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CYCLICAL AND EQUILIBRIUM RELATIONS AMONG VARIABLES IN  
DYNAMIC MODELS: THE PHILLIPS CURVE AND THE REAL WAGE

Mark Gersovitz\*

Abstract

Some economically relevant properties of linear dynamic systems subject to sinusoidal forcing functions are analyzed. A test for the direction of rotations about the elliptical relation between any two variables in the system is provided. A least squares line through these ellipses and its properties are discussed. The connection between the cyclical and equilibrium properties of these systems is analyzed. These properties and a model of the adjustment of employment, money wage and price inflation and the real wage are used to discuss the cyclical pattern of the rates of inflation and the real wage with respect to employment.

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## I. Introduction

This paper seeks to provide a framework within which to examine relations among variables over a cycle. A number of studies addressing different problems in macroeconomics can be analyzed in this context. In [13], Tobin discusses the cyclical relation between changes in the supply of money and nominal income. Two models--ultra-Keynesian and ultra-monetarist--are presented and the cyclical behavior of these systems is induced by exogenous cycles in investment and the rate of change in the money stock, respectively. Grossman [6] provides an analysis in which exogenous cycles in a component of aggregate demand induce cyclical rotations about a Phillips curve. Insight is then provided into the direction of these rotations first observed as empirical phenomena in [9].

The common element in these models is their susceptibility of formulation as linear dynamic systems subject to sinusoidal forcing functions. In sections II and III of this paper, we provide a general mathematical analysis of economically relevant properties of this type of system. In this way, we avoid the special methods of analysis employed in previous studies with their inability to be easily generalized.

The type of system under discussion implies that the pairwise relation between variables takes the form of an ellipse. While a system with a general coefficient matrix can be quite complicated to solve, we derive a simple solution for a system with a symmetric coefficient matrix. Returning to the general case, we provide a simple algebraic test for the direction of rotation about an ellipse generated by the dynamic system.

In addition, we introduce the concept of a least squares line through an ellipse to provide a measure of the cyclical relationship between any pair of variables in our system. This measure is chosen since it has an obvious econometric

analogue and consequent usefulness in interpreting findings of various empirical studies. A simple algebraic test to determine the sign of the slope of the least squares line is presented and a geometric interpretation is provided for the least squares line.

One special concern is the connection between the cyclical and equilibrium properties of a dynamic system. Our major conclusion is that there is little correspondence between these two sets of properties. We emphasize the consequent problems associated with making inferences about the equilibrium properties models should possess to be consistent with the stylized facts of cyclical data. We conclude our mathematical investigation by presenting the analysis for the two dimensional case where our general results are somewhat simpler.

In section IV, we provide a new, economically important example of the efficacy of this type of cyclical analysis in our discussion of the cyclical behavior of the real wage with respect to aggregate employment. This issue has recently been given new prominence by writers such as Grossman [5]. First we give a brief statement of a macrodynamic model which explains movements in aggregate employment, the real wage and the rates of money wage and price inflation. We then apply the techniques of section II to a discussion of the Phillips [9] phenomena and the cyclical pattern of the real wage using this model. We interpret the findings of a number of empirical studies on the cyclical behavior of the real wage in the context of the cyclical behavior of our macrodynamic model. In particular, we show that the procyclical pattern of the real wage is consistent with a model which has an inverse equilibrium relation between employment and the real wage. This result suggests that the use of the cyclical pattern of the real wage as a touchstone of macro models may well be inappropriate.

## II. Cyclical and Equilibrium Relations in Dynamic Systems

Consider the dynamic system

$$(1) \quad \dot{x}(t) = Ax(t) - Bu(t)$$

where  $\dot{x}$ ,  $x$  and  $B$  are  $N \times 1$  column vectors,  $A$  is an  $N \times N$  matrix and  $u$  is a scalar. Let  $A$  and  $B$  have elements which are constants. We consider three types of analysis of the system (1) which can be undertaken: stability, comparative equilibria and cyclical.

Stability is analyzed by setting  $u$  constant and investigating the eigenvalues of  $A$ . As is well known, the system is neutrally stable if the eigenvalues of  $A$  all have non-positive real parts. The conditions on the coefficients of  $A$  which are necessary and sufficient for neutral stability are the familiar Routh-Hurwitz conditions. In what follows, we naturally restrict our discussion to models which are at least neutrally stable and we denote the equilibrium values of the system by  $x^*$ .

Comparative equilibria analysis is undertaken by totally differentiating (1) when  $u$  is a constant and solving for  $dx^*/du$ . If  $A$  is non-singular (in which case the system has eigenvalues with negative real parts if it is stable), we have

$$(2) \quad \frac{dx^*}{du} = A^{-1} B.$$

The vector equation (2) defines the equilibrium or comparative equilibria relations among the variables of the system since it is possible to solve from (2) for  $dx_j^*/dx_i^*$  for any elements  $x_i^*$ ,  $x_j^*$ , of  $x^*$ :

$$\frac{dx_j^*}{dx_i^*} = \frac{dx_j^*/du}{dx_i^*/du}$$

The cyclical behavior of the system may be investigated by setting  $u = \cos t$ . In this case, the system (1) has a particular solution given by

$$(3) \quad x_p(t) = \text{REAL}\{(A - iI)^{-1} B e^{it}\}$$

where  $\text{REAL}\{ \}$  denotes the real parts of the expression in brackets.

The general solution, denoted  $x_g(t)$  is given as a linear combination of the particular solution (3) and the general solution of the homogeneous system  $\dot{x} = Ax$ .

As a consequence of the stability property, we have

$$(4) \quad x(\infty) \equiv \lim_{t \rightarrow \infty} x_g(t) = \lim_{t \rightarrow \infty} x_p(t) + C$$

where  $C$  is a constant dependent on the initial conditions of the system. In the case where  $A$  is nonsingular,  $C = 0$ . Each element of  $x(\infty)$  is a sinusoidal function of the form

$$(5) \quad x_i(\infty) = m_i \cos t + n_i \sin t, \quad i = 1, \dots, N$$

and is defined as the cyclical values of  $x_i$  induced by the cycle  $u = \cos t$ . In the special case where  $A$  is symmetric and of full rank, we have

Theorem. If  $A$  is real symmetric then

$$x_p(t) = (A + A^{-1})^{-1} B \cos t - A^{-1}(A + A^{-1})B \sin t.$$

Proof. Since  $A$  is real symmetric, there exist matrices  $P$  orthogonal and  $D$  diagonal with real elements such that

$$A = PDP^{-1}$$

$$\begin{aligned} (A - iI)^{-1} &= P^{-1}(D - iI)^{-1}P \\ &= P^{-1}(D + D^{-1})^{-1}P + iP^{-1}(D^2 + I)^{-1}P \\ &= (A + A^{-1})^{-1} + i(A^2 + I)^{-1} \end{aligned}$$

$$\text{REAL}\{(A - iI)^{-1} B e^{it}\} = (A + A^{-1})^{-1} B \cos t - A^{-1}(A + A^{-1})^{-1} B \sin t.$$

It is straightforward to eliminate the parameter  $t$  from any pair of equations of the form (5) to produce an ellipse between  $x_i^{(\infty)}$  and  $x_j^{(\infty)}$  of the form

$$(6) \quad (m_j x_i - m_i x_j)^2 + (n_j x_i - n_i x_j)^2 = (n_i m_j - m_i n_j)^2.$$

Now consider a line in  $(x_i, x_j)$  space

$$(7) \quad x_j = a + b x_i.$$

Define the least squares line as that line (7) with  $a, b$  chosen to minimize the sum of squared residuals between points on the line and the corresponding points on the ellipse (6).

Theorem. The least squares line for (6) has

$$a = 0, \quad b = \frac{m_j m_i + n_j n_i}{m_i^2 + n_i^2}.$$

Proof. Consider the ellipse  $Sy^2 + Txy + Ux^2 + V = 0$  with  $S, T, U, V$  constant and  $k$  is  $\max x$  on the ellipse

$$\begin{aligned} SSR \equiv & \int_{-k}^k \left( a + bx + \frac{Tx + \sqrt{(Tx)^2 - 4S(Ux^2 + V)}}{2S} \right)^2 dx \\ & + \int_{-k}^k \left( a + bx + \frac{Tx - \sqrt{(Tx)^2 - 4S(Ux^2 + V)}}{2S} \right)^2 dx \end{aligned}$$

$$\frac{\partial SSR}{\partial a} = \int_{-k}^k 4 \left( a + bx + \frac{Tx}{2S} \right) dx = 8ak = 0$$



hence  $a = 0$

$$\frac{\partial \text{SSR}}{\partial b} = \int_{-k}^k 4 \left( a + bx + \frac{Tx}{2S} \right) x \, dx = \frac{8}{3} \left( b + \frac{T}{2S} \right) k^3 = 0$$

$$b = -\frac{T}{2S} .$$

Expanding (6) and making the correspondence yields

$$b = \frac{\frac{m_i m_i}{2} + \frac{n_i n_i}{2}}{m_i + n_i} .$$

Geometrically, the least squares line can be shown to pass through the two points on the ellipse for which the slope of the tangent line to the ellipse is zero (Figure 1). This property is proved by finding  $dx_j/dx_i$  from (6), setting  $dx_j/dx_i = 0$  and solving for  $x_j$  in terms of  $x_i$ .

We define the least squares line corresponding to the ellipse between  $x_i$  and  $x_j$  to be the cyclical relation between the two variables. Note that the sign of the slope of the least squares line is invariant to changes in the scale of  $x_i$  or  $x_j$ . Our major point of concern is that the slope of the least squares line need not correspond to the slope of the equilibrium relation between two variables. Further, the two lines need not even have slopes of the same sign. As a result, it is very hazardous to make inferences about equilibrium relations from data which were cyclically generated. A precise discussion of this non-correspondence between the cyclical and equilibrium relations between variables is presented for the two dimensional case in the next section.

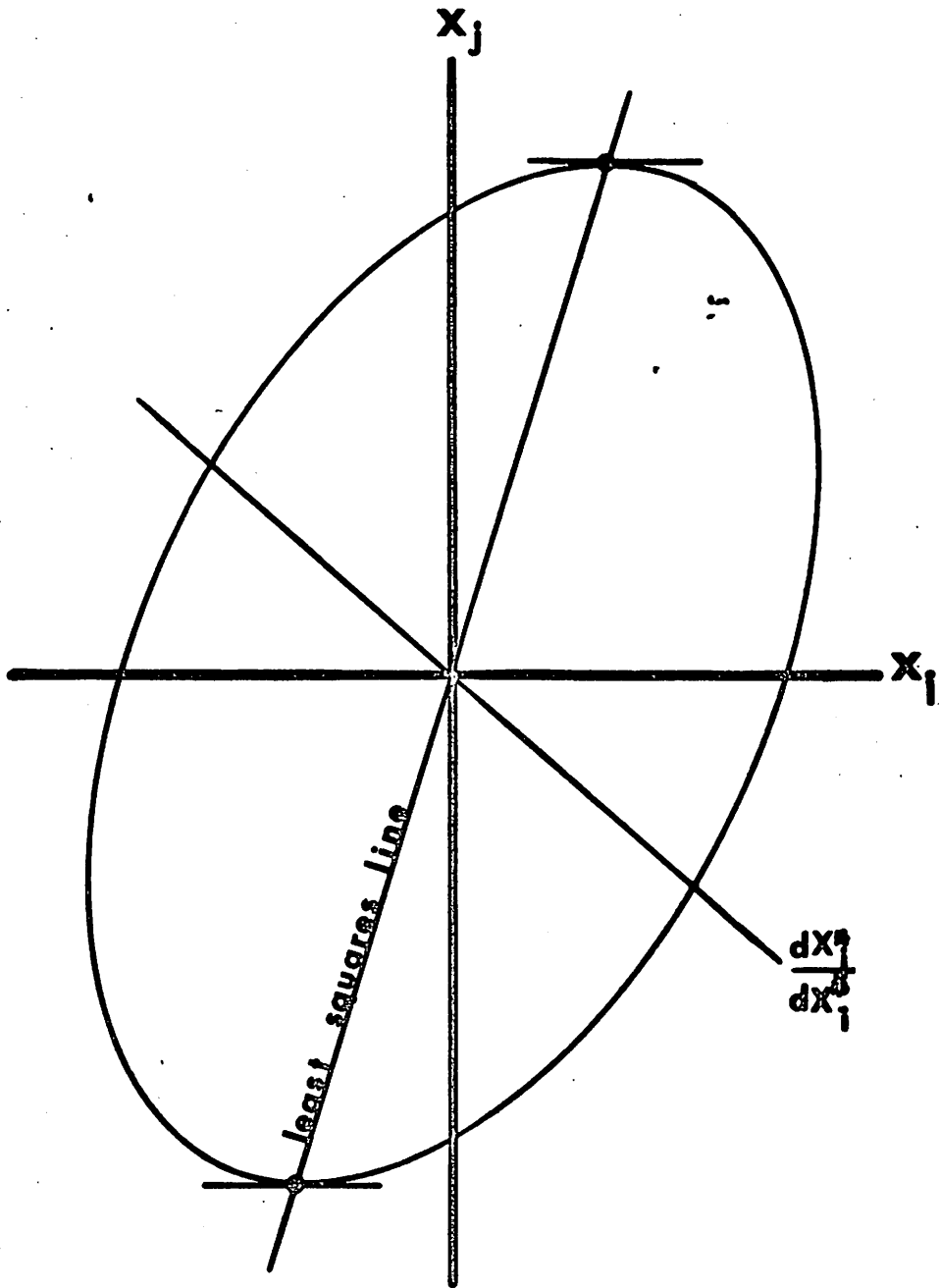


FIGURE 1

Once again, consider the general problem of two sinusoidal functions

$$x_i = m_i \cos t + n_i \sin t$$

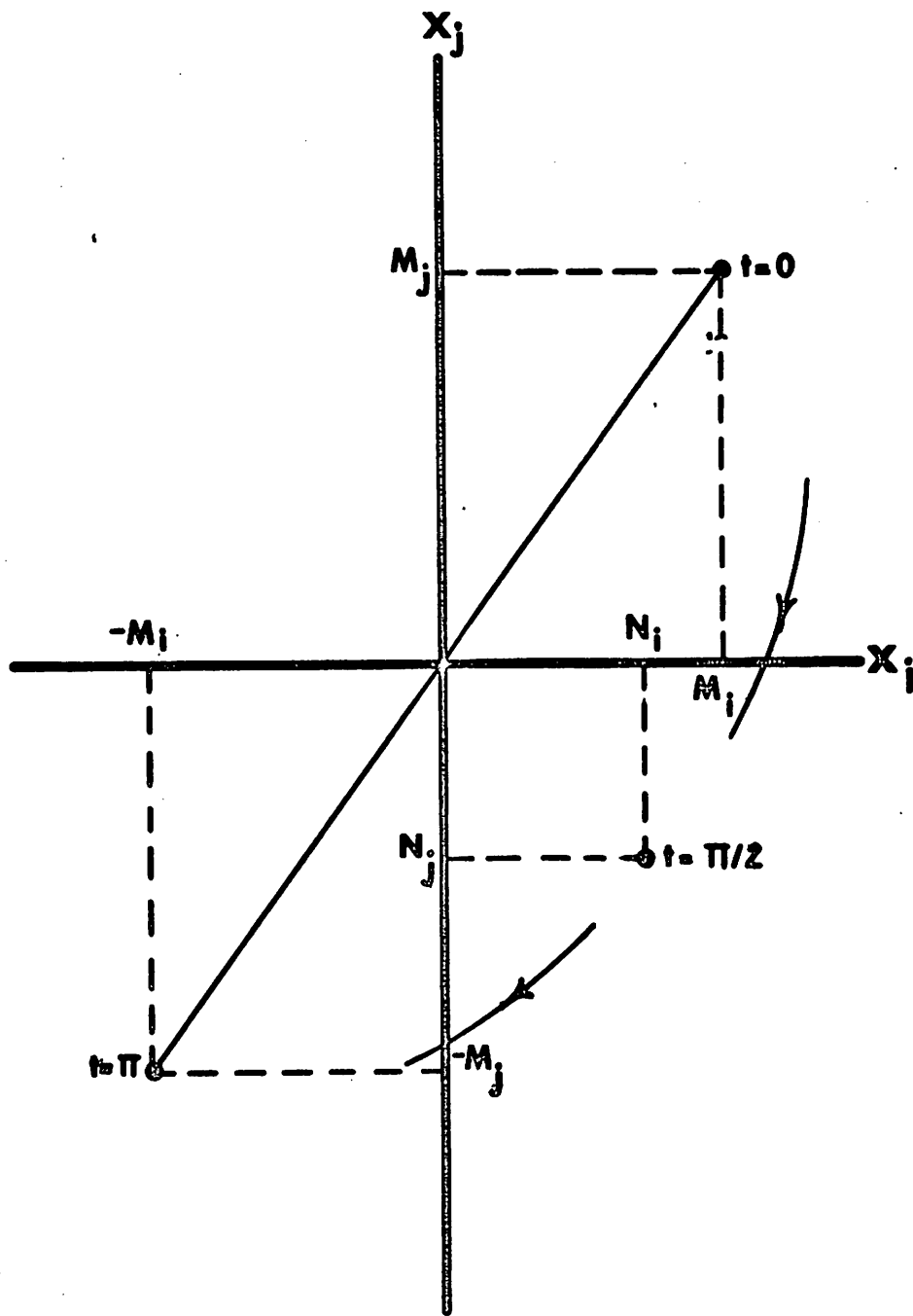
$$x_j = m_j \cos t + n_j \sin t$$

and the associated ellipse. To establish the direction of rotation around the ellipse plot the pairs  $(x_i(0), x_j(0))$ ,  $(x_i(\pi/2), x_j(\pi/2))$ , and  $(x_i(\pi), x_j(\pi))$  which are given by  $(m_i, m_j)$ ,  $(n_i, n_j)$  and  $(-m_i, -m_j)$  respectively. Consider the case where  $m_i, m_j > 0$ . It is clear (Figure 2) that clockwise motion requires

$$(8) \quad n_j < \frac{m_i}{m_j} n_i$$

i.e.,  $m_i n_j < m_j n_i$ . The same condition is easily seen to hold regardless of the signs of  $m_i$  and  $m_j$ , so that clockwise motion always requires

$$(9) \quad m_i n_j - m_j n_i < 0.$$



**FIGURE 2**

### III. The Two Dimensional Case

Consider a system of the form:

$$(10) \quad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u$$

where the coefficient matrix,  $A$ , has full rank and eigenvalues with negative real parts. Maintaining our notation of the preceding section, we have

$$(11) \quad x(\infty) = \text{REAL}\{(A - iI)^{-1} B e^{it}\}$$

that is

$$\begin{bmatrix} x_1(\infty) \\ x_2(\infty) \end{bmatrix} = \begin{bmatrix} k_1 k_3 + b_1 k_2 & k_1 b_1 - k_2 k_3 \\ k_1 k_4 + b_2 k_2 & k_1 b_2 - k_2 k_4 \end{bmatrix} \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} \frac{1}{\Delta \bar{\Delta}}$$

where

$$(12) \quad k_1 = (a_{11} a_{22} - a_{12} a_{21} - 1)$$

$$(13) \quad k_2 = a_{11} + a_{22}$$

$$(14) \quad k_3 = b_1 a_{22} - b_2 a_{12}$$

$$(15) \quad k_4 = b_2 a_{11} - b_1 a_{21}$$

$$(16) \quad \Delta = k_1 - k_2 i$$

and

$$(17) \quad \bar{\Delta} = k_1 + k_2 i .$$

Now, the characteristic equation of  $A$  is given by

$$(18) \quad \lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21})$$

so that our assumption of stability implies

$$(19) \quad k_1 + 1 > 0$$

$$(20) \quad k_2 < 0 .$$

Further, the sign of the slope of the least squares line of the ellipse determined by  $x_1(\infty)$  and  $x_2(\infty)$  is given by the sign of

$$(21) \quad (k_1^2 + k_2^2)(k_3k_4 + b_1b_2) .$$

Now, our interest in comparative equilibrium results is expressed by a desire to know the signs of the elements of

$$(22) \quad \frac{dx^*}{du} = A^{-1}B = \begin{bmatrix} k_3 \\ k_4 \end{bmatrix} .$$

It is clear that the sign of (21) provides very little knowledge about the signs of  $k_3$  and  $k_4$ , or their ratio  $k_3/k_4$ . Hence, it is difficult to make inferences about comparative equilibria properties of even the simplest dynamic models from the slope of the least squares line. Conversely, a knowledge of the signs of comparative equilibria properties provides very little information about the cyclical behavior of the system.

IV. An Interpretation of Cyclical Movements  
in the Real Wage

Notation:

$p$  : price level

$p^e$  : expected price level

$\epsilon$  : rate of price inflation ( $\dot{p}/p$ )

$\epsilon^e$  : expected rate of price inflation

$G$  : desired price gap

$w$  : money wage level

$w^e$  : expected money wage level

$\theta$  : rate of money wage inflation ( $\dot{w}/w$ )

$\theta^e$  : expected rate of money wage inflation

$g$  : desired wage gap

$V$  : real wage ( $w/p$ )

$V^e$  : expected real wage

$N$  : volume of aggregate employment

$I$  : exogenous component of aggregate demand

In this section, we sketch out a macrodynamic model which explains changes in aggregate employment, money wage and price inflation and the real wage and then discuss the implied cyclical behavior of the real wage. Our formulation is derived from the search literature (most especially [8]). Firms, as monopsonistic competitors in the labour market, attempt to attract recruits from the unemployed pool and from other firms and to hold their own employees by setting their own money wage relative to the money wage they expect to prevail economy-wide. Similarly, firms as monopolistic competitors in the goods market attempt to modulate the demand for their output by setting their own price relative to what they expect the economy-wide average price to be.

For convenience, we speak of the firm as setting its wage and price relative to its expectation of the respective economy-wide values by setting a gap. This gap may be positive, zero or negative. Thus, for the  $i^{\text{th}}$  firm

$$(23) \quad p_i(t) = (1 + G_i) p_i^e(t)$$

$$(24) \quad w_i(t) = (1 + g_i) w_i^e(t)$$

Of course, if we maintain the convention of the representative firm, we can suppress the subscript  $i$ .

The setting of its price and wage gaps by a firm is a simultaneous decision; variables affecting either the labour or the goods market influence both the price and wage decision. An integrated analysis, beyond the scope of this paper is presented in [4]. However, it is plausible that the most important variables affecting the firm's price and wage decisions are: the level of aggregate demand (represented by the exogenous component), the aggregate volume of employment (through its effect on the number of workers employed by the representative firm and on the size of the unemployed pool) and the expected real wage. In particular, we write

$$(25) \quad G(t) = G(V^e, N, I)$$

$$(26) \quad g(t) = g(V^e, N, I)$$

with the partial derivatives  $G_1, G_2, G_3 > 0$ ,  $g_1, g_2 < 0$  and  $g_3 > 0$ .

Of course, as is well documented in the search approach to inflation, the representative firm's attempt to maintain a wage or price gap different from zero leads to frustration and an acceleration of wage or price inflation beyond what the representative firm expected. Let  $\theta$  and  $\theta^e$  be the actual and expected rate of money wage inflation prevailing over the next  $h$  periods.



Then, if  $h$  is small

$$(27) \quad w(t) = (1 + \theta(t)) w(t-h) \approx \exp(\theta) w(t-h)$$

$$(28) \quad w^e(t) = (1 + \theta^e(t)) w(t-h) \approx \exp(\theta^e) w(t-h)$$

Consequently

$$(29) \quad \theta = \ln(1+g) + \theta^e$$

We adopt a simple adaptive model of expectations formation:

$$(30) \quad \dot{\theta}^e = d_1(\theta - \theta^e)$$

with  $0 < d_1 < 1$  a constant. Substituting (29) into (30)

$$(31) \quad \dot{\theta}^e = d_1 \ln(1+g) = Y(V^e, N, I)$$

as the first equation of the macrodynamic model. A parallel equation is derived for  $\dot{\epsilon}^e$

$$(32) \quad \dot{\epsilon} = \ln(1+G) + \dot{\epsilon}^e$$

$$(33) \quad \dot{\epsilon}^e = d_2 \ln(1+G) = Z(V^e, N, I)$$

We assume that aggregate employment responds in the direction which firms desire, so that

$$(34) \quad \dot{N} = W(V^e, N, I)$$

with  $N_1, N_2 < 0$  and  $N_3 > 0$ . Finally, we have the identity

$$(35) \quad \dot{V}^e = V^e (\theta^e - \epsilon^e)$$

so that equations (32) - (35) provide a dynamic model in  $V^e$ ,  $\theta^e$ ,  $\epsilon^e$  and  $N$  while equations (29) and (32) allow us to trace the time path of  $\theta$  and  $\epsilon$ .

$I$  is assumed exogenous to the system, representing the effects of a monetary-fiscal policy mix.

We shall assume that there is at least one set of values  $(V^{e*}, N^*, I^*)$  such that  $\dot{\theta}^e = \dot{\epsilon}^e = \dot{N} = 0$ . If this set of values is unique, then we may denote  $N^*$  as the natural rate of employment. Elsewhere [4] we have argued that various impediments to the search process may yield more than one set  $(V^{e*}, N^*, I^*)$ , but those complications need not detain us here.

Note that setting equations (32) - (35) equal to zero does not determine the equilibrium values  $\theta^{e*} = \epsilon^{e*}$ . This failure of the equilibrium conditions to determine the equilibrium values of the rate of inflation is, of course, typical of a search-natural-rate formulation. As is well known, any rate of inflation is consistent with equilibrium in these models. This aspect of the model is evidenced in the linearized version by the existence of a zero eigenvalue. The equilibrium values of  $\theta^{e*}$  and  $\epsilon^{e*}$  are determined by the dynamic system (32) - (35) with the specification of initial conditions and a time path of  $I$  which must eventually be set at  $I^*$  for equilibrium.

Linearizing the model about  $(V^{e*}, N^*, I^*)$  yields

$$(36) \quad \begin{bmatrix} \dot{N} \\ \dot{\theta}^e \\ \dot{\epsilon}^e \\ \dot{V}^e \end{bmatrix} = \begin{bmatrix} a_{11} & 0 & 0 & a_{14} \\ a_{21} & 0 & 0 & a_{24} \\ a_{31} & 0 & 0 & a_{34} \\ 0 & a_{42} & -a_{42} & 0 \end{bmatrix} \begin{bmatrix} N \\ \theta^e \\ \epsilon^e \\ V^e \end{bmatrix} - \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ 0 \end{bmatrix} I$$

with the sign pattern

$$(37) \quad \begin{bmatrix} \phantom{\dot{N}} \\ \phantom{\dot{\theta}^e} \\ \phantom{\dot{\epsilon}^e} \\ \phantom{\dot{V}^e} \end{bmatrix} = \begin{bmatrix} - & 0 & 0 & - \\ - & 0 & 0 & - \\ + & 0 & 0 & + \\ 0 & + & - & 0 \end{bmatrix} \begin{bmatrix} \phantom{N} \\ \phantom{\theta^e} \\ \phantom{\epsilon^e} \\ \phantom{V^e} \end{bmatrix} - \begin{bmatrix} - \\ - \\ - \\ 0 \end{bmatrix} I$$

Make the correspondence  $X_1 = N$ ,  $X_2 = \theta^e$ ,  $X_3 = \epsilon^e$ ,  $X_4 = V^e$  and  $u = I$  so that (36) is given as

$$(38) \quad \dot{X} = AX - Bu$$

It is easily shown using the Routh-Hurwitz conditions that the homogeneous system corresponding to (38) is stable if

$$(39) \quad a_{11}(a_{34} - a_{24}) + a_{14}(a_{21} - a_{31}) < 0$$

given the sign pattern of coefficients in (37).

We interpret Phillips curve theorists ([7], [10] and [12]) as stating that there exists a one-to-one equilibrium relationship between the rate of unemployment and the rate of money wage and product price inflation. The relation is of the familiar inverse form. Phillips curve theory arose from attempts to explain the empirical findings presented by Phillips [9]. The interpretation of the Phillips curve which we present is similarly a rationalization of the empirical results within the context of our macrodynamic model, rather than a justification of Phillips curve theory.

We view the phenomena which Phillips originally presented as resulting from a history of the movements in  $I$ . In particular, we believe that Phillips' observations are connected with movements in the variables relative to each other over the business cycle. A complete investigation of this approach would require the expansion of our model into a theory of the business cycle. However, an initial examination of this question may be made by assuming that the impulse problem of Frisch [3] is solved by setting  $I = \cos t$ .

A second important issue which we investigate is the cyclical pattern of the real wage. The empirical evidence ([1], [2] and [11]) on the subject is not especially conclusive, but tends to support a pro-cyclical

movement in the real wage. This result is in opposition to the Keynesian prediction of a counter-cyclical pattern based on the assumption that the real wage equals the marginal product of labor at every instant of time. Using our model of macro-dynamics, we show that the pro-cyclical pattern of the real wage is consistent with a model which has an inverse equilibrium relation between employment and the real wage. This result underscores our concern that inferences about equilibrium relationships should not be made from disequilibrium, cyclically generated data.

This identification of the cyclical pattern of the real wage with the least squares line of the ellipse between employment and the real wage is especially attractive in the context of Bodkin's study which reports low Durbin-Watson statistics for the regression of the real wage on unemployment rates. It is just such high autocorrelation which would be expected if the least squares line were estimated from cyclical data. Note that the problem is not corrected by an autocorrelation transform-- rather a difference equation specification corresponding to our differential equations should be estimated.

To develop our view of the Phillips curve and the cyclical pattern of the real wage, we recall our basic model of macrodynamics (36) and let

(I) = cost to yield the vector equation

$$(40) \quad \dot{X} = AX - B \text{ cost}$$

We shall assume that the model (36) is stable, i.e., (39) holds, and that the coefficient matrices A and B have the sign pattern of (37). Further, we shall assume that prices are more flexible than money wages, although this condition is by no means necessary to our analysis

$$(41) \quad |a_{31}| > |a_{21}| \quad \text{and} \quad |a_{34}| > |a_{24}| .$$

Finally, we assume, as is conventional, that own rate adjustment coefficients are less than 1 in absolute value and hence that

$$(42) \quad |a_{11}| < 1.$$

Solving (40) and using (29) and (30) yields

$$(43) \quad \begin{bmatrix} N \\ \theta \\ \epsilon \\ v \end{bmatrix} = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \\ q_{31} & q_{32} \\ q_{41} & q_{42} \end{bmatrix} \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$$

a result which, along with the appropriate values of  $q_{ij}$ , is developed in the Appendix.

Any two of the four equations contained in the system (43) permit the expression of one variable in terms of the other as discussed in Section A. Our particular concern is with the slope of the least squares line through the ellipses between  $N$  and  $\theta$ ,  $N$  and  $\epsilon$ , and  $N$  and  $v$ . The first provides the mirror image of our explanation of Phillips' data, while the last presents the cyclical relation between employment and the real wage. For these ellipses to conform to the empirical picture, all three ellipses should have least squares lines with positive slopes. The conditions required for these results are

$$(44a) \quad q_{11}q_{21} + q_{12}q_{22} > 0$$

$$(44b) \quad q_{11}q_{31} + q_{12}q_{32} > 0$$

$$(44c) \quad q_{11}q_{41} + q_{12}q_{42} > 0.$$

These conditions need not hold for all A and B which satisfy our restrictions on the signs of  $a_{ij}$  and  $b_i$ , and the conditions (39), (41) and (42). However, it is possible to find sets of  $a_{ij}$ ,  $b_i$  and  $d_i$ ,  $i = 1, 2$  which do satisfy the conditions (44). One such set of coefficients is

$$(45) \quad A = \begin{bmatrix} -.76 & 0 & 0 & -.20 \\ -.83 & 0 & 0 & -.02 \\ 2.27 & 0 & 0 & .93 \\ 0 & 1.3 & -1.3 & 0 \end{bmatrix} \quad B = \begin{bmatrix} -1 \\ -2.8 \\ -3.5 \\ 0 \end{bmatrix}$$

$$d_1 = d_2 = .8$$

implying

$$(46) \quad Q = \begin{bmatrix} -.26 & .10 \\ -3.30 & 1.35 \\ -.31 & 2.64 \\ -1.68 & 3.90 \end{bmatrix}$$

Finally, it is important to note that these results are possible despite there being an inverse relation between equilibrium values of the real wage and employment since

$$(47) \quad \frac{dv^*}{dI} = \frac{dv^{e^*}}{dI} = \frac{b_1 a_{34} - b_2 a_{14}}{a_{11} a_{34} - a_{14} a_{31}}$$

$$(48) \quad \frac{dN^*}{dI} = \frac{a_{11} b_2 - a_{31} b_1}{a_{11} a_{34} - a_{14} a_{31}}$$

$$(49) \quad \frac{dN^*}{dv^*} < 0 .$$

Appendix

The solution of (40) as given by (3) is

$$(50) \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \\ p_{31} & p_{32} \\ p_{41} & p_{42} \end{bmatrix} \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$$

where

$$(50a) \quad p_{11} = k_2 [(b_2 - b_3)a_{14}a_{42} + a_{42}(a_{34} - a_{24})b_1 - b_1]$$

$$(50b) \quad p_{12} = \frac{k_1}{k_2} p_{11}$$

$$(50c) \quad p_{21} = k_1 k_3 - k_2 k_4$$

$$(50d) \quad p_{22} = -k_2 k_3 - k_1 k_4$$

$$(50e) \quad p_{31} = k_1 k_5 + k_2 k_6$$

$$(50f) \quad p_{32} = -k_2 k_5 + k_1 k_6$$

$$(50g) \quad p_{41} = -k_2 k_7 + k_1 k_8$$

$$(50h) \quad p_{42} = -k_1 k_7 - k_2 k_8$$

$$(51a) \quad k_1 = 1 + a_{24}a_{42} - a_{34}a_{42}$$

$$(51b) \quad k_2 = -a_{11}k_1 + a_{14}a_{42}(a_{21} - a_{31})$$

$$(51c) \quad k_3 = [a_{21}(1 - a_{34}a_{42}) + a_{31}a_{24}a_{42}]b_1 + [a_{11}a_{34}a_{42} - a_{11} - a_{14}a_{42}a_{31}]b_2 \\ + [a_{42}a_{14}a_{21} - a_{42}a_{24}a_{11}]b_3$$

$$(51d) \quad k_4 = (1 - a_{34}a_{42})b_2 + a_{42}a_{24}b_3$$

$$(51e) \quad k_5 = [a_{31}(1 + a_{34}a_{42}) - a_{21}a_{34}a_{42}]b_1 + [a_{11}a_{34}a_{42} - a_{24}a_{31}a_{42}]b_2 \\ + [a_{14}a_{21}a_{42} - a_{11} - a_{11}a_{24}a_{42}]b_3$$

$$(51f) \quad k_6 = a_{34}a_{42}b_2 - (1 + a_{34}a_{42})b_3$$

$$(51g) \quad k_7 = a_{42}(a_{31} - a_{21})b_1 + a_{11}a_{42}(b_2 - b_3)$$

$$(51h) \quad k_8 = a_{42}(b_2 - b_3) .$$

Finally, recall from (29) and (32) that

$$(52) \quad \theta = \theta^e + \frac{a_{21}}{d_1}N + \frac{a_{24}}{d_1}v^e + \frac{b_2}{d_1}I$$

$$(53) \quad \epsilon = \epsilon^e + \frac{a_{31}}{d_2}N + \frac{a_{34}}{d_2}v^e + \frac{b_3}{d_2}I$$

$$(54) \quad v = a_{42} \int (\theta - \epsilon) dt .$$

Hence, from (50)-(54) we have

$$(55) \quad \begin{bmatrix} N \\ \theta \\ \epsilon \\ v \end{bmatrix} = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \\ q_{31} & q_{32} \\ q_{41} & q_{42} \end{bmatrix} \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$$

where

$$(56a) \quad q_{11} = p_{11}$$

$$(56b) \quad q_{12} = p_{12}$$



$$(56c) \quad q_{21} = p_{21} + \frac{a_{21}}{d_1} p_{11} + \frac{a_{24}}{d_1} p_{41} + \frac{b_2}{d_1}$$

$$(56d) \quad q_{22} = p_{22} + \frac{a_{21}}{d_1} p_{12} + \frac{a_{24}}{d_1} p_{42}$$

$$(56e) \quad q_{31} = p_{31} + \frac{a_{31}}{d_2} p_{11} + \frac{a_{34}}{d_2} p_{41} + \frac{b_3}{d_2}$$

$$(56f) \quad q_{32} = p_{32} + \frac{a_{31}}{d_2} p_{12} + \frac{a_{34}}{d_2} p_{42}$$

$$(56g) \quad q_{41} = a_{42}(q_{22} - q_{23})$$

$$(56h) \quad q_{42} = a_{42}(q_{13} - q_{12}) .$$

References

- [1] Bodkin, R. G., "Real Wages and Cyclical Variations in Employment: A Re-Examination of the Evidence," Canadian Journal of Economics (August 1969), pp. 353-374.
- [2] Dunlop, J. T., "The Movement of Real and Money Wage Rates," Economic Journal (September 1938), pp. 413-434.
- [3] Frisch, R., "Propagation Problems and Impulse Problems in Dynamic Models" in Gordon, R. A. and Klein, L. R., eds., A.E.A. Readings in Business Cycles (Irwin, 1965).
- [4] Gersovitz, M., An Essay on Macrodynamics: Unemployment and Inflation, unpublished Ph.D. thesis, Yale University, 1975.
- [5] Grossman, H. I., "Aggregate Demand, Job Search, and Employment," Journal of Political Economy (November/December 1973), pp. 1353-1369.
- [6] Grossman, H. I., "The Cyclical Pattern of Unemployment and Wage Inflation," Economica (November 1974), pp. 403-413.
- [7] Lipsey, R. G., "The Relation Between Unemployment and the Rate of Change of Money Wage Rates in the United Kingdom, 1862-1957," Economica (February 1960), pp. 1-31.
- [8] Phelps, E. S., ed., Microeconomic Foundations of Employment and Inflation Theory, (Norton, 1969).
- [9] Phillips, A. W., "The Relation Between Unemployment and the Rate of Change of Money Wage Rates in the United Kingdom, 1861-1957," Economica (November 1958), pp. 283-299.
- [10] Samuelson, P. A. and Solow, R. M., "Analytical Aspects of Anti Inflation Policy," American Economic Review (May 1961), pp. 177-194.
- [11] Tarshis, L., "Changes in Real and Money Wages," Economic Journal (March 1939), pp. 150-154.

- [12] Tobin, J., "Inflation and Unemployment," American Economic Review March 1972), pp. 1-18.
- [13] Tobin, J., "Money and Income: Post Hoc Ergo Propter?" Quarterly Journal of Economics (May 1970), pp. 301-317.