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by

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1. INTRODUCTION

In the literature on econometric theory, the problem of specifying and estimating a firm's production function has received some attention. Marschak and Andrews [13], in a classic paper, have shown that under the assumption of perfect competition and profit maximizing conditions the Ordinary Least Squares (OLS) method does not provide consistent estimates of the parameters of the Cobb-Douglas production function. Hoch [8], Kmenta [11] and Mundlak [15] provided consistent estimators of the parameters under the assumption that the "technical" disturbance (in the production function) and the "economic" disturbances (in the profit maximizing equations) are uncorrelated. Further, in an important paper, Zellner, Kmenta and Dréze [24], showed that under the assumption of perfect competition and expected profit maximizing conditions OLS method does provide consistent and unbiased estimates of the parameters. Thus, the literature concerning the estimation of the firm's production function is based either on the notion that firms, interested in maximizing profits, formulate input-output decisions under certainty conditions or on the assumption that firms, encountering uncertainty, are risk-neutral and interested in maximizing expected profits. However, in the recent developments concerning the theory of the firm facing uncertainty, it has been emphasized that the firms in general evince aversion to risk. These developments have found expression in the contributions by Horowitz [9], Baron [2], Sandmo [17], Leland [12] and Batra and Ullah [3,4,5] who all conclude, among other things, that the optimal output of such a firm is smaller than the optimal output of the firm operating in a certainty environment. All these developments clearly point out the need for a further specification and estimation of the production function.

The aim of this paper is two-fold. Firstly, we analyze the exact sampling distribution of the estimators of the parameters of the Cobb-Douglas production function in the case of Marschak and Andrews model and Zellner *et al.* model. This has been done in Sections 2 and 3. Also in Section 3 we suggest an iterative method of estimating parameters under a more general assumption that technical and economic disturbances are correlated. Secondly, in view of the recent developments in the theory of firm under uncertainty, we attempt to specify and estimate the model of a risk-averse competitive firm under output uncertainty.[1] The situation of price uncertainty can be treated similarly. The specified model turns out to be similar to Zellner *et al.* This has been analyzed in Section 4. An iterative method of estimating coefficient of variation of the marginal utility from profit and of output of the firm has also been suggested.

2. THE MARSCHAK-ANDREWS MODEL

We begin with a brief review of the famous Marschak-Andrews model of estimation which stresses the simultaneous equation character of the estimation procedure. Consider a competitive firm which seeks to maximize profit (π), which is defined as[2]

(1) $$\pi = pX - wL - rK$$

where p = the price of the product, X = output, L = labor, K = capital, w = wage rate, and r = price of capital. Facing a production function

$$X = f(L, K)$$

and given product and input prices, the competitive firm chooses the optimal quantities of inputs by setting $\partial\pi/\partial L$ and $\partial\pi/\partial K$ to zero, so that

(2) $$pf_L = w$$

and

(3) $$pf_K = r$$

where $f_L = \partial f/\partial L$ and $f_K = \partial f/\partial K$. Since the choice of inputs is governed by economic considerations, the estimation of the parameters of any production function should be based not only on the form of the function but also on the profit maximizing conditions. This is the logic behind the Marschak-Andrews procedure that emphasizes the simultaneous equation character of the model.

If the production function is of the Cobb-Douglas type, then

(4) $$X = f(L, K) = AL^{\alpha_1}K^{\alpha_2}$$

where A, α_1, and α_2 are some parameters. Differentiating equation (1) partially with respect to L and K and substituting in equations (2) and (3) yields the following optimality conditions:

(5) $$\frac{X}{L} = \frac{w}{p\alpha_1}$$

and

(6) $$\frac{X}{K} = \frac{r}{p\alpha_2}$$

When it comes to estimation, equations (4), (5) and (6) are modified by the introduction of stochastic disturbances. This, along with the logarithms of equations (4), (5) and (6), furnishes the following system of equations:

(7) $$x_{0i} - \alpha_1 x_{1i} - \alpha_2 x_{2i} = \lambda_0^* + v_{0i}$$

(8) $$x_{0i} - x_{1i} = \lambda_1^* + v_{1i}$$

and

(9) $$x_{0i} - x_{2i} = \lambda_2^* + v_{2i}$$

where the subscript i denotes the i^{th} firm, $i=1,\cdots,N$, $x_{0i} = \log X_i$, $x_{1i} = \log L_i$, $x_{2i} = \log K_i$, $\lambda_0^* = \log A$, v_{0i} and v_{qi} $(q=1,2)$ are stochastic disturbances, and

$$\lambda_1^* = \log \frac{wR_1}{p\alpha_1} \quad \text{and} \quad \lambda_2^* = \log \frac{rR_2}{p\alpha_2}$$

which are the same for all firms, because p, w, and r are assumed to be given to competitive firms. The parameters R_1 and R_2 are introduced to reflect that firms may commit systematic errors; otherwise $R_1 = R_2 = 1$.[3]

It can be noted from the reduced form for x_0, x_1 and x_2 (using (7) to (9)) that x_1 and x_2 are correlated with v_0. Thus the OLS estimators for α_1 and α_2 in (7) will be inconsistent. Under the assumption that v_1 and v_2 are uncorrelated with v_0, Hoch [8] provided a consistent estimator of the parameters α_1 and α_2 in (7) which, as shown by Kmenta [11], is indirect least squares estimator (ILS). The ILS estimator of α_1 and α_2 can be written as[4]

$$\hat{\alpha}_q = \frac{\hat{b}_q}{1 + \hat{b}_1 + \hat{b}_2} \quad ; \; q = 1,2$$

where $\hat{b}_1$ and $\hat{b}_2$ are OLS estimators of b_1 and b_2 in the reduced form equation for x_{0i} as below

$$x_{0i} = b_0 + b_1(x_{1i} - x_{0i}) + b_2(x_{2i} - x_{0i}) + \ell_i$$

where

$$b_q = \frac{\alpha_q}{1 - \alpha_1 - \alpha_2} \quad \text{and} \quad \ell_i = \frac{v_{0i}}{1 - \alpha_1 - \alpha_2} .$$

It has been shown by Wu [23] and Ullah and Agarwal [20] that the exact sampling distribution of $\hat{\alpha}_q$ is a ratio of t distributions. Further, the moments of any order of $\hat{\alpha}_q$ do not exist.[5]

3. THE ZELLNER-KMENTA-DRÉZE MODEL

Zellner, Kmenta and Dréze modify the Marschak-Andrews model on the grounds that the random term v_{0i} makes the production function stochastic, so that first profit-maximization is not possible, because profits become random due to the randomness of the production function, and second, the entrepreneurial choice of the labor and capital inputs i.e., x_{1i} and x_{2i}, is not independent of v_{0i}, so that the classical least-squares estimates of the production function parameters are, in general, biased and inconsistent.

Zellner, et al., then propose an alternative model where the stochastic terms v_{0i} enters directly in the production function and where firms are interested in maximizing expected profits. That is, the production is given by

(10) $$X^* = A L^{\alpha_1} K^{\alpha_2} e^{v_0} = X e^{v_0}$$

and

(11) $$E(\pi) = pE[X^*] - wL - rK$$

where

(12) $$E[X^*] = A L^{\alpha_1} K^{\alpha_2} e^{(1/2)\sigma_{oo}} ,$$

σ_{oo} is the variance of the production function disturbance v_0 and where

$$E[e^{v_0}] = e^{(1/2)\sigma_{oo}}$$

assuming that v_0 is normally distributed with mean zero.

Expected profit maximization leads to

(13) $$\frac{\partial E[\pi]}{\partial L} = e^{(1/2)\sigma_{oo}} \alpha_1 \frac{X}{L} - \frac{w}{p} = 0$$

and

(14) $$\frac{\partial E[\pi]}{\partial K} = e^{(1/2)\sigma_{oo}} \alpha_2 \frac{X}{K} - \frac{r}{p} = 0$$

Equations (12), (13) and (14) in logarithms are written as:

(15) $$x_{0i} - \alpha_1 x_{1i} - \alpha_2 x_{2i} = \alpha_0 + v_{0i}$$

(16) $$x_{0i} - x_{1i} = k_1 + v_{0i} + v_{1i}$$

(17) $$x_{0i} - x_{2i} = k_2 + v_{0i} + v_{2i}$$

where $\alpha_0 = \log A$, $k_1 = \log(\frac{wR_1}{p\alpha_1}) - \frac{1}{2}\sigma_{oo}$, $k_2 = \log(\frac{rR_2}{p\alpha_2}) - \frac{1}{2}\sigma_{oo}$

and v_{0i}, v_{1i} and v_{2i} are, as before, the disturbance terms. Further $x_{0i} = \log X_i^*$ and x_{1i} and x_{2i} are as defined in Section 2.

An important implication of this model is that inputs no longer depend on the disturbance term in the production function. This can be seen by solving for the three variables x_{0i}, x_{1i} and x_{2i} in the three equations (15)-(17), to obtain

$$x_{0i} = \frac{\alpha_0 - \alpha_1 k_1 - \alpha_2 k_2 + (1-\alpha_1-\alpha_2)v_{0i} - \alpha_1 v_{1i} - \alpha_2 v_{2i}}{1 - \alpha_1 - \alpha_2}$$

$$x_{1i} = \frac{[\alpha_0 + (\alpha_2-1)k_1 - \alpha_2 k_2 + (\alpha_2-1)v_{1i} - \alpha_2 v_{2i}]}{1 - \alpha_1 - \alpha_2}$$

and

$$x_{2i} = \frac{\alpha_0 - \alpha_1 k_1 + (\alpha_1-1)k_2 - \alpha_1 v_{1i} + (\alpha_1-1)v_{2i}}{1 - \alpha_1 - \alpha_2}$$

and it is clear that v_{0i} in no way enters into the determination of x_{1i} and x_{2i}. Thus, if we assume that v_{0i} is independent of v_{1i} and v_{2i}, then the ordinary least-square estimates of equation (15) will no longer be inconsistent and biased.

This is the important result derived by Zellner, et al., who also show that under the normality assumption of disturbances the maximum likelihood estimators are the same as OLS estimates of the parameters α_1 and α_2 in (15). We shall analyze below the sampling distribution of the OLS (Maximum Likelihood) estimators of α_1 and α_2.

3.1 The Exact Sampling Distribution of the Maximum Likelihood Estimator

Let us write the equations (15)-(17) as

$$x_{0i} - \alpha_1 x_{1i} - \alpha_2 x_{2i} = \alpha_0 + v_{0i}$$

(18) $$(\alpha_1 - 1)x_{1i} + \alpha_2 x_{2i} = k_1 - \alpha_0 + v_{1i}$$

$$\alpha_1 x_{1i} + (\alpha_2 - 1)x_{2i} = k_2 - \alpha_0 + v_{2i}$$

where we assume that

$$Ev_{0i} = Ev_{1i} = Ev_{2i} = 0$$

$$Ev_{0i}^2 = \sigma_{oo}, E\, v_{qi}^2 = \sigma_{qq} \quad \text{and} \quad E\, v_{1i}v_{2i} = \sigma_{12}\ , \ q=1,2$$

(19)

$$Ev_{0i}v_{0j} = E\, v_{qi}v_{qj} = E\, v_{1i}v_{2j} = 0 \quad \text{for} \quad i \neq j\ ;\ \text{and}$$

$$Ev_{0i}v_{qj} = 0 \text{ for both } j=i \text{ and } j\neq i\ ,\ i,j=1,\cdots,N\ .$$

In the matrix notation we can write it as

$$x_0 = \alpha_0 + X\alpha + v_0$$

(20)

$$XA = V + K$$

where

(21) $$\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}, \quad A = \begin{bmatrix} \alpha_1 - 1 & \alpha_1 \\ \alpha_2 & \alpha_2 - 1 \end{bmatrix}$$

and

(22) $$X = [x_1\ x_2]\ ,\ \iota = [v_1\ v_2]\ ,\ K = [(k_1-\alpha_0)\iota \quad (k_2-\alpha_0)\iota]$$

are each N x 2 matrices; x's and v's being each N x 1 vectors and ι is a N x 1 vector of unit elements. Further x_0 and v_0 are each N x 1 vectors. It should be noted from (19) that under the assumption of the normality of disturbances the vector

$$[v_{0i}\quad v_{1i}\quad v_{2i}]\ ;\ i=1,\cdots,N$$

is multivariate normal with a common mean vector 0 and covariance matrix

$$\Sigma = \begin{bmatrix} \sigma_{oo} & 0 \\ 0 & \Sigma_{11} \end{bmatrix} \; ; \; \Sigma_{11} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} . \tag{23}$$

Further, using (20), we can show that (x_{0i}, x_{1i}, x_{2i}) are independently distributed as multivariate normal with a common mean vector and covariance matrix

$$\Sigma^* = \begin{bmatrix} \sigma^*_{oo} & B'\Sigma_{11}B\,\alpha \\ \alpha'B'\Sigma_{11}B & B'\Sigma_{11}B \end{bmatrix} , \; \sigma^*_{oo} = \sigma_{oo} + \alpha'B'\Sigma_{11}B\alpha \; , \; B=A^{-1} \tag{24}$$

The OLS estimator of α_o and α can now be written as[6]

$$\begin{aligned} \hat{\alpha} &= (X'CX)^{-1}X'C\,x_0 \\ \hat{\alpha}_o &= \frac{1}{N}\,\iota'(x_0 - X\hat{\alpha}) \end{aligned} \tag{25}$$

where

$$C = I - \frac{\iota\iota'}{N} \tag{26}$$

is an indempotent matrix of rank N-1.

Using (20), we can write[7]

$$\hat{\alpha} = A(V'CV)^{-1}V'C\,v_0 + \alpha \tag{27}$$

Since C is an indempotent matrix of rank N-1 we note that the matrix

$$\begin{bmatrix} x_0'Cx_0 & x_0'CX \\ X'Cx_0 & X'CX \end{bmatrix} \tag{28}$$

follows a central Wishart distribution with parameter set (3, Σ^*, N-1). Thus, using Theorem 1 given in [21, p. 243], $\hat{\alpha}$ is distributed as bivariate student t, i.e.,[8]

$$\hat{\alpha} \sim t\{\alpha, B'\Sigma_{11}B, \sigma_{oo}, N-2, 2\} . \tag{29}$$

The density function of $\hat{\alpha}$ can be written as

$$f(\hat{\alpha}) = a[\sigma_{oo} + (\hat{\alpha}-\alpha)'B'\Sigma_{11}B\,(\hat{\alpha}-\alpha)]^{-\frac{N}{2}} \tag{30}$$

where

$$(31) \qquad a = |B'\Sigma_{11}B|^{\frac{1}{2}} (\sigma_{oo})^{\frac{N-2}{2}} \left[\prod_{q=1}^{2} B(\tfrac{1}{2}, \tfrac{N-q}{2})\right]^{-1} ,$$

B( ), represents beta function. Also, the mean vector and covariance matrix of $\hat{\alpha}$ are[9]

$$E\hat{\alpha} = \alpha$$

(32)

$$V(\hat{\alpha}) = E(\hat{\alpha}-\alpha)(\hat{\alpha}-\alpha)' = \frac{1}{N-4} \sigma_{oo}(B'\Sigma_{11}B)^{-1} = \frac{1}{N-4} \sigma_{oo} A \Sigma_{11}^{-1} A'$$

Further using Wegge [21], an unbiased estimator of the covariance matrix of $\hat{\alpha}$ is given by

$$(33) \qquad \hat{V}(\hat{\alpha}) = \frac{1}{N-3} [x_0' M x_0][X'CX]^{-1}$$

where

$$(34) \qquad M = C[I - X(X'CX)^{-1}X']C .$$

3.2 <u>Estimation of α when v_0 is Correlated with v_1 and v_2</u>

It may be noted that both the indirect least squares and OLS estimators of the parameters of the Cobb-Douglas production function are obtained under the assumption that the "technical" disturbance v_0 is uncorrelated with "economic" disturbances v_1 and v_2. While serial independence among the various disturbances is not a great problem, since it could be ensured by means of the well-known mechanical devices,[10] the independence of technical and economic disturbances is hard to reconcile, particularly when the profit maximizing conditions are obtained subject to the technical relationship--the production function.

In this section, therefore, we propose an iterative method of estimating α when the "technical" and "economic" disturbances are correlated. Thus, let us specify the relationship between the technical and economic disturbances as

(35) $$v_{0i} = \theta_1 v_{1i} + \theta_2 v_{2i} + e_{0i}$$

and between the two economic disturbances as,

(36) $$v_{2i} = \theta v_{1i} + e_{2i}$$

where we assume that e_{0i}, for $i=1,\cdots,N$ are independently distributed with mean 0 and variance σ_0^2 and it is also distributed independently of v_{1i}, v_{2i} and e_{2i}. Further e_{2i}, for $i=1,\cdots,N$ are independently distributed with mean 0 and variance σ_2^2 and it is also distributed independently of v_{1i}.

Under the above specification we can write the equation (15) as

(37) $$x_{0i} = \alpha_0 + \alpha_1 x_{1i} + \alpha_2 x_{2i} + \theta_1 v_{1i} + \theta_2 v_{2i} + e_{0i}$$

$$= \alpha_0 + \alpha_1 x_{1i} + \alpha_2 x_{2i} + (\theta_1 + \theta\theta_2) v_{1i} + e_{0i}$$

We have to estimate now three additional parameters θ, θ_1 and θ_2 which will create the identification problem. In fact this problem was overcome by Zellner et al. [24] by assuming v_0 to be uncorrelated with v_1 and v_2 and then estimating α's with OLS which is consistent. However, in the above equation a simple application of OLS would not provide estimates of α's and θ's. We shall, therefore, estimate it in different steps as outlined below.

3.3 The Estimation Procedure

First, applying OLS (Maximum Likelihood) to equation (15) we obtain consistent and unbiased estimators of α_0, α_1 and α_2 as given by Zellner et al. and presented in (25). This will then provide consistent and unbiased estimates of x_{0i} and v_{0i}, viz., $\hat{x}_{0i}$ and $\hat{v}_{0i}$. To know the estimates of individual parameters in (37) we may proceed with the iterative estimation technique as below.

Step 1: We begin with the estimates $\hat{x}_{0i}$ and $\hat{v}_{0i}$ as obtained above. This may be treated as an initial proxy for x_{0i} which may amount to using $\theta_1 = \theta_2 = \theta = 0$ as their corresponding proxies. Given this we may write from (16)

(38) $$x_{1i} = -k_1 + (\hat{x}_{0i} - \hat{v}_{0i}) + v_{1i}\ .$$

From equation (38), we can then obtain a consistent estimate of v_{1i}, say, $\hat{v}_{1i}$ by OLS.

Step II: Given $\hat{v}_{1i}$ we can write (17) by using (36) as

(39) $$x_{2i} = -k_2 + (\hat{x}_{0i} - \hat{v}_{0i}) + \theta\hat{v}_{1i} + e_{2i}\ .$$

We can again apply OLS and obtain the estimates for x_{2i} and θ as $\hat{x}_{2i}$ and $\hat{\theta}$.

Step III: Given $\hat{x}_{1i}$, $\hat{v}_{1i}$ from (38) and $\hat{x}_{2i}$, $\hat{\theta}$ from (39) we can write (37) as

(40) $$x_{0i} = \alpha_0 + \alpha_1\hat{x}_{1i} + \alpha_2\hat{x}_{2i} + \theta_1\hat{v}_{1i} + \theta_2\hat{v}_{2i} + e_{0i}$$

Thus, applying OLS to (40) we can obtain estimates of α_0, α_1, α_2 and θ_1 and θ_2. Since this provides second round estimates of x_{0i} and v_{0i} we can switch back and forth between these steps until the estimates converge to some stable values.

4. SPECIFICATION AND ESTIMATION OF THE MODEL FOR RISK-AVERSE FIRM

Uncertainty in the deterministic competitive model of the firm as discussed in Section 2 may be incorporated in a variety of plausible ways. The firm may be uncertain about the price at which it will be able to sell its product or there may be uncertainty in the production function where the supply of the finished product may be sensitive to factors beyond the firm's control, and so on. In the following, we develop a two-input model of the competitive firm which regards the level of its finished product to be stochastic. Results are qualitatively unmodified, if the firm is uncertain about the product price. We assume that the firms are (1) risk-averse (2) they base their current input-output decisions on the given input prices and a probability distribution of X* and (3) they are interested in maximizing the expected utility from profits and their attitude towards risk can be described by a Von Neumann-Morgenstern utility function.[11]

Let U stand for utility from profits. The firm's utility function is given by

$$U = U(\pi) = U[p\,X^* - wL - rK]$$

where $U'(\pi) = dU(\pi)/d\pi > 0$ and $U''(\pi) = dU'(\pi)/d\pi \lesseqgtr 0$, depending on whether the firm is risk-averse, risk-neutral, or a risk-preferrer and

$$X^* = AL^{\alpha_1}K^{\alpha_2}e^{v_0} = X\,e^{v_0}$$

as given in (10). The expected utility from profit is given by

$$E[U(\pi)] = E[U\{pX^* - wL - rK\}]$$

The first-order conditions for expected-utility maximization are given by

(41) $$\frac{\partial E[U(\pi)]}{\partial L} = E[U'(\pi)\,\frac{\partial \pi}{\partial L}] = E[U'(\pi)(pf_L e^{v_0} - w)] = 0$$

and

(42) $$\frac{\partial E[U(\pi)]}{\partial K} = E[U'(\pi)\,\frac{\partial \pi}{\partial K}] = E(U'(\pi)(pf_K e^{v_0} - r)] = 0$$

Since w and f_L are non-random, equations (41) and (42) yield

(43) $$\frac{E[U'(\pi)e^{v_0}]}{E(U'(\pi)]}\,f_L p = w$$

and

(44) $$\frac{E[U'(\pi)e^{v_0}]}{E[U'(\pi)]}\,f_K p = r$$

If the firms are risk-neutral, so that $U'' = 0$, then the expression $E[U'(\pi)e^{v_0}]/E[U'(\pi)]$ reduces to $E\,e^{v_0}$, in which case the equations (43) and (44) reduce to equations (13) and (14) of Zellner, et al. model.

However, in the general case of risk-averse behavior of the firm, it can be shown that

(45) $$E[U'(\pi)e^{v_0}] = \frac{1}{X}\,E[U'(\pi)X^*]$$
$$= \frac{1}{X}[E\,U'(\pi)E\,X^* + \operatorname{cov}(U',X^*)]$$

where cov(U′,X*) is the covariance between $U'(\pi)$ and X*. For risk-averse firms, this covariance is negative.[12]

In view of (45), (43) and (44) can be written as[13]

$$(46) \qquad \frac{P}{X}\left[EX^* + \frac{\text{cov}(U',X^*)}{E[U'(\pi)]}\right] f_L = w$$

$$(47) \qquad \frac{P}{X}\left[EX^* + \frac{\text{cov}(U',X^*)}{E[U'(\pi)]}\right] f_K = r$$

and with cov(U′,X*) < 0 and $E[U'(\pi)] > 0$, it is clear that

$$p\ e^{\frac{1}{2}\sigma_{oo}} f_L > w$$

and

$$p\ e^{\frac{1}{2}\sigma_{oo}} f_K > r$$

In other words, the risk-averse competitive firm hires inputs in such a way that the expected marginal value product of each factor exceeds the given input price. On the other hand, a risk-neutral firm, for which the covariance is zero, behaves like a firm operating under certainty conditions in that $p\ e^{\frac{1}{2}\sigma_{oo}} f_L = w$ and $p\ e^{\frac{1}{2}\sigma_{oo}} f_K = r$. Clearly then the optimal input choice in the presence of risk-aversion is not governed by the same principles as in the risk-neutrality or the certainty case. This in turn has repercussions for the firm's optimal output.

4.1 Estimation of the Model

Let us rewrite the complete model by using (46) and (47) as

$$(48) \qquad X^* = AL^{\alpha_1}K^{\alpha_2}e^{v_{0i}}$$

$$(49) \qquad p\alpha_1\ e^{\frac{1}{2}\sigma_{oo}} \frac{X}{L}[1 + \rho\ \beta\ \delta] = w$$

$$(50) \qquad p\alpha_2\ e^{\frac{1}{2}\sigma_{oo}} \frac{X}{K}[1 + \rho\ \beta\ \delta] = r$$

where

(51) $$\beta = \frac{\sigma_{U'}}{EU'}, \quad \delta = \frac{\sigma_{X*}}{EX*} = e^{-\frac{1}{2}\sigma_{oo}} \sqrt{e^{\sigma_{oo}}(e^{\sigma_{oo}} - 1)}$$

and

(52) $$\rho = \text{cov}(U', X*)/\sigma_{U'}\sigma_{X*}$$

such that $-1 \leq \rho \leq 0$ under the assumption of risk-averse firm. Further β and δ may be defined as the coefficient of variation in the marginal utility and output of the firm, respectively.

Using (51) and (52) in (49) and (50) we have

(53) $$AL^{\alpha_1 - 1} K^{\alpha_2} = \frac{w}{p\alpha_1 (1 + \rho\beta\ \delta)} e^{-\frac{1}{2}\sigma_{oo}}$$

(54) $$AL^{\alpha_1} K^{\alpha_2 - 1} = \frac{r}{p\alpha_2 (1 + \rho\beta\ \delta)} e^{-\frac{1}{2}\sigma_{oo}}$$

Taking logarithms of (48), (53) and (54) we have

(55) $$x_{0i} - \alpha_1 x_{1i} - \alpha_2 x_{2i} = A + v_{0i}$$

(56) $$x_{0i} - x_{1i} = \lambda_1 + v_{0i} + v_{1i}$$

(57) $$x_{0i} - x_{2i} = \lambda_2 + v_{0i} + v_{2i}$$

where $x_{0i} = \log X_i^*$, $x_{1i} = \log L_i$, and $x_{2i} = \log K_i$, and

(58) $$\lambda_1 = \log \frac{w}{p\alpha_1 (1 + \rho\beta\delta)} - \frac{1}{2}\sigma_{oo}$$

(59) $$\lambda_2 = \log \frac{r}{p\alpha_2 (1 + \rho\beta\delta)} - \frac{1}{2}\sigma_{oo}$$

The system of equations (55) through (57) takes the same form as the Zellner-Kmenta-Dréze model does. Therefore, it can be easily shown that x_{1i} and x_{2i} are independent of v_{0i} in which case application of OLS will yield consistent and unbiased estimates of α_1 and α_2. Further the sampling distribution of $\hat{\alpha}_1$ and $\hat{\alpha}_2$ will be bivariate student t as discussed in Section 3. Once $\hat{\alpha}_1$ and $\hat{\alpha}_2$ are known, the other parameters, namely ρ, β and δ involved nonlinearly in λ_1 and λ_2 can be obtained as follows:

Let the estimates of λ_1 and λ_2 be given by

(60) $$\hat{\lambda}_1 = \bar{x}_0 - \bar{x}_1 \; , \; \hat{\lambda}_2 = \bar{x}_0 - \bar{x}_2 \; .$$

Further, as given in [24], the maximum likelihood estimate of σ_{oo} can be obtained as[14]

(61) $$\hat{\sigma}_{oo} = \frac{1}{N} \sum^{N} \hat{v}_{0i}^2$$

Then using (58) and (59) we may write

(62) $$\hat{\rho} = \left\{ \frac{r}{p\hat{\alpha}_2 \hat{\lambda}_1^*} - 1 \right\} \frac{1}{\hat{\delta}\hat{\beta}}$$

and

(63) $$\hat{\beta} = \left\{ \frac{r}{p\hat{\alpha}_2 \hat{\lambda}_2^*} - 1 \right\} \frac{1}{\hat{\delta}\hat{\rho}}$$

where $\hat{\lambda}_1^*$ and $\hat{\lambda}_2^*$ are antilog of $\hat{\lambda}_1 + \frac{1}{2}\hat{\sigma}_{oo}$ and $\hat{\lambda}_2 + \frac{1}{2}\hat{\sigma}_{oo}$, respectively.

Since w, r and p are known constants (because of the assumption of perfect competition) and δ may be estimated by using (61) we may estimate ρ and β iteratively from equations (62) and (63). Starting with $\rho = -1$ as the initial proxy for ρ we may estimate β from (63) and estimate ρ using the value of β so obtained. Thus we may switch back and forth between (62) and (63) until the estimates converge to a "fixed point", say $\{\tilde{\rho}, \tilde{\beta}\}$ which is unique and invariant to the initial conditions.

Footnotes

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[1]Feldstein [7] does in passing mention the case of the risk-averse behavior, but his formulation is not based on entrepreneurial choice or decision-making, but on the decision-making of a non-profit making organization such as a hospital, etc. Consequently his model is different from ours.

[2]Marschak and Andrews examined the case of imperfect competition. However, over the years, attention has been restricted to the case of a competitive model. See, for example, Zellner et al. and Kmenta [11].

[3]The rationale behind the disturbance terms has been eloquently provided by Zellner et al., among others.

[4]Ullah and Agarwal [20] have generalized Hoch's estimator for Q inputs and have shown that it is identical with the indirect least squares estimator.

[5]Wu [23] has analyzed the approximate distribution of $\hat{\alpha}_q$ and discussed the test for constant returns to scale.

[6]Cf. Theil [19, p. 39].

[7]The expressions for $\hat{\alpha}_1$ and $\hat{\alpha}_2$ from this can be shown to be identical with those of Kmenta [11].

[8]Note that $\Sigma_{22\cdot 1}$ in Wegge's is equal to σ_{oo} in our case.

[9]It was earlier shown by Zellner et al. [24] that $\hat{\alpha}$ is unbiased. However, the expression for variance was not obtained. Further, they had indicated that the posterior distribution of $\hat{\alpha}$ will be student t.

[10]See [6, 10, 18, 22].

[11]Postulation of this utility function implies the postulation of the axioms underlying the Neumann-Morgenstern utility function. Among other things, these axioms require that the decision-making is done by one individual or a group of individuals with identical attitudes. This is perhaps a serious weakness of this utility function, but it is central to much of the recent work on the theory of the firm under uncertainty. The reason perhaps lies in the fact that this utility function enables one to characterize the decision-maker's risk-attitude in a simple manner. Also see Arrow [1] and Pratt [16].

[12] The covariance between $U'(\pi)$ and X^* is negative if the firms are risk-averse, because

$$\text{sign Cov } [U'(\pi),\ X^*] = \text{sign } \frac{dU'(\pi)}{d\pi} \frac{d\pi}{dX^*}$$

$$= \text{sign } U''(\pi) \frac{d\pi}{\partial X^*}$$

and with $U''(\pi) < 0$ and $d\pi/dX^* > 0$. With Cov $[U'(\pi),\ X^*] < 0$, it is clear that $E[U'(\pi)X^*] < E[U'(\pi)]EX^*$.

[13] If uncertainty is because output price is a random variable with $Ep = \mu$ and $X = AL^{\alpha_1}K^{\alpha_2}$ then we will have

$$\left[\mu + \frac{\text{cov } (U',p)}{E[U'(\pi)]}\right] f_L = w$$

$$\left[\mu + \frac{\text{cov}(U',p)}{E[U'(\pi)]}\right] f_K = r.$$

One can easily obtain the above two equations in the case where both p and X^* are random variables.

[14] An unbiased estimate of σ_{oo} can be written as

$$\hat{\sigma}_{oo} = \frac{1}{N-3} x_0' M x_0$$

by using Wegge [21, p. 243], where M is defined in (34).

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