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Zvi Safra

Uzi Segal

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by

Zvi Safra Uzi Segal

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Department of Economics
Social Science Centre
University of Western Ontario
London, Ontario, Canada
N6A 5C2
econref@sscl.uwo.ca

DUAL BETWEENNESS*

Zvi Safra† Uzi Segal‡

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Abstract

A key feature of the rank dependent model for decision making under risk is that the weighting of an outcome depends on its relative rank. This theory received numerous axiomatizations, however, all these sets of axioms need to make an explicit reference to the ranking of the outcomes. This situation is unsatisfactory, as it seems to be desirable to get the ranking property of this model as a consequence of the model, rather than as an assumption. Yaari [9] offered a special version of this model (called dual theory), where the utility function is linear. This paper offers a set of axioms implying a generalization of Yaari's dual theory, without making any reference to the order of the outcomes. The main axiom is called dual betweenness, which, unlike the usual case, is made on random variables rather than distribution functions.

*We thank SSHRCC and the Israel Institute of Business Research for financial support.

†Faculty of Management, Tel Aviv University, Tel Aviv 69978, Israel. E-mail: safraz@post.tau.ac.il

‡Department of Economics, University of Western Ontario, London ON N6A 5C2 Canada. E-mail: segal@sscl.uwo.ca

1 Introduction

The rank dependent model for decision making under risk established itself during the past two decades as a major alternative to expected utility theory. Similarly to the latter model, it assumes that outcomes are evaluated by the utility level the decision maker receives from them. But whereas the traditional theory evaluates lotteries with respect to the expected value of these utilities, the rank dependent model takes the expected value of these utilities with respect to a transformation of the distribution function. Formally, we assume the existence of a continuous, increasing, and onto transformation function $g : [0,1] \to [0,1]$ such that

$$V(F) = \int u(x) dg(F(x))$$

where F is a cumulative distribution function.

A key feature of this theory is that the weighting of an outcome depends on its relative rank. This theory received numerous axiomatizations (see Weymark [8], Quiggin [3], Chew and Epstein [1], Segal [6], Quiggin and Wakker [4], or Wakker [7]). However, all these sets of axioms need to make an explicit reference to the ranking of the outcomes. This situation is unsatisfactory, as it seems to be desirable to get the ranking property of this model as a consequence of the model, rather than as an assumption.

Yaari [9] offered a special version of this model (called dual theory), where the utility function u is linear. In [5] we offered, among other things, a set of axioms that imply a very restrictive version of Yaari's model, where the transformation function g is quadratic. In this paper we offer a different set of axioms, implying the general form of Yaari's dual theory, without making any reference to the order of the outcomes. We present our axioms in the next section, and the main results in section 3. Our main axiom is called dual betweenness, and assumes the following. For every two random variables X and Y that are equally attractive there are X' and Y' such that X and X' have the same distribution, as do Y and Y', and such that for all $\alpha \in [0,1]$, $X' \sim \alpha X' + (1-\alpha)Y'$. Together with continuity, this axiom implies an "implicit dual theory" functional, where for each indifference curve of the preferences $\succeq$ over random variables there is a dual theory functional with this indifference curve. However, the probability transformation function g changes from one indifference curve to another. Adding constant (absolute or relative) risk aversion to the above axioms implies Yaari's [9] dual theory. Section4 concludes the paper with a simple application of the implicit dual theory functional to insurance.

2 Dual Betweenness

Let $\Omega = (S, \Sigma, P)$ be a probability measure space and let $\mathfrak{X}$ be the set of real bounded random variables on it.[1] For $X \in \mathfrak{X}$, let F_X be the distribution function of X. Denote by $\mathfrak{F}$ the set of distribution functions obtained from elements of $\mathfrak{X}$. With a slight abuse of notations, we denote by a the constant random variable with the value a, and its distribution function by δ_a. Two random variables X and Y are comonotonic if for $s_1, s_2 \in S$, $[X(s_1) - X(s_2)][Y(s_1) - Y(s_2)] \geqslant 0$.

On $\mathfrak{X}$ we assume the existence of a complete and transitive preference relation $\succeq$. We assume throughout the paper that if $F_X = F_Y$, then $X \sim Y$. Therefore, $\succeq$ induces a preference relation on $\mathfrak{F}$, which we also denote $\succeq$. Assume further that $\succeq$ is continuous (with respect to the weak topology), and monotonic (with respect to first order stochastic dominance). It then follows that every $X \in \mathfrak{X}$ has a unique certainty equivalent $c(X) \in \mathbb{R}$, satisfying $X \sim c(X)$ (recall that for every $F \in \mathfrak{F}$ there exists x such that $F(x) = 1$). For $F \in \mathfrak{F}$, let $\mathfrak{X}(F) = \{X \in \mathfrak{X} : F_X = F\}$ be the set of random variables with distribution function F. For $X, Y \in \mathfrak{X}$, let $[X, Y] = \{\alpha X + (1 - \alpha)Y : \alpha \in [0, 1]\}$. Also, for $X \in \mathfrak{X}$, let $I(X) = \{Y : Y \sim X\}$ be the indifference curve of $\succeq$ through X and let $E(X)$ be the expected value of X.

Many axiomatizations of models for decision making under risk make assumptions about mixtures of distribution function. For example, the independence axiom states that $F \succeq G$ iff $\forall \alpha \in [0, 1)$ and $\forall H$, $\alpha F + (1 - \alpha)H \succeq \alpha G + (1 - \alpha)H$.[2] A weaker version of this axiom, called betweenness, assumes that $F \succeq G$ iff $\forall \alpha \in [0, 1]$, $F \succeq \alpha F + (1 - \alpha)G \succeq G$.

Similar axioms can be made with respect to random variables. For example, we can assume the following dual version of the independence ax-

[1] This does not imply that all random variables are *uniformally* bounded. In section 4 we explicitly use the fact that if $X \in \mathfrak{X}$, then for every a, $X + a \in \mathfrak{X}$.

[2] Note that for a distribution function F, αF is the function F multiplied by the number α. This function is not a distribution function (unless $\alpha = 1$). For a random variable X, αX is a random variable, which is obtained from X by multiplying its outcomes by α.

iom: $X \succeq Y$ iff $\forall \alpha \in [0,1)$ and $\forall Z$, $\alpha X + (1-\alpha)Z \succeq \alpha Y + (1-\alpha)Z$. Clearly this axiom implies expected value maximization (see below). Similarly, the dual version of the betweenness axiom is: $X \succeq Y$ iff $\forall \alpha \in [0,1]$, $X \succeq \alpha X + (1-\alpha)Y \succeq Y$. This axiom too implies expected value maximization. (To see why, let $P(s_1) = \cdots P(s_n) = \frac{1}{n}$. Then $(x_1, s_1; \ldots; x_n, s_n) \sim (x_2, s_1; \ldots; x_n, s_{n-1}; x_1, s_n) \sim \cdots \sim (x_n, s_1; x_1, s_2; \ldots; x_{n-1}, s_n)$. The sum of these random variables, multiplied by $\frac{1}{n}$, is their expected value. Hence, $(x_1, s_1; \ldots; x_n, s_n) \sim \sum x_i$.) The fact that the unrestricted version of the dual independence axiom implies expected value maximization lead Yaari to restrict it to comonotonic random variables [9, Axiom A5].[3]

As stated above, dual independence (or betweenness) without restrictions is too strong, and in our view, comonotonicity assumes too much of the desired properties of the rank dependent functional. The following axiom seems to be a natural compromise between the two approaches.

Dual Betweenness $X \sim Y$ implies the existence of $X' \in \mathfrak{X}(F_X)$ and $Y' \in \mathfrak{X}(F_Y)$ such that $[X', Y'] \subset I(X)$.

Alternatively, this axiom states that $G \sim H$ implies the existence of X and Y such that $F_X = G$ and $F_Y = H$, and $\forall \alpha \in [0,1]$, $F_{\alpha X + (1-\alpha)Y} \sim G$.

3 Representation Theorems

Theorem 1 *If $\succeq$ satisfies dual betweenness, then for each certain outcome a, there exists a probability transformation function g_a such that $X \sim a$ iff*

$$\int x dg_a(F_X) = a \tag{1}$$

In other words, each indifference curve of $\succeq$ coincides with an indifference curve of a Yaari's [9] dual theory preference relation. This functional was suggested by Chew and Epstein [1] under the name "implicit rank-dependent

[3]Formally speaking, Yaari does not assume comonotonicity. His dual independence axiom relates to mixtures of the inverses of the cumulative distribution functions. In terms of random variables, this operation is equivalent to mixtures of comonotonic random variables.

mean value." Their axioms are restricted to comonotonic random variables.

Proof We restrict attention first to random variables of the form $(x_1, s_1; \ldots; x_n, s_n)$ such that $P(s_1) = \cdots = P(s_n) = \frac{1}{n}$. Denote this set $\mathfrak{X}_n$. Let Π be the set of permutations on $\{1, \ldots, n\}$. For $X \in \mathfrak{X}_n$ and $\pi \in \Pi$, let $\pi(X) = (x_{\pi(1)}, s_1; \ldots; x_{\pi(n)}, s_n)$.

Fact 1

1. *$\forall \pi \in \Pi$ and $X \in \mathfrak{X}_n$, $F_X = F_{\pi(X)}$.*
2. *If for $X, Y \in \mathfrak{X}_n$, $F_X = F_Y$, then $\exists \pi \in \Pi$ such that $Y = \pi(X)$.*

Claim 1 *$\forall X, Y \in \mathfrak{X}_n$ such that $X \sim Y$ there exists $\pi \in \Pi$ such that $[Y, \pi(X)] \subset I(Y)$.*

Proof From Fact 1 it follows that there are $\hat{\pi}$ and $\bar{\pi}$ such that $[\hat{\pi}(Y), \bar{\pi}(X)] \subset I(Y)$. Define $\pi = \hat{\pi}^{-1}\bar{\pi}$. ■

Fact 2 $[X, c(X)] \subset I(X)$.

Denote $e = (1, \ldots, 1)$.

Claim 2 *Let I be an indifference curve of $\succeq$, and let L be a ray of the form $L = \{\lambda \bar{x} + ke : \lambda \geqslant 0\}$ for some $\bar{x} \in \mathbb{R}^n$ and $k \in \mathbb{R}$. Then $I \cap L$ is either empty, or L, or a singleton.*

Proof Let $X, Y \in I \cap L$ such that $X \neq Y$, and let $a = c(X) = c(Y)$. By Fact 2, $[X, a] \subset I$ and $[a, Y] \subset I$. If X, Y, a are not on one ray (that is, if $a \notin L$), then for some $\alpha_1, \alpha_2 \in [0, 1]$, and for some $\delta \neq 0$, $\alpha_1 X + (1 - \alpha_1)a = \alpha_2 Y + (1 - \alpha_2)a + \delta e$. A violation of monotonicity, since $\alpha_1 X + (1 - \alpha_1)a \sim \alpha_2 Y + (1 - \alpha_2)a \sim a$.

Let W be a point on the ray through X and Y, so W is also on the ray through X and a. By monotonicity, $\exists \delta^*$ such that $W + \delta^* e \in I$. The two chords $[X, a]$ and $[W + \delta^* e, a]$ belong to the same indifference curve I. Since W, X, and a are on the same ray, these observations violate monotonicity, unless $\delta^* = 0$ and $W \in I$. ■

For $X, Y \in \mathcal{X}_n$, define

$$\Phi(X,Y) = \{\pi \in \Pi : [X, \pi(Y)] \subset I(Y)\}$$

and let $\mathcal{N}(X,Y) = \#(\Phi(X,Y))$ be the size of the set $\Phi(X,Y)$. Note that if $X \sim Y$, then by dual betweenness $\Phi(X,Y) \neq \varnothing$. Also, $\Phi(Y,X) = \{\pi^{-1} : \pi \in \Phi(X,Y)\}$, hence $\mathcal{N}(X,Y) = \mathcal{N}(Y,X)$. By the continuity of $\succeq$, we obtain:

Fact 3

1. *Φ is upper-hemi continuous. That is, if $X_k \to X$ and $Y_k \to Y$, and if $\forall k$, $\pi \in \Phi(X_k, Y_k)$, then $\pi \in \Phi(X,Y)$.*

2. *$\mathcal{N}$ is upper-semi continuous.*

Claim 3 *Let I be an indifference set of $\succeq$ in $\mathcal{X}_n$, and let $\bar{X}, \bar{Y} \in I$. For every open (relative to I) neighborhoods $V(\bar{X})$ of $\bar{X}$ and $V(\bar{Y})$ of $\bar{Y}$, there are $X \in V(\bar{X})$ and $Y \in V(\bar{Y})$ and open neighborhoods $V(X) \subset V(\bar{X})$, $V(Y) \subset V(\bar{Y})$ such that Φ is constant on $V(X) \times V(Y)$.*

Proof By Fact 3, for each $X, Y \in I$, there are neighborhoods $V(X), V(Y) \subset I$ such that $\Phi(X', Y') \subset \Phi(X,Y)$, for all $X' \in V(X)$ and $Y' \in V(Y)$. The claim follows from the fact that $\Phi(\bar{X}, \bar{Y})$ is finite. ■

Claim 4 *Let V_1 and V_2 be open neighborhoods in I. If $\Phi = \Pi^* \subset \Pi$ constantly on $V_1 \times V_2$, then there are hyperplanes H_1 and H_2 in $\mathbb{R}^n$ such that V_i is the intersection of H_i with an open set in $\mathbb{R}^n$, $i = 1, 2$. Moreover, for every $\pi \in \Pi^*$, the convex hull of $V_1 \cup \pi(V_2)$ is contained in H_1.*

Proof We prove that if $X_1, X_2 \in V_1$, then $[X_1, X_2] \subset V_1$. Suppose not. Then $\exists Z \in V_1$, $W \in [X_1, X_2]$, and $\delta \neq 0$ such that $Z = W + \delta e$ (see Fig. 1). Choose $\pi \in \Pi^*$ and $Y \in V_2' := \pi(V_2)$. By definition, $[X_1, Y], [X_2, Y], [Z, Y] \subset I$. Choose $Y' \in [X_2, Y] \cap V_2'$, $Y' \neq Y$, and obtain that $[X_1, Y'] \subset I$. Define $W' = [W, Y] \cap [X_1, Y']$, and let $\delta' \neq 0$ such that $Z' = W' + \delta' e \in [Z, Y]$. Since $W', Z' \in I$, we obtain a violation of monotonicity, hence $[X_1, X_2] \subset V_1$. Since V_1 is convex, it follows once more by monotonicity that it is contained in a hyperplane. By monotonicity and continuity it follows that V_1 is not contained in any lower dimensional plane. Therefore, V_1 is the intersection

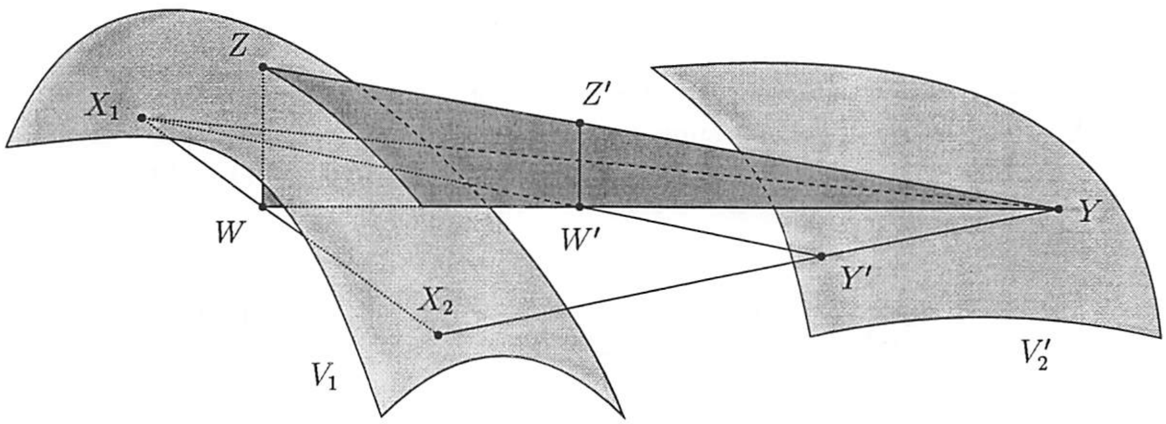

Figure 1: Proof of Claim 4

of a hyperplane with an open set in $\mathbb{R}^n$. By construction it follows that $\pi(V_2) \subset H_1$. The proof for V_2 is similar. ■

Summarizing, let I be an indifference set of $\succeq$. For every $\bar{X}, \bar{Y} \in I$ and for every open (relative to I) neighborhoods V_1 of $\bar{X}$ and V_2 of $\bar{Y}$, there are nonempty open (relative to I) sets $V_i' \subset V_i$, $i = 1, 2$, a hyperplane H, and a permutation π, such that the convex hull of $V_1' \cup \pi(V_2')$ is the intersection of H with an open set in $\mathbb{R}^n$.

For any permutation $\pi \in \Pi$, let $\mathbb{R}^n_\pi = \{x \in \mathbb{R}^n : x_{\pi(1)} \leqslant \cdots \leqslant x_{\pi(n)}\}$.

Claim 5 *Let H be a hyperplane in $\mathbb{R}^n$ that does not contain the main diagonal $D = \{ke : k \in \mathbb{R}^n\}$, and let V and V' be open neighborhoods in H such that $V \subset \mathrm{Int}(\mathbb{R}^n_\pi)$ and $V' \subset \mathrm{Int}(\mathbb{R}^n_{\pi'})$ for some $\pi \neq \pi'$. Then there are $X \in V$ and $X' \in V'$ such that every $Y = (y_1, \ldots, y_n) \in [X, X']$ has at most one pair of equal co-ordinates.*

Proof Let X and X' be as in the statement of claim, and assume that for some $Y \in [X, X']$, $y_i = y_j$ and $y_k = y_\ell$ (j may equal k, but then $i \neq \ell$). The linear subspace $J = \{Z \in \mathbb{R}^n : z_i = z_j \text{ and } z_k = z_\ell\}$ is of dimension $n-2$, and it contains the main diagonal D. Since H does not contain D, the intersection of J and H is of dimension $n-3$. Therefore there are $V_1 \subset V$ and $V_1' \subset V'$ such that for every $X_1 \in V_1$ and $X_1' \in V_1'$, for all $Y \in [X_1, X_1']$, if $y_i = y_j$ then $y_k \neq y_\ell$, and vice versa.

The claim now follows by repeating the argument a finite number of times. ■

Claim 6 *For every $\pi \in \Pi$ and for every indifference set I, $I \cap \mathbb{R}^n_\pi$ is contained in a hyperplane.*

Proof Let I be an indifference set, and let $V \subset \mathbb{R}^n_\pi$ be an open neighborhood in I that is contained in a hyperplane H. Let $X \in I \cap \mathbb{R}^n_\pi$, and suppose that $X \notin H$. By the above discussion, there is a neighborhood $V' \subset I \cap \mathbb{R}^n_\pi$ close to X that is contained in a hyperplane $H' \neq H$, and there is a permutation π' such that $\pi'(V') \subset H$ (see Fig. 2). We show next that $\pi'(V) \subset H$.

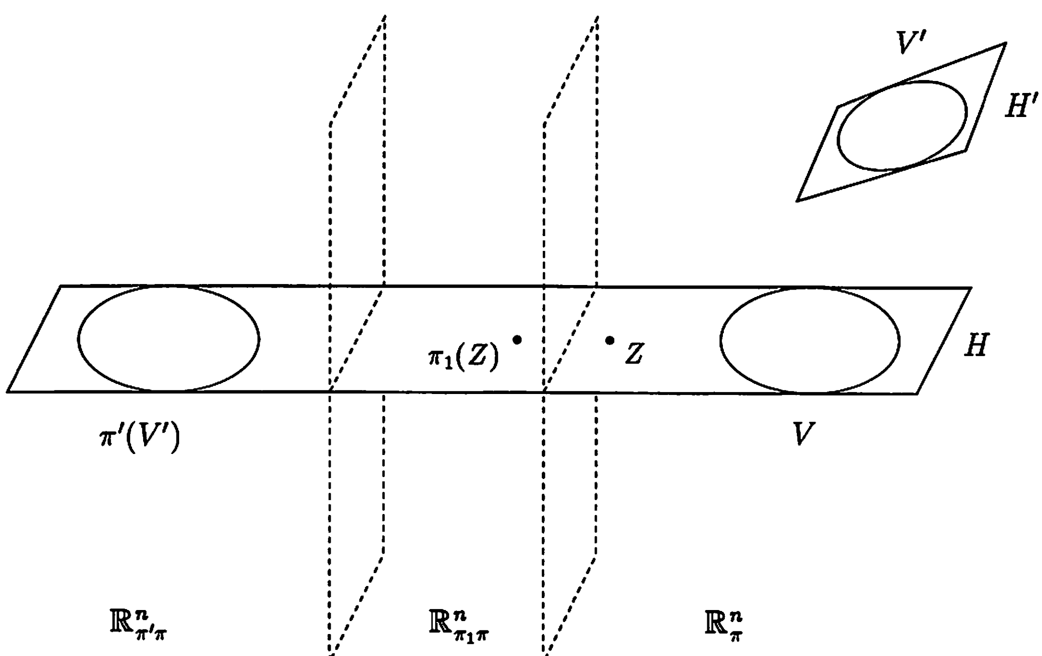

Figure 2: Proof of Claim 6

As was argued above, the convex hull of $V \cup \pi'(V')$ is contained in $H \cap I$, and its dimension is $n-1$. Denote this set C. By monotonicity, $D \not\subset H$.

Therefore, by Claim 5, we can assume, without loss of generality, that $\pi' = \pi_m \cdots \pi_1$, such that

1. For every k, π_k permutates only two numbers; and
2. For every $X_1 \in V$ and $X_1' \in \pi'(V')$, the chord $[X_1, X_1']$ goes sequentially through the sections $\mathbb{R}^n_\pi, \mathbb{R}^n_{\pi_1\pi}, \ldots, \mathbb{R}^n_{\pi_{m-1}\cdots\pi_1\pi}, \mathbb{R}^n_{\pi_m\cdots\pi_1\pi} = \mathbb{R}^n_{\pi'\pi}$.

We now show by induction that $\pi_k \cdots \pi_1(V) \subset H$. We prove the case of $k = 1$, the other steps are similarly proved. Let $H = \{Y : a \cdot Y = c\}$. Then $\pi_1(V) \subset \pi_1(H) = \{Y : \pi_1(a) \cdot Y = c'\}$. Denote $a' = \pi_1(a)$ and assume, for simplicity, that $\pi_1(1) = 2$ and $\pi_1(2) = 1$. Then $a_1' = a_2$, $a_2' = a_1$, and $a_i = a_i'$ for $i = 3, \ldots, n$. Pick $Z \in C \cap \mathbb{R}^n_\pi$ close to the common boundary of $\mathbb{R}^n_\pi$ and $\mathbb{R}^n_{\pi_1\pi}$ such that $\pi_1(Z) + \delta e \in C$ for some $\delta \in \mathbb{R}$. Since $\pi_1(Z) \in I$, it follows by monotonicity that $\delta = 0$. Hence $\pi_1(Z) \in H$, and it follows that $a_1 = a_2$. Therefore, $\pi_1(V) \subset H$, and, by induction, $\pi'(V) \subset H$.

Since $\pi'(V)$ and $\pi'(V')$ are both in H, V and V' are contained in the same hyperplane. Since $V \subset H$, so is V'. A contradiction, hence $X \in H$. ■

It now follows that for all π, $I \cap \mathbb{R}^n_\pi$ is the intersection of a hyperplane $H = \{X : a \cdot X = c\}$ with $\mathbb{R}^n_\pi$. By monotonicity, $a \in \mathbb{R}^n_{++}$. Let τ be the identity permutation, then on $\mathbb{R}^n_\tau$, there is a strictly increasing function $g^n_I : [0,1] \to [0,1]$ such that $I = \{X : \int x dg^n_I(F_X) = u^n_I\}$ for some u^n_I. The function g^n_I must satisfy $g^n_I(0) = 0$, and

$$g^n_I(\frac{i}{n}) = \frac{\sum_{j=1}^{i} a_j}{\sum_{j=1}^{n} a_j}$$

and is chosen to be linear on $[\frac{i}{n}, \frac{i+1}{n}]$, $j = 0, \ldots, n-1$. From the fact that for every X and π, $\pi(X) \sim X$ it follows that $X \in I \cap \mathfrak{X}_n$ iff $\int x dg^n_I(F_X) = u^n_I$.

For each n there exists such a function g^n_I, and for every j, n, and k,

$$g^{kn}_I\left(\frac{j}{n}\right) = g^n_I\left(\frac{j}{n}\right)$$

For every $x \in [0,1]$ there is a sequence $j_m/2^m \uparrow x$. Define

$$g_I(x) = \lim_{m\to\infty} g^{2^m}_I\left(\frac{j_m}{2^m}\right)$$

Since each X can be approached by a sequence X_n where for every n, $X_n \in \mathcal{X}_n$, the theorem follows by the continuity of $\succeq$. ■

Theorem 2 *If $\succeq$ satisfies dual betweenness and either constant absolute risk aversion or constant relative risk aversion, then $\succeq$ can be represented by a Yaari's [9] dual theory functional $\int x dg(F_X)$.*

Proof Since absolute risk aversion relates any two indifference sets, it follows that all g_I are the same. For the case of relative risk aversion, we observe that any two indifference sets that are better than 0 can be compared by this condition, as can any two indifference sets that are worse than zero. The argument follows by continuity at the common limit of these two sets, namely the indifference curve through zero. ■

4 Insurance

In this section we present a simple application of the functional form of eq. (1). It is based on the assumption that the domain of preferences is unbounded. In that case, since indifference curves do not intersect, higher indifference curves should have a smaller set of subgradients at their intersection with the certainty line.

Theorem 3 *Under the assumptions of Theorem 1,*

1. *If $a > a'$, then for every p, $g_a(p) \geqslant g_{a'}(p)$.*
2. *$\forall X$, $\rho(X, \alpha) := E(X + \alpha) - c(X + \alpha)$ is a non-decreasing function of α.*

Proof 1. Let $a > a'$, and suppose that for some $p \in (0,1)$, $g_a(p) < g_{a'}(p)$. Consider the set of lotteries $\{(x_1, 1-p; x_2, p) : x_2 \leqslant x_1\} \subset I(a)$. For such lotteries,

$$x_2 g_a(p) + x_1[1 - g_a(p)] = a$$

Similarly, for lotteries in $\{(x_1, 1-p; x_2, p) : x_2 \leqslant x_1\} \subset I(a')$,

$$x_2 g_{a'}(p) + x_1[1 - g_{a'}(p)] = a'$$

If the pair (x_1, x_2) solves these two equations, then

$$x_1 - x_2 = \frac{a' - a}{g_a(p) - g_{a'}(p)} > 0$$

Hence, the certainty equivalent of $(x_1, (1-p); x_2, p)$ is both a and a', a contradiction.

2. Let $a(\alpha) = c(X + \alpha)$. By eq. (1),

$$\begin{aligned} \rho(X,\alpha) &= \int x dF_{X+\alpha} - \int x dg_{a(\alpha)}(F_{X+\alpha}) = \\ & \int (x+\alpha) dF_X - \int (x+\alpha) dg_{a(\alpha)}(F_X) = \\ & \int x dF_X - \int x dg_{a(\alpha)}(F_X) \end{aligned}$$

For $\alpha > \alpha'$ we obtain

$$\rho(X,\alpha) - \rho(X,\alpha') = \int x dg_{a(\alpha')}(F_X) - \int x dg_{a(\alpha)}(F_X) \geqslant 0$$

where the last inequality follows from the first part of the theorem and from Yaari [9]. ■

The maximal insurance premium a decision maker is willing to pay to entirely avoid the risk X is the difference between the expected value of X and his certainty equivalent of X, which we denoted $\rho(X, 0)$. The second part of Theorem 3 implies that as income goes up, decision makers are willing to pay higher premium for full insurance (of a given risk X).

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