1975

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Research Report 7522

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October 1975

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Whenever a producer charges different prices for the same product in the same time period, he is said to be practicing price discrimination. Although any imperfectly competitive producer may follow this practice, most micro economic texts attribute the phenomenon of price discrimination to a profit maximizing monopolist who optimizes by charging different prices in different markets, provided each market has a different elasticity of demand.

The purpose of this paper is to extend the usual textbook discussion of the deterministic price discriminating behavior to the case of stochastic demand. As in the conventional theory, we assume that the total market for the monopolist's product can be subdivided into effectively separated markets each represented by its own demand function. However, the demand curves for the submarkets are not known at the time of decision making. As will be seen below, our analysis throws considerable light on the phenomenon of price discrimination. For example, for a price-setting monopolist we find that even if the elasticity of demand is the same in all markets, price discrimination will occur if the probability distributions of the various market demand curves differ. Conclusions also depend on the way in which the random elements enter into the demand functions. In the case of a multiplicative random term, it turns out that if the marginal cost is constant, the traditional result holds in entirety even if demand curves are stochastic. What is of greater interest is that this latter result is valid regardless of the attitude of the monopolist towards risk.

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1. Assumptions and Formulation of the Problem

In the certainty case, the decision making of the producer does not depend on whether the monopolist is a price-setter or a quantity setter. However, as recently shown by Leland, under uncertainty the choice of the behavioral made plays a critical role in determining the producer's behavior. For this reason, we will consider below the case of both the price-setting and the quantity-setting monopolist. However, regardless of the choice of the behavioral model, the following assumptions will be maintained throughout the analysis.

1. For the sake of simplicity we assume that the producer sells a homogeneous product in two effectively separated markets, each having an independent demand function which is not known at the time of decision making.

2. The demand function for each market is given by

$$g_i(p_i, q_i, u_i) = 0 \tag{1}$$

where p = price, q = quantity produced, u is a random variable with a subjective probability density $dF(u)$, mean $\bar{u}$ and variance σ^2, and i denotes the ith market (i = 1,2)[.1] Following Leland [1972] we will assume that for any u_i, p_i and q_i are negatively related and that higher values of u_i denote higher demand in the ith market. We will also invoke the principle of increasing uncertainty (PIU) which according to Leland ensures that marginal and expected marginal revenues respond to u_i in the same manner. Given that the submarkets are effectively separated, u_1 and u_2 and hence g_1 and g_2 are independent.

3. The monopolist seeks to maximize expected utility from profit derived from sale in the two markets. Let π = profit, U = utility, and E = expectations

operator and C(q) = the total cost function with $q = q_1 + q_2$. Then

$$\pi = p_1 q_1 + p_2 q_2 - C(q) \tag{2}$$

and

$$E[U(\pi)] = E[U\{p_1 q_1 + p_2 q_2 - C(q)\}] \tag{3}$$

where $U'(\pi) = dU(\pi)/d\pi$ is the marginal utility from profit. We assume that $U'(\pi) > 0$.

II. The Case of the Price-Setting Firm

In the case of the price-setting firm, the producer fixes the price before the demand function is known and lets the quantity adjust to the realized level of demand. When the monopolist faces two different markets, then the firm will set price in both markets and adjust the quantity to meet the actual demand in both markets. For the price-setting firm, the demand function for the ith market may be written as

$$q_i = q_i(p_i, u_i) \tag{4}$$

with $q_i' = \partial q_i/\partial p_i < 0$ and $\partial q_i/\partial u_i > 0$. The profit function then becomes

$$\pi = p_1 q_1(p_1, u_1) + p_2 q_2(p_2, u_2) - C[q_1(p_1, u_1) + q_2(p_2, u_2)] \tag{5}$$

The producer then maximizes $E[U(\pi)]$ with respect to the two decision variables, p_1 and p_2. The two first order conditions for the maximum are:

$$\frac{\partial E[U(\pi)]}{\partial p_1} = E[U'(\pi)\{R_1(p_1, u_1) - C'(q)q_1'(p_1, u_1)\}] = 0 \tag{6}$$

and

$$\frac{\partial E[U(\pi)]}{\partial p_2} = E[U'(\pi)\{R_2(p_2, u_2) - C'(q)q_2'(p_2, u_2)\}] = 0 \tag{7}$$

where $R_i(p_i, u_i) = q_i(p_i, u_i) + p_i \partial q_i(p_i, u_i)/\partial p_i$ = marginal revenue in the ith market.

The second-order conditions for the maximum are:[2]

$$\frac{\partial^2 E[U(\pi)]}{\partial p_1^2} = E[U''(R_1 - C'q_1')^2 + U'(\partial R_1/\partial p_1 - q_1'^2 C'' - C'\partial q_1'/\partial p_1)] = A_1 < 0 \quad (8)$$

$$\frac{\partial^2 E[U(\pi)]}{\partial p_2^2} = E[U''(R_2 - C'q_2')^2 + U'(\frac{\partial R_2}{\partial p_2} - q_1'^2 C'' - C'\frac{\partial q_1'}{\partial p_1})] = A_2 < 0 \quad (9)$$

and

$$A_1 A_2 - B^2 = D > 0 \quad (10)$$

where

$$B = \frac{\partial^2 E[U(\pi)]}{\partial p_1 \partial p_2} = E[U''(R_1 - C'q_1')(R_2 - C'q_2') - U'q_1'q_2'C'')] \quad (11)$$

With $\partial R_i/\partial p_i = 2q_i' + p_i q_i'' < 0$, it can be easily seen that the second-order conditions (8)-(10) may be satisfied regardless of the signs of $U''(\pi)$ and $C''(q)$. With $U''(\pi)$ determining the risk-attitude of the producer and $C''(q)$ determining the shape of the marginal cost curve, we conclude that the monopolist may achieve the optimum irrespective of the cost conditions and his attitude towards risk. In what follows, we assume that the second-order conditions are satisfied for all p_i.

As noted earlier, under certainty price discrimination occurs when the elasticity of demand at the optimum is different in each market. If the demand functions are random, it does not make sense to talk in terms of actual elasticity of demand, because the latter in most cases is random. However, the problem may be formulated in terms of the expected elasticity of demand or the elasticity of the expected demand function in each market. Two questions may now be raised. First, does price discrimination occur in the presence of uncertainty if expected elasticity of demand at the optimum is different in the two markets? Second, can price discrimination occur if at

the optimum expected elasticities of demand are the same. The following passages attempt to provide the answers.

The first order conditions (6) and (7) may be written as

$$E[U'(q_i + p_i q_i')] = E[U'C'q_i'] \qquad (12)$$

where for R_i we have written $q_i + p_i q_i'$. Using the definition of covariance between any two variables,(12) may be written as [3]

$$E[U']E[q_i + p_i q_i'] + \text{Cov}(U', R_i) = E[U']E[C'q_i'] + \text{Cov}(U', C'q_i')$$

whence

$$p_i[1 - \frac{1}{\bar{\varepsilon}_i}] = \frac{E[C'q_i']}{\bar{q}_i'} + \frac{\text{Cov}(U', C'q_i')}{\bar{q}_i'E[U']} - \frac{\text{Cov}(U', R_i)}{\bar{q}_i'E[U']} \qquad (13)$$

where $\bar{\varepsilon}_i$ is the elasticity of the expected demand function in the ith market and is defined as [4]

$$\bar{\varepsilon}_i = - \frac{\partial \bar{q}_i(p_i, \bar{u}_i)}{\partial p_i} \frac{p_i}{\bar{q}_i(p_i, \bar{u}_i)} = - \bar{q}_i' \frac{p_i}{\bar{q}_i}$$

It is clear from (13), that the traditional result will hold only if the expressions on the right hand side are the same in both markets. In general, these expressions need not be identical everywhere and we then have some other explanations for the phenomenon of price discrimination. Under certainty,

$$p_i q_i' (1 - \frac{1}{\varepsilon_i}) = C'q_i'$$

or

$$p_i(1 - \frac{1}{\varepsilon_i}) = C' \qquad (i = 1,2)$$

and since C' is the same in both markets, it is clear that $p_1 > p_2$ if $\epsilon_1 < \epsilon_2$, that is to say, at the optimum the monopolist charges the higher price in the market with relatively inelastic demand. Under uncertainty all the three expressions in the right hand side of (13) may be different in different markets. Hence even if $\bar{\epsilon}_1 < \bar{\epsilon}_2$, the price discrimination may not occur or at the optimum p_1 may actually be lower than p_2, that is, the monopolist may select a lower price in a market with low expected elasticity of demand; and even if $\bar{\epsilon}_1 = \bar{\epsilon}_2$, price discrimination may occur.[5]

Let us now look at some special cases where the traditional result does hold. Suppose the monopolist is neutral towards risk, so that $U'(\pi)$ is constant. In this case (13) reduces to

$$p_i[1 - \frac{1}{\epsilon_i}] = \frac{E[C'q_i']}{\overline{q_i'U'}} = \overline{C'} + \frac{Cov(C',q_i')}{\overline{q_i'U'}} \tag{14}$$

If $\partial q_i'/\partial u_i = 0$, or if $C''(q) = 0$ so that marginal cost is constant, then $Cov(C',q_i') = 0$ and the traditional result clearly holds. The following theorem may now be derived:

> Theorem 1: <u>If the monopolist is risk-neutral, then he charges higher price in the market with relatively low expected elasticity of demand provided</u> (i) <u>the demand functions have constant second derivative or</u> (ii) <u>the marginal cost of production is constant.</u>

In the general case, the results depend on the covariance between U' and R_i and between U' and $C'q_i'$. These covariances in turn depend on the expected values of the random variables u_1 and u_2 and their variance.

Therefore, in order to obtain some more specific results, we have to examine the particular way in which the stochastic terms enter into the demand functions.

A. Multiplicative Stochastic Terms:

In this case the demand function may be written as

$$q_i = u_i f_i(p_i)$$

with $f_i'(p_i) < 0$ and $u_i > 0$. The profit function becomes

$$\pi = p_1 u_1 f_1(p_1) + p_2 u_2 f_2(p_2) - C[u_1 f_1(p_1) + u_2 f_2(p_2)] \qquad (14')$$

The first-order conditions are given by

$$E[U'(\pi)\{u_i(p_i f_i'(p_i) + f_i(p_i) - C'(q)u_i f_i'(p_i)\}] = 0 \qquad (15)$$

so that

$$p_i(1 - \frac{1}{\varepsilon_i}) = \frac{E[U'(\pi)C'(q)u_i]}{E[U'(\pi)u_i]} \qquad (16)$$

Note that in the case of the multiplicative random term, the elasticity of demand is non-random, because

$$\varepsilon_i = \frac{\partial[u_i f_i(p_i)]}{\partial p_i} \frac{p_i}{u_i f_i(p_i)} = p_i \frac{f_i'(p_i)}{f_i(p_i)}$$

Equation (16) enables us to make an intuitively obvious statement: If the subjective probability distributions of u_1 and u_2 are the same, then the term in the right hand side of (16) is the same in both markets, in which case the expected elasticity of demand will determine the optimal

price charged by the monopolist in each market and the traditional results will hold in entirety. However, if u_1 and u_2 follow different distributions, then the results will depend on the sign of $U''(\pi)$, $C''(q)$ as well as the parameters of the two probability distributions. Before we demonstrate this, the following theorem can be derived immediately:

Theorem 2: <u>If the random term in the demand function is multiplicative,</u> <u>then the monopolist charges a higher price in the market with</u> <u>relatively low elasticity of demand</u>, <u>provided the marginal cost</u> <u>is constant.</u>[6]

What is interesting is that theorem (2) holds irrespective of whether the monopolist is neutral, averse or preferring towards risk, because with C' constant, (16) reduces to

$$p_i[1 - \frac{1}{\epsilon_i}] = C'$$

Until now our concern has been with finding sufficient conditions which ensure the validity of the traditional results concerning the theory of price discrimination. We will now consider the case where the expected demand function is the same in both markets and then see if any specific results concerning price discrimination can be derived.

From (15), we may write

$$p_i = \frac{E[U'(\pi)C'(q)u_i]}{E[U'(\pi)u_i]} - \frac{f_i(p_i)}{f_i'(p_i)} \tag{17}$$

Let us define the identity of the expected demand function in both markets by

$$f(p) = f_1(p_1) = f_2(p_2) \text{ and } \bar{u}_1 = \bar{u}_2 = \bar{u}.$$

In view of this definition and (17),

$$P_1 - P_2 = \frac{E[U'(\pi)C'(q)u_1]}{E[U'(\pi)u_1]} - \frac{E[U'(\pi)C'(q)u_2]}{E[U'(\pi)u_2]} \qquad (18)$$

If marginal cost, i.e., $C'(q)$ is constant then it can be easily seen that $P_1 - P_2 = 0$ in (18). The following theorem is then immediate.

Theorem 3: If the random term is multiplicative, then optimal price is the same in both markets, provided that marginal cost is constant, so that $C'' = 0$.

The interesting part of this theorem is that it is independent of the variance of the distributions of u_1 and u_2 as well as the attitude of the producer towards risk.

If, however, marginal cost is not con̲s̲i̲s̲t̲ent we first write equation (18) as

$$P_1 - P_2 = \frac{(EU'C'u_1)(EU'u_2) - (EU'C'u_2)(EU'u_1)}{(EU'u_1)(EU'u_2)} \qquad (19)$$

Using Taylor series expansion, the numerator on the right hand side of the equation (19) has been simplified in (A.10) of the Appendix A. It can be noted from (19) and (A.11) that if variance of $u_1 = \sigma_1^2$ is identical with the variance of $u_2 = \sigma_2^2$ then $P_1 = P_2$, provided expected demand functions in both markets are the same. If $\sigma_1^2 \neq \sigma_2^2$, price discrimination will occur, but the sign of $P_1 - P_2$ cannot be determined.

B. <u>Additive Stochastic Term</u>:

In the case of additive stochastic term, the demand function may be written as

$$q_i = f_i(p_i) + u_i$$

The profit function now becomes

$$\pi = p_1[f_1(p_1 + u_1] + p_2[f_2(p_2) + u_2] - C[f_1(p_1) + u_1 + f_2(p_2) + u_2] \qquad (19')$$

so that the first-order conditions are given by

$$E[U'(\pi)\{f_i(p_i) + p_i f_i'(p_i) + u_i - C'(q)f_i'(p_i)\}] = 0 \qquad (20)$$

From (20), we can obtain

$$E[U'(\pi)(1 - \frac{1}{\varepsilon_i})] \ p_i f_i'(p_i) = E[U'(\pi)C'(q)]f_i'(p_i) \qquad (21)$$

where here

$$\varepsilon_i = \frac{f_i(p_i) + u_i}{p_i f_i'(p_i)}$$

From (21),

$$E[U'(\pi)(1 - \frac{1}{\varepsilon_1})]p_1 = E[U'(\pi)(1 - \frac{1}{\varepsilon_2})]p_2 \qquad (22)$$

Under certainty, $\varepsilon_1 > \varepsilon_2$ implies that $p_1 < p_2$. When demand function in each market is stochastic and the stochastic term is additive, then ε_i is random and it may not be possible to compare ε_1 with ε_2. However, if the stochastic

terms are such that $\varepsilon_1 > \varepsilon_2$ for all u_1 and u_2, then it is clear from (22) that $p_1 < p_2$. Thus, <u>when the stochastic term is additive, then the traditional price discrimination result continues to hold provided it is possible to make comparisons between stochastic elasticities of demand.</u>

In the case where $(\varepsilon_1-\varepsilon_2)$ is not unique, we write (20) as

$$P_i = \frac{E[U'(\pi)C'(q)]}{E[U'(\pi)]} - \frac{E[U'(\pi)u_i]}{E[U'(\pi)]f_i'(p_i)} - \frac{f_i(p_i)}{f_i'(p_i)} \tag{23}$$

If we now assume that the expected demand function is the same in both markets, such that $\bar{u}_1 = \bar{u}_2 = \bar{u}$ and $f_1(p_1) = f_2(p_2) = f(p)$, then in view of (23)

$$P_1 - P_2 = \frac{E[U'(\pi)(u_2-u_1)]}{E[U'(\pi)]f'(p)} \tag{24}$$

Expanding up to second moment the term $U'(\pi)(u_2-u_1)$ in Taylor series around the means $\bar{u}_1$ and $\tilde{u}_2$, we obtain (as given in (A.9) of Appendix A),

$$P_1 - P_2 = \frac{\{\bar{U}''(\pi)\}}{E[U'(\pi)]f'(p)} [\{p_2 - C'(\bar{q})\}\sigma_2^2 - \{p_1 - C'(\bar{q})\}\sigma_1^2] \tag{25}$$

Unfortunately, equation (25) does not yield the definite results. All we can say is that if $\sigma_1^2 \neq \sigma_2^2$, price discrimination will occur, but the sign of (p_1-p_2) cannot be determined a priori, because as is clear from (25), $p_1=p_2$ if $\sigma_1^2 = \sigma_2^2$. However, if $\sigma_1^2 \gtrless \sigma_2^2$, the sign of $p_1 - p_2$ is indeterminate, although it is clear that p_1 and p_2 cannot be equal.

III. The Price Setting Term and the Effect of a Change in Uncertainty in
     One Market

By now we have proved that price discrimination may occur even if
expected demand functions are the same in both markets. We will now examine
the effect of a marginal change in uncertainty in one market on the optimal
prices in both markets. The marginal rise in uncertainty will be defined in
terms of the "mean-preserving spread" of the distribution of the demand
function in the first market. Let us define, q_1^* as

$$q_1^*(p_1,u_1) = \gamma q_1(p_1,u_1) + \theta \tag{26}$$

where γ and θ are the shift parameters. Initially, $\gamma = 1$ and $\theta = 1$.
A marginal increase in uncertainty in market 1 is then defined by

$$d\gamma > 0 \text{ and } \frac{d\theta}{d\gamma} = -\bar{q}_1 \tag{27}$$

Substituting $q_1^*(p_1,u_1)$ for $q_1(p_1,u_1)$ in (5), (6) and (7) and $\gamma q_1'(p_1,u_1)$
for $q_1'(p_1,u_1)$ in (6) and (7), then differentiating (6) and (7), and utilizing
(27) we obtain

$$A_1 \frac{\partial p_1}{\partial \gamma} + B \frac{\partial p_2}{\partial \gamma} = H_1 \tag{28}$$

$$B \frac{\partial p_1}{\partial \gamma} + A_2 \frac{\partial p_2}{\partial \gamma} = H_2 \tag{29}$$

where A_1, A_2 and B have been defined before in (8), (9) and (11) and where

$$H_1 = -E[U''(q_1-\bar{q}_1)(p_1-C')\{q_1+q_1'(p_1-C')\}]$$

$$-E[U'\{(q_1-q_1)(1-C''q_1') + (p_1-C')q_1'\}] \tag{30}$$

and

$$H_2 = - E[U''(q_1-\bar{q}_1)(p_1-C')\{q_2+q_2'(p_2-C')\}]$$

$$+ E[U'C''q_2'(q_1-\bar{q}_1)] \tag{31}$$

The solution of (28) and (29) yields

$$\frac{\partial p_1}{\partial \gamma} = \frac{A_2 H_1 - B H_2}{D} \tag{32}$$

$$\frac{\partial p_2}{\partial \gamma} = \frac{A_1 H_2 - B H_1}{D} \tag{33}$$

where $D = A_1 A_2 - B^2 > 0$ from (10), and where from (11)

$$B = E[U''(R_2-C'q_1')(R_1-C'q_2') - U'C''q_2'q_1'] \tag{34}$$

In (32) and (33), A_1 and A_2 are negative and D is positive. Unfortunately, without further specifications H_1, H_2 and B are indeterminate.

Suppose marginal cost is constant and the disturbance term u_i is additive so that q_i' is nonstochastic. Then it can be shown that H_1 , H_2 and B are positive, if following Arrow [1965] we assume that absolute risk-aversion is a non-increasing function of profit.[7] Under these specifications, it is evident that both $\partial p_1/\partial \gamma$ and $\partial p_2/\partial \gamma$ are negative. The following theorem is then immediate.

Theorem 4: <u>If disturbance terms in the demand functions are additive, marginal cost is constant and absolute risk-aversion is non-increasing in profits,</u> then an increase in uncertainty in one market causes a decline in the optimal price in both markets.

The same result is available in the case where $U'' = 0$ and the marginal cost function satisfies some further requirements.

$$H_1 = -U'E[(q_1-\bar{q}_1)(1-C''q_1') + (p_1-C')q_1']$$

$$= -U'[Cov\{q_1-\bar{q}_1, 1-C''q_1'\} + E(p_1-C')q_1'] \qquad (35)$$

From (6), with $U''=0$, it is clear that

$$E[(p_1-C')q_1'] = -E[q_1] < 0.$$

Also $\partial(q_1-\bar{q}_1)/\partial u_1 = \partial q_1/\partial u_1 > 0$ and

$$\frac{\partial(1-C''q_1')}{\partial u_1} = -[C''\frac{\partial q_1'}{\partial u_1} + q_1' C''' \frac{\partial q}{\partial u_1}] \leq 0$$

if $C'' \leq 0$ and $C''' \leq 0$, because $\partial q_1'/\partial u_1 \leq 0$ and $\partial q/\partial u_1 = \frac{\partial q_1}{\partial u_1} > 0$. Under these

conditions then, the covariance term in (35) is non-positive, in which case $H_1 > 0$. In the same way, $U'' = 0$, $C'' \leq 0$ and $C''' \leq 0$ are sufficient to ensure that $H_2 \geq 0$. Similarly, $B \geq 0$. Clearly, then $\partial p_i/\partial \gamma < 0$.

> Theorem 5: <u>An increase in uncertainty in one market causes a decline in the optimal price in both markets</u>, <u>provided the monopolist is risk-neutral</u>, <u>marginal-cost is non-increasing and as output expands it declines at a non-increasing rate (i.e. $C''' \leq 0$).</u>

However, if the conditions specified in theorems 5 and 6 are not specified, the effect of a change in uncertainty in one market on optimal prices is indeterminate.

IV. The Case of the Quantity Setting Firm

In the case of the quantity-setting monopolist, results turn out to be much more general than those available from the analysis of the price-setting firm. This is not surprising, because with the quantity-setting firm the cost function is no longer random.

The demand function is now written as

$$p_i = p_i(q_i, u_i)$$

with $p_i' = \partial p_i/\partial q_i < 0$ and $\partial p_i/\partial u_i > 0$. The profit function now becomes

$$\pi = q_1 p_1(q_1, u_1) + q_2 p_2(q_2, u_2) - C(q) \tag{36}$$

Expected utility maximization with respect to the two decision variables, q_1 and q_2, now leads to the following first order condition.

$$E[U'(\pi)\{R_i(q_i, u_i) - C'(q)\}] = 0 \tag{37}$$

where now $R_i = p_i + q_i p_i'$.

The second-order conditions are now given by

$$E[U''(R_i-C')^2 + U'(R_{ii}-C'')] = A_i \qquad (i=1,2). \tag{38}$$

$$B = E[U''(R_1-C')(R_2-C') - U'C''] \tag{39}$$

and

$$A_1 A_2 - B^2 = D > 0 \tag{40}$$

An examination of the first-order conditions (37) reveals that a quantity-setting monopolist operates in such a way that

$$E[U'(R_1 - R_2)] = 0 \tag{41}$$

which means that in the case of risk-neutrality, the monopolist produces an

output in such a way that the expected marginal revenues in the two markets are equalized. This, however, need not be the case in the case of risk-aversion or preference.

One result available from the deterministic model is this: If the demand function has the same slope in each market, then a larger output is sold in a market with the higher elasticity of demand.

To see whether this result holds under uncertainty, let us assume that the disturbance term in the demand function is additive, so that p_i' is non-random and it makes sense to speak in terms of the equality between p_1' and p_2'. Then from (41)

$$E[U'(1-\varepsilon_i)]q_i p_i' = 0$$

which with $p_1' = p_2'$ implies that

$$E[U'(1-\varepsilon_1)]q_1 = E[U'(1-\varepsilon_2)]q_2$$

If $\varepsilon_1 > \varepsilon_2$ for all values of u_1 and u_2, then it is clear that $q_1 > q_2$ for all u_i.

Theorem 6: <u>If the demand function has the same slope in the two markets and $(\varepsilon_1-\varepsilon_2)$ has a unique sign for all u_1 and u_2, then even under uncertainty, a larger output is sold in the market with the higher elasticity of demand.</u>

To get some results which are attributable purely to uncertainty, let us assume that expected demand function is the same in both markets. In the case of additive distrubance term, we define this by,

$$f_1(q_1) = f_2(q_2) = f(q) \text{ and } \bar{u}_1 = \bar{u}_2 = \bar{u}$$

where

$$p_i = f_i(q_i) + u_i$$

In this case, from the first-order condition, we get

$$f_1'(q_1 - q_2) = E[U'(u_2 - u_1)]/E U'$$

which can be simplified (using (A.9) for the case of quantity setting firm) to

$$q_1 [1 + \frac{\overline{U}''}{f_1' E U'} \sigma_1^2] = q_2 [1 + \frac{\overline{U}''}{f_1' E U'} \sigma_2^2]$$

so that if $\sigma_1^2 > \sigma_2^2$, $q_1 < q_2$, because $\overline{U}'' < 0$, $f_1' < 0$ and $E U' > 0$.

In the case of the multiplicative disturbance term where,

$$p_i = u_i f_i(q_i)$$

we find (under the assumption that $\overline{u}_i$ and $f_i(q_i)$ are the same in both markets) that

$$f E U'(u_2 - u_1) = f_1'(q_1 E U' u_1 - q_2 E U' u_2) \tag{41'}$$

where, noting that $C'(q)$ is not a function of u_1 and u_2 in case of quantity setting firm and using (A.6) and (A.7) of the Appendix A

$$E U' u_i = a + \sigma_i^2 b_i, \qquad i=1,2,$$

$$a = \overline{U}' \overline{u} + \frac{f^2}{2} \overline{u} \sum_{i=1}^{2} \sigma_i^2 [\overline{U}''' q_i^2 - \overline{U}'' \overline{C}'] \tag{42}$$

$$b_i = \overline{U}'' f q_i$$

It can be noted from (41') and (42) that a definite result about the sign of $q_1 - q_2$ cannot be determined in this case as compared to the additive disturbance case. However, it is clear from (41') and (42) that $q_1 = q_2$ if $\sigma_1^2 = \sigma_2^2$.

The following theorem is then immediate.

Theorem 7:   If the demand function has additive disturbance term and ex-
pected demand function is the same in both markets, then a
quantity-setting monopolist sells a smaller output in the
market with the larger variance demand.

What about price discrimination?   Since the quantity-setting producer
selects the quantity and lets price adjust to the actual demand, prices in
both markets are random.   Here then we cannot speak in terms of discrimina-
tion about actual prices.   However, we may still analyze the effect of un-
certainty on expected prices.   In view of theorem 7, this implication is
straightforward.

If the expected demand function is given by

$$P_i = f_i(q_i) + \bar{u}_i$$

then it is clear that with $f_i'(q_i) < 0$, $\bar{u}_1 = \bar{u}_2$, $f_1(q_1) = f_2(q_2)$ and
$f_1'(q_1) = f_2'(q_2)$,

$$\bar{P}_1 > \bar{P}_2 \quad \text{if} \quad q_1 < q_2$$

which in view of theorem (8) implies that

$$\bar{P}_1 > \bar{P}_2 \quad \text{if} \quad \sigma_1^2 > \sigma_2^2.$$

This result is also valid in the case of multiplicative disturbance term.
The following theorem may now be derived

Theorem 8:   Expected price is higher in the market with the larger
variance of demand, given that the expected demand function is
the same everywhere.

V. The Quantity Setting Firm and the Effect of a Change in Uncertainty in

One Market

In this section we examine the question of how a marginal change in uncertainty in one market affects the output sold in the two markets. Using the concept of the mean-preserving spread, let us define

$$p_1^*(q_1,u_1) = \gamma p_1(q_1,u_1) + \Theta$$

where the marginal increase in uncertainty in the first market is defined by

$$d\gamma > 0 \text{ and } \frac{d\Theta}{d\gamma} = -\overline{p}_1 \tag{42'}$$

Substituting $p_1^*(q_1,u_1)$ for $p_1(q_1,u_1)$ in (36) and (37) and $\gamma p_1'(q_1,u_1)$ for $p_1'(q_1,u_1)$ in (37), then differentiating (37) with respect to γ and utilizing (42'), we will obtain two equations. The solution of these two yields

$$\frac{\partial q_1}{\partial \gamma} = \frac{\Pi_1 A_2}{D} \tag{43}$$

and

$$\frac{\partial q_2}{\partial \gamma} = -\frac{\Pi_1 B}{D} \tag{44}$$

where A_2, B and D have been defined in (38), (39) and (40) and where

$$\Pi_1 = -E[U''(R_1-C')(p_1-\overline{p}_1)q_1 + U'(p_1'q_1 + p_1-\overline{p}_1)] \tag{46}$$

We show in the appendix that if marginal revenue equals the extended marginal revenue, then in the presence of the hypothesis of decreasing or constant absolute risk-aversion, $\Pi_1 > 0$. Since $A_2 < 0$ and $D > 0$, it is clear that $\partial q_1/\partial \gamma < 0$, <u>so that a marginal increase in uncertainty in market causes a decline in the output sold in that market.</u>

As regards the output sold in the second market, the results are not predictable without adding additional specifications. From (39),

$$B = E[U''(R_1-C')(R_2-C') - U'C'']$$

$$= E[\frac{U''}{U'} U'(R_1-C')(R_2-C') - U'C'']$$

If we assume that the Arrow-Pratt index of absolute risk-aversion—given by $-U''/U'$—is constant, then

$$B = \frac{U''}{U'} \{[U'(R_1-C')]E[R_2-C'] + Cov[U'(R_1-C'),(R_2-C')]\} - E[U']C''$$

$$= \frac{U''}{U'} Cov[U'(R_1-C'),(R_2-C')] - E[U']C''$$

because from (37), $E[U'(R_1-C')] = 0$. Now $Cov[U'(R_1-C'),(R_2-C')]$ is negative, because any rise in R_2 is accompanied by a constant (R_1-C') (because C' is non-random and $R_1(U_1)$ is independent of $R_2(U_2)$) as well as a rise in profits. So that with $U'' < 0$ in the presence of risk-aversion, this covariance is negative. If in addition $C'' \leq 0$, then it is clear that $B > 0$. With these specifications then, $\partial q_2/\partial \gamma < 0$ <u>and the marginal increase in uncertainty in market 1 leads to a decline in the output sold in the second market as well</u>.

APPENDIX A

Let us write the Taylor series expansion of a function, $f(u_1, u_2)$ in two random variables u_1, u_2 around their mean values $\bar{u}_1$, $\bar{u}_2$ as [8]

$$(A.1) \qquad f(u_1, u_2) = f(\bar{u}_1, \bar{u}_2) + (u_1 - \bar{u}_1)\, f'_1(\bar{u}_1, \bar{u}_2) + (u_2 - \bar{u}_2)\, f'_2(\bar{u}_1, \bar{u}_2)$$

$$+ \frac{(u_1 - \bar{u}_1)^2}{2}\, f''_1(\bar{u}_1, \bar{u}_2) + \frac{(u_2 - \bar{u}_2)^2}{2}\, f''_2(\bar{u}_1, \bar{u}_2)$$

$$+ (u_1 - \bar{u}_1)(u_2 - \bar{u}_2)\, f''_{12}(\bar{u}_1, \bar{u}_2) + \ldots\ldots$$

where $f'_i(\bar{u}_1, \bar{u}_2) = \frac{\partial}{\partial u_i} f(u_1, u_2)$ at $u_1 = \bar{u}_1$; $u_2 = \bar{u}_2$, $i = 1,2$; $f''_i(\bar{u}_1, \bar{u}_2) = \frac{\partial^2}{\partial u_i^2} f(u_1, u_2)$, at $u_1 = \bar{u}_1$, $u_2 = \bar{u}_2$, $i = 1,2$, and $f''_{12}(\bar{u}_1, \bar{u}_2) = \frac{\partial^2}{\partial u_1 \partial u_2} f(u_1, u_2)$

at $u_1 = \bar{u}_1$ and $u_2 = \bar{u}_2$.

Now we note that $U'(\pi)$ and $C'(q)$ in equation (18) of the text are both functions of u_1 and u_2. Also, therefore, $U'(\pi)C'(q)u_1$ and $U'(\pi)C'(q)u_2$ are functions of u_1 and u_2. Thus, expanding $U'(\pi)C'(q)u_i$, $i = 1,2$ by using (A.1), taking the expectations on both sides and retaining terms up to second moments of u_1 and u_2 we obtain

$$(A.2) \qquad E\, U'\, C'\, u_i = A + \sigma_i^2\, B_i$$

where

$$(A.3) \qquad A = \bar{U}'\, \bar{C}'\, \bar{u} + \tfrac{1}{2}\, \bar{U}'''\, \bar{C}'\, (f(p))^2\, \bar{u} \sum_{i=1}^{2} (p_i - \bar{C}')^2 \sigma_i^2$$

$$+ \bar{U}''\, \bar{C}''(f(p))^2\, \bar{u} \sum_{i=1}^{2} (p_i - \bar{C}')\sigma_i^2 + [\bar{C}'''\, \bar{U}' - \bar{U}''\bar{C}''C'](f(p))^2\bar{u}$$

$$\times \sum_{i=1}^{2} \sigma_i^2 / 2$$

$$(A.4) \qquad B_i = f(p)\, [\bar{U}''(p_i - \bar{C}')\bar{C}' + \bar{U}'\bar{C}''],$$

$\bar{u} = \bar{u}_1 = \bar{u}_2$, $f(p) = f_1(p_1) = f_2(p_2)$, $\bar{C}' = \bar{C}'(\bar{q})$, and $\bar{U}' = U'$ at $u_1 = \bar{u}_1$ and $u_2 = \bar{u}_2$.

In deriving the result in (A.2) we have noted that

(A.5) $\quad \dfrac{d}{du_1}U' = U'' \dfrac{d\pi}{du_1} = U''(p_1 - C')f(p)$

$\qquad \dfrac{d}{du_1}C(q) = C' \dfrac{dq}{du_1} = C'f(p)$

where $\pi = p_1 u_1 f_1(p_1) + p_2 u_2 f_2(p_2) - C[u_1 f_1(p_1) + u_2 f_2(p_2)]$ as

given in equation (14').

Using (A.1), it can be similarly seen that

(A.6) $\quad E U' u_i = a + \sigma_i^2 b_i, \qquad i=1,2$

where

$$a = \overline{U}' \overline{u} + \frac{(f(p))^2 \overline{u}}{2} \sum_{i=1}^{2} \sigma_i^2 [\overline{U}'''(p_i - \overline{C}')^2 - \overline{U}'' \overline{C}'']$$

(A.7) $\quad b_i = \overline{U}'' f(p) (p_i - \overline{C}')$

Thus, it can be verified that

(A.8) $\quad E U' u_2 - E U' u_1 = \overline{U}'' f(p) [(p_2 - \overline{C}')\sigma_2^2 - (p_1 - \overline{C}')\sigma_1^2]$.

In case, however, the disturbance term is additive in the demand function,

i.e., $q_i = f_i(p_i) + u_i$ as given in equation (19'), we obtain

(A.9) $\quad E U' u_2 - E U' u_1 = \overline{U}''[(p_2 - \overline{C}')\sigma_2^2 - (p_1 - \overline{C}')\sigma_1^2]$.

Finally, using (A.2) and (A.6) we obtain

(A.10) $\quad (E U' C' u_1)(E U' u_2) - (E U' C' u_2)(E U' u_1) = a(B_1 \sigma_1^2 - B_2 \sigma_2^2)$

$\qquad + A(b_2 \sigma_2^2 - b_1 \sigma_1^2) + \sigma_1^2 \sigma_2^2 (B_1 b_2 - B_2 b_1)$.

If $\sigma_1^2 = \sigma_2^2 = \sigma^2$ the equation (A.10) reduces to

(A.11) $\quad (E U' C' u_1)(E U' u_2) - (E U' C' u_2)(E U' u_1)$

$\qquad = \sigma^2 (p_2 - p_1)\overline{U}'' f(p) [\sigma^2 f(p)\overline{U}'\overline{C}'' + A - a \overline{C}']$.

APPENDIX B

We note that R_1 and R_2 in (34) are functions of u_1 and u_2, respectively. Further U'' is a function of u_1 and u_2. Thus, using the conditional expectations we can write[9]

$$(B.1) \qquad E(R_1 - C'q_1')(R_2 - C'q_2')U'' = E[(R_1 - C'q_1')E\{U''(R_2 - C'q_2')\}/u_1].$$

Now, if C' is constant and q_2' is nonstochastic it is well known that[10]

$$EU''(R_2 - C'q_2')/u_1 \geq 0$$

provided absolute risk-aversion is a non-increasing function of profit. It, therefore, follows that

$$E(R_1 - C'q_1')(R_2 - C'q_2')U'' \geq 0.$$

FOOTNOTES

[1] For simplicity, our analysis is confined to the case of two sub-markets.

[2] Whenever the equation is long, we do not write the variable in its functional form. For example, $U'(\pi)$ is simply written as U'.

[3] The covariance between any two random variables a and b is defined as

$$Cov(a,b) = E[a,b] - E[a]E[b]$$

[4] In what follows, the bar or any variable denotes its expected value.

[5] Note that since at present we are dealing with the case of a price-setting monopolist, who sets the prices before the realization of actual demand, we can speak in terms of discrimination in actual rather than expected prices.

[6] Remember that in this case the actual and the expected elasticity of demand are the same.

[7] See the Appendix B.

[8] See, e.g., Goldberger [1964].

[9] If $f(u_1,u_2) = g(u_1)g(u_1,u_2)$ and $h(u_1,u_2)$ is the density function of u_1 and u_2, then

$$E f(u_1,u_2) = \iint f(u_1,u_2)h(u_1,u_2)du_1 \, du_2$$

$$= \int g(u_1)\{ \int g(u_1,u_2)h(u_2/u_1)du_2 \}h(u_1)du_1$$

$$= \int g(u_1)\{ E g(u_1,u_2)/u_1 \} h(u_1)du_1$$

$$= E[g(u_1) E\{ g(u_1,u_2) \}/u_1]$$

[10] See Batra and Ullah [1974], Leland [1972].

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