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1. Introduction

Economists often produce models which are stated as regression equations with the lagged value of the dependent variable on the right side. The coefficients of such models are estimated by either ordinary least squares (OLS), on the assumption that the disturbances are serially independent, or instrumental variables (IV) if it is felt that the disturbances are autocorrelated. This paper presents exact small sample properties of these two estimators in the case that the disturbances follow a specific, easily justified, autoregressive scheme.

To obtain the exact moments of the OLS estimator we require the moments of the noncentral chi-square distribution and its partial derivatives with respect to its noncentrality parameter. These have been given in Appendix B. The method is straightforward and can be useful in obtaining moments in a wide class of situations. The exact and approximate distribution of the IV estimator have been analysed by using the results of Fieller (1932) and Carter (1976).

The main results of the paper can be summarized as follows. First, the relative bias of the OLS estimator, for a given sample size, lies between -1 and 0. Also, the absolute value of the relative bias is a decreasing function of one of the parameters of the distribution and the mean squared error is a monotonically decreasing function of the same parameter. Further, the OLS estimator converges to the true value of the parameter if the noncentrality parameter of the distribution increases indefinitely. With regard to the IV estimator we note that its exact moments, to any order, do not exist. An approximation to the exact distribution has been obtained which is centered on the true parameter.

Thus in section 2 we present the model and its assumptions. Then in section 3 we analyse the exact and approximate moments of the OLS estimator. Finally, in section 4 we analyse the exact distribution of the IV estimator.

2. The Model

We begin with the assumption that the values of y_t are independent drawings from normal populations with constant variances but varying means.

(2.1) $$y_t \sim N(\mu_t, \sigma^2) \text{ for } t = 1,\ldots,T$$

Next, we assume that μ_t is determined by a linear function analogous to a regression equation

(2.2) $$\mu_t = \gamma\, \mu_{t-1} + X_t'\beta$$

where X_t' is a non-random 1xK vector, β is a Kx1 vector of unknown coefficients, γ is an unknown scalar coefficient, and μ_t and μ_{t-1} are unknown means of y_t and y_{t-1}. To ensure that the process described by (2.2) is stable we assume

(2.3) $$|\gamma| < 1.$$

Now y_t can be written as

(2.4) $$\begin{aligned} y_t &= \mu_t + \eta_t = \gamma\, \mu_{t-1} + X_t'\beta + \eta_t \\ &= \gamma(\mu_{t-1} + \eta_{t-1}) + X_t'\beta + \eta_t - \gamma\, \eta_{t-1} \\ &= \gamma\, y_{t-1} + X_t'\beta + \varepsilon_t, \quad t=2,\ldots,T, \end{aligned}$$

where $\varepsilon_t = \eta_t - \gamma\, \eta_{t-1}$ and η_t is an independent drawing from $N(0,\sigma^2)$. Therefore,

(2.5) $\quad E\, \varepsilon_t = 0$ and

(2.6) $$\text{var}(\varepsilon_t) = E(\eta_t - \gamma\, \eta_{t-1})^2 = (1 + \gamma^2)\sigma^2$$

To see whether ε_t is independent of $\varepsilon_s (s < t)$ consider

(2.7) $$E(\varepsilon_t, \varepsilon_s) = E(\eta_t - \gamma\, \eta_{t-1})(\eta_s - \gamma\, \eta_{s-1})$$

$$= \begin{cases} -\gamma\, \sigma^2 & \text{if } s = t - 1 \\ 0 & \text{if } s < t - 1 \end{cases}$$

Therefore, ε_t has first order autocorrelation only. Furthermore, the coefficient of autocorrelation is

(2.8) $$r(\varepsilon_t\, \varepsilon_{t-1}) = \frac{E(\varepsilon_t\, \varepsilon_{t-1})}{\sigma^2} = \frac{-\gamma}{1+\gamma^2}$$

We can also view ε_t as the result of an autoregessive process

(2.9) $$\varepsilon_t = \eta_t - \gamma\, \eta_{t-1} = -\gamma\, \varepsilon_{t-1} - \gamma^2\, \eta_{t-2} + \eta_t = -\sum_{j=1}^{t} \gamma^j\, \varepsilon_{t-j} + \eta_t.$$

The autoregressive process (2.9) may seem to be unduly arbitrary. However, it is solely the result of assumptions (2.1) and (2.2), and (2.2) can be obtained in at least two appealing ways. First, consider a Koyck (1954) type distributed lag model which has been specified in terms of μ_t (instead of in terms of the random y_t)

(2.10) $$\mu_t = \alpha_o + \alpha_1 z_t + \alpha_1 \lambda z_{t-1} + \alpha_1 \lambda^2 z_{t-2} + \dots\ .$$

This model says that the mean response, μ_t, is a function of the present value and all past values of an exogenous variable, z_t, where α_i and λ, $0 < \lambda < 1$, are unknown coefficients. Then, applying the Koyck transformation we have

$$(2.11) \qquad \mu_t = \lambda\, \mu_{t-1} + [1, z_t] \begin{bmatrix} (1-\lambda)\alpha_o \\ \alpha_1 \end{bmatrix}$$

which is of the form (2.2) with $\gamma = \lambda$, $X_t' = [1 \; z_t]$ and $\beta = \begin{bmatrix} (1-\lambda)\alpha_o \\ \alpha_1 \end{bmatrix}$

A second justification for (2.2) is an adaptive expectations type of model.

$$(2.12) \qquad \mu_t = \alpha_o + \alpha_1 \, p_t^*$$

$$(2.13) \qquad p_t^* - p_{t-1}^* = \delta(p_{t-1} - p_{t-1}^*) \qquad 0 < \delta \le 1$$

which says that the mean response, μ_t, depends on expectations about the future, p_t^*, and that these expectations are adjusted by some fraction of the extent to which past expectations were in error. By substitution and transformation, this model can be reduced to

$$(2.14) \qquad \mu_t = (1-\delta)\mu_{t-1} + [1 \; p_{t-1}] \begin{bmatrix} \alpha_o \delta \\ \alpha_1 \delta \end{bmatrix}$$

which is of the form of (2.2) with $\gamma = (1-\delta)$, $X_t' = [1 \; p_{t-1}]$ and $\beta = \begin{bmatrix} \alpha_o \delta \\ \alpha_1 \delta \end{bmatrix}$.

3. Least Squares Estimation

It is convenient, at this point, to rewrite equation (2.4) in matrix notation .

$$(3.1) \qquad y = \gamma\, y_{-1} + X\, \beta + \varepsilon$$

where y is an nxl vector of independent random variables $y_t \sim N(\mu_t \; \sigma^2)$, $n = T - 1$, y_{-1} is an nxl vector of independent random variables $y_{t-1} \sim N(\mu_{t-1}, \sigma^2)$, X is an nxK matrix with non-random rows X_t', and ε is an nxl vector of autoregressive disturbances. The OLS estimates of γ and β are obtained by solving the equations

(3.2) $c\, y_{-1}'y_{-1} + y_{-1}'X\, b = y_{-1}'y$

(3.3) $c\, X'y_{-1} + X'X\, b = X'y$

OLS will not be consistent in this application because

(3.4) $y_{-1}'\varepsilon = -\gamma\, \eta_{-1}'\eta_{-1} - \gamma\, \mu_{-1}'\, \eta_{-1}' + \mu_{-1}'\eta + \eta_{-1}'\, \eta,$

(where μ_{-1}, η and η_{-1} are nx1 vectors of elements μ_{t-1}, η_t and η_{t-1}) and we cannot reasonably expect to have plim $n^{-1}\eta_{-1}'\eta_{-1} = 0$. Nevertheless, we will consider the OLS estimator of γ, c, and present its exact moments.

Equation (3.3) can be solved for b which can then be substituted into (3.2) to obtain:

(3.5) $c = \dfrac{y_{-1}'\, M\, y}{y_{-1}'\, M\, y_{-1}} = \dfrac{z_{-1}'\, M\, z}{z_{-1}'\, M\, z_{-1}}$ where $z_t = \dfrac{y_t}{\sigma}$

and $M = I - X(X'X)^{-1}X'$, an idempotent matrix of rank $L = n - K$.

At this point we simplify the model (3.1) by letting[1] $X = 0$. Then, (3.1) becomes

(3.6) $y = \gamma\, y_{-1} + \varepsilon$

and the OLS estimator of γ can then be written as

(3.7) $c = \dfrac{y'\, y_{-1}}{y_{-1}'\, y_{-1}} = \dfrac{z'\, z_{-1}}{z'_{-1}\, z_{-1}}$

where $z = \frac{1}{\sigma}y$ and $z_{-1} = \frac{1}{\sigma}y_{-1}$ and both vectors have T - 1 elements. Also, using (2.1), we have

$$z_t \sim N(\mu_t/\sigma,\ 1) \quad , \quad t=1,\ldots,T,$$

so that $z_{-1}'z_{-1}$ has a noncentral chi-square distribution with T - 1 degrees of freedom and a noncentrality parameter

(3.8) $$\theta = \frac{\mu'_{-1}\mu_{-1}}{2\sigma^2}$$

which is of order T in magnitude.

We can now write (3.7) as

(3.9) $$c = \frac{\sum_2^T z_t z_{t-1}}{W} = \sum_2^{T-1} \left(\frac{z_t z_{t-1}}{W}\right) + \frac{z_T z_{T-1}}{W}$$

where

(3.10) $$W = \sum_2^T z_{t-1}^2 .$$

Then, taking expectations on both sides and using the results (C.10) and (C.12) (for $\bar{z}_t = \frac{\mu_t}{\sigma}$ and r=1) given in Appendix C we obtain

(3.11) $$E\ c = \sum_2^{T-1} E\left(\frac{z_t z_{t-1}}{W}\right) + \frac{\mu_T}{\sigma} E\left(\frac{z_{T-1}}{W}\right)$$

$$= \frac{1}{2}\left[\frac{\mu'\mu_{-1}}{\sigma^2} f_{1,2} + \frac{\mu_T \mu_{T-1}}{\sigma^2}(f_{o,1} - f_{1,2})\right]$$

where

(3.12) $$f_{\delta,\nu} = \frac{\Gamma(n/2 + \delta)}{\Gamma(n/2 + \nu)} e^{-\theta} \ {}_1F_1(n/2 + \delta,\ n/2 + \nu; \theta),$$

${}_1F_1$ ( ) represents the confluent hypergeometric function (see Appendix A) and n = T - 1.

If we use (2.2) (with $X'_t = 0$) and the recurrence relations (A.3) and (A.4) in Appendix A it can be verified that

(3.13) $$E(c-\gamma) = \frac{\gamma}{2}\left(\frac{\mu_{T-1}^2}{\sigma^2} f_{o,2} - n\ f_{o,1}\right).$$

For given n, we note that[2]

(3.14) $$\lim_{\theta\to\infty} E(c-\gamma) = 0$$

and

(3.15) $$\lim_{\theta\to 0} E(c-\gamma) = -\gamma$$

Thus, we conclude that the relative bias of the OLS estimator lies between 0 and -1 for given n, i.e.,

(3.16) $$-1 \le E(\frac{c-\gamma}{\gamma}) \le 0.$$

Using the asymptotic expansion of the confluent hypergeometric functions involved in (3.13), for given n and large[3] θ, we obtain

(3.17) $$E(c-\gamma) = -\gamma\left[\frac{n}{2\theta} + \frac{n(2-n)}{4\theta^2} - \frac{\mu_{T-1}^2}{2\sigma^2}\frac{1}{\theta^2}\right] + o(1/\theta^2)$$

where $o(1/\theta^2)$ means terms of smaller order than $1/\theta^2$.

We can conclude from (3.17) that the absolute value of the relative bias, up to order $1/\theta$, is a decreasing function of the noncentrality parameter θ.

Now, using the asymptotic expansion of the confluent hypergeometric functions for large n and large[4] θ we can write (3.13) as

(3.18) $$E(c-\gamma) = -\frac{\gamma n}{n+2\theta}[1 - \frac{8}{n}(\frac{\theta}{n+2\theta})^2] + o(1/n)$$

where $o(1/n)$ represents terms of smaller order than $1/n$. It follows from (3.18) that

(3.19) $$\lim_{n\to\infty} E(c-\gamma) = -\frac{\gamma}{1+2q}, \text{ where } q = \lim_{n\to\infty}\frac{\theta}{n},$$

which is the well known asymptotic bias of the OLS estimator.

To obtain the second moment of c we write the square of (3.9) as

$$(3.20)\qquad c^2 = \frac{\left(\sum_2^{T-1} z_t z_{t-1}\right)^2}{W^2} + \frac{z_T^2 z_{T-1}^2}{W^2} + 2\, z_T z_{T-1} \sum_2^{T-1} \frac{z_t z_{t-1}}{W^2} .$$

Then

$$(3.21)\qquad Ec^2 = E\left[\frac{\left(\sum_2^{T-1} z_t z_{t-1}\right)^2}{W^2}\right] + Ez_T^2\, E\left(\frac{z_{T-1}^2}{W^2}\right) + 2Ez_T\left[\sum_2^{T-2} E\left(\frac{z_{T-1} z_t z_{t-1}}{W^2}\right) + E\left(\frac{z_{T-1}^2 z_{T-2}}{W^2}\right)\right].$$

Now the first term on the right side of (3.21) is:

$$(3.22)\qquad E\left[\frac{\left(\sum_2^{T-1} z_t z_{t-1}\right)^2}{W^2}\right] = \sum_2^{T-1} E\left(\frac{z_t^2 z_{t-1}^2}{W^2}\right) + 2\sum_2^{T-2} E\left(\frac{z_t^2 z_{t-1} z_{t+1}}{W^2}\right)$$

$$+\, 2\sum_{j=2}^{T-3}\sum_{t=2}^{T-1-j} E\left(\frac{z_t z_{t-1} z_{t+j} z_{t-1+j}}{W^2}\right)$$

$$= \frac{1}{4}\left[\left(\sum_2^{T-1} \bar{z}_t \bar{z}_{t-1}\right)^2 f_{2,4} + (T-2) f_{o,2} + \left\{\sum_2^{T-1} (\bar{z}_t^2 + \bar{z}_{t-1}^2)\right.\right.$$

$$\left.\left. +\, 2\sum_2^{T-2} \bar{z}_{t-1}\bar{z}_{t+1}\right\} f_{1,3}\right]$$

where $\bar{z}_t = \frac{\mu_t}{\sigma}$ and (C.14), (C.18) and (C.20) have been used with r=2. The second term on the right side of (3.21) is, using (C.11),

$$(3.23)\qquad E\, z_T^2\, E\left(\frac{z_{T-1}^2}{W^2}\right) = \frac{1}{4}\,(1 + \bar{z}_T^2)(\bar{z}_{T-1}^2 f_{o,2} + f_{-1,1})$$

and the third term is

$$(3.24)\qquad 2Ez_T\left[\sum_2^{T-2} E\left(\frac{z_{T-1} z_t z_{t-1}}{W^2}\right) + E\left(\frac{z_{T-1}^2 z_{T-2}}{W^2}\right)\right] = \frac{1}{2}\left[\left\{\bar{z}_{T-1}\bar{z}_T \sum_2^{T-1} \bar{z}_t \bar{z}_{t-1}\right\} f_{1,3} + \bar{z}_T \bar{z}_{T-2} f_{o,2}\right]$$

using (C.14) and (C.15). Then (3.21) can be written as the sum of (3.22), (3.23) and (3.24) which is

$$\text{(3.25)}\qquad Ec^2 = \frac{1}{4}\gamma^2(2\theta - \frac{\mu_{T-1}^2}{\sigma^2})^2 f_{2,4} + \{n-1 + \frac{\mu_{T-1}^2}{\sigma^2}(3 + \gamma^2\frac{\mu_{T-1}^2}{\sigma^2})\}f_{o,2}$$
$$+ (1+\gamma^2\frac{\mu_{T-1}^2}{\sigma^2})f_{-1,1} + \{(1 + 3\gamma^2 + 2\gamma^2\frac{\mu_{T-1}^2}{\sigma^2})(2\theta - \frac{\mu_{T-1}^2}{\sigma^2}) - 2\frac{\mu_{T-1}^2}{\sigma^2}\}f_{1,3}\Big] .$$

Using the asymptotic expansion, for large θ given n, of the confluent hypergeometric functions in (3.25) we obtain the mean squared error of c, to order $\frac{1}{\theta^2}$, as

$$\text{(3.26)}\qquad E(c - \gamma)^2 = \frac{1-\gamma^2}{2\theta} + \frac{\gamma^2}{4\theta^2}[n(n+4) + 6\bar{z}_{T-1}^2]$$

Since $\frac{dE(c-\gamma)^2}{d\theta} < 0$ because of (2.3), $E(c-\gamma)^2$ is a monotonically decreasing function of θ.

Now holding n constant, using (3.17) and (3.26),

$$\text{(3.27)}\qquad \lim_{\theta\to\infty} Ec = \gamma$$

$$\text{(3.28)}\qquad \lim_{\theta\to\infty} Ec^2 = \gamma^2$$

so that c converges to γ as θ grows large, even though c is not consistent. That is, as $\mu_{-1}'\mu_{-1} \to \infty$ or $\sigma^2 \to 0$, $c \to \gamma$.

The first two moments of c in the general case, when $M \neq I$, are given in Appendix D. A conclusion of that Appendix is that (3.27) and (3.28) hold so that, in general, $c \to \gamma$ as $\theta \to \infty$.

Our findings about the distribution of c allow us to derive the moments of b. Using (3.3) we have

$$\text{(3.29)}\qquad b = (X'X)^{-1}X'y - c(X'X)^{-1}X'y_{-1}.$$

Since $(X'X)^{-1}X' M = 0$, $(X'X)^{-1}X'y_{-1}$ is independent of both the numerator and denominator of c as given in (3.5). Therefore, c and $(X'X)^{-1}X'y_{-1}$ are independent and

(3.30) $$Eb = (X'X)^{-1}X'(\mu - Ec\,\mu_{-1}).$$

Of course, b is biased because c is biased.

The sampling error of b is given by

(3.31) $$\begin{aligned} b - \beta &= (X'X)^{-1}X'(y - cy_{-1} - y + \gamma y_{-1} + \varepsilon) \\ &= (X'X)^{-1}X'(\varepsilon - (c - \gamma)y_{-1}). \end{aligned}$$

Now

(3.32) $$E\varepsilon\varepsilon' = \sigma^2 \begin{bmatrix} 1+\gamma^2 & -\gamma & 0 \cdots 0 \\ -\gamma & 1+\gamma^2 & \vdots \\ 0 & \ddots & -\gamma \\ 0 \cdots 0 & -\gamma & 1+\gamma^2 \end{bmatrix} = \sigma^2 V,$$

(3.33) $$E\varepsilon y_{-1}' = \sigma^2(J - \gamma I)$$

where $J = \begin{bmatrix} 0 & 1 & 0 \cdots & 0 \\ \cdot & 0 & 1 & \vdots \\ \vdots & \vdots & 0 & 1 \\ 0 & 0 & 0 \cdots & 0 \end{bmatrix}$ and

(3.34) $$E\, y_{-1}y_{-1}' = \mu_{-1}\mu_{-1}' + \sigma^2 I.$$

Therefore, the mean squared error of b is given by

(3.35) $$\begin{aligned} MSE(b) = E(b-\beta)(b-\beta)' &= \sigma^2 AVA' - \sigma^2 AUA'\, E(c-\gamma) \\ &+ [A\mu_{-1}\mu_{-1}'A' + \sigma^2(X'X)^{-1}]\, E(c-\gamma)^2 \end{aligned}$$

where $A = (X'X)^{-1}X'$ and

$$U = \begin{bmatrix} -2\gamma & 1 & 0 & \cdots & 0 \\ 1 & -2\gamma & 1 & \ddots & \vdots \\ 0 & 1 & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 1 & -2\gamma \end{bmatrix}.$$

When $\theta \to \infty$, with n constant, $c \to \gamma$ and the distribution of $b-\beta$, in (3.31), approach that of $A\varepsilon$ so that

(3.36) $$(b-\beta) \to (X'X)^{-1}X'\varepsilon \sim N[0,\ \sigma^2(X'X)^{-1}X'VX(X'X)^{-1}].$$

That is, for large values of θ b is approximately normal and unbiased.

4. Consistent Estimation of γ

Liviatan (1963) has proposed two consistent estimators for models like (2.4). The simplest of them uses a lagged exogenous variable,[5] w, as an instrument to produce the normal equations

(4.1) $$\hat{\gamma}\, w'y_{-1} + w'X\hat{\beta} = w'y$$

(4.2) $$\hat{\gamma}\, X'y_{-1} + X'X\hat{\beta} = X'y$$

from which

(4.3) $$\hat{\gamma} = \frac{w'My}{w'My_{-1}} = \frac{u}{u_1}$$

where $u \sim N(w'M\mu, \sigma^2 w'Mw) = N(\bar{u}, \omega^2)$, $u_1 \sim N(w'M\mu_{-1}, \sigma^2 w'Mw) = N(\bar{u}_1, \omega^2)$ and $E(u-\bar{u})(u_1-\bar{u}_1)' = \sigma^2 w'MJMw = \rho\,\omega^2$ where ρ is coefficient of correlation between u and u_1. Under (2.1), this ratio has been shown (Carter (1976)) to have a distribution of the type described by Fieller (1932) which has no moments of any order. However, if $\frac{\omega}{\bar{u}_1} < 1/3$ the distribution of $\hat{\gamma}$ can be approximated by (Geary (1930), Fieller (1932)).

(4.4) $$f(\hat{\gamma}) \doteq \frac{-\omega^2[(\bar{u}\rho - \bar{u}_1) + (\bar{u}_1\rho - \bar{u})\hat{\gamma}]}{\sqrt{2\pi[\omega^2(\hat{\gamma}^2 - 2\hat{\gamma}\rho + 1)]^3}} \exp\left\{\frac{-(\bar{u} - \bar{u}_1\hat{\gamma})^2}{2\omega^2(\hat{\gamma}^2 - 2\hat{\gamma}\rho + 1)}\right\}$$

We can use the results given by Merrill (1928) to approximate the mean and variance of (4.4) as

$$(4.5) \qquad E(\hat{\gamma}) \doteq \frac{\bar{u}}{\bar{u}_1}\left[1 + \left\{\left(\frac{\omega}{\bar{u}_1}\right)^2 - \frac{\rho\omega^2}{\bar{u}\,\bar{u}_1}\right\}\left\{1 + 3\left(\frac{\omega}{\bar{u}_1}\right)^2 + 15\left(\frac{\omega}{\bar{u}_1}\right)^4 + 105\left(\frac{\omega}{\bar{u}_1}\right)^6\right\}\right]$$

and

$$(4.6) \qquad \sigma^2(\hat{\gamma}) \doteq \frac{\bar{u}^2}{\bar{u}_1^2}\,\delta\left[1 + \left(\frac{\omega}{\bar{u}_1}\right)(5\lambda + 3) + \left(\frac{\omega}{\bar{u}_1}\right)^4(54\lambda + 15) + \left(\frac{\omega}{\bar{u}_1}\right)^6(591\lambda + 105)\right]$$

where $\lambda = \dfrac{(\bar{u} - \rho\,\bar{u}_1)^2}{\bar{u}^2 - 2\rho\,\bar{u}\,\bar{u}_1 + \bar{u}_1^2}$ and $\delta = \dfrac{\omega^2}{\bar{u}^2\bar{u}_1^2}(\bar{u}^2 - 2\rho\,\bar{u}\,\bar{u}_1 + \bar{u}_1^2)$. The shape of the distribution is given by the skewness coefficient

$$(4.7) \qquad \sqrt{\beta_1} \doteq \lambda\left[\frac{\omega}{\bar{u}_1}\right]^{\frac{1}{2}}\left[36 - \left[\frac{\omega}{\bar{u}_1}\right]^2(12\lambda - 540) + 4\left[\frac{\omega}{\bar{u}_1}\right]^4(-146\lambda^2 + 450\lambda + 1917)\right]^{\frac{1}{2}}$$

and the kurtosis coefficient

$$(4.8) \qquad \beta_2 = 3 + 3\left[\frac{\omega}{\bar{u}_1}\right]^2(20\lambda + 4) + \left[\frac{\omega}{\bar{u}_1}\right]^4(-14\lambda^2 + 404\lambda + 42).$$

The matrix form of (2.2) is

$$(4.9) \qquad \mu = \gamma\,\mu_{-1} + X\beta$$

from which we find, using (4.3), that

$$(4.10) \qquad \gamma = \frac{w'M\mu}{w'M\mu_{-1}} = \frac{\bar{u}}{\bar{u}_1}$$

which is approximately equal to $E(\hat{\gamma})$ when the approximation (4.4) is justified; that is, γ is nearly unbiased when $\frac{\omega}{\bar{u}_1} < \frac{1}{3}$. Further, since $\bar{u}$, $\bar{u}_1$, and ω^2 are $O(n)$, $\frac{\bar{u}}{\bar{u}_1}$ is $O(1)$

and $\frac{\omega^2}{\bar{u}_1^2}$ is $O(n^{-1})$ so that the bias of $\hat{\gamma}$ becomes even smaller as n grows. Also δ is $O(n^{-1})$, so that the $\sigma^2(\hat{\gamma})$ shrinks as n grows. These same effects could be achieved for fixed, small, n if $\frac{\omega}{\bar{u}_1} \to 0$; which would also serve to make approximation (4.4) better. To obtain this result it would be sufficient to have $\sigma^2 \to 0$. This is analogous to the result (3.27) and (3.28) on the OLS estimator. Furthermore, as $\frac{\omega}{\bar{u}_1} \to 0$ we see from (4.7) and (4.8) that the distribution of $\hat{\gamma}$ approaches normality.

To analyze the moments of $\hat{\beta}$ we use (4.2) to obtain

(4.11) $$\hat{\beta} = (X'X)^{-1}X'(y - \hat{\gamma}y_{-1}) = Ay - \hat{\gamma}Ay_{-1}; \quad A = (X'X)^{-1}X'$$

Because $AM=0$, Ay_{-1} is independent of both the numerator and denominator of (4.3) so it is, therefore, independent of $\hat{\gamma}$. Then in the special case (4.4) we find

(4.12) $$E\hat{\beta} \doteq A\mu - E\hat{\gamma}A\mu_{-1} \doteq A(\mu - \gamma\mu_{-1}) = \beta$$

so that $\hat{\beta}$ is, approximately, unbiased.

The sampling error of $\hat{\beta}$ is

(4.13) $$\hat{\beta} - \beta = A[\epsilon - (\hat{\gamma} - \gamma)y_{-1}]$$

so that

(4.14) $$MSE(\hat{\beta}) \doteq \sigma^2 AVA' + \sigma^2 AUA' E(\hat{\gamma} - \gamma) + [A\mu_{-1}\mu_{-1}' A' + \sigma^2(X'X)^{-1}]E(\hat{\gamma} - \gamma)^2$$

$$\doteq \sigma^2 AVA' + [A\mu_{-1}\mu_{-1}' A' + \sigma^2(X'X)^{-1}]\sigma^2(\hat{\gamma})$$

since $\hat{\gamma}$ is approximately unbiased.

If $\frac{\omega}{\bar{u}_1} \to 0$, $\hat{\gamma} \to \gamma$ and the distribution of $\hat{\beta} - \beta$ approaches that of $A\epsilon$ which is the same as that of the OLS estimator b given in (3.36). We note that the conditions for $\frac{\omega}{\bar{u}_1} \to 0$ are the same as the conditions for $\theta \to \infty$ in the OLS case.

5. Conclusion

The coefficients of a regression equation containing the lagged dependent variables on the right side can be estimated by least squares or by instrumental variables. This paper presents some exact properties of these estimators in the case that the disturbance follows a certain autoregressive scheme.

The results in (3.27), and (3.28) indicate that the OLS estimator of the coefficient of the lagged dependent variable possesses desirable properties as θ, a noncentrality parameter of the distribution, grows large. Also the OLS estimator of the coefficient of the exogenous variables approaches normality and unbiasedness as θ grows. Equations (4.5) to (4.11) show that all the IV coefficient estimators become normal and unbiased as one of the parameters, $\frac{\omega}{\bar{u}_1}$, of its distribution approaches zero. It is noteworthy that the conditions under which $\frac{\omega}{\bar{u}_1}$ approach zero are the same as the conditions under which θ grows large and that as this happens the two estimators of the coefficients of the exogenous variables take on the same distribution.

APPENDIX

A. Confluent Hypergeometric Functions

A confluent hypergeometric function is defined as

(A.1) $$ {}_1F_1(a;\ c;\ x) = \frac{\Gamma(c)}{\Gamma(a)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{\Gamma(c+n)} \frac{x^n}{n!} $$

and it is an "absolutely convergent" series for all values of a, c and x excluding c = 0, -1, -2,... [See Slater (1960, p. 2)].

Since ${}_1F_1(a;\ c;\ x)$ is absolutely convergent we can differentiate the right hand side of (A.1) term by term. Thus

(A.2) $$ \frac{d^s}{dx^s} [e^{-x}\ {}_1F_1(a;\ c;\ x)] = $$

$$ = (-1)^s \frac{\Gamma(c-a+s)}{\Gamma(c-a)} \frac{\Gamma(c)}{\Gamma(c+s)} e^{-x}\ {}_1F_1(a;\ c+s;\ x) $$

for s = 1,2,3,... . [Cf. Slater (1960, p. 15)].

The following recurrence relations should be noted:

(A.3) $$ c[{}_1F_1(a;\ c;\ x) - {}_1F_1(a-1;\ c;\ x)] = x\ {}_1F_1(a;\ c+1;\ x), $$

(A.4) $$ (1+a-c)\ {}_1F_1(a;\ c;\ x) + (c-1)\ {}_1F_1(a;\ c-1;\ x) $$
$$ = a\ {}_1F_1(a+1;\ c;\ x), $$

(A.5) $$ (1+a-c)(a-c)\ {}_1F_1(a;\ c+1;\ x) + 2(1+a-c)c\ {}_1F_1(a;\ c;\ x) $$
$$ + c(c-1)\ {}_1F_1(a;\ c-1;\ x) = a(a+1)\ {}_1F_1(a+2;\ c+1;\ x), $$

(A.6) $$(1 + a - c)(a - c)(a - c - 1)\,{}_1F_1(a;\ c+2;\ x) +$$
$$+ 3(1 + a - c)(a - c)(c + 1)\,{}_1F_1(a;\ c+1;\ x) +$$
$$+ 3(1 + a - c)(c + 1)c\,{}_1F_1(a;\ c;\ x) +$$
$$+ (c + 1)\,c(c - 1)\,{}_1F_1(a;\ c-1;\ x)$$
$$= a(a + 1)(a + 2)\,{}_1F_1(a+3;\ c+2;\ x),$$

(A.7) $$(1 + a - c)(a - c)(a - c - 1)(a - c - 2)\,{}_1F_1(a;\ c+3;\ x)$$
$$+ 4(1 + a - c)(a - c)(a - c - 1)(c+2)\,{}_1F_1(a;\ c+2;\ x)$$
$$+ 6(1 + a - c)(a - c)(c + 2)(c+1)\,{}_1F_1(a;\ c+1;\ x)$$
$$+ 4(1 + a - c)(c + 2)(c + 1)c\,{}_1F_1(a;\ c;\ x)$$
$$+ (c + 2)(c + 1)\,c\,(c - 1)\,{}_1F_1(a;\ c-1;\ x)$$
$$= a(a + 1)(a + 2)(a + 3)\,{}_1F_1(a+4;\ c+3;\ x).$$

The relations (A.3) and (A.4) are given in Slater (1960), p. 19--equations (2.2.4) and (2.2.3), respectively. The relations (A.5) - (A.7) can be verified by substituting relevant confluent hypergeometric series and equating coefficients of $x^n/n!$ on both sides of the respective relations.

B. Expectation of the Noncentral Chi-Square and its Partial Derivatives

Let $z_1,\ldots,z_n$ be independent normal variates with

(B.1) $$E\ z_i = \bar{z}_i \text{ and } \operatorname{Var} z_i = 1 \qquad i=1,\ldots,n.$$

Then we know that the distribution of

(B.2) $$W = \sum_{i=1}^{n} z_i^2$$

is a 'noncentral chi-square' with n degrees of freedom and the parameter of noncentrality as

$$(B.3) \qquad \theta = \tfrac{1}{2} \sum_{i=1}^{n} \bar{z}_i^2 .$$

The density function of W is given by

$$(B.4) \qquad f(W) = e^{-\theta} \sum_{m=o}^{\infty} \frac{\theta^m}{m!} \frac{W^{\frac{1}{2}(n+2m)-1} e^{-\frac{1}{2}W}}{2^{\frac{1}{2}(n+2m)} \Gamma[(n+2m)/2]} , \quad 0 \le W < \infty .$$

Therefore, if $n/2 > r$

$$(B.5) \qquad E\, W^{-r} = \int_o^{\infty} W^{-r} f(W)\, dW$$

$$= 2^{-r} \frac{\Gamma(n/2 - r)}{\Gamma(n/2)} e^{-\theta} {}_1F_1(n/2 - r;\ n/2;\ \theta)$$

for $r=1,2,\ldots$,[see Ullah (1974, p. 147)]. Then the partial derivatives of $E\,W^{-r}$ in (B.5) with respect to $\bar{z}_i$ can be written as[6]

$$(B.6) \qquad \frac{\partial}{\partial \bar{z}_i} E\, W^{-r} = 2^{-r}\, r\, \bar{z}_i\, f_{-r,1}$$

where

$$(B.7) \qquad f_{\delta,\nu} = \frac{\Gamma(n/2 + \delta)}{\Gamma(n/2 + \nu)} e^{-\theta} {}_1F_1(n/2 + \delta;\ n/2 + \nu;\ \theta)$$

and writing $\delta = -r$, $\nu = 1$, we get $f_{-r,1}$. Further, we have

$$(B.8) \qquad \frac{\partial^2}{\partial \bar{z}_i^2} E\, W^{-r} = -2^{-r}[(r+1)\bar{z}_i^2 f_{-r,2} - f_{-r,1}]$$

$$(B.9) \qquad \frac{\partial^2}{\partial \bar{z}_i \partial \bar{z}_j} E\, W^{-r} = 2^{-r}\, r(r+1)\bar{z}_i \bar{z}_j f_{-r,2}$$

(B.10) $$\frac{\partial^3}{\partial\bar{z}_i^2\,\partial\bar{z}_j} E\,W^{-r} = 2^{-r}\, r(r+1)\bar{z}_j\,[f_{-r,2} - (r+2)\bar{z}_i^2\, f_{-r,3}]$$

(B.11) $$\frac{\partial^3}{\partial\bar{z}_i\,\partial\bar{z}_j\,\partial\bar{z}_k} E\,W^{-r} = -2^{-r}\, r(r+1)(r+2)\,\bar{z}_i\,\bar{z}_j\,\bar{z}_k\, f_{-r,3}$$

(B.12) $$\frac{\partial^4}{\partial\bar{z}_i^3\partial\bar{z}_j} E\,W^{-r} = 2^{-r}\, r(r+1)(r+2)\bar{z}_i\bar{z}_j[(r+3)\bar{z}_i^2\, f_{-r,4} - 3\, f_{-r,3}]$$

(B.13) $$\frac{\partial^4}{\partial\bar{z}_i^2\,\partial\bar{z}_j^2} E\,W^{-r} = 2^{-r}\, r(r+1)[(r+2)(r+3)\bar{z}_i^2\bar{z}_j^2\, f_{-r,4} - (r+2)(\bar{z}_i^2+\bar{z}_j^2)f_{-r,3} + f_{-r,2}]$$

(B.14) $$\frac{\partial^4}{\partial\bar{z}_i^2\,\partial\bar{z}_j\partial\bar{z}_k} E\,W^{-r} = 2^{-r}\, r(r+1)(r+2)\,\bar{z}_j\bar{z}_k[(r+3)\bar{z}_i^2\, f_{-r,4} - f_{-r,3}]$$

(B.15) $$\frac{\partial^4}{\partial\bar{z}_i\,\partial\bar{z}_j\,\partial\bar{z}_k\,\partial\bar{z}_\ell} E\,W^{-r} = 2^{-r}\, r(r+1)(r+2)(r+3)\bar{z}_i\bar{z}_{i-1}\bar{z}_{i+1}\bar{z}_{i+2}\, f_{-r,4}$$

To use these results in the text we let $j=i-1$, $k=i+1$ and $\ell=i+2$ such that $i\neq j\neq k\neq \ell$.

C. Evaluation of Certain Mathematical Expectations

First, we note that[7]

(C.1) $$E(z_i - \bar{z}_i)W^{-r} = (2\pi)^{-n/2}\int_{z_1}\cdots\int_{z_n}(z_i - \bar{z}_i)\,W^{-r}\exp[-\tfrac{1}{2}\sum_1^n (z_i-\bar{z}_i)^2]\,dz_1,\ldots,dz_n$$

$$= \frac{\partial}{\partial\bar{z}_i} E\,W^{-r}$$

where we require $E\,W^{-r}$ and its partial derivative with respect to $\bar{z}_i$ which is given in (B.6).

Similarly, we can obtain the following results in a straightforward manner

(C.2) $$E(z_i - \bar{z}_i)^2 W^{-r} = \left(\frac{\partial^2}{\partial \bar{z}_i^{\,2}} + 1\right) E W^{-r} ,$$

(C.3) $$E(z_i - \bar{z}_i)(z_j - \bar{z}_j) W^{-r} = \frac{\partial^2}{\partial \bar{z}_i \, \partial \bar{z}_j} E W^{-r}$$

(C.4) $$\underset{i \neq j}{E} (z_i - \bar{z}_i)^2 (z_j - \bar{z}_j) W^{-r} = \left(\frac{\partial^3}{\partial \bar{z}_i^2 \, \partial \bar{z}_j} + \frac{\partial}{\partial \bar{z}_j}\right) E W^{-r}$$

(C.5) $$E(z_i - \bar{z}_i)(z_j - \bar{z}_j)(z_k - \bar{z}_k) W^{-r} = \frac{\partial^3}{\partial \bar{z}_i \, \partial \bar{z}_j \, \partial \bar{z}_k} E W^{-r}$$

(C.6) $$\underset{i \neq j}{E} (z_i - \bar{z}_i)^3 (z_j - \bar{z}_j) W^{-r} = \left(\frac{\partial^4}{\partial \bar{z}_i^3 \, \partial \bar{z}_j} + 3 \frac{\partial^2}{\partial \bar{z}_i \, \partial \bar{z}_j}\right) E W^{-r}$$

(C.7) $$\underset{i \neq j}{E} (z_i - \bar{z}_i)^2 (z_j - \bar{z}_j)^2 W^{-r} = \left(\frac{\partial^4}{\partial \bar{z}_i^2 \, \partial \bar{z}_j^2} + \frac{\partial^2}{\partial \bar{z}_i^2} + \frac{\partial^2}{\partial \bar{z}_j^2} + 1\right) E W^{-r}$$

(C.8) $$\underset{i \neq j \neq k}{E} (z_i - \bar{z}_i)^2 (z_j - \bar{z}_j)(z_k - \bar{z}_k) W^{-r} = \left(\frac{\partial^4}{\partial \bar{z}_i^2 \, \partial \bar{z}_j \, \partial \bar{z}_k} + \frac{\partial^2}{\partial \bar{z}_j \, \partial \bar{z}_k}\right) E W^{-r}$$

and

(C.9) $$\underset{i \neq j \neq k \neq \ell}{E} (z_i - \bar{z}_i)(z_j - \bar{z}_j)(z_k - \bar{z}_k)(z_\ell - \bar{z}_\ell) W^{-r} = \frac{\partial^4}{\partial \bar{z}_i \, \partial \bar{z}_j \, \partial \bar{z}_k \, \partial z_\ell} E W^{-r}$$

The expectations required in Section 3 may now be stated as follows.

(C.10) $$E(z_i W^{-r}) = 2^{-r} \bar{z}_i f_{-r+1,1}$$

Next

$$(C.11)\qquad E(z_i^2\ W^{-r}) = E(z_i - \bar{z}_i)^2\ W^{-r} + 2\ \bar{z}_i\ E(z_i - \bar{z}_i)W^{-r} + \bar{z}_i^2\ E\,W^{-r}$$

$$= \left(\frac{\partial^2}{\partial \bar{z}_i^2} + 1\right) E\,W^{-r} + 2\ \bar{z}_i\ \frac{\partial}{\partial \bar{z}_i} E\,W^{-r} + \bar{z}_i^2\ E\,W^{-r}$$

$$= 2^{-r}\ [\bar{z}_i^2\ f_{-r+2,2} + f_{-r+1,1}];$$

similarly, for $i \neq j$,

$$(C.12)\qquad E(z_i\ z_j\ W^{-r}) = 2^{-r}\ \bar{z}_i\ \bar{z}_j\ \ f_{-r+2;2}\ .$$

Further we have

$$(C.13)\qquad E(z_i^3\ W^{-r}) = 2^{-r}\ \bar{z}_i^3\ \ f_{-r+3,3} + 3 \times 2^{-r}\ \bar{z}_i\ \ f_{-r+2,2}\ ,$$

$$(C.14)\qquad \underset{i \neq j}{E}\,(z_i^2\ z_j\ W^{-r}) = 2^{-r}\ \bar{z}_i^2\ \bar{z}_j\ \ f_{-r+3,3} + 2^{-r}\ \bar{z}_j\ \ f_{-r+2,2}$$

$$(C.15)\qquad \underset{i \neq j \neq k}{E}\,(z_i\ z_j\ z_k\ W^{-r}) = 2^{-r}\ \bar{z}_i\ \bar{z}_j\ \bar{z}_k\ \ f_{-r+3,3}$$

$$(C.16)\qquad E(z_i^4\ W^{-r}) = 2^{-r}\ \bar{z}_i^4\ f_{-r+4,4} + 6 \times 2^{-r}\ \bar{z}_i^2\ \ f_{-r+3,3} + 3 \times 2^{-r}\ \ f_{-r+2,2}$$

$$(C.17)\qquad \underset{i \neq j}{E}\,(z_i^3\ z_j\ W^{-r}) = 2^{-r}\ \bar{z}_i^3\ \bar{z}_j\ \ f_{-r+4,4} + 3 \times 2^{-r}\ \bar{z}_i\ \bar{z}_j\ \ f_{-r+3,3}$$

$$(C.18)\qquad \underset{i \neq j}{E}\,(z_i^2\ z_j^2\ W^{-r}) = 2^{-r}\ \bar{z}_i^2\ \bar{z}_j^2\ \ f_{-r+4,4} + 2^{-r}(\bar{z}_i^2 + \bar{z}_j^2)\ f_{-r+3,3} + 2^{-r}\ f_{-r+2,2}$$

$$(C.19)\qquad \underset{i \neq j \neq k}{E}\,(z_i^2\ z_j\ z_k\ W^{-r}) = 2^{-r}\ \bar{z}_i^2\ \bar{z}_j\ \bar{z}_k\ f_{-r+4,4} + 2^{-r}\ \bar{z}_j\ \bar{z}_k\ f_{-r+3,3}$$

$$(C.20)\qquad \underset{i \neq j \neq k \neq \ell}{E}\,(z_i\ z_j\ z_k\ z_\ell\ W^{-r}) = 2^{-r}\ \bar{z}_i\ \bar{z}_j\ \bar{z}_k\ \bar{z}_\ell\ \ f_{-r+4,4}$$

D. The Evaluation of Useful Sums and Their Expectations

In deriving the first two moments of c we will find it necessary to expand to the numerator of (3.5) and its square. To begin we write

$$\text{(D.1)} \qquad z_{-1}' \, M \, z = [z_1, z_2, \ldots, z_n][m_{ij}] \begin{bmatrix} z_2 \\ z_3 \\ \vdots \\ z_n \\ z_{n+1} \end{bmatrix}$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} m_{ij} \, z_i z_{j+1} ,$$

where m_{ij} is the ij^{th} element of M, and

$$\text{(D.2)} \qquad z_{-1}' \, M \, z_{-1} = \sum_{i=1}^{n} \sum_{j=1}^{n} m_{ij} \, z_i z_j = W$$

Since z_{n+1} does not appear in W, the denominator of (3.5), it is convenient to separate terms involving z_{n+1} out of the sum (D.1). It is also convenient to separate out terms involving z_i^2. These separations yield

$$\text{(D.3)} \qquad z_{-1}' \, M z = \sum_{\substack{i=1 \\ i \neq j+1}}^{n} \sum_{j=1}^{n} m_{ij} \, z_i \, z_{j+1} + \sum_{i=1}^{n} m_{in} \, z_i z_{n+1} + \sum_{i=2}^{n} m_{i,i-1} \, z_i^2$$

The numerator of the square c is

$$\text{(D.4)} \qquad (z_{-1}' \, M \, z)^2 = \left[\sum_{\substack{i=1 \\ i \neq j+1}}^{n} \sum_{j=1}^{n-1} m_{ij} \, z_i \, z_{j+1} \right]^2 + \left[z_{n+1} \sum_{i=1}^{n} m_{in} \, z_i \right]^2$$

$$+ \left[\sum_{i=2}^{n} m_{i,i-1} \, z_i^2 \right]^2 + 2 z_{n+1} \sum_{\substack{i=1 \\ i \neq j+1}}^{n} \sum_{j=1}^{n-1} m_{ij} \, z_i z_{j+1} \sum_{i=1}^{n} m_{in} \, z_i$$

$$+ 2 \sum_{\substack{i=1 \\ i \neq j+1}}^{n} \sum_{j=1}^{n-1} m_{ij} \, z_i z_{j+1} \sum_{i=2}^{n} m_{i,i-1} \, z_i^2$$

$$+ 2 \, z_{n+1} \sum_{i=1}^{n} m_{in} z_i \sum_{i=2}^{n} m_{i,i-1} \, z_i^2$$

The first term on the right of (D.4) can be written as

$$(D.5)\quad \Big[\sum_{\substack{i=1\\ i\neq j+1}}^{n} (\sum_{j=1}^{n-1} m_{ij}\, z_i z_{j+1})\Big]^2 = \sum_{\substack{i=1\\ i\neq j+1}}^{n} \Big(\sum_{j=1}^{n-1} m_{ij}\, z_i z_{j+1}\Big)^2$$

$$+\, 2 \sum_{i=1}^{n} \sum_{\substack{k=1\\ k<i,\ i\neq j+1,\ k\neq j+1}}^{n} \Big(\sum_{j=1}^{n-1} m_{ij}\, z_i z_{j+1}\Big)\Big(\sum_{j=1}^{n-1} m_{kj}\, z_k\, z_{j+1}\Big)$$

We now expand the first term on the right of (D.5) keeping in mind that the z values in this term have different subscripts. This expansion yields

$$(D.6)\quad \sum_{\substack{i=1\\ i\neq j+1}}^{n} (\sum_{j=1}^{n-1} m_{ij}\, z_i z_{j+1})^2 = \sum_{\substack{i=1\\ i\neq j+1}}^{n} \Big[\sum_{j=1}^{n-1} m_{ij}^2\, z_i^2 z_{j+1}^2 + 2 \sum_{j=1}^{n-1} \sum_{\substack{\ell=1\\ \ell<j,\ i\neq \ell+1}}^{n-1} m_{ij}\, m_{i\ell}\, z_i^2\, z_{j+1}\, z_{\ell+1}\Big]$$

The second term on the right of (D.5) becomes

$$(D.7)\quad 2 \sum_{i=1}^{n} \sum_{\substack{k=1\\ k<i,\ i\neq j+1,\ k\neq j+1}}^{n} (\sum_{j=1}^{n-1} m_{ij}\, z_i z_{j+1})(\sum_{j=1}^{n-1} m_{kj}\, z_k z_{j+1}) = 2 \sum_{i=1}^{n} \sum_{\substack{k=1\\ k<i,\ i\neq j+1,\ k\neq j+1}}^{n} \Big[\sum_{j=1}^{n-1} m_{ij} m_{kj}\, z_i z_k z_{j+1}^2$$

$$+ \sum_{j=1}^{n-1} \sum_{\substack{\ell=1\\ \ell\neq j,\ k\neq j+1}}^{n-1} m_{ij}\, m_{k\ell}\, z_i\, z_{j+1}\, z_k\, z_{\ell+1}\Big]$$

$$= 2 \sum_{i=1}^{n} \sum_{\substack{k=1\\ k<i,\ i\neq j+1,\ k\neq j+1}}^{n} \Big[\sum_{j=1}^{n-1} m_{ij} m_{kj}\, z_i z_k z_{j+1}^2$$

$$+ \sum_{j=1}^{n-1} \sum_{\substack{\ell=1\\ \ell\neq j,\ k\neq \ell+1,\ i=\ell+1}}^{n-1} m_{ij} m_{k,i-1}\, z_i^2 z_{j+1} z_k + \sum_{j=1}^{n-1} \sum_{\substack{\ell=1\\ \ell\neq j,\ k\neq \ell+1,\ i\neq \ell+1}}^{n-1} m_{ij} m_{k,\ell}\, z_i z_{j+1} z_k z_{\ell+1}\Big]$$

where the second equation above accounts for the difference between cases where $i = \ell+1$ and $i \neq \ell+1$.

We now expand the second term on the right of (D.4)

$$\left[z_{n+1} \sum_{i=1}^{n} m_{in} z_i\right]^2 = z_{n+1}^2 \left[\sum_{i=1}^{n} m_{in}^2 z_i^2 + 2 \sum_{i=1}^{n} \sum_{\substack{k=1 \\ k<i}}^{n} m_{in} m_{kn} z_i z_k\right] \tag{D.8}$$

Similarly, the third term in (D.4) is

$$\left[\sum_{i=2}^{n} m_{i,i-1} z_i^2\right]^2 = \sum_{i=2}^{n} m_{i,i-1}^2 z_i^4 + 2 \sum_{i=2}^{n} \sum_{\substack{k=2 \\ k<i}}^{n} m_{i,i-1} m_{k,k-1} z_i^2 z_k^2 \tag{D.9}$$

The fourth term in (D.4) is

$$2z_{n+1} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j+1\neq i}}^{n-1} m_{ij} z_i z_{j+1} \sum_{i=1}^{n} m_{in} z_i = 2z_{n+1}\left[\sum_{i=1}^{n} \sum_{\substack{j=1 \\ j+1\neq i}}^{n-1} m_{ij} m_{in} z_i^2 z_{j+1} + \sum_{i=1}^{n} \sum_{\ell=1}^{n} \sum_{j=1}^{n-1} m_{ij} m_{\ell n} z_i z_{j+1} z_\ell\right] \tag{D.10}$$

$$\ell\neq i,\ j+1\neq i,\ j+1\neq \ell$$

The second sum on the right of (D.10) can be expanded to reflect the difference between cases where $\ell=j+1$ and $\ell\neq j+1$. This expansion of (D.10) results in

$$2z_{n+1} \sum_{i=1}^{n} \Big(\sum_{\substack{j=1 \\ j+1\neq i}}^{n-1} m_{ij} z_i z_{j+1}\Big) \sum_{i=1}^{n} m_{in} z_i = 2z_{n+1}\left[\sum_{i=1}^{n} \sum_{\substack{j=1 \\ j+1\neq i}}^{n-1} m_{ij} m_{in} z_i^2 z_{j+1} \right. \tag{D.11}$$

$$\left. + \sum_{i=1}^{n} \sum_{\ell=1}^{n} \sum_{j=1}^{n-1} m_{ij} m_{\ell n} z_i z_{j+1}^2 + \sum_{i=1}^{n} \sum_{\ell=1}^{n} \sum_{j=1}^{n-1} m_{ij} m_{\ell n} z_i z_{j+1} z_\ell\right].$$

$$\ell\neq i,\ j+1\neq i,\ \ell=j+1 \qquad\qquad \ell\neq i,\ j+1\neq i,\ \ell\neq j+1$$

In order to expand the fifth term on the right of (D.4) we must first separate the first sum in that product into two components, the second of which has a subscript running from 2 to n.

$$2 \sum_{\substack{i=1 \\ i\neq j+1}}^{n} \sum_{j=1}^{n-1} m_{ij} z_i z_{j+1} \sum_{i=2}^{n} m_{i,i-1} z_i^2 = 2\left[\sum_{\substack{j=1 \\ j\neq 2}}^{n-1} m_{1j} z_1 z_{j+1} + \sum_{i=2}^{n} \sum_{j=1}^{n-1} m_{ij} z_i z_{j+1}\right] \sum_{i=2}^{n} m_{i,i-1} z_i^2 \tag{D.12}$$

$$= 2 \sum_{\substack{j=1 \\ j\neq 2}}^{n-1} m_{ij}\, z_1 z_{j+1} \sum_{i=2}^{n} m_{i,i-1}\, z_i^2$$

$$+ 2 \sum_{\substack{i=2 \\ i\neq j+1}}^{n} \sum_{j=1}^{n-1} m_{ij}\, z_i\, z_{j+1} \sum_{i=2}^{n} m_{i,i-1}\, z_i^2$$

Then the first term on the right of (D.12) is

$$\text{(D.13)} \qquad 2 \sum_{\substack{j=1 \\ j\neq 2}}^{n-1} m_{1j}\, z_1 z_{j+1} \sum_{i=2}^{n} m_{i,i-1}\, z_i^2 = 2 \sum_{i=2}^{n} \sum_{\substack{j=1 \\ j\neq 2}}^{n-1} m_{1j}\, m_{i,i-1}\, z_1 z_i^2 z_{j+1}$$

and the second term on the right of (D.12) can be expanded as

$$\text{(D.14)} \qquad 2 \sum_{\substack{i=2 \\ j+1\neq i}}^{n} \sum_{j=1}^{n-1} m_{ij}\, z_i z_{j+1} \sum_{i=2}^{n} m_{i,i-1}\, z_i^2 = 2 \sum_{\substack{i=2 \\ j+1\neq i}}^{n} \sum_{j=1}^{n-1} m_{ij}\, m_{i,i-1}\, z_i^3\, z_{j+1}$$

$$+ 2 \sum_{i=2}^{n} \sum_{\ell=2}^{n} \sum_{\substack{j=1 \\ \ell\neq i,\ j+1\neq i,\ \ell=j+1}}^{n-1} m_{ij}\, m_{\ell,\ell-1}\, z_i\, z_{j+1}^3$$

$$+ 2 \sum_{i=2}^{n} \sum_{\ell=2}^{n} \sum_{\substack{j=1 \\ \ell\neq i,\ j+1\neq i,\ \ell\neq j+1}}^{n-1} m_{ij}\, m_{\ell,\ell-1}\, z_i\, z_{j+1}\, z_\ell^2 \quad .$$

The technique used to expand the fifth term in (D.4) is used again to expand the sixth term in (D.4)

$$\text{(D.15)} \qquad 2z_{n+1} \sum_{i=1}^{n} m_{in}\, z_i \sum_{i=2}^{n} m_{i,i-1}\, z_i^2 = 2z_{n+1}\Big[\sum_{i=2}^{n} m_{1n}\, m_{i,i-1}\, z_1\, z_i^2$$

$$+ \sum_{i=2}^{n} m_{in}\, z_i \sum_{i=2}^{n} m_{i,i-1}\, z_i^2\Big]$$

The second term on the right of (D.15) is

$$\text{(D.16)} \qquad \sum_{i=2}^{n} m_{in}\, z_i \sum_{i=2}^{n} m_{i,i-1}\, z_i^2 = \sum_{i=2}^{n} m_{in}\, m_{i,i-1}\, z_i^3 + \sum_{\substack{i=2 \\ \ell\neq i}}^{n} \sum_{\ell=2}^{n} m_{in}\, m_{\ell,\ell-1}\, z_i z_\ell^2$$

Finally we combine the results of (D.6) to (D.16) with (3.5) to obtain

(D.17)
$$c^2 = \frac{1}{W^2} \sum_{\substack{i=1\\ i\neq j+1}}^{n} \left[\sum_{j=1}^{n-1} m_{ij}^2 \; z_i^2 z_{j+1}^2 + 2 \sum_{j=1}^{n-1} \sum_{\substack{\ell=1\\ \ell<j,\; i\neq \ell+1}}^{n-1} m_{ij}\, m_{i\ell}\, z_i^2\, z_{j+1}\, z_{\ell+1} \right]$$

$$+ \frac{2}{W^2} \sum_{i=1}^{n} \sum_{\substack{k=1\\ k<i,\; i\neq j+1}}^{n} \left[\sum_{j=1}^{n-1} m_{ij}\, m_{kj}\, z_i z_k z_{j+1}^2 + \sum_{j=1}^{n-1} \sum_{\substack{\ell=1\\ \ell\neq j,\; k\neq j+1,\; i=\ell+1}}^{n-1} m_{ij}\, m_{k,i-1}\, z_i^2 z_{j+1} z_k \right.$$

$$\left. + \sum_{j=1}^{n-1} \sum_{\substack{\ell=1\\ \ell\neq j,\; k\neq \ell+1,\; i\neq \ell+1}}^{n-1} m_{ij}\, m_{k\ell}\, z_i z_{j+1} z_k z_{\ell+1} \right]$$

$$+ \frac{z_{n+1}^2}{W^2} \left[\sum_{i=1}^{n} m_{in}^2\, z_i^2 + 2 \sum_{i=1}^{n} \sum_{\substack{k=1\\ k<i}}^{n-1} m_{in}\, m_{kn}\, z_i z_k \right]$$

$$+ \frac{1}{W^2} \left[\sum_{i=2}^{n} m_{i,i-1}^2\, z_i^4 + 2 \sum_{i=2}^{n} \sum_{\substack{k=2\\ k<i}}^{n-1} m_{i,i-1}\, m_{k,k-1}\, z_i^2 z_k^2 \right]$$

$$+ \frac{2z_{n+1}}{W^2} \left[\sum_{i=1}^{n} \sum_{\substack{j=1\\ j+1\neq i}}^{n-1} m_{ij}\, m_{in}\, z_i^2 z_{j+1} + \sum_{i=1}^{n} \sum_{\ell=1}^{n} \sum_{\substack{j=1\\ \ell\neq i,\; j+1\neq i,\; \ell=j+1}}^{n-1} m_{ij}\, m_{\ell n}\, z_i z_{j+1}^2 \right.$$

$$\left. + \sum_{i=1}^{n} \sum_{\ell=1}^{n} \sum_{\substack{j=1\\ \ell\neq i,\; j+1\neq i,\; \ell\neq j+1}}^{n-1} m_{ij}\, m_{\ell n}\, z_i z_{j+1} z_\ell \right] + \frac{2}{W^2} \left[\sum_{i=2}^{n} \sum_{\substack{j=1\\ j\neq 2}}^{n-1} m_{1j}\, m_{i,i-1}\, z_1 z_i^2 z_{j+1} \right.$$

$$+ \sum_{i=2}^{n} \sum_{\substack{j=1\\ j+1\neq i}}^{n-1} m_{ij}\, m_{i,i-1}\, z_i^3 z_{j+1} + \sum_{i=2}^{n} \sum_{\ell=2}^{n} \sum_{\substack{j=1\\ \ell\neq i,\; j+1\neq i,\; \ell=j+1}}^{n-1} m_{ij}\, m_{\ell,\ell-1}\, z_i z_{j+1}^3$$

$$\left. + \sum_{i=2}^{n} \sum_{\ell=2}^{n} \sum_{\substack{j=1\\ \ell\neq i,\; j+1\neq i,\; \ell\neq j+1}}^{n-1} m_{ij}\, m_{\ell,\ell-1}\, z_i\, z_{j+1}\, z_\ell^2 \right] + \frac{2z_{n+1}}{W^2} \left[\sum_{i=2}^{n} m_{1n}\, m_{i,i-1}\, z_1 z_i^2 \right.$$

$$\left. + \sum_{i=2}^{n} m_{in}\, m_{i,i-1}\, z_i^3 + \sum_{\substack{i=2\\ \ell\neq i}}^{n} \sum_{\ell=2}^{n} m_{in}\, m_{\ell,\ell-1}\, z_i\, z_\ell^2 \right]$$

We can now find the first two moments of c. Using (C.10), (C.11), (C.12) and (D.3) and recalling that z_{n+1} is independent of W, we have

$$\text{(D.18)} \quad Ec = \frac{1}{2\sigma^2} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ i\neq j+1}}^{n-1} m_{ij}\, \mu_i \mu_{j+1}\, f_{1,2} + \frac{1}{2\sigma^2} \sum_{i=1}^{n} m_{in}\, \mu_i \mu_{n+1}\, f_{0,1} + \frac{1}{2\sigma^2} \sum_{i=2}^{n} m_{i,i-1} (\mu_i^2 f_{1,2} + f_{0,1})$$

$$= \frac{1}{2\sigma^2} \left[\mu_{-1}' M \mu\, f_{1,2} + \sum_{i=1}^{n} m_{in}\, \mu_i \mu_{n+1} (f_{0,1} - f_{1,2}) + \sum_{i=2}^{n} m_{i,i-1}\, f_{0,1} \right]$$

Using (C.11) to (C.20) and (D.17) we have

$$\text{(D.19)} \quad Ec^2 = \frac{1}{4} \sum_{\substack{i=1 \\ i\neq j+1}}^{n} \left[\sum_{j=1}^{n-1} m_{ij}^2 \left\{ \frac{\mu_i^2 \mu_{j+1}^2}{\sigma^4} f_{2,4} + \left(\frac{\mu_i^2}{\sigma^2} + \frac{\mu_{j+1}^2}{\sigma^2} \right) f_{1,3} + f_{0,2} \right\} \right.$$

$$\left. + 2 \sum_{j=1}^{n-1} \sum_{\substack{\ell=1 \\ \ell<j,\, i\neq \ell+1}}^{n-1} m_{ij}\, m_{i\ell} \left(\frac{\mu_i^2\, \mu_{j+1}\, \mu_{\ell+1}}{\sigma^4} f_{2,4} + \frac{\mu_{j+1}\, \mu_{\ell+1}}{\sigma^2} f_{1,3} \right) \right]$$

$$+ \frac{1}{2} \sum_{i=1}^{n} \sum_{k=1}^{n} \left[\sum_{j=1}^{n-1} m_{ij}\, m_{kj} \left(\frac{\mu_i \mu_k \mu_{j+1}^2}{\sigma^4} f_{2,4} + \frac{\mu_i \mu_k}{\sigma^2} f_{1,3} \right) \right. \qquad k<i,\; i\neq j+1,\; k\neq j+1$$

$$+ \sum_{j=1}^{n-1} \sum_{\ell=1}^{n-1} m_{ij}\, m_{k,i-1} \left(\frac{\mu_i^2\, \mu_{j+1}\, \mu_k}{\sigma^4} f_{2,4} + \frac{\mu_{j+1}\, \mu_k}{\sigma^2} f_{1,3} \right) \qquad \ell\neq j,\; k\neq \ell+1,\; i=\ell+1$$

$$\left. + \sum_{j=1}^{n-1} \sum_{\ell=1}^{n-1} m_{ij}\, m_{k\ell} \frac{\mu_i\, \mu_{j+1}\, \mu_k\, \mu_{\ell+1}}{\sigma^4} f_{2,4} \right] \qquad \ell\neq j,\; k\neq \ell+1,\; i\neq \ell+1$$

$$+ \frac{1}{4}\left(1 + \frac{\mu_{n+1}^2}{\sigma^2}\right)\left[\sum_{i=1}^{n} m_{in}^2 \left(\frac{\mu_i^2}{\sigma^2} f_{0,2} + f_{-1,1}\right)\right.$$

$$\left. + 2 \sum_{i=1}^{n} \sum_{\substack{k=1 \\ k<i}}^{n-1} m_{in}\, m_{kn}\left(\frac{\mu_i\, \mu_k}{\sigma^2} f_{0,2}\right)\right]$$

$$+ \frac{1}{4}\left[\sum_{i=2}^{n} m_{i,i-1}^2 \left(\frac{\mu_i^4}{\sigma^4} f_{2,4} + 6\,\frac{\mu_i^2}{\sigma^2} f_{1,3} + 3\, f_{0,2}\right)\right.$$

$$\left. + 2 \sum_{i=2}^{n} \sum_{\substack{k=2 \\ k<i}}^{n-1} m_{i,i-1}\, m_{k,k-1} \left\{\frac{\mu_i^2\, \mu_k^2}{\sigma^4} f_{2,4} + \left(\frac{\mu_i^2}{\sigma^2} + \frac{\mu_k^2}{\sigma^2}\right) f_{1,3} + f_{0,2}\right\}\right]$$

$$+ \frac{1}{2}\,\frac{\mu_{n+1}}{\sigma}\left[\sum_{i=1}^{n} \sum_{\substack{j=1 \\ j+1\neq i}}^{n-1} m_{ij}\, m_{in}\left(\frac{\mu_i^2\, \mu_{j+1}}{\sigma^3} f_{1,3} + \frac{\mu_{j+i}}{\sigma} f_{0,2}\right)\right.$$

$$+ \sum_{\substack{i=1 \\ \ell\neq i,\ j+1\neq i,\ \ell=j+1}}^{n} \sum_{\ell=1}^{n} \sum_{j=1}^{n-1} m_{ij}\, m_{\ell n}\left(\frac{\mu_i \mu_{j+1}^2}{\sigma^3} f_{1,3} + \frac{\mu_i}{\sigma} f_{0,2}\right)$$

$$\left. + \sum_{\substack{i=1 \\ \ell\neq i,\ j+1\neq i,\ \ell\neq j+1}}^{n} \sum_{\ell=1}^{n} \sum_{j=1}^{n-1} m_{ij}\, m_{\ell n}\, \frac{\mu_i\, \mu_{j+1}\, \mu_\ell}{\sigma^3} f_{1,3}\right]$$

$$+ \frac{1}{2}\left[\sum_{i=2}^{n} \sum_{\substack{j=1 \\ j\neq 2}}^{n-1} m_{1j}\, m_{i,i-1}\left(\frac{\mu_1\, \mu_i^2\, \mu_{j+1}}{\sigma^4} f_{2,4} + \frac{\mu_1\, \mu_{j+1}}{\sigma^2} f_{1,3}\right)\right.$$

$$+ \sum_{i=2}^{n} \sum_{\substack{j=1 \\ j+1\neq i}}^{n-1} m_{ij}\, m_{i,i-1}\left(\frac{\mu_i^3\, \mu_{j+1}}{\sigma^4} f_{2,4} + 3\,\frac{\mu_i\, \mu_{j+1}}{\sigma^2} f_{1,3}\right)$$

$$+ \sum_{\substack{i=2 \\ \ell\neq i,\ j+1\neq i,\ \ell=j+1}}^{n} \sum_{\ell=2}^{n} \sum_{j=1}^{n-1} m_{ij}\, m_{\ell,\ell-1}\left(\frac{\mu_i\, \mu_{j+1}^3}{\sigma^4} f_{2,4} + 3\,\frac{\mu_i\, \mu_{j+1}}{\sigma^2} f_{1,3}\right)$$

$$+\sum_{\substack{i=2\\ \ell\neq i,}}^{n}\sum_{\substack{\ell=2\\ j+1\neq i,}}^{n}\sum_{\substack{j=1\\ \ell\neq j+1}}^{n-1} m_{ij}\, m_{\ell,\ell-1}\left(\frac{\mu_i\,\mu_{j+1}\,\mu_\ell^2}{\sigma^4} f_{2,4} + \frac{\mu_i\,\mu_{j+1}}{\sigma^2} f_{1,3}\right)\Bigg]$$

$$+\frac{1}{2}\,\frac{\mu_{n+1}}{\sigma}\left[\sum_{i=2}^{n} m_{1n}\, m_{i,i-1}\left(\frac{\mu_1\,\mu_i^2}{\sigma^3} f_{1,3} + \frac{\mu_1}{\sigma} f_{0,2}\right)\right.$$

$$+\sum_{i=2}^{n} m_{in}\, m_{i,i-1}\left(\frac{\mu_i^3}{\sigma^3} f_{1,3} + \frac{\mu_i}{\sigma} f_{0,2}\right)$$

$$\left.+\sum_{\substack{i=2\\ \ell\neq i}}^{n}\sum_{\ell=2}^{n} m_{in}\, m_{\ell,\ell-1}\left(\frac{\mu_i\,\mu_\ell^2}{\sigma^3} f_{1,3} + \frac{\mu_i}{\sigma} f_{0,2}\right)\right] .$$

If we gather together terms in (D.18) and (D.19) and use the equations $M\mu = \gamma M\mu_{-1}$ and $\theta = \frac{1}{2\sigma^2}\mu'_{-1} M \mu_{-1}$ we obtain

$$\text{(D.20)}\qquad Ec = \gamma\,\theta\, f_{1,2} + \frac{1}{2\sigma^2}\left[(\mu_{n+1}\sum_{i=1}^{n} m_{in}\,\mu_i)(f_{0,1} - f_{1,2}) + (\sum_{i=2}^{n} m_{i,i-1})f_{0,1}\right]$$

and

$$\text{(D.21)}\qquad Ec^2 = \gamma^2\theta^2 f_{2,4} - \frac{\mu_{n+1}}{2\sigma^2}\left[\frac{\mu_{n+1}}{2}\left(\sum_{i=1}^{n} m_{in}\,\mu_i\right)^2 + \sum_{\substack{i=1\\ j+1\neq i}}^{n}\sum_{j=1}^{n-1} m_{ij}\, m_{in}\,\mu_i^2\,\mu_{j+1}\right.$$

$$+\sum_{\substack{i=1\\ \ell\neq i,}}^{n}\sum_{\ell=1}^{n}\sum_{\substack{j=1\\ j+1\neq i}}^{n-1} m_{ij}\, m_{\ell n}\,\mu_i\,\mu_{j+1}\,\mu_\ell$$

$$\left.+ (m_{1n}\,\mu_1 + \sum_{i=2}^{n} m_{in}\,\mu_i)\sum_{i=2}^{n} m_{i,i-1}\,\mu_i^2\right] f_{2,4}$$

$$+\frac{1}{2\sigma^2}\left[\frac{1}{2}\sum_{\substack{i=1\\ i\neq j+1}}^{n}\sum_{j=1}^{n-1} m_{ij}^2\,(\mu_i^2 + \mu_{j+1}^2) + \sum_{\substack{i=1\\ i\neq j+1,}}^{n}\sum_{\substack{j=1\\ \ell<j,}}^{n-1}\sum_{\substack{\ell=1\\ i\neq \ell+1}}^{n-1} m_{ij}\, m_{i\ell}\,\mu_{j+1}\,\mu_{\ell+1}\right.$$

$$+\sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{n-1} m_{ij}\,\mu_k(m_{k,i-1}\,\mu_{j+1} + m_{kj}\,\mu_i)$$

$$k<i,\ i\neq j+1,\ k\neq j+1$$

$$+ \sum_{i=2}^{n} m_{i,i-1} \left\{(3m_{i,i-1} + \sum_{\substack{k=2\\k<i}}^{n} m_{k,k-1})\mu_i^2 + \sum_{\substack{k=2\\k<i}}^{n} m_{k,k-1} \mu_k^2\right\}$$

$$+ \frac{\mu_{n+1}}{\sigma^2} \sum_{i=1}^{n} \sum_{\substack{j=1\\j+1\neq i}}^{n-1} m_{ij} \mu_i (m_{in} \mu_i \mu_{j+1} + \sum_{\substack{\ell=1\\\ell\neq i}}^{n} m_{\ell n} \mu_{j+1} \mu_\ell)$$

$$+ \sum_{i=2}^{n} m_{i,i-1} (\mu_1 \sum_{\substack{j=1\\j\neq 2}}^{n-1} m_{1j} \mu_{j+1} + 3 \sum_{\substack{j=1\\j+1\neq i}}^{n-1} m_{ij} \mu_i \mu_{j+1})$$

$$+ \sum_{i=2}^{n} \sum_{\ell=2}^{n} \sum_{j=1}^{n-1} m_{ij} m_{\ell,\ell-1} \mu_i \mu_{j+1} + \frac{\mu_{n+1}}{\sigma^2} m_{1n} \mu_1 \sum_{i=2}^{n} m_{i,i-1} \mu_i^2$$
$$\ell\neq i,\ j+1\neq i$$

$$+ \frac{\mu_{n+1}}{\sigma^2} \sum_{i=2}^{n} m_{in} \mu_i \sum_{i=2}^{n} m_{i,i-1} \mu_i^2 \Big] f_{1,3} + \frac{1}{4}\Big[\sum_{\substack{i=1\\i\neq j+1}}^{n} \sum_{j=1}^{n-1} m_{ij}^2$$

$$+ \left(1 + \frac{\mu_{n+1}^2}{\sigma^2}\right)\left(\sum_{i=1}^{n} m_{in} \frac{\mu_i}{\sigma}\right)^2 + 3 \sum_{i=2}^{n} m_{i,i-1}^2 + 2 \sum_{i=2}^{n} \sum_{\substack{k=2\\k<i}}^{n-1} m_{i,i-1} m_{k,k-1}$$

$$+ \frac{2\mu_{n+1}}{\sigma^2} \sum_{i=1}^{n} \sum_{\substack{j=1\\j+1\neq i}}^{n-1} m_{ij} (m_{in} \mu_{j+1} + \mu_i \sum_{\substack{\ell=1\\\ell\neq i,\ \ell=j+1}}^{n} m_{\ell n})$$

$$+ \frac{2\mu_{n+1}}{\sigma^2} \left(m_{1n} \mu_1 \sum_{i=2}^{n} m_{i,i-1} + \sum_{i=2}^{n} m_{in} \mu_i \sum_{i=2}^{n} m_{i,i-1}\right) \Big] f_{0,2}$$

$$+ \frac{1}{4}\Big[\left(1 + \frac{\mu_{n+1}^2}{\sigma^2}\right) \sum_{i=1}^{n} m_{in}^2\Big] f_{-1,1}$$

If we use footnote 3 to analyze $f_{1,2}$ and $f_{0,1}$, we see that the second term on the right of (D.20) vanishes as $\theta\to\infty$ and the first term tends to γ; that is, $Ec \to \gamma$ as $\theta\to\infty$. Similarly, all terms of (D.21) beyond the first vanish as $\theta\to\infty$ and the first term tends to γ^2; that is, $Ec^2 \to \gamma^2$ as $\theta\to\infty$.

Footnotes

[1]This simplification is not really a loss of generality since we give the exact moments of the general case in Appendix D.

[2]The result in (3.14) follows from (3.17). To get $\theta \to 0$ as in (3.15), it is sufficient to have $\sigma^2 \to \infty$.

[3]If $\theta > 0$ and a, c > 0, then, using Sawa's (1972, p. 667) results, we have

$$ {}_1F_1(a;c;\theta) = \frac{\Gamma c}{\Gamma a} e^{\theta} \theta^{-(c-a)}\left[\sum_{j=0}^{p-1} \frac{(c-a)_j (1-a)_j}{j!} \theta^{-j} + 0(\theta^{-P})\right]. $$

θ will grow if $\mu_{-1}'\mu_{-1} \to \infty$ or if $\sigma^2 \to 0$. Kadane (1970), (1971) has analyzed the behavior of estimators as $\sigma^2 \to 0$.

[4]For large a and b, with $\theta > 0$,

$$ {}_1F_1(a;b;bx) = e^{bx}(1+x)^{a-b}\left[1 - \frac{(b-a)(b-a+1)}{2b}\left(\frac{x}{1+x}\right)^2 + 0\left(\frac{1}{|b|^2}\right)\right] $$

so long as (b-a) and x are bounded; Slater (1960, p. 66).

[5]Liviatan considered the case where X has only one column so the choice of w was obvious.

[6]In deriving the result (B.6), we have used the result

$$ \text{(i)} \qquad \frac{d^s}{d\theta^s} EW^{-r} = 2^{-r}(-1)^s \frac{\Gamma(r+s)}{\Gamma(r)} \frac{\Gamma(n/2-r)}{\Gamma(n/2+s)} e^{-\theta} \; {}_1F_1(n/2-r;n/2+s;\theta) $$

from (A.2). Further, it has been noted that

$$ \frac{\partial}{\partial \bar{z}_i} EW^{-r} = \left(\frac{d}{d\theta} EW^{-r}\right) \frac{\partial \theta}{\partial \bar{z}_i} $$

$$ \frac{\partial^2}{\partial \bar{z}_i^2} EW^{-r} = \left(\frac{d^2}{d\theta^2} EW^{-r}\right)\left(\frac{\partial \theta}{\partial \bar{z}_i}\right)^2 + \left(\frac{d}{d\theta} EW^{-r}\right)\frac{\partial^2 \theta}{\partial \bar{z}_i^2} $$

and so on.

[7]See Baranchik (1973, p. 314) and Ullah (1974, p. 146).

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