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Abstract

Constant risk aversion means that adding the same constant to all outcomes of two distributions, or multiplying all their outcomes by the same positive constant, will not change the preference relation between them. In this paper we prove several representation theorems, where constant risk aversion is combined with some other known axioms to imply specific functional forms. Among other things, we obtain a form of disappointment aversion theory without using the concept of reference point in the axioms. and a form of the rank dependent model without making any reference to the ranking of the outcomes. This axiomatization leads to a natural generalization of the Gini inequality index. Our analysis also establishes a connection between constant risk aversion. Fréchet differentiability, and orders of risk aversion.

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1 Introduction

Constant risk aversion means that adding the same constant to all outcomes of two distributions, or multiplying all their outcomes by the same positive constant, will not change the preference relation between them. Within the expected utility framework, this assumption implies expected value maximization. But there are many (non expected utility) functionals that satisfy this requirement, for example, Yaari's [48] dual theory, or functions offered by Roberts [37] and by Smorodinsky [44] (see Section 2 below).

In this paper we prove several representation theorems, where constant risk aversion is combined with some other known axioms to imply specific functional forms. We first show that non-trivial (that is, non expected value) functionals that satisfy constant risk aversion cannot be Fréchet differentiable. This differentiability is the key assumption in Machina's [34] analysis, and is widely used in the decision theoretic literature. As we show in Appendix A, this assumption is not without economic significance, as it implies second order attitude towards risk (see Segal and Spivak [41]). Since they are not Fréchet differentiable, constant risk aversion functionals cannot be approximated (in the L_1 norm) by expected utility preferences. These results are presented in Section 3.

A possible relaxation of Fréchet differentiability is the requirement that representation functionals are only Gâteaux differentiable. This requires the derivative with respect to α of $V((1-\alpha)F+\alpha G)$ to exist, and to be continuous and linear in $G-F$. Many constant risk aversion functionals satisfy this assumption. but as we show in Section 4. adding betweenness to the list of axioms ($F \sim G$ implies $\alpha F+(1-\alpha)G \sim F$ for all $\alpha \in [0,1]$) permits only one functional form, which is Gul's [24] disappointment aversion theory with a linear utility function u. According to this theory, the decision maker evaluates outcomes that are better than the certainty equivalent of a lottery by using an expected utility functional with a utility function u. He similarly evaluates outcomes that are worse than the certainty equivalent (with the same function u). Finally, the value of a lottery is a weighted sum of these two evaluations. In this theory, the certainty equivalent serves as a natural reference point. to which Gul's axioms make an explicit reference. Our axioms do not make any such explicit dependency, and the reference point is obtained as part of the results of the model, rather than as part of its assumptions.

One of the most popular alternatives to expected utility is the rank dependent model, given by $V(F) = \int u(x)dg(F(x))$. (This model has several different versions, see Weymark [47], Quiggin [35], and other citations in Section 5 below). This functional form is consistent with constant risk aversion whenever u is a linear function, which is Yaari's [48] dual theory. Although many axiomatizations of the rank dependent model exist,[1] they all depend on one crucial assumption, namely, that the value of an outcome depends on its relative position. This is of course a key feature of the rank dependent model, from which its name is derived. But it would be nice to be able to obtain this property as a result, rather than as an assumption of the model. In Section 5 we offer an axiomatization of a non-trivial special case of Yaari's functional, where none of the axioms makes an explicit appeal to the relative position of any of the outcomes. The key added axiom is mixture symmetry, which states that if $F \sim G$, then for all $\alpha \in [0,1]$, $\alpha F + (1-\alpha)G \sim (1-\alpha)F + \alpha G$ (see Chew, Epstein, and Segal [12]). The functional form we obtain is a weighted average of the expected value functional, and the Gini inequality index. We discuss the relevance of our results to the theory of income distribution in Section 6.

Two recent works on bargaining with non expected utility preferences make use of homogeneity of preferences with respect to probabilities (see Rubinstein, Safra, and Thomson [38] and Grant and Kajii [22]). We offer a slightly stronger assumption, where $F \sim G$ iff for every $\alpha \in [0,1]$, $\alpha F + (1-\alpha)\delta_0 \sim \alpha G + (1-\alpha)\delta_0$ (δ_0 is the distribution of the degenerate lottery that yields the outcome zero with probability one). Together with constant risk aversion, this axiom implies Yaari's representation with a probability transformation function of the form $g(p) = 1-(1-p)^t$. We prove this result in Section 7. The paper ends with a brief discussion of the empirical validity of some of the axioms we use (see Section 8).

2 Definitions

Let $\Omega = (S, \Sigma, P)$ be a measure space and let $\mathcal{X}$ be the set of real bounded random variables with non-negative outcomes on it. For $X \in \mathcal{X}$, let F_X be the distribution function of X. Denote by $\mathcal{F}$ the set of distribution functions obtained from elements of $\mathcal{X}$. With a slight abuse of notations, we denote

[1] We have counted more than ten axiomatizations of this model.

by a the constant random variable with the value a, and its distribution function by δ_a. For $X \in \mathcal{X}$, let $\underline{X}$ be the lowest possible value of X (that is, $\underline{X}$ is the supremum of the values of x for which $F_X(x) = 0$). Observe that for $X \in \mathcal{X}$ and $a > -\underline{X}$, $X + a \in \mathcal{X}$, and the distribution F_{X+a} is given by $F_{X+a}(x) = F_X(x - a)$. Throughout the paper, when we use the notation $X + a$ or $F + a$ we assume that $a > -\underline{X}$. Also, for $X \in \mathcal{X}$ and $\lambda > 0$, $\lambda X \in \mathcal{X}$, and we define the distribution $\lambda \times F_X := F_{\lambda X}$ by $(\lambda \times F_X)(x) = F_X(x/\lambda)$.

On $\mathcal{X}$ we assume the existence of a complete and transitive preference relation $\succeq$. We assume throughout the paper that if $F_X = F_Y$, then $X \sim Y$. Therefore, $\succeq$ induces an order on $\mathcal{F}$, which we also denote $\succeq$. Assume further that $\succeq$ is continuous (with respect to the weak topology), and monotonic (with respect to first order stochastic dominance). It then follows that every $F \in \mathcal{F}$ has a unique certainty equivalent $x \in [0, \infty)$, satisfying $F \sim \delta_x$ (recall that for every $F \in \mathcal{F}$ there exists x such that $F(x) = 1$). We restrict attention to preference relations satisfying the following assumption.

Constant Risk Aversion $X \succeq Y$ iff for every $a > \max\{-\underline{X}, -\underline{Y}\}$ and for every $\lambda > 0$, $\lambda(X + a) \succeq \lambda(Y + a)$. Or equivalently, $F \succeq G$ iff for every such a and λ, $\lambda \times (F + a) \succeq \lambda \times (G + a)$.

Note that $\succeq$ satisfies constant risk aversion iff it satisfies both constant absolute risk aversion and constant relative risk aversion. Also, its representation functional $V : \mathcal{F} \to \mathbb{R}$ which is defined implicitly by $F \sim \delta_{V(F)}$ (that is, $V(F)$ is the certainty equivalent of F), satisfies $V(\lambda \times (F + a)) = \lambda[V(F) + a]$. In such a case we say that V satisfies constant risk aversion. The following are examples for such functionals.

- $V(F) = \int x dg(F(x))$ (Yaari's [48] dual theory).
- $V(F) = \mu_F + \sigma_F W(\frac{1}{\sigma_F} \times [F - \mu_F])$ for some functional W, where μ_F is the expected value of F and σ_F is its standard deviation (Roberts [37]).
- $V(F) = \text{argmin}_t \int |x - t|^{c+1} dF(x)$ for some $c > 0$ (Smorodinsky [44]).

The next lemma shows that the set of functionals satisfying constant risk aversion is much larger than the above list. Moreover, from two such functionals more functionals can be created.

Lemma 1 *Let $\mathcal{V}$ be the set of all functionals that satisfy constant risk aversion. Then for every $\mathcal{V}' \subseteq \mathcal{V}$, the functionals V_1 and V_2 are in $\mathcal{V}$, where $V^1(F) = \inf\{V(F) : V \in \mathcal{V}'\}$ and $V^2(F) = \sup\{V(F) : V \in \mathcal{V}'\}$.*

Proof For a given F, there is a sequence V_i in $\mathcal{V}$ such that $V^1(F) = \lim V_i(F)$. Hence $V^1(\lambda\times(F+a)) \leqslant \lim V_i(\lambda\times(F+a)) = \lim \lambda(V_i(F)+a) = \lambda(V^1(F)+a)$. On the other hand, there is a sequence V_i' in $\mathcal{V}$ such that for $G = \lambda \times (F + a)$, $V^1(G) = \lim V_i'(G)$, hence $V^1(F) \leqslant \lim V_i'(F) = \lim V_i'(\lambda^{-1} \times (G - \lambda a)) = \lim \lambda^{-1}(V_i'(G) - \lambda a) = \lambda^{-1}(V^1(G) - \lambda a)$. These two inequalities imply $V^1(F) \leqslant \lambda^{-1}(V^1(\lambda\times(F+a)) - \lambda a) \leqslant V^1(F)$, hence $V^1(\lambda\times(F+a)) = \lambda(V^1(F) + a)$.

The proof of the sup case is similar. ∎

3 Smooth Preferences

Machina [34] introduced the concept of smooth representations, that is, representations that are Fréchet differentiable.

Definition 1 *The function $V : \mathcal{F} \to \mathbb{R}$ is Fréchet differentiable if for every $F \in \mathcal{F}$ there exists a "local utility" function $u(\cdot; F) : \mathbb{R} \to \mathbb{R}$ such that for every $G \in \mathcal{F}$,*

$$V(G) - V(F) = \int u(x; F)d[G(x) - F(x)] + o(\| G - F \|) \tag{1}$$

where $\| \cdot \|$ is the L_1-norm.

In other words, V is Fréchet differentiable if for every F, the functional V behaves like an expected utility representation with the von Neumann & Morgenstern utility function $u(\cdot; F)$. Obviously, if $V(F)$ always equals $\mathrm{E}[F]$, the expected value of F, then V is smooth and satisfies constant risk aversion. But we have seen above that many other functionals satisfy constant risk aversion. It is therefore natural to ask whether any of them is Fréchet differentiable.

Theorem 1 *The following two conditions are equivalent.*

1. *V is an expected value functional, that is, $V(F) = \mathrm{E}[F] = \int x dF(x)$.*

2. V satisfies constant risk aversion and is Fréchet differentiable.

Proof Obviously, (1) $\Longrightarrow$ (2). To see why (2) $\Longrightarrow$ (1), consider first the family of local utilities $u(\cdot;\delta_x)$. Since local utility functions are unique up to scalar addition,[2] we assume that for every x, $u(x;\delta_x) = x$.

Consider now the lottery $(y,p;1,1-p)$ for arbitrary y and p. Using the local utility $u(\cdot;\delta_1)$ we obtain (recall that $u(1;\delta_1)=1$)

$$V(y,p;1,1-p) - V(\delta_1) = [pu(y;\delta_1) + 1 - p] - 1 + o(p(y-1)) \tag{2}$$

Similarly, for $(y+\alpha,p;1+\alpha,1-p)$ we obtain

$$\begin{aligned} &V(y+\alpha,p;1+\alpha,1-p) - V(\delta_{1+\alpha}) = \\ &\quad [pu(y+\alpha;\delta_{1+\alpha}) + (1-p)(1+\alpha)] - (1+\alpha) + o(p(y-1)) \end{aligned} \tag{3}$$

By constant risk aversion, the left hand sides of eqs. (2) and (3) are the same. Subtract eq. (2) from eq. (3) to obtain

$$0 = p\left[u(y+\alpha;\delta_{1+\alpha}) - u(y;\delta_1) - \alpha + \frac{o(p(y-1))}{p}\right]$$

Divide by p, and then take the limit as $p \to 0$ to obtain

$$u(y+\alpha;\delta_{1+\alpha}) = u(y;\delta_1) + \alpha$$

Set $x = y+\alpha$ and $k = 1+\alpha$, and obtain for $x \geqslant k-1$,

$$u(x;\delta_k) = u(x-k+1;\delta_1) + k - 1 \tag{4}$$

Similarly to eq. (2) we obtain for $\lambda > 0$

$$V(\lambda y,p;\lambda,1-p) - V(\delta_\lambda) = [pu(\lambda y;\delta_\lambda) + \lambda(1-p)] - \lambda + o(p\lambda(y-1)) \tag{5}$$

By constant risk aversion, the left hand side of eq. (5) equals λ times the left hand side of eq. (2). Hence

$$pu(\lambda y;\delta_\lambda) - \lambda p + o(p\lambda(y-1)) = \lambda[pu(y;\delta_1) - p + o(p(y-1))]$$

[2] For this and other statements concerning local utilities, see Machina [34].

Divide both sides of the last equation by p, and then take the limit as $p \to 0$ to obtain

$$u(\lambda y; \delta_\lambda) = \lambda u(y; \delta_1)$$

Set $x = \lambda y$ and $k = \lambda$ to obtain for $x \geqslant k - 1$,

$$u(x; \delta_k) = ku\left(\frac{x}{k}; \delta_1\right) \tag{6}$$

Let $h(\cdot) = u(\cdot; \delta_1)$ and obtain from eqs. (4) and (6)

$$h(x - k + 1) + k - 1 = kh\left(\frac{x}{k}\right) \tag{7}$$

Since h is increasing, it is almost everywhere differentiable. Pick a point $x^* > 1$ at which h' exists. Differentiate both sides of eq. (7) and obtain that for $x = kx^*$,

$$h'(k(x^* - 1) + 1) = h'(x^*)$$

Since $x^* > 1$, it follows that $h'(z)$ is constant for $z > 1$.

By similar arguments, there is $x^* < 1$ at which h is differentiable. We now obtain that h is differentiable on $(0, 1)$, and that its derivative there is constant. In other words, there are two numbers, s and t such that

$$\frac{\partial u(x; \delta_1)}{\partial x} = \begin{cases} s & x < 1 \\ t & x > 1 \end{cases}$$

From eq. (6) it now follows that

$$\frac{\partial u(x; \delta_k)}{\partial x} = \begin{cases} s & x < k \\ t & x > k \end{cases}$$

But since V is Fréchet differentiable, it follows from Theorem 5 in the appendix that for almost all k, $u(\cdot; \delta_k)$ is differentiable with respect to the first argument at k. Hence $s = t$, and since $V(\delta_k) = k$, it follows that $u(x; \delta_k) = sx + (1 - s)k$.[3]

[3]Up to this point, we only used Gâteaux, rather than Fréchet, differentiability (see Section 4 bellow).

Fix the probabilities $p_1, \ldots, p_n$, and consider the space of lotteries $(x_1, p_1; \ldots; x_n, p_n)$. These lotteries can be represented as vectors of the form $(x_1, \ldots, x_n) \in \mathbb{R}^n$. For $y \geqslant 0$, let $d_y = (y, \ldots, y)$ be the point on the main diagonal corresponding to the lottery δ_y. Pick a point x^* not on the main diagonal of $\mathbb{R}^n$ and y such that $x^* \sim \delta_y$, and let H^* be the two dimensional plane containing x^* and the main diagonal. It follows from Roberts [37, p. 430] that indifference curves on H^* below the main diagonal are linear and parallel to each other, and so are indifference curves above the main diagonal. In other words, $x^* \sim d_y$ iff for all $\alpha \in [0,1]$, $\alpha x^* + (1-\alpha)d_y \sim d_y$. Note that this mixture is with respect to outcomes, not with respect to distributions.

The local utility function at δ_y is given by $u(x;\delta_y) = sx + (1-s)y$, hence

$$\begin{aligned} 0 &= V(x^*) - V(d_y) = V(\alpha x^* + (1-\alpha)d_y) - V(d_y) = \\ &\sum_{i=1}^{n} p_i[s(\alpha x_i^* + (1-\alpha)y) + (1-s)y] - [sy + (1-s)y] + o(\alpha) = \\ &\alpha s[\sum_i p_i x_i^* - y] + o(\alpha) \end{aligned}$$

Divide by αs, and then let $\alpha \to 0$ to obtain $\sum_{i=1}^{n} p_i x_i^* = y$, which is the expected value functional. By continuity, $V(F)$ is the expected value functional for all F. ■

Following this result, Chambers and Quiggin [8] proved that, assuming differentiability, decreasing absolute risk aversion implies increasing relative risk aversion.

Fréchet differentiability is not without economic meaning. As is shown in Appendix A, it implies that the preference relation generically represents second order attitude towards risk. That is, for almost all x^* and for every random variable $\tilde{\varepsilon}$ with expected value zero, the risk premium $\pi(t; x^*, \tilde{\varepsilon})$ the decision maker with wealth level x^* is willing to pay to avoid receiving $t \cdot \tilde{\varepsilon}$ goes to zero faster than t (so it is of order $o(t)$; see Segal and Spivak [41]). The next lemma further explores the connection between constant risk aversion and orders of risk aversion.

Proposition 1 *Assume constant risk aversion. Then the following three conditions are equivalent.*

1. *V is an expected value functional.*

2. *There is a positive wealth level x^* at which the decision maker's attitude towards risk is of order 2.*

3. *At all positive wealth levels, the decision maker's attitude towards risk is of order 2.*

Proof Obviously, (1) $\Longrightarrow$ (3) $\Longrightarrow$ (2). By constant risk aversion, it is easy to verify that (2) $\Longrightarrow$ (3). To see why (3) $\Longrightarrow$ (1), fix, as in the proof of Theorem 1, the probabilities $p_1, \ldots, p_n$. Suppose $\sum p_i x_i = x^*$, but $x = (x_1, \ldots, x_n) \not\sim d_{x^*}$. If x is sufficiently close to x^*, then there is $\gamma \neq 0$ such that $x+\gamma = (x_1+\gamma, \ldots, x_n+\gamma) \sim d_{x^*}$. Define $\tilde{\varepsilon} = (x_1 - x^*, p_1; \ldots; x_n - x^*, p_n)$ and obtain by constant risk aversion that the risk premium the decision maker is willing to pay to avoid $t \cdot \tilde{\varepsilon}$ is $t\gamma$. This contradicts the assumption that the preference relation satisfies second order risk aversion. ∎

4 Betweenness and Gâteaux Differentiability

The last section suggests that at the presence of constant risk aversion, the assumption of Fréchet differentiability is too strong. Weaker notions of differentiability exist, and at least one of them got special attention in the literature.

Definition 2 *A functional V is Gâteaux differentiable at F (Zeidler [50, p. 191]) if for every G,*

$$\delta V(F, G-F) := \frac{\partial}{\partial t} V((1-t)F + tG)\Big|_{t=0}$$

exists and if $\delta V(F, G-F)$ is a continuous linear function of $G - F$. V is Gâteaux differentiable if it is Gâteaux differentiable at F for every F.

If V is Fréchet differentiable, then it is also Gâteaux differentiable, but the opposite is not true. For example, the rank dependent model is Gâteaux, but not Fréchet differentiable (see Chew, Karni, and Safra [13]).[4] In this

[4]The minimum of two Gâteaux differentiable functionals is not necessarily differentiable. Suppose that for $\alpha \in [0, \alpha^*)$, $V^1(\alpha F + (1-\alpha)G) > V^2(\alpha F + (1-\alpha)G)$, but for $\alpha \in (\alpha^*, 1]$, $V^1(\alpha F + (1-\alpha)G) < V^2(\alpha F + (1-\alpha)G)$. Let $H = \alpha^* F + (1-\alpha^*)G$ and let $V^* = \min\{V^1, V^2\}$ to obtain that $\delta V^*(H, F-H) \neq -\delta V^*(H, G-H)$.

section we assume that preferences are Gâteaux differentiable, and that they satisfy the following betweenness assumption.

Betweenness $F \succeq G$ implies that for every $\alpha \in [0,1]$, $F \succeq \alpha F+(1-\alpha)G \succeq G$ (see Chew [9, 10] and Dekel [16]).

In this section we characterize preferences that satisfy constant risk aversion, Gâteaux differentiability, and betweenness. It turns out that the only functional to satisfy these three axioms is a special case of Gul [24] disappointment aversion theory.

Definition 3 *V is a Disappointment Aversion functional (see Gul [24]) if it can be represented by a functional V, given by*

$$V(F) = \frac{\gamma(\alpha)}{\alpha} \int_{x>C(F)} u(x)dF(x) + \frac{1-\gamma(\alpha)}{1-\alpha} \int_{x<C(F)} u(x)dF(x) \qquad (8)$$

where α is the probability that F yields an outcome above its certainty equivalent $C(F)$, and $\gamma(\alpha) = \alpha/[1+(1-\alpha)\beta]$ for some number β.

According to this theory, the decision maker evaluates outcomes that are better than the certainty equivalent of a lottery by using an expected utility functional with a utility function u. He similarly evaluates outcomes that are worse than the certainty equivalent. Finally, the value of a lottery is a weighted sum of these two evaluations.

Theorem 2 *The following two conditions on V are equivalent.*

1. *It is Gâteaux differentiable, and satisfies constant risk aversion and betweenness.*

2. *It can be represented by Gul's disappointment aversion functional with the linear utility $u(x) = x$.*

Proof If $\succeq$ satisfies betweenness, then each indifference curve can be obtained from an expected utility functional. Assuming as before that for every k, $V(\delta_k) = k$, it follows that there are utility functions $u_k : [0,\infty) \to \mathbb{R}$ such that $F \sim \delta_k$ iff

$$\int u_k(x)dF(x) = u_k(k) = k \qquad (9)$$

Choose $k > m > 0$, and let $(x, p; 0, 1-p) \sim (k, 1)$. Then $(mx/k, p; 0, 1-p) \sim (m, 1)$. By eq. (9), $pu_k(x) = k$ and $pu_m(mx/k) = m$, hence

$$u_k(x) = \frac{k}{m} u_m\left(\frac{mx}{k}\right) \tag{10}$$

Let $\mathcal{F}_{k,m} = \{F \sim \delta_k : F(k-m) = 0\}$. That is, $\mathcal{F}_{k,m}$ consists of those distributions whose certainty equivalent is k, and whose lowest outcome is not less than $k-m$. On $\mathcal{F}_{k,m}$, the preference relation $\succeq$ satisfies $\int u_k(x)dF = k$ and $\int u_m(x-k+m)dF = m$, or equivalently, $\int u_m(x-k+m)dF + k - m = k$. The reason is that $F \sim \delta_k$ iff $F - k + m \sim \delta_m$. Consider the two expected utility preferences on $\{F : F(k-m) = 0\}$ that are represented by $\int u_k(x)dF(x)$ and $\int v(x)dF(x)$, where $v(x) = u_m(x-k+m) + k - m$. Since they share an indifference curve (the one that goes through δ_k), they are the same, hence v is a linear transformation of u_k. Also, since $u_k(k) = v(k) = k$, there exists $\theta > 0$ such that

$$v(x) = u_m(x-k+m) + k - m = \theta u_k(x) + (1-\theta)k$$

Together with eq. (10), this implies

$$u_m(x-k+m) + k - m = \frac{\theta k}{m} u_m\left(\frac{mx}{k}\right) + (1-\theta)k$$

Let $y = x - k + m$ and obtain

$$u_m(y) = \frac{\theta k}{m} u_m\left(\frac{my + mk - m^2}{k}\right) + m - \theta k \tag{11}$$

We want to show that $\theta = 1$.

Since V is Gâteaux differentiable, it follows from the proof of Theorem 1 (see fn. 2) that

$$\frac{\partial u_m(x)}{\partial x} = \begin{cases} s & x < m \\ t & x > m \end{cases}$$

For $y \neq m$ we obtain

$$u'_m(y) = \theta u'_m\left(\frac{my + mk - m^2}{k}\right)$$

But $m + (m/k)(y - m) \in (y, m)$ (or $\in (m, y)$), hence $\theta = 1$. This is the case of disappointment aversion theory, where $u(x) = x$ and $\beta = (s/t) - 1$.

In Appendix B we show that disappointment aversion (eq. (8)) with linear utility function u is Gâteaux differentiable. ∎

Remark Theorem 2 strongly depends on the assumption that the functional is Gâteaux differentiable. For a functional that satisfies betweenness, is not disappointment aversion, (and therefore, by Theorem 2, is not Gâteaux differentiable), see Smorodinsky [44].

In disappointment aversion theory, the certainty equivalent of a lottery serves as a natural reference point, which Gul's axioms explicitly use. Our axioms do not refer to any special point, and the reference point is obtained as part of the results of the model, rather than as part of its assumptions, even if only for a special case of this theory.

5 Mixture Symmetry

As mentioned in the introduction, Yaari's [48] dual theory, given by $V(F) = \int x dg(F(x))$, satisfies constant risk aversion. This functional is a special case of the rank dependent model, $V(F) = \int u(x) dg(F(x))$ where the utility function $u : \mathbb{R}_+ \to \mathbb{R}$ and the probability transformation function $g : [0, 1] \to [0, 1]$ are strictly monotonic, $g(0) = 0$ and $g(1) = 1$ (see Weymark [47] and Quiggin [35]). This family of functionals received many different axiomatizations (e.g., Chew and Epstein [11], Segal [40], Quiggin and Wakker [36], or Wakker [45]), but all these axiomatizations make use of the order of the outcomes. For example, Quiggin's [35] axiom 4 implies expected utility if non-ordered outcomes are allowed.

Since rank dependent functionals evaluate outcomes not only by their value, but also by their relative rank as compared to other possible outcomes, axioms that presuppose attitudes that are based on outcomes' relative rank are arguably less convincing than axioms that do not make an explicit appeal to such ranks. The aim of this section is to offer what we believe to be the first axiomatization of a non-trivial set of rank dependent functionals where none of the axioms refers to the order of the outcomes, or treats an outcome differently based on its rank. We will use the following terms.

Mixture symmetry $F \sim G$ implies for all $\alpha \in [0, 1]$, $\alpha F + (1 - \alpha)G \sim$

$(1-\alpha)F+\alpha G$ (see Chew, Epstein, and Segal [12]. We discuss the normative appeal of this axiom in Section 6 below).

Non-Betweenness There exist $F \sim G$ and $\alpha \in [0,1]$ such that $F \nsim \alpha F + (1-\alpha)G$.

Quasi Concavity / Quasi convexity $F \succeq G$ implies that for every $\alpha \in [0,1]$, $\alpha F + (1-\alpha)G \succeq G$ / $F \succeq \alpha F + (1-\alpha)G$.

The next lemma shows that quasi concavity / betweenness / quasi convexity along one indifference curve implies global quasi concavity / betweenness / quasi convexity. Formally,

Lemma 2 *Suppose that $\succeq$ satisfies constant risk aversion. If there is $a^* > 0$ such that $F \sim G \sim \delta_{a^*}$ implies for all $\alpha \in (0,1)$, 1. $\alpha F + (1-\alpha)G \succ F$; 2. $F \sim \alpha F + (1-\alpha)G$; 3. $F \succ \alpha F + (1-\alpha)G$, then the preference relation $\succeq$ 1. is quasi concave; 2. satisfies the betweenness assumption; 3. is quasi convex, respectively.*

Proof We prove the case of betweenness; the other two cases are similar. Let $F \sim G \sim \delta_a$ for $a \neq 0$. Then

$$\frac{a^*}{a} \times F \sim \frac{a^*}{a} \times G \sim \frac{a^*}{a} \times \delta_a = \delta_{a^*}$$

Hence. by the betweenness assumption.

$$\forall\alpha,\ \ \frac{a^*}{a} \times F \sim \alpha\left[\frac{a^*}{a} \times F\right] + (1-\alpha)\left[\frac{a^*}{a} \times G\right] \Longrightarrow$$

$$\forall\alpha,\ \ \frac{a}{a^*} \times \left(\frac{a^*}{a} \times F\right) \sim \frac{a}{a^*} \times \left(\alpha\left[\frac{a^*}{a} \times F\right] + (1-\alpha)\left[\frac{a^*}{a} \times G\right]\right) \Longrightarrow$$

$$\forall\alpha,\ \ F \sim \alpha F + (1-\alpha)G$$

Since all outcomes are non-negative, $F \sim \delta_0$ implies $F = \delta_0$, hence the lemma. ■

Theorem 3 *The following two conditions on $\succeq$ are equivalent.*

1. *It satisfies constant risk aversion, non-betweenness, and mixture symmetry.*

2. *It can be represented by a rank dependent functional with linear utility (that is, Yaari's [48] dual theory functional) and quadratic probability transformation function of the form* $g(p) = p + cp - cp^2$ *for some* $c \in [-1, 0) \cup (0, 1]$.

Proof Obviously, (2) implies (1), so we show that (1) implies (2). As is proved in [12], mixture symmetry implies that the domain of $\succeq$ can be divided into three regions A, B, and C, $A \succ B \succ C$, such that on B, $\succeq$ satisfies betweenness, on A and on C, $\succeq$ can be represented by (not necessarily the same) quadratic functional of the form

$$V(x_1, p_1; \ldots; x_n, p_n) = \sum_{i=1}^{n} \sum_{j=1}^{n} p_i p_j \varphi_\ell(x_i, x_j), \quad \ell = A, C \tag{12}$$

where φ_ℓ is symmetric, $\ell = A, C$. Moreover, V is quasi concave on A and quasi convex on C.

By Lemma 2, only one of the three regions is not empty, and by the non-betweenness assumption, V is quadratic throughout. In other words, $\succeq$ can be represented by a strictly quadratic function $\sum_i \sum_j p_i p_j \varphi(x_i, x_j)$. Define a function $v : \mathbb{R}_+ \times [0, 1] \to \mathbb{R}_+$ implicitly by

$$(x, p; 0, 1 - p) \sim (v(x, p), 1) \tag{13}$$

For $\lambda > 0$, $v(\lambda x, p) = \lambda v(x, p)$. So for a fixed p the function v is homogeneous of degree 1, hence

$$v(x, p) = \rho(p)x \tag{14}$$

Assume without loss of generality that $\varphi(0, 0) = 0$. Let $q(x) := \varphi(x, x)$ and $r(x) := \varphi(x, 0)$, hence $q(0) = 0$. Then from Eqs. (12) and (13) it follows that

$$p^2 q(x) + 2p(1 - p)r(x) = q(x\rho(p)) \tag{15}$$

In Lemma 3 below we prove that the two functions q and ρ are trice differentiable. Differentiate both sides three times with respect to p to obtain

$$0 \equiv \rho'''(p)xq'(x\rho(p)) + 3\rho''(p)\rho'(p)x^2 q''(x\rho(p)) + [\rho'(p)]^3 x^3 q'''(x\rho(p)) \tag{16}$$

Since this equation holds for every x, it follows that all the coefficients of x on the right-hand side of Eq. (16) are zero. In particular, $\rho'''(p)q'(x\rho(p)) \equiv 0$. By monotonicity, $q' > 0$, hence ρ is quadratic. Since by Eq. (13) $\rho(0) = 0$ and $\rho(1) = 1$, it follows that

$$\rho(p) = cp^2 + (1-c)p \tag{17}$$

Also, $[\rho']^3 q''' \equiv 0$. Since ρ is not constant, it follows that $q''' \equiv 0$, hence q is quadratic. In other words, together with the assumption that $q(0) = 0$, we obtain

$$\varphi(x,x) = ax^2 + bx \tag{18}$$

From Eqs. (13) and (14) it follows that $(x,p;0,1-p) \sim (\rho(p)x, 1)$. By constant absolute risk aversion we obtain for every $x > y$

$$\begin{aligned}
&(x-y,p;0,1-p) \sim (\rho(p)[x-y],1) \Longrightarrow \\
&(x,p;y,1-p) \sim (\rho(p)x + (1-\rho(p))y, 1) \Longrightarrow \\
&p^2\varphi(x,x) + (1-p)^2\varphi(y,y) + 2p(1-p)\varphi(x,y) = \\
&\qquad \varphi(\rho(p)x + (1-\rho(p))y, \rho(p)x + (1-\rho(p))y) \Longrightarrow \\
&2p(1-p)\varphi(x,y) = \\
&\qquad a[\rho(p)x + (1-\rho(p))y]^2 + b[\rho(p)x + (1-\rho(p))y] - \\
&\qquad\qquad p^2[ax^2+bx] - (1-p)^2[ay^2+by]
\end{aligned}$$

Substitute Eq. (17) into this last equality to obtain

$$\begin{aligned}
&(2p-2p^2)\varphi(x,y) = \\
&\qquad a([cp^2+(1-c)p]x + [1-cp^2-(1-c)p]y)^2 + \\
&\qquad b([cp^2+(1-c)p]x + [1-cp^2-(1-c)p]y) - \\
&\qquad p^2[ax^2+bx] - (1-p)^2[ay^2+by]
\end{aligned} \tag{19}$$

Comparing the coefficients of powers of p on both sides of this last equation we get for p^1

$$0 = ac^2(x-y)^2$$

Hence either $a = 0$ or $c = 0$. Suppose first that $c = 0$, then $\rho(p) = p$. Comparing the coefficients of p^2 in Eq. (19) we obtain

$$\varphi(x,y) = axy + \frac{b}{2}(x+y)$$

It follows from Eq. (12) that for $X = (x_1, p_1; \ldots; x_n, p_n)$,

$$V(F_X) = a\sum_i\sum_j p_ip_jx_ix_j + \frac{b}{2}\sum_i\sum_j p_ip_j(x_i+x_j) = a(\mathrm{E}[F_X])^2 + b\,\mathrm{E}[F_X]$$

hence V is an expected value functional, but this contradicts the non-betweenness assumption.

On the other hand, if $c \neq 0$ and $a = 0$, then by eq. (18) $b \neq 0$. By comparing the coefficients of p^2 in Eq. (19) we get for $x > y$

$$\varphi(x,y) = \frac{b}{2}[(1-c)x + (1+c)y]$$

Similarly, for $y > x$,

$$\varphi(x,y) = \frac{b}{2}[(1-c)y + (1+c)x]$$

By the monotonicity of φ we may assume, without loss of generality, that $b = 1$. We thus obtain that

$$\varphi(x,y) = \frac{1}{2}(1-c)(x+y) + c\min\{x,y\}$$

Following Chew, Epstein, and Segal [12, Section 3, especially footnote 5], we obtain

$$V(F_X) = (1-c)\mathrm{E}[F_X] + c\int x dg^*(F_X(x)) \tag{20}$$

where $g^*(p) = 1-(1-p)^2$. Alternatively, $V(F_X) = \int x dg(F_X(x))$, where $g(p) = p + cp - cp^2$. ∎

Lemma 3 *The functions q and ρ of the proof of Theorem 3 are trice differentiable.*

Proof By monotonicity, both q and ρ are increasing functions, hence almost everywhere differentiable. The left hand side of eq. (15) is always differentiable with respect to p, hence so is the right hand side of this equation. Suppose ρ is not differentiable at p^*. Since q is almost everywhere differentiable, there is x such that q is differentiable at $x\rho(p^*)$, a contradiction. Since ρ is differentiable, it follows by the differentiability of the left hand side of eq. (15) that so is q. Differentiating both sides of eq. (15) with respect to p we thus obtain

$$2pq(x) + (2 - 4p)r(x) = \rho'(p)xq'(x\rho(p)) \tag{21}$$

The right-hand side of eq. (15) is differentiable with respect to x, and since q is differentiable, so is r. It follows that the left-hand side of eq. (21) is differentiable with respect to x, hence so is the right-hand side of this equation, in other words, q'' exists. Since ρ is differentiable, $\partial q'(\rho(p))/\partial p$ exists, and since the left-hand side of eq. (21) is differentiable with respect to p, so must be $\rho'(p)$. In other words, ρ is twice differentiable. Differentiating both sides of eq. (21) with respect to p we obtain

$$2q(x) - 4r(x) = [\rho'(p)]^2x^2q''(x\rho(p)) + \rho''(p)xq'(x\rho(p)) \tag{22}$$

Similarly to the above analysis, since both sides of eq. (22) are differentiable with respect to x, q''' exists, and since both sides are differentiable with respect to p, ρ''' exists. ■

6 Gini

Consider again eq. (20). When $c = -1$, $V(F_X) = \int xd\bar{g}(F_X(x))$, where $\bar{g}(p) = p^2$, and when $c = 1$, $V(F_X) = \int xdg^*(F_X(x))$, which is the Gini measure of income inequality. Since $\rho(p)$ is monotonic (see eq. (14)), it follows by eq. (17) that $c \in [-1, 1]$. Since g^* is concave it represents risk aversion (see Yaari [48] and Chew, Karni, and Safra [13]), hence $\int xdg^*(F(x)) < \mathrm{E}[F_X]$. In other words, the Gini measure is the lower bound of all the monotonic functionals that satisfy the assumptions of Theorem 3.

The axioms of constant risk aversion and mixture symmetry were made in the context of decision theory, but they are meaningful in the analysis of income distribution. In this literature, arguments were made for inequality indices that are not affected either by adding the same amount to each person's income or by multiplying all incomes by a positive scalar. Indices in the first class are called absolute measures of inequality (see Blackorby and Donaldson [6]), while indices in the second class are called relative measures (see Atkinson [3], Kolm [29], and Sen [43]). The Gini index is both an absolute and a relative measurement. Weymark [47] suggested the linear-utility rank dependent as a generalized Gini index that still carries these properties. Another axiomatization for a generalization of the Gini index is provided by Ben Porath and Gilboa [4], where the representing function is an additively separable function of the expected value and the Gini index. (See also Ben Porath, Gilboa. and Schmeidler [5], where indices are assumed to be in both absolute and relative measures).

The adaptation of the mixture symmetry axiom to income distributions imposes some restriction on the domain of distributions. One way in which this axiom was used in situations involving allocations of resources was to assume the existence of lotteries over possible allocations (see Epstein and Segal [19] and Border and Segal [7]). Here we offer two different approaches. The domain of preferences in the first approach is the set of income distributions in continuum economies, while the second assumes preferences over income distributions in *all* finite economies.

For the first approach consider the set of all income distributions in a continuum economy, where the distribution F means that for income level x, an $F(x)$ part of the population receives x or less. The mixture $\alpha F+(1-\alpha)G$ is the distribution obtained by using the distribution F for an α part of the economy, and the distribution G for the remaining $1-\alpha$ part. Mixture symmetry state that if $F \sim G$, then for every $\alpha \in [0,1]$, is does not matter if the α part receives F and the $1-\alpha$ part receives G, or if the α part receives G and the $1-\alpha$ part receives F.

For the alternative approach we need preferences to be over income distributions in all finite economies, but preferences still depend only on the distribution. So for example, the income distribution of $(50,100)$ in the two person economy is as good as the distribution $(50,50,100,100)$ in the four person economy.[5] Mixtures of lotteries mean the following. Let F and G be income distributions for an n person economy. Then for every k and ℓ, the distribution $\frac{k}{k+\ell}F+\frac{\ell}{k+\ell}G$ is a distribution for the $(k+\ell)n$ person economy,

[5] This property is called the Dalton [15] principle of population.

where k groups of n people receive the distribution F and ℓ groups of n people receive the distribution G. Mixture symmetry states that if $F \sim G$, then $\frac{k}{k+\ell}F + \frac{\ell}{k+\ell}G \sim \frac{\ell}{k+\ell}F + \frac{\ell}{k+\ell}G$.

The justification for the mixture symmetry in both approaches is similar, and we offer an explicit one for the first. If $F \sim G$, this does not mean that a distribution where an α part of the economy receives F and $1-\alpha$ receives G is equally attractive as F or G. (In other words, we do not assume betweenness here). However, given that an α part is going to receive one distribution, and a $1-\alpha$ part is to receive an equally attractive distribution, it does not matter which of the two distributions does each part receive.

This analysis does not extend itself to finite economies if the sizes of the economies for the F and G distributions are not the same, for example, if F is the income distribution of an n person economy, and G is the income distribution of an m person economy. In such a case we can start with an $n \times m$ person economy that will be able to support both the F and the G distributions (recall that for this approach we assumed that the preference relation $\succeq$ depends on the distribution of outcomes and is independent of the size of the economies). In this case mixture symmetry says that having k times this economy with the distribution F and ℓ times the same economy with G is indifferent to ℓ times the economy with F and k times the economy with G. For example, if $k > \ell$, then in both situations ℓ groups of n people get F, ℓ groups get G, and either all the remaining $k - \ell$ groups get F, or they all get G. Since $F \sim G$, the mixture symmetry assumptions suggests that the two income distributions $\frac{k}{k+\ell}F + \frac{\ell}{k+\ell}G$ and $\frac{\ell}{k+\ell}F + \frac{k}{k+\ell}G$ are also equally attractive.

7 Zero Independence

Suppose $F \sim G$. What will be the relation between $qF + (1-q)\delta_0$ and $qG + (1-q)\delta_0$? Some existing empirical evidence suggests that if $x > y > 0$ and $(x, p; 0, 1-p) \sim (y, 1)$, then $(x, qp; 0, 1-qp) \succ (y, q; 0, 1-q)$ (this is called the common ratio effect—see Allais [2], MacCrimmon and Larsson [33], or Kahneman and Tversky [27]). Assuming the rank dependent model, Segal [39. 40] connected this effect to the elasticity of the probability transformation function g, and showed that if $F \sim G$ iff $qF+(1-q)\delta_0 \sim qG+(1-q)\delta_0$ for all F and G with two outcomes at most. then $g(p) = 1-(1-p)^t$ for some

$t > 0$. Stronger results were achieved by Grant and Kajii [23], who proved the same for a wider set of initially possible functionals. In this section we prove that to a certain extent, it is enough to assume constant risk aversion (although in that case the utility function will have to be linear). The formal assumption we use is the following.

Zero Independence $F \sim G$ iff for all $q \in [0,1]$, $qF + (1-q)\delta_0 \sim qG + (1-q)\delta_0$.

Slightly weaker assumptions were introduced by Rubinstein, Safra, and Thomson [38], and later by Grant and Kajii [22], for the derivation of a preference-based Nash bargaining solution that applies to generalized expected utility preferences. The outcome of zero is considered there to be the disagreement outcome (the outcome that the bargainers receive if they fail to reach an agreement). In [38] the similar assumption is called homogeneity and it requires that zero independence should hold for $G = \delta_x$. In [22] it is called weak homogeneity and it requires that zero independence should hold for $G = \delta_x$ and for $F = p\delta_y + (1-p)\delta_0$. These assumptions play a crucial role in establishing the existence and the uniqueness of the ordinal Nash solution.

To prove Theorem 4 we will modify a result from Gilboa and Schmeidler [21] and assume that preferences satisfy diversification (see below). As in Section 2, let $\Omega = (S, \Sigma, P)$ be a measure space and let $\mathcal{X}$ be the set of all measurable bounded random variables on it.

Diversification If $X \sim Y$, then $\alpha X + (1-\alpha)Y \succeq X$ for all $0 < \alpha < 1$. Equivalently, if $G \sim H$, then for all X and Y such that $G = F_X$ and $H = F_Y$, and for all $0 < \alpha < 1$, $F_{\alpha X + (1-\alpha)Y} \succeq G$.

Lemma 4 *The following two conditions are equivalent.*

1. *V satisfies constant risk aversion and diversification.*

2. *There exists a unique set T of increasing, concave and onto functions over $[0,1]$ such that $\succeq$ can be represented by*

$$V(F_X) = \min_{g \in T} \left\{ \int x dg(F_X(x)) \right\}$$

Proof Clearly (2) implies (1) (see Lemma 1 in Section 2 above), so we prove here that (1) implies (2). First note that for each $X \in \mathcal{X}$ with certainty equivalent x, the set $\{Y : Y \succeq X\}$ is a convex cone with a vertex at x. The reason is that for $X \sim x$, constant risk aversion implies that for all $\alpha \geqslant 0$, $\alpha X + (1-\alpha)x \sim x$, and diversification implies convexity of supper sets. Constant risk aversion also implies that all these cones are parallel shifts of each other.

The conditions in (1) imply the axioms of Gilboa and Schmeidler [21]. Therefore, by their Theorem 1 and Proposition 4.1, there exists a compact set $\mathcal{C}$ of finitely additive measures on Ω such that $X \succeq Y$ iff $\min_{Q\in\mathcal{C}}\{\int X dQ\} \geqslant \min_{Q\in\mathcal{C}}\{\int Y dQ\}$. For a given X, let F_X^Q denote the distribution function of X with respect to the measure Q (F_X^P is denoted by F_X, as before) and let $V : \mathcal{F} \to \mathbb{R}$ be defined by $V(F_X) = \min_{Q\in\mathcal{C}}\{\int z dF_X^Q(z)\}$. Then $F_X \succeq F_Y$ iff $V(F_X) \geqslant V(F_Y)$.

Claim 1 *Every $Q \in \mathcal{C}$ is absolutely continuous with respect to P. That is, for all Q and $E \subset S$, $P(E) = 0 \Longrightarrow Q(E) = 0$.*

Proof of Claim 1 Let E be such that $P(E) = 0$ and let $[z; w]$ stand for the random variable $[z$ on E; w on $E^c]$ (E^c is the complement of E). By monotonicity with respect to first-order stochastic dominance, $w_1 > w_2$ implies $[z_1; w_1] \succ [z_2; w_2]$. Let $q = \max_{Q\in\mathcal{C}}\{Q(E)\}$. If there exists $Q \in \mathcal{C}$ such that $Q(E) > 0$, then $q > 0$ and $[4; 5] \succ [5 - q; 5 - \frac{q}{2}]$.

On the other hand, for $z \leq w$,

$$\min_{Q\in\mathcal{C}}\{zQ(E) + w(1 - Q(E))\} = \min_{Q\in\mathcal{C}}\{w + (z - w)Q(E)\} = w + (z - w)q$$

Therefore

$$\min_{Q\in\mathcal{C}}\left\{(5 - q)Q(E) + (5 - \frac{q}{2})(1 - Q(E))\right\} = 5 - \frac{q}{2} - \frac{q^2}{2}$$

On the other hand,

$$\min_{Q\in\mathcal{C}}\{4Q(E) + 5(1 - Q(E))\} = 5 - q$$

Hence $[5 - q; 5 - \frac{q}{2}] \succeq [4; 5]$, a contradiction. □

Claim 1 implies that, for all X, Q, z_1, and z_2,

$$F_X(z_1) = F_X(z_2) \Longrightarrow F_X^Q(z_1) = F_X^Q(z_2)$$

Consider now a given $X \in \mathcal{X}$ and a measure $Q \in \mathcal{C}$, and define a function $g_{Q,X} : [0,1] \to [0,1]$ by

$$g_{Q,X}(p) = \begin{cases} F_X^Q(F_X^{-1}(p)) & p \in \text{Image}(F_X) \\ 0 & p = 0 \\ l(p) & \text{otherwise} \end{cases}$$

where l is the piece-wise linear function, defined on the complement of the image of F_X, that makes $g_{Q,X}$ continuous. By the claim, $g_{Q,X}$ is well defined. Clearly, it is onto and non-decreasing, and it satisfies $\int z dF_X^Q(z) = \int z dg_{Q,X}(F_X(z))$.

For each $X \in \mathcal{X}$ there exists $Q(X) \in \mathcal{C}$ such that

$$V(F_X) = \min_{Q \in \mathcal{C}} \left\{ \int z dF_X^Q(z) \right\} = \int z dF_X^{Q(X)}(z) = \int z dg_X(F_X(z))$$

where $g_X = g_{Q(X),X}$.

Let $\text{Com}(X) = \{Y : \forall s_1, s_2 \in S,\ (X(s_1) - X(s_2))(Y(s_1) - Y(s_2)) \geqslant 0\}$ (that is, the set of all random variables that are comonotone with X). Restricted to $\text{Com}(X)$, indifference sets of $\int z dg_Z(F_Y(z))$ are hyperplanes in $\mathcal{X}$. Therefore, for $Y \in \text{Com}(X)$,

$$\begin{aligned} V(F_Y) &= \min_{Z \in \text{Com}(X), Z \sim Y} \left\{ \int z dF_Y^{Q(Z)}(z) \right\} = \\ & \min_{Z \in \text{Com}(X), Z \sim Y} \left\{ \int z dg_Z(F_Y(z)) \right\} \end{aligned}$$

Next, we discuss the case of non-comonotone random variables. Define $W^* : \mathcal{F} \to \mathbb{R}$ by

$$W^*(F_X) = \min_{Z \in \mathcal{X}, Z \sim X} \left\{ \int z dg_Z(F_X(z)) \right\}$$

By definition, $W^*(F_X) \leqslant V(F_X)$. Suppose there exists X such that $W^*(F_X) = \tilde{x} < x = V(F_X)$. Then there exists $\overline{X} \notin \text{Com}(X)$ such that $W^*(F_X) = \int z dg_{\overline{X}}(F_X(z))$.

Let $\vec{\mathcal{X}}^n$ be the set of all random variables $Y \in \mathcal{X}$ that have at most n different outcomes $y_1 \leqslant \cdots \leqslant y_n$ and satisfy $\Pr(Y = y_i) = \frac{1}{n}$. Assume first that there exist n and $\hat{X}$ such that $X, \hat{X} \in \vec{\mathcal{X}}^n$ and $F_{\hat{X}} = F_{\overline{X}}$. Clearly, $Q(\hat{X}) = Q(\overline{X})$. Therefore, $g_{\hat{X}} = g_{\overline{X}}$, which implies $W^*(F_X) = V(F_X)$.

If there is no such n then, by continuity, there exists n large enough for which there exist $X^n, \hat{X}^n \in \vec{\mathcal{X}}^n$ that satisfy $W^*(F_{X^n}) = \int z dg_{\hat{X}^n}(F_{X^n}(z)) < \tilde{x} + \frac{1}{2}(x - \tilde{x})$ and $V(F_{X^n}) > \tilde{x} + \frac{1}{2}(x - \tilde{x})$. A contradiction. Hence, for $\mathcal{T} = \{g_Z : Z \in \mathcal{X}\}$,

$$V(F_X) = \min_{g \in \mathcal{T}} \left\{ \int z dg(F_X(z)) \right\}$$

It remains to show that the functions g_Z are concave. Assume, without loss of generality, that there exist n and $X \in \operatorname{int}(\vec{\mathcal{X}}^n)$ such that, for some $i_0 \in \{2, \ldots, n-1\}$,

$$2g_X\left(\frac{i_0}{n}\right) < g_X\left(\frac{i_0 - 1)}{n}\right) + g_X\left(\frac{i_0 + 1)}{n}\right)$$

Let $Q = Q(X)$ and denote $q_i = Q(X = x_i)$. By definition, $g_X(\frac{i}{n}) = \sum_{j=1}^{i} q_j$. Therefore, $q_{i_0} < q_{i_0+1}$. Take $\varepsilon > 0$ small enough and consider $X(\varepsilon) \in \operatorname{int}(\vec{\mathcal{X}}^n)$ with the values $x_1 < \cdots < x_{i_0} - \varepsilon < x_{i_0+1} + \varepsilon < \cdots < x_n$. By the construction of Q, $X(\varepsilon) \succ X$. This, however, contradicts risk aversion (note that risk aversion is implied by diversification, see Dekel [17]). ■

Theorem 4 *The following two conditions are equivalent.*

1. *V satisfies constant risk aversion, diversification, and zero independence.*

2. *$V(F) = \int x dg(F(x))$, where $g(p) = 1 - (1-p)^t$ for some $t \geqslant 1$.*

Proof Obviously (2) $\Longrightarrow$ (1). We prove that (1) $\Longrightarrow$ (2) for finite lotteries (that is, for lotteries with a finite number of different outcomes) by induction on the number of the nonzero outcomes. Continuity is then used to get the desired representation for all $F \in \mathcal{F}$.

By Lemma 4 there is a family of probability transformation functions $\{g_\alpha : \alpha \in \mathcal{A}\}$ such that for every F, $V(F) = \min_\alpha \int x dg_\alpha(F(x))$. For lotteries of the form $(x, p; 0, 1-p)$ we obtain

$$V(x, p; 0, 1-p) = \min_\alpha x[1 - g_\alpha(1-p)]$$

Define $f_\alpha(p) = 1 - g_\alpha(1-p)$ and $h(p) = \min_\alpha f_\alpha(p)$ and obtain that $V(x, p; 0, 1-p) = xh(p)$.

By zero independence, $(x, p; 0, 1-p) \sim \delta_y$ implies for all $q \in [0,1]$, $(x, pq; 0, 1-pq) \sim (y, q; 0, 1-q)$, hence $xh(p) = y$ and $xh(pq) = yh(q)$. Combining the two we obtain

$$h(pq) = h(p)h(q)$$

The solution of this functional equation is $h(p) = p^t$ (see Aczél [1, p. 41]).

Suppose we have already proved that for lotteries with at most n prizes $V(F) = \int x dg(F(x)) = \int x d(1-(1-F(x))^t) = \int x d(-(1-F(x))^t)$. That is, for $x_1 \leqslant \cdots \leqslant x_n$,

$$V(x_1, p_1; \ldots; x_n, p_n) = x_n p_n^t + \sum_{i=1}^{n-1} x_i \left[\left(\sum_{j=i}^{n} p_j \right)^t - \left(\sum_{j=i+1}^{n} p_j \right)^t \right]$$

We will now prove it for $n+1$. Let $x_1 \leqslant \cdots \leqslant x_{n+1}$, and consider the lottery $X = (x_1, p_1; \ldots; x_{n+1}, p_{n+1})$. By constant risk aversion,

$$V(F_X) = x_1 + V(F_X - x_1) = x_1 + V(0, p_1; \ldots; x_{n+1} - x_1, p_{n+1})$$

By continuity there is y such that $F_X - x_1 \sim (y, 1-p_1; 0, p_1)$. By zero independence $F_{X^*} \sim \delta_y$, where

$$X^* = \left(x_2 - x_1, \frac{p_2}{1-p_1}; \ldots; x_{n+1} - x_1, \frac{p_{n+1}}{1-p_1} \right)$$

By the induction hypothesis, $V(F_{X^*}) = V(\delta_y)$ implies

$$(x_{n+1} - x_1) \left(\frac{p_{n+1}}{1-p_1} \right)^t + \sum_{i=2}^{n} (x_i - x_1) \left[\left(\sum_{j=i}^{n+1} \frac{p_j}{1-p_1} \right)^t - \left(\sum_{j=i+1}^{n+1} \frac{p_j}{1-p_1} \right)^t \right] = y \tag{23}$$

Also, since $F_X - x_1 \sim (y, 1-p_1; 0, p_1)$, it follows by the constant risk aversion properties of V that $V(F_X) = x_1 + y(1-p_1)^t$. Substitute into eq. (23) to obtain

$$
\begin{aligned}
V(F_X) &= x_1 + (1-p_1)^t \left\{ (x_{n+1} - x_1) \left(\frac{p_{n+1}}{1-p_1} \right)^t + \right. \\
&\left. \sum_{i=2}^{n} (x_i - x_1) \left[\left(\sum_{j=i}^{n+1} \frac{p_j}{1-p_1} \right)^t - \left(\sum_{j=i+1}^{n+1} \frac{p_j}{1-p_1} \right)^t \right] \right\} = \\
&x_{n+1} p_{n+1}^t + \sum_{i=1}^{n} x_i \left[\left(\sum_{j=i}^{n+1} p_j \right)^t - \left(\sum_{j=i+1}^{n+1} p_j \right)^t \right]
\end{aligned}
$$

which proves the induction hypothesis. ∎

The functional form of Theorem 4 has been previously appeared in the literature on income distribution under the name of "the S-Gini family." (see Donaldson and Weymark [18] and Yitzhaki [49]).

8 Some Experimental Evidence

Throughout the paper we used several axioms, like Fréchet and Gâteaux differentiability, constant risk aversion, betweenness, mixture symmetry, and zero independence. In this section we survey the experimental validity of some of these axiom.

As is explained in Appendix A. Fréchet differentiability implies (generically) second order attitude towards risk. Direct examination of differentiability is probably impossible, but Loomes and Segal [32] found out that in their experiments about one third of their subjects showed a clear first order attitude. (About one third of the subjects showed a clear second order attitude, while the results concerning the rest were less clear). In other words, a significant minority does not satisfy the Fréchet differentiability assumption.

The assumption of constant, absolute, and relative, risk aversion was tested by levy [31], who found out that at a 5% significance level, 24 out of 62 participants satisfied neither increasing nor decreasing absolute risk aversion (p. 296). The results regarding relative risk aversion are not explicitly given but, in Levy's words (p. 298) "The other 51 [out of 62] subjects reveal either constant relative risk aversion or decreasing relative risk aversion." Kreps [30,

p. 76] too claims that when prizes are not too large, one should be very comfortable with constant absolute risk aversion.

The zero independence axiom is related to the common ratio effect (Allais [2]). According to this effect, if $x < y$ and $\delta_x \sim (y, p; 0, 1-p)$, then for $\lambda < 1$, $(y, \lambda p; 0, 1-\lambda p) \succ (x, \lambda; 0, 1-\lambda)$. Many experiments show the existence of such an effect (see, for example, MacCrimmon and Larsson [33] and Kahneman and Tversky [27]). Such a behavior obviously contradicts zero independence, but recently Cubitt, Starmer, and Sugden [14] showed that the existence of this effect may well depend on the design of the experiment. When properly designed (that is, when each participant is asked only one question and issues of dynamic consistency are eliminated), the common ratio effect appears less frequently and zero independence may be satisfied.

For experimental tests of betweenness, quasi concavity / convexity, and mixture symmetry, see Harless and Camerer [25], Hey and Orme [26], and references there.

A Orders of Risk Aversion and Fréchet Differentiability

Machina [34] introduced the concept of smooth representations of preferences over risky assents and argued that since many problems in economics involve only local analysis (for example, optimization and comparative statics analysis), and since by eq. (1) the functional V can be locally approximated by an expected utility functional, it should follow that the economic results of expected utility apply to all (smooth) nonexpected utility functionals.

It turns out that some well known functionals are not Fréchet differentiable (see Chew, Karni, and Safra [13]), but they too have local utility approximations, although only in L_p for $p > 1$ (see Wang [46]). It thus seems that although the Fréchet differentiability assumption rules out some models, it does not have any effect on our ability to analyze local behavior under risk. In this appendix we show that this, however, is not true, and assuming Fréchet differentiability has some economic meaning.

At points where the increasing utility function $u : \mathbb{R} \to \mathbb{R}$ is differentiable (that is, almost everywhere), the expected utility functional $\int u(x)dF(x)$ behaves locally like expected value. Extending this property to nonexpected utility preferences, Segal and Spivak [41] defined the concept of second order risk aversion as follows. For a random variable $\tilde{\varepsilon}$ with expected value zero and for $x^* \in \mathbb{R}$, define the risk premium function π implicitly by $\delta_{x^*-\pi(x^*,\tilde{\varepsilon})} \sim x^* + \tilde{\varepsilon}$. The preference relation $\succeq$ is said to satisfy second order risk aversion at x^* if for every $\tilde{\varepsilon}$ with expected value zero,

$$\frac{\partial}{\partial t}\pi(x^*, t \cdot \tilde{\varepsilon})\bigg|_{t=0^+} = 0$$

Similarly, the preference relation $\succeq$ is said to represent first order attitude towards risk at x^* if for every such $\tilde{\varepsilon}$,

$$\frac{\partial}{\partial t}\pi(x^*, t \cdot \tilde{\varepsilon})\bigg|_{t=0^+} \neq 0$$

The concept of orders of risk aversion is not without economic significance. A second order risk averter will buy full insurance if and only if its price is "fair," that is, when the price of a dollar insurance equals the probability of loss. On the other hand, a second order risk averter will buy full insurance even at the face of some marginal loading. (For this, see Segal and Spivak [41]. See also Karni [28] for other results concerning insurance and orders of risk aversion. For other applications, see Epstein and Zin [20]).

Segal and Spivak [42] show that under the assumption of Fréchet differentiability of the representation functional V, the preference relation $\succeq$ satisfies first [second] order risk aversion at a point x^* iff the local utility $u(x; \delta_{x^*})$ is not differentiable [differentiable] with respect to its first argument at $x = x^*$. Using these results, Theorem 5 below states a connection between Fréchet differentiability and orders of risk aversion.

Theorem 5 *If the monotonic preference relation $\succeq$ can be represented by a Fréchet differentiable functional V, then for almost all x^*, $\succeq$ satisfies second order risk aversion at x^*. In other words, the set of points where $\succeq$ satisfies first order risk aversion is of measure zero.*

Proof By monotonicity, the functional V satisfies $V(\delta_x) > V(\delta_y) \iff x > y$, hence the set of points where $\partial V(\delta_x)/\partial x$ does not exist is of measure zero. The theorem now follows from the equivalence of the following three conditions.

1. The derivative $\partial V(\delta_x)/\partial x$ exists at $x = x^*$.
2. The preference relation $\succeq$ satisfies second order risk aversion at $x = x^*$.
3. The local utility $u(x;\delta_{x^*})$ is differentiable with respect to its first argument at $x = x^*$.

The equivalence of (2) and (3) is proved in [42]. To see why (1) and (3) are equivalent, note that

$$V(\delta_{x^*+\varepsilon}) - V(\delta_{x^*}) = u(x^* + \varepsilon; \delta_{x^*}) - u(x^*; \delta_{x^*}) + o(\varepsilon)$$

Divide both sides by ε and let $\varepsilon \to 0$ to obtain that $V(\delta_x)$ is differentiable with respect to x at $x = x^*$ iff $u(x;\delta_{x^*})$ is differentiable with respect to its first argument at $x = x^*$. ■

The theorem implies that all models that always have kinked indifference curves along the main diagonal in a states-of-the-world representation are not (L_1) Fréchet differentiable. Such is the rank dependent model (for a direct proof that this model is not Fréchet differentiable, see Chew, Karni and Safra [13]. For a proof that this model satisfies first order risk aversion, see Segal and Spivak [41]). Another example is Gul's [24] disappointment aversion model. Indeed, since for two-outcome lotteries it behaves like the rank dependent model, disappointment aversion implies first order behavior at all x (see [32]). By Theorem 5, this functional is not Fréchet differentiable.

The theorem also implies that there is an economic behavioral difference between preferences that are L_1 Fréchet differentiable and preferences that are L_p Fréchet differentiable for $p > 1$, as the latter may display first order risk aversion everywhere.

B Disappointment Aversion Theory and Gâteaux Differentiability

In this appendix we prove that the disappointment aversion with linear utility is Gâteaux differentiable. In our case, we assume that $V(\delta_k) = k$. Therefore, for fixed s and t, the value of V at F is the number k that solves the implicit equation

$$\int_0^k [sx + (1-s)k]dF(x) + \int_k^\infty [tx + (1-t)k]dF(x) - k = 0$$

For given F and G, let $H_\alpha = \alpha G + (1-\alpha)F$, and obtain that the value of $V(H_\alpha)$ is the number $k(\alpha)$ that solves

$$\int_0^{k(\alpha)} \varphi(x, k(\alpha))dH_\alpha(x) + \int_{k(\alpha)}^\infty \psi(x, k(\alpha))dH_\alpha(x) - k(\alpha) = 0$$

where $\varphi(x, k(\alpha)) = sx + (1-s)k(\alpha)$ and $\psi(x, k(\alpha)) = tx + (1-t)k(\alpha)$. Define

$$\begin{aligned} J(\alpha, k) &= \int_0^k \varphi(x, k)dH_\alpha(x) + \int_k^\infty \psi(x, k)dH_\alpha(x) - k = \\ &\int_0^\infty \psi(x, k)dH_\alpha(x) + (s-t)\int_0^k (x-k)dH_\alpha(x) - k \end{aligned}$$

By the implicit function theorem, $\partial k/\partial\alpha = -J_\alpha/J_k$. Trivially,

$$\frac{\partial J}{\partial \alpha} = \int_0^k \varphi(x, k)d[G(x) - F(x)] + \int_k^\infty \psi(x, k)d[G(x) - F(x)] \qquad (24)$$

We compute next the derivative of J with respect to k. We need only the derivative at $\alpha = 0$, in which case $H_\alpha = F$. Therefore, the derivative of J with respect to k at $\alpha = 0$ is given by

$$\begin{aligned} &(1-t)\int_0^\infty dF(x) + (s-t)\lim_{\varepsilon \to 0}\frac{1}{\varepsilon}\left[\int_0^{k+\varepsilon}(x-k-\varepsilon)dF(x) - \right. \\ &\quad \left. \int_0^k (x-k)dF(x)\right] - 1 = \\ &-t + (s-t)\lim_{\varepsilon \to 0}\frac{1}{\varepsilon}\left[\int_0^{k+\varepsilon}(x-k-\varepsilon)dF(x) - \int_0^{k+\varepsilon}(x-k)dF(x)\right] + \\ &\quad (s-t)\lim_{\varepsilon \to 0}\frac{1}{\varepsilon}\int_k^{k+\varepsilon}(x-k)dF(x) = \\ &-t - (s-t)\lim_{\varepsilon \to 0}F(k+\varepsilon) + (s-t)\lim_{\varepsilon \to 0}\frac{1}{\varepsilon}\int_k^{k+\varepsilon}(x-k)dF(x) \end{aligned}$$

The expression

$$\int_k^{k+\varepsilon}(x-k)dF(x) \qquad (25)$$

is bounded from above by the maximal change in $x - k$ multiplied by the maximal change in the cumulative probability, that is, by $\varepsilon[F(k+\varepsilon) - F(k)]$.

The integral at eq. (25) is bounded from below by ε multiplied by the minimal possible change in the value of F, namely by $\lim_{\varepsilon\to 0} F(k+\varepsilon) - F(k)$. Hence, ε^{-1} times the integral at eq. (25) is between $\lim_{\varepsilon\to 0} F(k+\varepsilon) - F(k)$ and $F(k+\varepsilon) - F(k)$. It thus follows that

$$(s-t)\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_k^{k+\varepsilon}(x-k)dF(x) = (s-t)[\lim_{\varepsilon\to 0} F(k+\varepsilon) - F(k)]$$

and

$$\left.\frac{\partial J}{\partial k}\right|_{\alpha=0} = -t-(s-t)F(k) = -t(1-F(k)) - sF(k) < 0$$

From eq. (24) it now follows that

$$\left.\frac{\partial k}{\partial \alpha}\right|_{\alpha=0} = \frac{\int_0^k \varphi(x,k)d[G(x)-F(x)] + \int_k^\infty \psi(x,k)d[G(x)-F(x)]}{t+(s-t)F(k)}$$

Obviously, this expression is continuous in $G-F$ and also linear in $G-F$. ■

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