Xi \mid \xi' \geq \xi\}$ with root $\xi \in \Xi$. $\Xi^+(\xi) = \{\xi' \in \Xi(\xi) \mid \xi' > \xi\} = \{\xi' \in \Xi \mid \xi' > \xi\}$ is the set of strict successors of ξ. The subset of vertices of $\Xi(\xi)$ at date τ is denoted by $\Xi_\tau(\xi)$ and the subset of vertices between dates $t(\xi)$ and τ by $\Xi^\tau(\xi)$

$$\Xi_\tau(\xi) = \{\xi' \in \Xi(\xi) \mid t(\xi') = \tau\}$$

$$\Xi^\tau(\xi) = \{\xi' \in \Xi(\xi) \mid t(\xi) \leq t(\xi') \leq \tau\}$$

If ξ is the initial vertex the notation is simplified to Ξ^+, Ξ_τ, Ξ^τ.

The number of immediate successor vertex of any $\xi \in \Xi$ is denoted $\#\xi$. A vertex $\xi \in \Xi$ is terminal if $\#\xi = 0$, and otherwise non-terminal. $\xi^+ = \{\xi' \in \Xi(\xi) \mid t(\xi') = t(\xi) + 1\}$ is the set of immediate successors of ξ. Every $\xi \in \Xi$ with $t(\xi) \geq 1$ has the unique predecessor vertex ξ^- of ξ.

$$\Xi = \{\xi_t \mid \xi_t \in \mathcal{F}_t,\ t = 0, 1, \cdots, T\}$$

$$\Xi^+ = \{\xi \in \Xi \mid t(\xi) \neq 0\}$$

$$\Xi_T = \{\xi \in \Xi \mid t(\xi) = T\}$$

$$\Xi^{T-1} = \{\xi \in \Xi \mid t(\xi) \neq T\}$$

2.2 Security Dividends and Security Prices

For any integer $N = 1, 2, \cdots$, let E^N denote the space of $\mathcal{R}^N$-valued functions on Ξ: $E^N = \mathcal{R}^{\Xi \times \mathcal{N}}$ where $\mathcal{N} = \{1, \cdots, N\}$. We assume there exist $J \geq 1$ securities in our model, all of them are issued at time 0. A security is a process $D_j \in E^1$ of spot market "dividends" for $j = 1, \cdots, J$. The security dividend process is the vector $D = (D_1, \cdots, D_J)$. The j-th column of the matrix $D(\xi)$ denotes the vector of accounts that an agent receives at vertex ξ if he held one unit of the j-th security at vertex ξ^-. The securities thus are characterized by a collection of matrices: $D = (D(\xi), \xi \in \Xi)$. Payment for any security is made at the date of purchase. Each security D_j is assigned a real-valued price process $P_j \in E^1$ for $j = 1, \cdots, J$. In other words, $P_j(\xi)$ is the market value of D_j at vertex ξ for $j = 1, \cdots, J$. It will be convenient to treat $P_j(\xi)$ as the market value of D_j after the dividend $D_j(\xi)$ has been "declared", (that is, after vertex ξ occurs) and has been paid for $j = 1, \cdots, J$. Let $P = (P(\xi), \xi \in \Xi)$ where $P(\xi)$ denotes the vector of prices for the securities at vertex ξ. The security price process is the vector $P = (P_1, \cdots, P_J)$. We assume that $D(\xi_0) = 0$ where ξ_0 is the initial vertex; and $P(\xi) = 0$ for $\xi \in \Xi_T$. The pair $(D, P) \in E^J \times E^J$ is a complete characterization of trading opportunities, or a market system.

2.3 Security Trading Strategies

A trading strategy is an element $\theta = (\theta_1, \cdots, \theta_J)$ of the space E^J. Let $\theta = (\theta(\xi), \xi \in \Xi^{T-1})$ and $\theta(\xi)$ denote the number of those securities which are held at vertex ξ. Clearly $\theta(\xi)$ lies in the Euclidean space $\mathcal{R}^J$ where J is a "big" number indicating the total number of securities traded through the vertex ξ. The scalar $\theta_j(\xi)$ represents the number of units of security j held at vertex ξ when strategy θ is followed. We adopt the convention that $\theta(\xi)$ represents the portfolio held after trading at vertex ξ has occurred and dividends $D(\xi) \in \mathcal{R}^J$ are paid. $\theta(\xi^-)[P(\xi) + D_\tau(\xi)]$ is the number of units of account paid in gains at vertex ξ, $\theta(\xi)P(\xi)$ is the market value of the trading strategy θ bought vertex ξ. The dividend process $\delta^\theta \in E^1$ generated by a trading strategy θ is defined by

$$\delta^\theta(\xi) = \theta(\xi^-)[P(\xi) + D(\xi)] - \theta(\xi)P(\xi) = \theta(\xi^-)D(\xi) - \Delta\theta(\xi)P(\xi), \qquad \xi \in \Xi$$

The market value $\theta(\xi^-)D(\xi)$ accrues to trading strategy $\theta \in E^J$ at vertex $\xi \in \Xi$. The portfolio of securities held by $\theta \in E^J$ at any vertex $\xi \in \Xi$ is denoted by $\Delta\theta(\xi) = \theta(\xi) - \theta(\xi^-)$. The market value of the portfolio of securities purchased at vertex $\xi \in \Xi$ under trading strategy $\theta \in E^J$ is then $\Delta\theta(\xi)P(\xi)$.

3. Security Markets with Transaction Costs

3.1 Transaction Costs

We consider security markets with transaction costs: the coefficients $B^j(\xi) \in [0, \infty)$ and $S^j(\xi) \in [0, 1)$ are the transaction cost rates for purchasing and selling the security j at vertex ξ, respectively. Then the transaction cost rates for purchasing and selling are the processes $B = (B^1, \cdots B^J)$ and $S = (S^1, \cdots S^J)$, respectively. Then the algebraic cost induced by (buying) a position $\theta^j(\xi) - \theta^j(\xi^-) \geq 0$ units of security j at vertex ξ is $P^j(\xi)[1 + B^j(\xi)][\theta^j(\xi) - \theta^j(\xi^-)]$ and the algebraic gain induced by (selling) a position $\theta^j(\xi) - \theta^j(\xi^-) \leq 0$ units of security j at vertex ξ is $q^j(\xi)[1 - S^j(\xi)][\theta^j(\xi) - \theta^j(\xi^-)]$. For any $\xi \in \Xi$, we introduce the functions $\Phi^j(\xi) : \mathcal{R} \to \mathcal{R}$ defined by

$$\Phi^j(\xi)(z) = \begin{cases} P^j(\xi)[1 + B^j(\xi)]z, & z \geq 0 \\ P^j(\xi)[1 - S^j(\xi)]z, & z \leq 0 \end{cases}$$

and the functions $\phi^j(\xi) : \mathcal{R} \to \mathcal{R}$ defined by

$$\phi^j(\xi)(z) = \begin{cases} [1 + B^j(\xi)]z, & z \geq 0 \\ [1 - S^j(\xi)]z, & z \leq 0 \end{cases}$$

Then $\Phi^j(\xi)(z) = P^j(\xi)\phi^j(\xi)(z)$. It is obvious that the function $\phi^j(\xi)$ is sublinear, and hence convex. Therefore the function $\Phi^j(\xi)$ is also sublinear, and hence convex.

For any $\xi \in \Xi$, the total cost or gain induced by (trading) a portfolio $\theta(\xi) - \theta(\xi^-) \in \mathcal{R}^J$ is

$$\sum_{j=1}^{J} \Phi^j(\xi)(\theta^j(\xi) - \theta^j(\xi^-)) = \sum_{j=1}^{J} P^j(\xi)\phi^j(\xi)(\theta^j(\xi) - \theta^j(\xi^-))$$

We define the function $\psi(\xi) : \mathcal{R}^J \to \mathcal{R}$ by

$$\psi(\xi)(z) = \sum_{j=1}^{J} \Phi^j(\xi)(z^j) = \sum_{j=1}^{J} P^j(\xi)\phi^j(\xi)(z^j)$$

Then the total cost or gain induced by (trading) a portfolio $\theta(\xi)-\theta(\xi^-) \in \mathcal{R}^J$ is $\psi(\xi)(\theta(\xi)-\theta(\xi^-))$. As we know, the function $\psi(\xi)$ is sublinear, and hence convex. The total cost or gain process $\Psi^{\Delta\theta} \in E^1$ induced by (trading) a trading strategy θ is defined by

$$\Psi^{\Delta\theta}(\xi) = \psi(\xi)(\theta(\xi) - \theta(\xi^-)) = \sum_{j=1}^{J} \Phi^j(\xi)(\theta^j(\xi) - \theta^j(\xi^-)) = \sum_{j=1}^{J} P^j(\xi)\phi^j(\xi)(\theta^j(\xi) - \theta^j(\xi^-)),\ \xi \in \Xi$$

Thus the total cost or gain process $\Psi^{\Delta\theta}$ is sublinear for the trading strategy $\Delta\theta$. In the frictional security markets, the dividend process $\delta^\theta \in E^1$ generated by a trading strategy θ is defined by

$$\delta^\theta(\xi) = \theta(\xi^-)D(\xi) - \Psi^{\Delta\theta}(\xi), \qquad \xi \in \Xi$$

that is,

$$\delta^\theta(\xi) = \begin{cases} -\Psi^{\Delta\theta}(\xi_0), & \xi = \xi_0 \\ \theta(\xi^-)D(\xi) - \Psi^{\Delta\theta}(\xi), & \xi \in \Xi^{T-1} \setminus \xi_0 \\ \theta(\xi^-)D(\xi), & \xi \in \Xi_T \end{cases}$$

Thus the total cost or gain process δ^θ is continuous and superlinear for the trading strategy θ.

3.2 Definitions of Arbitrage-freeness

Definition 1. *The security market (P, D, B, S) is* **weakly arbitrage-free** *if any trading strategy $\theta \in E^J$ has a positive algebraic cost or gain at the root vertex, $\Psi^{\Delta\theta}(\xi_0) \geq 0$, whenever it has a positive net dividend after root vertex, $\theta(\xi^-)D(\xi) - \Psi^{\Delta\theta}(\xi) \geq 0,\ \xi_t \in \Xi^{T-1} \setminus \xi_0$ and $\theta(\xi^-)D(\xi) \geq 0,\ \xi \in \Xi_T$.*

Definition 2. *The security market (P, D, B, S) is* **strongly arbitrage-free** *if there is no trading strategy $\theta \in E^J$ such that $\delta^\theta \geq 0$ and $\delta^\theta \neq 0$, that is*

$$\{\theta \in E^J \mid \delta^\theta \geq 0 \ and\ \delta^\theta \neq 0\} = \emptyset$$

It is impossible to generate positive non-zero dividends. More specifically, the security market (P, D, B, S) is strongly arbitrage-free if
(1) any trading strategy $\theta \in E^J$ has a positive non-zero algebraic cost or gain at the root vertex, $\Psi^{\Delta\theta}(\xi_0) > 0$, whenever it has a positive non-zero net dividend after root vertex, $\theta(\xi^-)D(\xi) - \Psi^{\Delta\theta}(\xi) \geq 0,\ \xi_t \in \Xi^{T-1} \setminus \xi_0$ and $\theta(\xi^-)D(\xi) \geq 0,\ \xi \in \Xi_T$, and a positive non-zero net dividend at, at least, one vertex;
(2) any trading strategy $\theta \in E^J$ has a positive algebraic cost or gain at the root vertex, $\Psi^{\Delta\theta}(\xi_0) \geq 0$, whenever it has a positive net dividend after the root vertex, $\theta(\xi^-)D(\xi) - \Psi^{\Delta\theta}(\xi) \geq 0,\ \xi_t \in \Xi^{T-1} \setminus \xi_0$ and $\theta(\xi^-)D(\xi) \geq 0,\ \xi \in \Xi_T$.

We define the subset M in the space E^1 as follows

$$M = \{\delta \in E^1 \mid \delta \leq \delta^\theta \ for\ \theta \in E^J\}$$

Lemma. *M is a closed and convex cone in the space E^1.*

Proof: M is a closed cone obviously since the total cost or gain process δ^θ is continuous and superlinear for the trading strategy θ.

For any $\delta^1 \in M$ and $\delta^2 \in M$, there exist $\theta^1 \in E^J$ and $\theta^2 \in E^J$ such that $\delta^1 \leq \delta^{\theta^1}$ and $\delta^2 \leq \delta^{\theta^2}$. Thus $\delta^1 + \delta^2 \leq \delta^{\theta^1} + \delta^{\theta^2} \leq \delta^{\theta^1+\theta^2}$ since the total cost or gain process δ^θ is continuous and superlinear for the trading strategy θ. Therefore $\delta^1 + \delta^2 \in M$. Since M is a cone, M is convex. Q.E.D.

3.3 Some Notations

For simplicity, we use the following notations in the subsequent sections. $\delta \in E^1$, then $\delta = (\delta(\xi_0), \tilde{\delta})$, where $\delta(\xi_0) \in \mathcal{R}$ and $\tilde{\delta} \in \mathcal{R}^{\Xi \setminus \xi_0} = \mathcal{X}$. Thus we can rewrite the definition of arbitrage-free security market (P, D, B, S) as follows.

Definition 1. *The security market (P, D, B, S) is* **weakly arbitrage-free** *if any trading strategy $\theta \in E^J$ has a positive algebraic cost or gain at the root vertex, $\Psi^{\Delta\theta}(\xi_0) \geq 0$, whenever it has a positive net dividend after the root vertex, $\tilde{\delta}^\theta \in \mathcal{X}_+$.*

Definition 2. *The security market (P, D, B, S) is* **strongly arbitrage-free** *if there is no trading strategy $\theta \in E^J$ such that $\delta^\theta \geq 0$ and $\delta^\theta \neq 0$, that is*

$$\{\theta \in E^J \mid \delta^\theta \geq 0 \text{ and } \delta^\theta \neq 0\} = \emptyset$$

It is impossible to generate positive non-zero dividends. Specifically, the security market (P, D, B, S) is strongly arbitrage-free if
(1) any trading strategy $\theta \in E^J$ has a positive non-zero algebraic cost or gain at the root vertex, $\Psi^{\Delta\theta}(\xi_0) > 0$, whenever it has a positive non-zero net dividend after root vertex, $\tilde{\delta}^\theta \in \mathcal{X}_+ \setminus \{0\}$;
(2) any trading strategy $\theta \in E^J$ has a positive algebraic cost or gain at root vertex, $\Psi^{\Delta\theta}(\xi_0) \geq 0$, whenever it has a positive net dividend after root vertex, $\tilde{\delta}^\theta \in \mathcal{X}_+$.

For any integer $N = 1, 2, \cdots$, we define the box product of two vectors $y_1 \in \mathcal{R}^N$ and $y_2 \in \mathcal{R}^N$ by

$$y_1 \,\square\, y_2 = \begin{pmatrix} y_1^1 y_2^1 \\ \vdots \\ y_1^N y_2^N \end{pmatrix}$$

4. Weakly arbitrage-free security markets

Proposition 1. *The security market (P, D, B, S) is weakly arbitrage-free if and only if $M \cap E^1_+ = \{\delta \in M \cap E^1_+ \mid \delta(\xi_0) = 0\}$.*

Proof: $\delta \in M$ implies that there exists $\theta \in E^J$ such that $\delta \leq \delta^\theta$. $\delta \in M \cap E^1_+$ implies $\delta \leq \delta^\theta$ and $\delta \in E^1_+$. That is, $\delta^\theta(\xi_0) = -\Psi^\theta(\xi_0) \geq 0$ and $\tilde{\delta}^\theta \in \mathcal{X}_+$. On the other hand, $\tilde{\delta}^\theta \in \mathcal{X}_+$ implies $\Psi^\theta(\xi_0) \geq 0$ from Definition 1 of the weakly arbitrage-free security market, then $\delta^\theta(\xi_0) \leq 0$. Thus $\delta(\xi_0) = \delta^\theta(\xi_0) = 0$. that is, $M \cap E^1_+ = \{\delta \in M \cap E^1_+ \mid \delta(\xi_0) = 0\}$.

Conversely, if there exists a trading strategy $\theta \in E^J$ such that it has a positive net dividend after the root vertex, $\tilde{\delta}^\theta \in \mathcal{X}_+$, and a strictly negative algebraic cost or gain at the root vertex, $\Psi^\theta(\xi_0) < 0$, then $\delta^\theta(\xi_0) = -\Psi^\theta(\xi_0) > 0$. Thus $\delta^\theta \in M \cap E^1_+$ and $\delta^\theta \notin \{\delta \in M \cap E^1_+ \mid \delta(\xi_0) = 0\}$. This is a contradiction! Therefore, any trading strategy $\theta \in E^J$ has a positive algebraic cost or gain at the root vertex, $\Psi^\theta(\xi_0) \geq 0$, whenever it has a positive net dividend after the root vertex, $\tilde{\delta}^\theta \in \mathcal{X}_+$. Q.E.D.

Theorem 1. *The security market (P, D, B, S) is weakly arbitrage-free if and only if there exists a possitive process $\lambda \in E^1_+$ with $\lambda(\xi_0) \in \mathcal{R}_{++}$ such that, for any $\xi \in \Xi^{T-1}$,*

$$[\lambda(\xi)P(\xi)] \,\square\, [\mathbb{1} - S(\xi)] \leq \sum_{\eta > \xi} \lambda(\eta) D(\eta) \leq [\lambda(\xi)P(\xi)] \,\square\, [\mathbb{1} + B(\xi)]$$

Proof: The security market (P, D, B, S) is weakly arbitrage-free if and only if $M \cap E^1_+ = \{\delta \in M \cap E^1_+ \mid \delta(\xi_0) = 0\}$ from Proposition 1. Thus $M \cap intE^1_+ = \emptyset$. In fact, if $\delta \in M \cap intE^1_+$, then

$\delta \in M \cap int E^1_+ \subseteq M \cap E^1_+ = \{\delta \in M \cap E^1_+ \mid \delta(\xi_0) = 0\}$ hence $\delta(\xi_0) = 0$; and $\delta \in M \cap int E^1_+ \subseteq int E^1_+$, hence $\delta(\xi_0) > 0$. A contradiction means that $M \cap int E^1_+ = \emptyset$.

Both M and E^1_+ is are closed and convex cones of E^1. The Separating Hyperplane Theorem states that M and E^1_+ can be separated by a closed hyperplane. There exists a continuous linear functional $f : E^1 \to \mathcal{R}$ such that $f(m) \leq 0$ for all $m \in M$ and $f(n) > 0$ for all $n \in E^1_+ \setminus M$.

Thus f is represented by some positive process $\lambda \in E^1_+$ with $\lambda(\xi_0) \in \mathcal{R}_{++}$ by

$$f(\delta) = \sum_{\xi \in \Xi} \lambda(\xi)\delta(\xi)$$

for any $\delta \in E^1$. In fact, f is represented by some process $\lambda \in E^1$. If $\delta(\xi_0) = 1$ and $\tilde{\delta} = 0$, then $\delta \in E^1_+ \setminus M$ and $f(\delta) > 0$, that is, $\lambda(\xi_0) \in \mathcal{R}_{++}$. If $\tilde{\lambda} \notin \mathcal{X}_+$, then there exists $\tilde{\delta}_0 \in \mathcal{X}_+$ such that $\sum_{\xi \in \Xi \setminus \xi_0} \lambda(\xi)\delta_0(\xi) < 0$, take

$$\delta_0(\xi_0) = -\frac{1}{2\lambda(\xi_0)} \sum_{\xi \in \Xi \setminus \xi_0} \lambda(\xi)\delta_0(\xi) > 0$$

then $\delta_0 \in E^1_+ \setminus M$ and

$$f(\delta_0) = \lambda(\xi_0)\left\{-\frac{1}{2\lambda(\xi_0)} \sum_{\xi \in \Xi \setminus \xi_0} \lambda(\xi)\delta_0(\xi)\right\} + \sum_{\xi \in \Xi \setminus \xi_0} \lambda(\xi)\delta_0(\xi) = \frac{1}{2} \sum_{\xi \in \Xi \setminus \xi_0} \lambda(\xi)\delta_0(\xi) < 0$$

which is a contradiction! Thus $\lambda \in E^1_+$ with $\lambda(\xi_0) \in \mathcal{R}_{++}$.

Since $\delta^\theta \in M$ for $\theta \in E^J$, then

$$\sum_{\xi \in \Xi} \lambda(\xi)[\theta(\xi^-)D(\xi) - \Psi^{\Delta\theta}(\xi)] = \sum_{\xi \in \Xi} \lambda(\xi)\delta^\theta(\xi) \leq 0$$

for all $\theta \in E^J$. For any $\xi \in \Xi$ and $j \in \mathcal{J}$, we define

$$\theta^{j'}(\xi') = \begin{cases} 1, & \xi' \in \Xi(\xi) \ and \ j' = j \\ 0, & otherwise \end{cases}$$

then

$$\Delta\theta^{j'}(\xi') = \begin{cases} 1, & \xi' = \xi \ and \ j' = j \\ 0, & otherwise \end{cases}$$

thus, for $\xi \in \Xi^{T-1}$ and $j \in \mathcal{J}$,

$$\sum_{\eta > \xi} \lambda(\eta)D^j(\eta) - \lambda(\xi)P^j(\xi)[1 + B^j(\xi)] \leq 0$$

that is, for $\xi \in \Xi^{T-1}$,

$$\sum_{\eta > \xi} \lambda(\eta)D(\eta) \leq [\lambda(\xi)P(\xi)] \square [\mathbb{1} + B(\xi)]$$

For any $\xi \in \Xi$ and $j \in \mathcal{J}$, we define

$$\theta^{j'}(\xi') = \begin{cases} -1, & \xi' \in \Xi(\xi) \ and \ j' = j \\ 0, & otherwise \end{cases}$$

then

$$\Delta\theta^{j'}(\xi') = \begin{cases} -1, & \xi' = \xi \text{ and } j' = j \\ 0, & \text{otherwise} \end{cases}$$

thus, for $\xi \in \Xi^{T-1}$ and $j \in \mathcal{J}$,

$$-\sum_{\eta > \xi} \lambda(\eta) D^j(\eta) + \lambda(\xi) P^j(\xi)[1 - S^j(\xi)] \leq 0$$

that is, for $\xi \in \Xi^{T-1}$,

$$[\lambda(\xi) P(\xi)] \square [\mathbb{1} - S(\xi)] \leq \sum_{\eta > \xi} \lambda(\eta) D(\eta)$$

Therefore, for $\xi \in \Xi^{T-1}$,

$$[\lambda(\xi) P(\xi)] \square [\mathbb{1} - S(\xi)] \leq \sum_{\eta > \xi} \lambda(\eta) D(\eta) \leq [\lambda(\xi) P(\xi)] \square [\mathbb{1} + B(\xi)]$$

This completes the proof of the necessary condition.

Conversely, if there exists a possitive process $\lambda \in E^1_+$ with $\lambda(\xi_0) \in \mathcal{R}_{++}$ such that, for any $\xi \in \Xi^{T-1}$,

$$[\lambda(\xi) P(\xi)] \square [\mathbb{1} - S(\xi)] \leq \sum_{\eta > \xi} \lambda(\eta) D(\eta) \leq [\lambda(\xi) P(\xi)] \square [\mathbb{1} + B(\xi)]$$

that is, for $\xi \in \Xi^{T-1}$ and $j \in \mathcal{J}$,

$$\lambda(\xi) P^j(\xi)[1 - S^j(\xi)] \leq \sum_{\eta > \xi} \lambda(\eta) D^j(\eta) \leq \lambda(\xi) P^j(\xi)[1 + B^j(\xi)]$$

thus, for any $\Delta\theta^j(\xi) \geq 0$,

$$\Delta\theta^j(\xi) \sum_{\eta > \xi} \lambda(\eta) D^j(\eta) \leq \lambda(\xi) P^j(\xi)[1 + B^j(\xi)] \Delta\theta^j(\xi)$$

and for any $\Delta\theta^j(\xi) \leq 0$,

$$\Delta\theta^j(\xi) \sum_{\eta > \xi} \lambda(\eta) D^j(\eta) \leq \lambda(\xi) P^j(\xi)[1 - S^j(\xi)] \Delta\theta^j(\xi)$$

Therefore, for $\xi \in \Xi^{T-1}$ and $j \in \mathcal{J}$,

$$\Delta\theta^j(\xi) \sum_{\eta > \xi} \lambda(\eta) D^j(\eta) \leq \lambda(\xi) P^j(\xi) \phi^j(\xi)(\theta^j(\xi) - \theta^j(\xi^-)) = \lambda(\xi) \Phi^j(\xi)(\theta^j(\xi) - \theta^j(\xi^-))$$

Summering over $j \in \mathcal{J}$, we then have

$$\Delta\theta(\xi) \sum_{\eta > \xi} \lambda(\eta) D(\eta) \leq \lambda(\xi) \psi(\xi)(\theta(\xi) - \theta(\xi^-)) = \lambda(\xi) \Psi^{\Delta\theta}(\xi), \qquad \xi \in \Xi^{T-1}$$

for all $\theta \in E^J$.

If a trading strategy $\theta \in E^J$ has a positive net dividend after the root vertex, $\tilde{\delta}^\theta \in \mathcal{X}_+$, that is,

$$\theta(\xi^-) D(\xi) - \Psi^{\Delta\theta}(\xi) \geq 0, \qquad \xi \in \Xi^{T-1} \setminus \xi_0$$

and

$$\theta(\xi^-)D(\xi) \geq 0, \qquad \xi \in \Xi_T$$

then

$$\Delta\theta(\xi)\sum_{\eta>\xi}\lambda(\eta)D(\eta) \leq \lambda(\xi)\Psi^{\Delta\theta}(\xi) \leq \lambda(\xi)\theta(\xi^-)D(\xi), \qquad \xi \in \Xi^{T-1}\setminus\xi_0$$

that is,

$$\theta(\xi)\sum_{\eta>\xi}\lambda(\eta)D(\eta) \leq \theta(\xi^-)\sum_{\eta\geq\xi}\lambda(\eta)D(\eta), \qquad \xi \in \Xi^{T-1}\setminus\xi_0$$

Thus

$$\begin{aligned}
&\lambda(\xi_0)\Psi^{\Delta\theta}(\xi_0)\\
\geq&\Delta\theta(\xi_0)\sum_{\eta>\xi_0}\lambda(\eta)D(\eta)\\
=&\theta(\xi_0)\sum_{\xi_1\in\xi_0^+}\sum_{\eta\geq\xi_1}\lambda(\eta)D(\eta)\\
=&\sum_{\xi_1\in\xi_0^+}\theta(\xi_1^-)\sum_{\eta\geq\xi_1}\lambda(\eta)D(\eta)\\
\geq&\sum_{\xi_1\in\xi_0^+}\theta(\xi_1)\sum_{\eta>\xi_1}\lambda(\eta)D(\eta)\\
=&\sum_{\xi_1\in\xi_0^+}\theta(\xi_1)\sum_{\xi_2\in\xi_1^+}\sum_{\eta\geq\xi_2}\lambda(\eta)D(\eta)\\
=&\sum_{\xi_1\in\xi_0^+}\sum_{\xi_2\in\xi_1^+}\theta(\xi_2^-)\sum_{\eta\geq\xi_2}\lambda(\eta)D(\eta)\\
\geq&\sum_{\xi_1\in\xi_0^+}\sum_{\xi_2\in\xi_1^+}\theta(\xi_2)\sum_{\eta>\xi_2}\lambda(\eta)D(\eta)\\
\geq&\cdots\cdots\cdots\cdots\\
\geq&\sum_{\xi_1\in\xi_0^+}\sum_{\xi_2\in\xi_1^+}\cdots\sum_{\xi_{T-1}\in\xi_{T-2}^+}\theta(\xi_{T-1})\sum_{\eta>\xi_{T-1}}\lambda(\eta)D(\eta)\\
\geq&\sum_{\xi_1\in\xi_0^+}\sum_{\xi_2\in\xi_1^+}\cdots\sum_{\xi_{T-1}\in\xi_{T-2}^+}\sum_{\xi\in\xi_{T-1}^+}\lambda(\xi)\theta(\xi^-)D(\xi)\geq 0
\end{aligned}$$

then $\Psi^{\Delta\theta}(\xi_0) \geq 0$, that is to say, the trading strategy $\theta \in E^J$ yields a positive market value at root vertex. Therefore the security market (P, D, B, S) is weakly arbitrage-free. Q.E.D.

Remark: Take $B = S = 0$, we then obtain the setting of frictionless security markets. The security market (P, D) is weakly arbitrage-free if and only if there exists a possitive process $\lambda \in E^1_+$ with $\lambda(\xi_0) \in \mathcal{R}_{++}$ such that

$$\lambda(\xi)P(\xi) = \sum_{\eta>\xi}\lambda(\eta)D(\eta), \qquad \xi \in \Xi^{T-1}$$

We also obtain a necessary condition as follows.

Theorem 1′. *If the security market* (P, D, B, S) *is weakly arbitrage-free, then there exists a possitive process* $\lambda \in E^1_+$ *with* $\lambda(\xi_0) \in \mathcal{R}_{++}$ *such that, for any* $\xi \in \Xi^{T-1}$,

$$-\sum_{\eta\in\xi^+}[\lambda(\eta)P(\eta)]\,\Box[1+B(\eta)] + [\lambda(\xi)P(\xi)]\,\Box[1-S(\xi)] \le \sum_{\eta\in\xi^+}\lambda(\eta)D(\eta)$$
$$\le -\sum_{\eta\in\xi^+}[\lambda(\eta)P(\eta)]\,\Box[1-S(\eta)] + [\lambda(\xi)P(\xi)]\,\Box[1+B(\xi)]$$

Proof: From the Proof of Theorem 1, If the security market (P, D, B, S) is weakly arbitrage-free, then there exists a continuous linear functional $f : E^1 \to \mathcal{R}$ such that $f(m) \le 0$ for all $m \in M$ and $f(n) > 0$ for all $n \in E^1_+ \setminus M$. The functional f is represented by some positive process $\lambda \in E^1_+$ with $\lambda(\xi_0) \in \mathcal{R}_{++}$ by

$$f(\delta) = \sum_{\xi\in\Xi}\lambda(\xi)\delta(\xi)$$

for any $\delta \in E^1$ (See the Proof of Theorem 1).

$$\sum_{\xi\in\Xi}\lambda(\xi)[\theta(\xi^-)D(\xi) - \Psi^{\Delta\theta}(\xi)] = \sum_{\xi\in\Xi}\lambda(\xi)\delta^\theta(\xi) \le 0$$

for all $\theta \in E^J$. For any $\xi \in \Xi$ and $j \in \mathcal{J}$, we define

$$\theta^{j'}(\xi') = \begin{cases} 1, & \xi' = \xi \text{ and } j' = j \\ 0, & \text{otherwise} \end{cases}$$

then, for $\xi \in \Xi^{T-1}$ and $j \in \mathcal{J}$,

$$\sum_{\eta\in\xi^+}\lambda(\eta)D^j(\eta) + \sum_{\eta\in\xi^+}\lambda(\eta)P^j(\eta)[1-S^j(\eta)] - \lambda(\xi)P^j(\xi)[1+B^j(\xi)] \le 0$$

that is,

$$\sum_{\eta\in\xi^+}\lambda(\eta)D^j(\eta) \le -\sum_{\eta\in\xi^+}\lambda(\eta)P^j(\eta)[1-S^j(\eta)] + \lambda(\xi)P^j(\xi)[1+B^j(\xi)]$$

thus

$$\sum_{\eta\in\xi^+}\lambda(\eta)D(\eta) \le -\sum_{\eta\in\xi^+}[\lambda(\eta)P(\eta)]\,\Box[1-S(\eta)] + [\lambda(\xi)P(\xi)]\,\Box[1+B(\xi)]$$

For any $\xi \in \Xi$ and $j \in \mathcal{J}$, we define

$$\theta^{j'}(\xi') = \begin{cases} -1, & \xi' = \xi \text{and } j' = j \\ 0, & \text{otherwise} \end{cases}$$

then, for $\xi \in \Xi^{T-1}$ and $j \in \mathcal{J}$,

$$-\sum_{\eta\in\xi^+}\lambda(\eta)D^j(\eta) - \sum_{\eta\in\xi^+}\lambda(\eta)P^j(\eta)[1+B^j(\eta)] + \lambda(\xi)P^j(\xi)[1-S^j(\xi)] \le 0$$

that is,

$$-\sum_{\eta\in\xi^+}\lambda(\eta)P^j(\eta)[1+B^j(\eta)] + \lambda(\xi)P^j(\xi)[1-S^j(\xi)] \le \sum_{\eta\in\xi^+}\lambda(\eta)D^j(\eta)$$

thus

$$-\sum_{\eta\in\xi^+}[\lambda(\eta)P(\eta)]\,\Box[1+B(\eta)] + [\lambda(\xi)P(\xi)]\,\Box[1-S(\xi)] \le \sum_{\eta\in\xi^+}\lambda(\eta)D(\eta)$$

This completes the proof. Q.E.D.

5. Strictly arbitrage-free security markets

Proposition 2. *The security market (P, D, B, S) is strongly arbitrage-free if and only if M and E^1_+ intersect precisely at $(0,0)$, that is, $M \cap E^1_+ = \{(0,0)\}$.*

Proof: $\delta \in M$ implies that there exists $\theta \in E^J$ such that $\delta \leq \delta^\theta$. $\delta \in M \cap E^1_+$ implies $\delta \leq \delta^\theta$ and $\delta \in E^1_+$. That is, $\delta^\theta(\xi_0) = -\Psi^\theta(\xi_0) \geq 0$ and $\tilde{\delta}^\theta \in \mathcal{X}_+$. If $\tilde{\delta}^\theta \in \mathcal{X}_+ \setminus \{0\}$, then $\Psi^\theta(\xi_0) > 0$ from the Definition 2 (1) of the strongly arbitrage-free security market, that is, $\delta^\theta(\xi_0) = -\Psi^\theta(\xi_0) < 0$, which is a contradiction! If $\tilde{\delta}^\theta = 0 \in \mathcal{X}_+$ (hence $\tilde{\delta} = 0 \in \mathcal{X}_+$), then $\Psi^\theta(\xi_0) \geq 0$ from the Definition 2 (2) of the strongly arbitrage-free security market, that is, $\delta^\theta(\xi_0) = -\Psi^\theta(\xi_0) \leq 0$. Thus $\delta^\theta(\xi_0) = 0$ hence $\delta = 0$, therefore $M \cap E^1_+ = \{(0,0)\}$.

Conversely, (1) if there exists trading strategy $\theta \in E^J$ such that $\tilde{\delta}^\theta \in \mathcal{X}_+ \setminus \{0\}$ and $\Psi^\theta(\xi_0) \leq 0$, then $\delta^\theta(\xi_0) = -\Psi^\theta(\xi_0) \geq 0$, hence $(0,0) \neq \delta^\theta \in M \cap E^1_+$, a contradiction means that any trading strategy $\theta \in E^J$ has a positive non-zero market value at root vertex, $\Psi^\theta(\xi_0) > 0$, whenever it has a positive non-zero net dividend after root vertex, $\tilde{\delta}^\theta \in \mathcal{X}_+ \setminus \{0\}$; (2) if there exists trading strategy $\theta \in E^J$ such that $\tilde{\delta}^\theta \in \mathcal{X}_+$ and $\Psi^\theta(\xi_0) < 0$, then $\delta^\theta(\xi_0) = -\Psi^\theta(\xi_0) > 0$, hence $(0,0) \neq \delta^\theta \in M \cap E^1_+$, a contradiction means that any trading strategy $\theta \in E^J$ has a positive market value at root vertex, $\Psi^\theta(\xi_0) \geq 0$, whenever it has a positive net dividend after root vertex, $\tilde{\delta}^\theta \in \mathcal{X}_+$. Q.E.D.

Theorem 2. *The security market (P, D, B, S) is strongly arbitrage-free if and only if there exists a strictly possitive process $\lambda \in E^1_{++}$ such that, for any $\xi \in \Xi^{T-1}$,*

$$P(\xi) \,\square[\mathbb{1} - S(\xi)] \leq \frac{1}{\lambda(\xi)} \sum_{\eta > \xi} \lambda(\eta) D(\eta) \leq P(\xi) \,\square[\mathbb{1} + B(\xi)]$$

Proof: The security market (P, D, B, S) is strongly arbitrage-free if and only if $M \cap E^1_+ = (0,0)$ from Proposition 2. Thus $M \cap [E^1_+ \setminus (0,0)] = \emptyset$. Both M and E^1_+ is are closed and convex cones of E^1. The Separating Hyperplane Theorem states that M and E^1_+ can be separated by a closed hyperplane. There exists a continuous linear functional $f : E^1 \to \mathcal{R}$ such that $f(m) \leq 0$ for all $m \in M$ and $f(n) > 0$ for all $n \in E^1_+ \setminus (0,0)$. Thus f is represented by some strictly positive process $\lambda \in E^1_{++}$ by

$$f(\delta) = \sum_{\xi \in \Xi} \lambda(\xi) \delta(\xi)$$

for any $\delta \in E^1$.

The rest is similar with the corresponding part of the proof of Theorem 1. Q.E.D.

Remark 1: Theorem 2 is the main result in this paper, which implies the first fundamental valuation theorem with proportional transaction costs. Theorem 2 means that the asset prices are not unique in strongly arbitrage-free security markets, we obtain the continuous pricing interval in this setting. The lower and upper boundaries of the pricing interval are decided by the transaction costs rates for purchasing and selling securities, respectively. That is, for any $\xi \in \Xi^{T-1}$ and $j \in \mathcal{J}$,

$$\frac{1}{\lambda(\xi)[1 + B^j(\xi)]} \sum_{\eta > \xi} \lambda(\eta) D^j(\eta) \leq P^j(\xi) \leq \frac{1}{\lambda(\xi)[1 - S^j(\xi)]} \sum_{\eta > \xi} \lambda(\eta) D^j(\eta)$$

Remark 2: Theorem 2 is different from the corresponding result in frictionless security markets. Take $B = S = 0$, we then obtain the setting of frictionless security markets. The security market (P, D) is strongly arbitrage-free if and only if there exists a strictly possitive process $\lambda \in E^1_{++}$ such that

$$P(\xi) = \frac{1}{\lambda(\xi)} \sum_{\eta > \xi} \lambda(\eta) D(\eta), \qquad \xi \in \Xi^{T-1}$$

thus the asset prices are uniquely determined by deflator $\lambda \in E^1_{++}$ in strongly arbitrage-free security markets (Duffie 1996).

We also obtain a necessary condition from Theorems 2 and 1′ as follows.

Theorem 2′. *If the security market (P, D, B, S) is strongly arbitrage-free, then there exists a possitive process $\lambda \in E^1_{++}$ such that, for any $\xi \in \Xi^{T-1}$,*

$$-\sum_{\eta\in\xi^+}[\lambda(\eta)P(\eta)]\,\square[1+B(\eta)] + [\lambda(\xi)P(\xi)]\,\square[1-S(\xi)] \le \sum_{\eta\in\xi^+}\lambda(\eta)D(\eta)$$
$$\le -\sum_{\eta\in\xi^+}[\lambda(\eta)P(\eta)]\,\square[1-S(\eta)] + [\lambda(\xi)P(\xi)]\,\square[1+B(\xi)]$$

6. Concluding Remarks

In Section 5, we proved the equivalent conditions (Proposition 2 and Theorem 2) of strongly arbitrage-free security market by using Definition 2. In fact, we have another definition of strongly arbitrage-free frictional market as follows.

Definition 3. *The security market (P, D, B, S) is* **strongly arbitrage-free** *if*
(1) any trading strategy $\theta \in E^J$ has a positive non-zero algebraic cost or gain at root vertex, $\Psi^\theta(\xi_0) > 0$, whenever it has a positive non-zero net dividend after root vertex, $\theta(\xi^-)D(\xi) - \Psi^\theta(\xi) \ge 0$, $\xi_t \in \Xi^{T-1}\setminus\xi_0$ and $\theta(\xi^-)D(\xi) \ge 0$, $\xi \in \Xi_T$, and a positive non-zero net dividend at, at least, one vertex;
(2) any trading strategy $\theta \in E^J$ has a zero algebraic cost or gain at root vertex, $\Psi^\theta(\xi_0) = 0$, whenever it has a zero net dividend after root vertex, $\theta(\xi^-)D(\xi) - \Psi^\theta(\xi) = 0$, $\xi_t \in \Xi^{T-1}\setminus\xi_0$ and $\theta(\xi^-)D(\xi) = 0$, $\xi \in \Xi_T$.

Take the simplex notations, we can rewrite the definition as following.

Definition 3. *The security market (P, D, B, S) is* **strongly arbitrage-free** *if*
(1) any trading strategy $\theta \in E^J$ has a positive non-zero algebraic cost or gain at root vertex, $\Psi^\theta(\xi_0) > 0$, whenever it has a positive non-zero net dividend after root vertex, $\tilde{\delta}^\theta \in \mathcal{X}_+ \setminus \{0\}$;
(2) any trading strategy $\theta \in E^J$ has a zero algebraic cost or gain at root vertex, $\Psi^\theta(\xi_0) = 0$, whenever it has a zero net dividend after root vertex, $\tilde{\delta}^\theta = 0$.

There are some difference between Definitions 2 and 3. Definition 3 obviously implies Definition 2. Definition 2 does not imply Definition 3 because of the presence of friction. Using Definition 3 of the strongly arbitrage-free security market, we can obtain the following (weaker) results for strongly arbitrage-free frictional market.

Proposition 3. *If the security market (P, D, B, S) is strongly arbitrage-free, then M and E^1_+ intersect precisely at $(0,0)$, that is, $M \cap E^1_+ = \{(0,0)\}$.*

Theorem 3. *If the security market (P, D, B, S) is strongly arbitrage-free, then there exists a strictly possitive process $\lambda \in E^1_{++}$ such that, for any $\xi \in \Xi^{T-1}$,*

$$P(\xi)\,\square[\mathbb{1} - S(\xi)] \le \frac{1}{\lambda(\xi)}\sum_{\eta>\xi}\lambda(\eta)D(\eta) \le P(\xi)\,\square[\mathbb{1} + B(\xi)]$$

The proofs of Proposition 3 and Theorem 3 are from the proofs of Proposition 2 and Theorem 2, respectively. Conversely, we can't check the sufficiency. In fact, if there exists a strictly possitive process $\lambda \in E^1_{++}$ such that, for any $\xi \in \Xi^{T-1}$,

$$P(\xi) \square [\mathbb{1} - S(\xi)] \leq \frac{1}{\lambda(\xi)} \sum_{\eta > \xi} \lambda(\eta) D(\eta) \leq P(\xi) \square [\mathbb{1} + B(\xi)]$$

then any trading strategy $\theta \in E^J$ has a positive non-zero algebraic cost or gain at the root vertex, $\Psi^{\Delta\theta}(\xi_0) > 0$, whenever it has a positive non-zero net dividend after the root vertex, $\tilde{\delta}^\theta \in \mathcal{X}_+ \setminus \{0\}$; and any trading strategy $\theta \in E^J$ has a positive algebraic cost or gain at the root vertex, $\Psi^{\Delta\theta}(\xi_0) \geq 0$, whenever it has a positive net dividend after the root vertex, $\tilde{\delta}^\theta \in \mathcal{X}_+$. This implies that any trading strategy $\theta \in E^J$ has a positive algebraic cost or gain at the root vertex, $\Psi^{\Delta\theta}(\xi_0) \geq 0$, whenever it has a zero net dividend after the root vertex, $\tilde{\delta}^\theta = 0$. But we can't obtain that any trading strategy $\theta \in E^J$ has a zero algebraic cost or gain at the root vertex, $\Psi^{\Delta\theta}(\xi_0) = 0$, whenever it has a zero net dividend after the root vertex, $\tilde{\delta}^\theta = 0$. Therefore the security market (P, D, B, S) is strongly arbitrage-free in the sense of Definition 2 and not in the sense of Definition 3.

We also obtain the following Theorem from Theorems 3 and 2′.

Theorem 3′. *If the security market (P, D, B, S) is strongly arbitrage-free, then there exists a possitive process $\lambda \in E^1_{++}$ such that, for any $\xi \in \Xi^{T-1}$,*

$$\begin{aligned} - \sum_{\eta \in \xi^+} [\lambda(\eta) P(\eta)] \square [1 + B(\eta)] + [\lambda(\xi) P(\xi)] \square [1 - S(\xi)] &\leq \sum_{\eta \in \xi^+} \lambda(\eta) D(\eta) \\ &\leq - \sum_{\eta \in \xi^+} [\lambda(\eta) P(\eta)] \square [1 - S(\eta)] + [\lambda(\xi) P(\xi)] \square [1 + B(\xi)] \end{aligned}$$

Frictional economies are fundamentally different from their frictionless counterparts. The theory of general economic equilibrium for frictional economies with incomplete financial markets would be a natural problem to study. We make the first step by establishing the corresponding no-arbitrage (that is, strongly arbitrage-free) pricing theory. From the first fundamental valuation theorems of asset pricing with transaction costs we have obtained here, one would further study the corresponding existence of general equilibrium for frictional economy with incomplete financial markets from finite-dimensional commodity space to infinite-dimensional one.

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