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## PROGRESSION AND RISK-TAKING: A FURTHER NOTE

Syed M. Ahsan<sup>\*</sup>

### I. INTRODUCTION

In a recent paper (Ahsan, 1974), this author has used a simple progressive tax schedule--a linear tax which has an exemption level (K) and a constant marginal rate ( $t$ ) that applies both above and below K--to analyze the effects of taxation on risk-taking in the context of a model of expected utility maximization. The analysis was carried out under the hypotheses on the risk-aversion functions suggested by Arrow (1965), namely, those of

- (i) decreasing absolute risk-aversion, and
- (ii) increasing relative risk-aversion.<sup>1</sup>

Arrow (1965) had earlier shown the following results regarding the relationship between risk-aversion and an individual's portfolio allocation decisions in response to changes in wealth:

- (a) The wealth-elasticity of the risky asset demand is greater than, equal to or less than zero as absolute risk-aversion is decreasing, constant or increasing;
- (b) The wealth elasticity of the risky asset demand is greater than, equal to or less than unity as relative risk-aversion is decreasing, constant or increasing.

In light of these results, it seems reasonable to assume that absolute risk-aversion is a decreasing function of wealth (and hence, risky assets are not inferior). But, Arrow's other hypothesis that relative risk-aversion is an increasing function of wealth may not be so obvious and Stiglitz (1969a)

has discussed some of the difficulties involved. More importantly, it turns out that these results on the relationship between hypotheses on risk-aversion and wealth elasticity of the risky asset demand, as obtained by Arrow,<sup>2</sup> are not invariant with respect to the tax schedules considered (see the appendix). In particular, we find that the second result (proposition (b) above) is modified in the presence of the progressive tax schedule considered by this author. We now obtain:

- (c) The wealth elasticity of the risky asset demand is less than unity if relative risk-aversion is constant or increasing.

Thus decreasing relative risk-aversion is no longer sufficient to guarantee a wealth elasticity of the risky asset demand greater than unity.

In view of this modification, some of the results of the previous paper (Ahsan (1974)) have to be restated, and in so doing we also generalize the analysis by not restricting ourselves to the specific assumptions on risk-aversion made earlier. Further, using a measure of private risk-taking suggested by Atkinson-Stiglitz (forthcoming) additional results on the effects of taxation on private risk-taking are obtained.<sup>3</sup>

## II. THE PORTFOLIO PROBLEM<sup>4</sup>

The household is assumed to allocate its portfolio to maximize expected utility. For simplicity, we assume that there are only two assets, a safe asset with a secure rate of return ( $r \geq 0$ ) and one risky asset with a random rate of return ( $x \geq -1$ ). In the absence of taxation, the portfolio allocation problem can be written as

$$\max_a E\{u(W)\} \tag{II-1}$$

where a denotes the amount invested in the risky asset and final wealth, W, is given by

$$W = (W_0 - a)(1+r) + a(1+x) \quad (\text{II-2})$$

where  $W_0$  is initial wealth and thus  $(W_0 - a)$  denotes the amount invested in the riskless asset. We further assume that  $u(W)$  is continuous and is at least twice continuously differentiable with positive and diminishing marginal utilities (i.e.,  $u'(W) > 0$ ,  $u''(W) < 0$ ). The restriction  $u''(W) < 0$  implies risk-aversion for gambles about final wealth.

### III. PROGRESSIVE INCOME TAXATION AND RISK-TAKING

In the case of a progressive tax on investment income with full loss offsets, after-tax wealth is given by

$$W = W_0 + (1-t) \{rW_0 + a(x-r) - K\} + K \quad (\text{III-1})$$

where  $t$  is the marginal tax rate on investment income and  $K$  is the level of exemption on such income. Maximization of (II-1) s.t. (III-1) yields, given  $u''(W) < 0$ , the following necessary and sufficient condition for an interior solution:

$$E\{u'(W)(x-r)\} = 0 \quad (\text{III-2})$$

Differentiating (III-2) w.r.t. ' $t$ ' and ' $K$ ' and after some manipulation, the effect of a progressive linear income tax on risk-taking is given by

$$\frac{\partial a}{\partial t} = \left( \frac{a}{1-t} \left[ 1 - \left\{ \frac{r(1-t)}{1+r(1-t)} \frac{W_0}{a} \frac{\partial a}{\partial W_0} \right\} \right] + \frac{K}{[1+r(1-t)]} \frac{\partial a}{\partial W_0} \right) \quad (\text{III-3})$$

Evidently, for  $r=0$ ,

$$\frac{\partial a}{\partial t} = \frac{a}{1-t} + K \frac{\partial a}{\partial W_0} \quad (\text{III-3a})$$

where the first term on the r.h.s. has the interpretation of the substitution effect and the second term is the usual income effect. Given

decreasing absolute risk-aversion (which implies  $\partial a / \partial W_0 > 0$ , see the appendix, section A.1), social risk-taking increases (a increases) and, if we follow Atkinson-Stiglitz and regard  $a(1-t)$  as an indicator of private risk-taking, this also increases since we can rewrite (III-3a) as

$$(1-t) \frac{\partial a}{\partial t} - a = \frac{K}{(1-t)} \frac{\partial a}{\partial W_0} \quad (\text{III-3b})$$

where the l.h.s. measures the change in private risk-taking.

For the general case of  $r > 0$ , it follows from (III-3) that the hypothesis of decreasing absolute risk-aversion alone (i.e.,  $\partial a / \partial W_0 > 0$ ) is not sufficient to determine the direction of changes in risk-taking. However, given decreasing absolute risk-aversion, total risk-taking increases where relative risk-aversion is constant or increasing<sup>5</sup> (this is proposition III.1(b) of Ahsan (1974)). For decreasing relative risk-aversion, relative risk-taking (as measured by  $(t/a)(\partial a / \partial t)$ ), would increase if the effective marginal tax rate does not exceed 50%. This is a new result and can be seen by comparing (III-3) when multiplied by  $(t/a)$ , and equation (A-3) of the appendix.

Our major conclusion then is the following:

An increase in the marginal tax rate, in a system with a linearly progressive income tax with full loss offsets, leads to an increased demand for the risky asset if

- (a) the wealth elasticity of the risky asset demand is positive, and is less than or equal to unity (or, alternatively, absolute risk-aversion is decreasing and relative risk-aversion is non-decreasing in wealth); or
- (b) relative risk-aversion is decreasing and the marginal tax rate does not exceed 50%.

The detailed results are presented in Tables 1 and 2. These results demonstrate the possibility that progressive income taxation may lead to increased total

risk-taking rather than a decrease as might have been speculated. An intuitive rationale for this outcome is that by allowing full loss offsets and an exemption on (risky) income, the government shares part of the risk. Thus, although taxation reduces the probability of large gains, it also reduces the probability of large losses. In other words, the size of the bet has been reduced by taxation.

Table 1: Effects of a Progressive Income Tax  
on Risk-Taking ( $r = 0$ )

Type of Absolute Risk-Aversion	Type of Risk-Taking	
	Social	Private
Decreasing	+	+
Constant	+	0
Increasing	?	-

Table 2: Effects of a Progressive Income Tax  
on Risk-Taking ( $r > 0$ )

Type of Absolute Risk-Aversion \ Type of Risk-Taking	Type of Relative Risk-Aversion					
	Decreasing		Constant		Increasing	
	Social	Private	Social	Private	Social	Private
Decreasing	+	?	+	?	+	?
Constant	NA*	NA	NA	NA	+	?
Increasing	NA	NA	NA	NA	?	?

\* It is impossible to have constant or increasing absolute risk-aversion with non-increasing relative risk-aversion.



APPENDIX

The relationship between the wealth elasticity of the demand for the risky asset and the alternative hypotheses on the risk-aversion functions, as mentioned in the introduction, was originally established by Arrow (1965) in a model without any taxes. In this appendix we extend these results for the case of linear progressive taxation of investment income.

A.1 Absolute Risk-Aversion and Wealth Elasticities

Differentiating equation (III-2) w.r.t.  $\underline{W}_0$  it is seen that the derivative  $\partial a / \partial \underline{W}_0$  has the sign of  $E\{u''(W)(x-r)\}$ , which can be rewritten as (using the definition of absolute risk-aversion):

$$-E\{A(W)u'(W)(x-r)\}, \quad (A-1)$$

where  $A(W)$  denotes absolute risk-aversion ( $\equiv [-u''(W)/u'(W)]$ ). Now, we show that the expression (A-1) has the opposite sign of  $\partial A(W) / \partial W$ . This is seen as follows. Define  $W^* = \underline{W}_0 [1 + r(1-t)] + tK$ . Then

$$\begin{aligned} -E\{A(W)u'(W)(x-r)\} &= E\{[A(W^*) - A(W)]u'(W)(x-r)\} \\ &\quad - A(W^*)E\{u'(W)(x-r)\}, \end{aligned}$$

and using the first-order condition (equation (III-2) of the text)

$$-E\{A(W)u'(W)(x-r)\} = E\{[A(W^*) - A(W)]u'(W)(x-r)\}.$$

When  $(x-r) > 0$ ,  $W > W^*$  (see equation (III-1) of the text). Hence if  $A(W)$  is increasing with  $W$ ,  $[A(W^*) - A(W)] < 0$ , and  $\{A(W^*) - A(W)\}(x-r) < 0$ . Similarly, when  $(x-r) < 0$ ,  $W < W^*$ , and if  $A(W)$  is increasing with  $W$ ,  $[A(W^*) - A(W)] > 0$ , so that  $\{A(W^*) - A(W)\}(x-r)$  is still negative. Thus we have shown that

$$-E\{A(W)u'(W)(x-r)\} \begin{matrix} \geq 0 \\ < 0 \end{matrix} \text{ as } A(W) \text{ is } \begin{cases} \text{decreasing,} \\ \text{constant,} \\ \text{increasing.} \end{cases}$$

In other words,

$$\frac{\partial a}{\partial W_0} \begin{matrix} \geq 0 \\ < 0 \end{matrix} \text{ as } A(W) \text{ is } \begin{cases} \text{decreasing,} \\ \text{constant,} \\ \text{increasing.} \end{cases}$$

## A.2 Relative Risk-Aversion and Wealth Elasticities

Using the same argument as above, we can show that

$$-E\{R(W)u'(W)(x-r)\} \begin{matrix} \geq 0 \\ < 0 \end{matrix} \text{ as } R(W) \text{ is } \begin{cases} \text{decreasing,} \\ \text{constant,} \\ \text{increasing;} \end{cases}$$

or,  $E\{u''(W)(W)(x-r)\} \begin{matrix} \geq 0 \\ < 0 \end{matrix} \text{ as } \frac{\partial R(W)}{\partial W} \begin{matrix} \leq 0 \\ \geq 0 \end{matrix},$

where  $R(W)$  denotes relative risk-aversion ( $\equiv [-u''(W)W/u'(W)]$ ). Using the definition of  $W$ , as given by equation (III-1) of the text, we can rewrite the above relation as

$$E\{u''(W)(x-r)[W_0 + (1-t)(rW_0 + a(x-r) - K) + K]\} \begin{matrix} \geq 0 \\ < 0 \end{matrix}$$

as  $\frac{\partial R(W)}{\partial W} \begin{matrix} \leq 0 \\ \geq 0 \end{matrix}. \tag{A-2}$

Using the expression for the derivative  $\partial a/\partial W_0$ , (A-2) can be rearranged as

$$1 \begin{matrix} \leq \\ \geq \end{matrix} \frac{W_0}{a} \frac{\partial a}{\partial W_0} + \frac{tK}{a[1+r(1-t)]} \frac{\partial a}{\partial W_0} \text{ as } \frac{\partial R(W)}{\partial W} \begin{matrix} \leq 0 \\ \geq 0 \end{matrix} \tag{A-3}$$

This result is different from that obtained by Arrow. In particular, even if relative risk-aversion is constant, wealth elasticity of the demand for the risky asset is strictly less than unity, and decreasing relative risk-aversion no longer guarantees that the investor allocates proportionately more of his portfolio to the risky asset as he becomes wealthier.

Footnotes

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<sup>1</sup> For a discussion of the risk-aversion functions, see Arrow (1965) and Pratt (1964).

<sup>2</sup> Although the original discussion by Arrow does not consider taxation, the later authors writing on taxation and risk-taking have taken the Arrow results for granted (e.g., Mossin (1968), Stiglitz (1969 b), and Ahsan (1974)). Most of these studies, however, involved simple flat rate proportional taxes and the Arrow results apply in such models without any changes.

<sup>3</sup> Total (social) risk-taking is measured by the demand for the risky asset (denoted by  $\underline{a}$  in Ahsan (1974) and here). The Atkinson-Stiglitz suggestion of a measure of private risk-taking is  $\underline{a}(1-t)$ .

<sup>4</sup> For a detailed discussion, see Ahsan (1974).

<sup>5</sup> As seen in the appendix (section A.2), for non-decreasing relative risk-aversion the elasticity  $(W_0/a)(\partial a/\partial W_0)$  is strictly less than unity.

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