1975

A Searching Tale or the Non-Adaptive Consumer's Stream of Consciousness

Antoni Bosch-Domenech

20244

Research Report 7508

A SEARCHING TALE OR THE NON-ADAPTIVE CONSUMER'S STREAM OF CONSCIOUSNESS

Antoni Bosch-Domènech

March 1975

Preliminary Draft: Not for Quotation

I. Introduction

The theory of equilibrium competitive analysis is an impressive and beautiful achievement of many generations of economists. But no satisfactory explanation exists of how equilibrium is reached, even though an hypothesis concerning the laws of adjustment was already described by Walras (as tatônnements), and the underlying principle can be traced back at least to Adam Smith. As is well known, this principle states that prices of commodities increase or decrease depending on whether the excess demand for them is positive or negative. When this principle is incorporated into a competitive model some odd situations appear. If the economic agents are price takers that accept their inability to change prices, who changes them, and why does the pressure of the market compel someone to change them? Those logical difficulties have always been acknowledged. What seems a new development of the last fifteen years[1] is the critical attitude towards the further idealization that was introduced to solve these difficulties: the referee who fixes prices according to some variant of the old principle stated above. There is no doubt that the perfectly-competitive-cum-referee framework has been used to discuss equilibrium economics with gratifying results. But the limitations of this paradigm when applied to stability analysis are too well known. Both the referee and the unawareness of the participants of trading in a market in disequilibrium are theoretical idealizations of doubtful usefulness. They have helped to identify and codify problems in the area of equilibrium economics. But, I believe, any paradigm that incorporates them will fare very poorly when used as the frame of reference for dealing with disequilibrium situations--situations whose empirical relevance offers no doubt (they are everyday economic reality), and whose theoretical elucidation is basic to the assessment of the merits of equilibrium economics.

Stirred, perhaps, by Hahn's remark that no unifying principle such as maximization seems available when dealing with disequilibrium situations,[2] a great deal of work has been done in building disequilibrium models in which the participants are maximizers.[3] But, as stated, the formal requirements of perfect competition are internally consistent at a cost of a further idealization, namely the postulation of the existence of a referee. But then the paradigm does not provide the adequate intellectual lens to recognize and solve disequilibrium problems. Therefore, if the maximization postulate had to be kept, something had to go. In particular, the assumption of perfect competition was emptied of two of its basic characteristics. First, individuals were considered to be aware of the disequilibrium situation, and second, they were postulated as having only imperfect information about the relevant data on which to base their decisions. Unfortunately it turned out that to dig under the Marshallian cross when the cross itself was only partially drawn was not a simple task. And so, we witnessed in the last few years the flourishing of a rather appalling variety of models to explain the behaviour of the market participants and the ultimate approach of the market to some kind of equilibrium, based on different degrees of information and rationality assigned to the individual participants.[4] It seems, therefore, necessary, sooner or later (see [39], Conclusions), to start building a systematic taxonomy of cases of rational behaviour based on what each participant knows of each other (what buyers know about sellers, what sellers know about other sellers) and what sorts of equilibria result from each case. Still, it seems to me premature to build such a taxonomy when most of the discussion on the participants behaviour has been conducted with the help of too many <u>ad hoc</u> assumptions introduced to mimic, instead of resulting from, the rational behaviour of the participants.

In particular, I believe that some of the assumptions of the behaviour of rational consumers who do not possess complete information about the market can be derived from more simple postulates. And this is my first task. I will make precise what buyers know about the market prices and about sellers' behaviour and I will postulate a maximizing behaviour.[5] The market in which the transactions take place has a very rudimentary institutional framework. Nonetheless, results are obtained which seem to provide a theoretical explanation of some of the basic peculiarities of markets that the standard models have to disregard. But, as I see it, this paper is especially relevant as the first step towards a more ambitious project, in which a similar treatment of sellers' behaviour with different amounts of information will be complemented with the results obtained here, to elucidate the market's evolution and its eventual convergence to some equilibrium position. "Ce ne sont pas les perles qui font le collier, c'est le fil."

II. Outline of the Model of Consumer Behaviour

We shall be concerned with an exchange economy with a finite number of homogeneous commodities[6] to be labelled 1,2,...,m. The quantities of the m commodities will be represented by an m-vector $x = (x_1, x_2, \ldots, x_m)$. We shall be dealing only with non-negative quantities $x \geqq 0$. These commodities are traded in abstract entities called markets, of which there is one for each good. Prices of all m commodities are assumed to be positive, and they are represented by an m-vector $\overline{P} \in R^m_+$.[7] Prices for the first m-1 commodities are known to the consumer and assumed to be fixed and equal in all the stores selling these goods. The m^{th} commodity is sold by a certain positive number of stores at prices which might differ from store to store and which are imperfectly known at least to a positive number of consumers. It will be convenient to partition the price vector $\overline{P} = (\tilde{P}, P)$, where $\tilde{P}$ is the m-1 vector of prices of the first m-1 commodities and P is the price of the m^{th} commodity.

Our purpose is to discuss the optimal behaviour of the consumer who has imperfect knowledge of the prices quoted for the m^{th} commodity. We are not concerned, though, at this stage of the analysis, with how the stores come to decide on the particular prices that they quote. The analysis will cover from the moment, which will be called time t=0, at which the consumer has to decide whether he searches for a price of the m^{th} good, until the moment he finishes his search. This time span will be divided into periods and we shall assume that in each period the searching consumer visits one store at random.[8]

It should be stressed that time does not enter into this model in any relevant way and that, therefore, there is no rate of discount. We talk about periods to convey the idea that the search is done in a sequential way. But we could as well talk of the i^{th} search instead of the search of period i.

Search involves a cost. And so, after visiting a store and observing the price quoted, the consumer has to decide whether to buy at this price or to postpone his purchase in the hope of finding, after paying the additional search cost, a store with a cheaper price. We shall construct two alternative models of a consumer's behaviour. In one we assume that once a price has been observed and rejected, the consumer may lose the opportunity of buying at this price forever, because he does not have any guarantee that the store will quote the same price in later periods. (We will call this case search without recall.) In the other we assume that the consumer has the privilege of going back to any store previously visited and buying at the price he had observed (we will call this case search with recall). What model is better fitted to describe a particular real market is a highly speculative question. It is customary to believe that a search with recall model mimics more accurately what happens on a market of frequently purchased goods, while a search without recall model more closely describes what happens on a market for certain durable goods.[9] Whether this is true or not should not concern us here too much. Indeed, a more realistic description of both types of markets would be one in which the probability of being able to buy at a price previously observed would lie between zero and one.

We shall assume, in addition, in a way that will be made precise below, that the consumer knows, with or without certainty, the m^{th} commodity price distribution,[10] i.e., the probability of observing a price $P \leq p$ when a store is visited at random. This might create a problem of interpretation in the case in which the recall privilege is not allowed. Indeed, in this case two apparently uncongenial postulates are made. One postulates the belief by the consumer of an unchanged price distribution. The other, the no recall postulate, seems to require that the prices quoted by the stores change from period to period, which, most likely, would change the price distribution itself. Is it possible to imagine a situation in which both postulates can coexist? A straightforward answer would be to suppose that the price distribution can change from period to period but that the consumers are not aware of it. This interpretation has serious drawbacks. In the first place, contrary to our proclaimed desires, it introduces an element of irrationality in the consumer's behaviour, and one cannot help wondering why the consumers would be so incapable of discovering the changes. But worse still, a sequential search without recall must be based on the assumption that they know that stores can change the price and, taken literally, that stores will change their prices every period. Otherwise the consumer should consider, when making a decision, going back to the store already visited. Clearly, to suppose that he cannot go back because he does not have a locational memory when he has a price memory seems too ridiculously implausible.

Another line of defense is to suppose that the consumer is aware that the stores change prices every period and, therefore, is aware that whenever he does not buy at an observed price he may have lost the opportunity. And yet, he behaves as if the price distribution was immutable. This assumption will appear acceptable if the robustness properties of the Bayesian analysis hold, that is, in particular (for a more detailed discussion see [7]), if the price distribution changes in a "smooth" way from period to period. The "smoothness" of the price changes could be justified in terms of the caution displayed by the store managers, fearful of competition and of retaliatory measures.

Another possible case is that in which the prices do not change in the short run and the consumer is aware of this. Then the model might be appropriate for a market of durable goods (houses, cars,...) in which to each unit of the good corresponds one price. The inability of the consumer to return to a price previously observed stems, then, from the fact that he has no guarantee that a previous offer remains available after he rejected it. Whatever is the case we will assume that the consumer knows the price distribution with some degree of certainty and discuss his behaviour when, alternatively, he is allowed or he is not allowed the recall privilege.

We shall assume that the individual consumer has a (direct) utility function

$$U(x_1, x_2, \ldots, x_m) = U(x)$$

which is strictly increasing and strictly quasi concave in x, and which he maximizes over the commodity bundles that his budget constraint $\bar{p}x \leq y$ permits him to attain, where y is the consumer's income. If all prices, as well as y, are known, we can obtain the consumer's demand function $x = h(\bar{p}, y)$ which specifies the relationship between quantities x of the m commodities to be consumed on the one hand, and the prices p and income y on the other. From the direct utility function and the demand function, a relation $\bar{V}$ can be obtained between utilities on one hand, and prices and income on the other, in the following way,

$$U(h(\bar{p}, y)) = \bar{V}(\bar{p}, y) \; .$$

This relation is called the indirect utility function, a function which gives the value of the maximized utility for specific values of prices and income.

Therefore--let me make, for the sake of clarity, this trivial point--whenever the consumer has to choose between two situations defined by two different sets of prices and income, as in the case we are analyzing, his decision can be described in two alternative but equivalent ways. Either by maximizing the (direct) utility function subject first to one set of prices and income and then to the other set of prices and income and, between the resulting two bundles of goods, choosing the one that yields a higher level of utility. Or by substituting, one after the other, in the indirect utility function, the two sets of prices and income, and choosing the one that yields the higher (indirect) utility. Note that this second approach will implicitly yield the bundle of goods that the consumer decides to buy as it is made clear by Roy's formula

$$x_i = -\frac{\partial\bar{V}/\partial p_i}{\partial\bar{V}/\partial y}, \quad i=1,2,\ldots,m$$

(see [44]). It seems clear then that being, as we are, more interested in whether the consumer will select a particular set of prices and income in preference to another (expected) one, that in solving for the bundle of goods that he will buy, the second approach is simpler and more elegant. We shall, therefore, discuss the consumer's optimal search in terms of the indirect utility function $\bar{V}(\bar{p},y)$.

From the assumptions on the (direct) utility function it follows (see [24]) that $\bar{V}$ is strictly decreasing in $\bar{p}$, strictly increasing in y and strictly quasiconvex in $\bar{p}$.

For simplicity, we want to restrict the indirect utility function to being additive in the following way

$$\bar{V}(\bar{p},y) = V^*(\bar{p}) + W(y) .$$

What we postulate, then, is a strictly increasing and quasi concave direct utility function which yields an indirect utility function additive in prices and income. Could it not be the case, we may ask, that both sets of assumptions were contradictory? Or, to put it this way, is the intersection of the family of strictly increasing and quasi concave (direct) utility functions and the family of (direct) utility functions that yield additive indirect utility functions empty?[11] And then, how restrictive is the additivity assumption? The following theorems answer these questions.

Theorem 1

$\bar{V}(\bar{p},y) = V^*(\bar{p}) + W(y)$, where $\bar{V}$ is an indirect utility function if and only if $\bar{V}(\bar{p},y) = A + \log \xi(\bar{p}) + b \log y$, where $\xi(p)$ is a homogeneous function of degree $-b$, $b > 0$, and A is a parameter.

Proof:

(a) Necessity

Lemma A $\bar{V}(\bar{p},y) = V^*(\bar{p}) + W(y) \Rightarrow W(y) = b \log y + A$, b and A being two parameters.

Proof: $\bar{V}(\bar{p},y)$ being homogeneous of degree zero it follows that its partial derivative with respect to Y, $\bar{V}_y = W_y$ is homogeneous of degree -1. Since, in addition, W_y is independent of the prices, if follows that $W_y = \frac{b}{y}$, b being a parameter. Then, integrating, $W(y) = b \log y + A$, A being a parameter.

Lemma B $\bar{V}(\bar{p},y) = V^*(\bar{p}) + b \log y + A \Rightarrow V^*(\bar{p}) = \log \xi(\bar{p})$, where ξ is a homogeneous function of degree $-b$.

Proof: $\bar{V}(\bar{p},y)$ being homogeneous of degree zero, by Euler's theorem, if follows that $\sum_{i=1}^{m} \frac{\partial \bar{V}}{\partial p_i} p_i + \frac{\partial \bar{V}}{\partial y} y = 0$ or, substituting, that $\sum_{i=1}^{m} \frac{\partial V^*}{\partial p_i} p_i = -b$. Now we make the following change of variables: Call $V^*(\bar{p}) = \log \xi(\bar{p})$. Then $\frac{\partial V^*}{\partial p_i} = \frac{1}{\xi(\bar{p})} \cdot \frac{\partial \xi}{\partial p_i}$ and $\sum_{i=1}^{m} \frac{\partial \xi}{\partial p_i} p_i = -b\,\xi(\bar{p})$, the Euler's relation. And since the Euler's relation is necessary and <u>sufficient</u> for the homogeneity of the function (see, e.g., Courant [14], p. 109), it follows that $\xi(\bar{p})$ must be homogeneous of degree $-b$. Therefore only if $V^*(\bar{p})$ is the log of a homogeneous function of degree $-b$, can $\bar{V}(\bar{p},y) = V^*(\bar{p}) + b \log y + A$.

Since by definition $\bar{V}$ is strictly decreasing in $\bar{p}$ and strictly increasing in y, $b > 0$.

(b) Sufficiency - Trivial

Note, in particular, that the function $V^*(\bar{p}) = -\sum_{i=1}^{m} b_i \log p_i \equiv \log(\prod_{i=1}^{m} p_i^{-b_i})$, $\sum_{i=1}^{m} b_i = b$, that we shall use below, satisfies Lemma B.

Now we want to determine the restrictions that the additivity assumption imposes on the direct utility function.

<u>Theorem 2</u>

$\bar{V}(\bar{p},y) = A + \log \xi(\bar{p}) + b \log y$, where ξ is a homogeneous function of degree $-b$, $b > 0$, only if the direct utility function is $U(x) = \log \rho(x)$, where ρ is a homogeneous function of degree b.

<u>Proof</u>:

From the homogeneity of ξ,

$$\sum_{i=1}^{m} \frac{\partial \bar{V}}{\partial p_i} p_i = \frac{1}{\xi} \sum_{i=1}^{m} \frac{\partial \xi}{\partial p_i} p_i = -b .$$

From Roy's equation,

$$\frac{\partial \bar{V}}{\partial p_i} = -\frac{b}{y} x_i , \qquad i=1,2,\ldots,m.$$

From the assumption of consumer's rationality, $p_i = \frac{\partial U}{\partial x_i} \cdot \frac{1}{\lambda}$, where $\lambda = \frac{b}{y}$ is the Lagrangean multiplier.

Therefore, $\sum_{i=1}^{m} \frac{\partial \bar{V}}{\partial p_i} p_i = \sum_{i=1}^{m} \frac{\partial U}{\partial x_i} x_i = b$, and by the argument developed in

Theorem 1, $U(x) = \log \rho(x)$ where ρ is a homogeneous function of degree b. From Theorems 1 and 2 it follows:

Corollary

$\bar{V}(\bar{p},y) = V^*(\bar{p}) + W(y)$, where $\bar{V}$ is the indirect utility function only if the direct utility function is $U(x) = \log \rho(x)$, ρ being a homogeneous function.

It is clear, therefore, that the additivity of the indirect utility function does not contradict the assumption of quasi concavity of the direct utility function.

Note that since we have assumed that the vector of prices $\tilde{P}$ is constant, and recalling that $\overline{P} = (\tilde{P},P)$, we can write

$$V^*(\bar{p}) \equiv V(p)$$

and therefore

$$\overline{V}(\bar{p},y) = V(p) + W(y) .$$

The consumer's decision to buy or to keep searching (or at time t=0 to start searching at all) will be made on the basis of his preferences (represented by the indirect utility function), his knowledge of the price distribution and the prices observed.[13] It will be therefore indispensable to formally specify the following.

1. Decision Space $D = \{0,1\}$.

   It consists of two elements 0 and 1, 0 if the consumer stops the search at any $t > 0$ or decides not to search at time t=0, and 1 if the consumer decides to continue searching at any $t > 0$ or to start searching at t=0.

2. Sample Space S, whose elements are the prices of the m^{th} good observed by the consumer during the search.

3. The set $\mathcal{P}$ of all probability measures on S.

4. Preference ordering among the elements of $\mathcal{P}$. If $Pr^1 \in \mathcal{P}$ and $Pr^2 \in \mathcal{P}$, $Pr^1 \geq^* Pr^2$ means that Pr^1 is at least as good as Pr^2.

Next we make the "expected utility hypothesis"

5. $\overline{V}$ is such that for any $Pr^1 \in \mathcal{P}$ and $Pr^2 \in \mathcal{P}$, $Pr^1 \geq^* Pr^2 \Leftrightarrow E(\overline{V}|Pr^1) \geqq E(\overline{V}|Pr^2)$ where E is the expectation operator.

The problem that the consumer confronts is: "When, on the basis of my knowledge and my preferences should I decide to buy instead of looking for another price?" Stated in this way this is a standard problem of optimal stopping. The consumer is faced with a multiple decision sequential problem: at each stage he will have to decide whether to stop or to continue searching. His decision must be based only on his knowledge of the price distribution up to that stage, on his preferences and on the cost of further search. The rule that tells the consumer when to stop is called a stopping rule. More formally, a stopping rule is a partition of all possible prices of the m^{th} good into two sets, the stopping set T and the continuation set $\sim T$. In addition, for the rule to be reasonable, the probability of entering T through the sequential search must be equal to 1.

<u>Definition</u>. For any $Pr \in \mathcal{P}$, a stopping rule is any set $T \subseteq S$ such that Pr (entering T through sequential search) = 1. The rule is stated as follows: Stop the first time that T is entered. That is

$$\begin{aligned} d(x) &= 0 \quad \text{if } x \in T \\ &= 1 \quad \text{if } x \in \sim T \end{aligned}$$

where $d: S \to D$.

Let us call Δ the set of all stopping rules as defined above. If $E_T \overline{V}$ is the expected utility (with respect to Pr) using the stopping rule T, then, if it exists, T* is the optimal stopping rule whenever

$$(\forall T \in \Delta)\ E_{T^*} \overline{V} \geq E_T \overline{V} \ .$$

For a survey of such problems see Breiman [9].

We have already mentioned that searching comports a cost. Let us make this precise.

6. a) The consumer has at time $t=0$ an income $y > 0$ and does not receive any additional income during the search.

   b) Every search involves a constant loss of income $\overline{e} > 0$.[14]

   c) $y - i\overline{e} > 0$ (where i indicates the number of searches), i.e., in no case will the consumer spend all his income in searching.[15]

Let us define

$$c_i = W[y - (i-1)e] - W[y - i\overline{e}] \ , \ i=1,2,\ldots \ ,$$

i.e., c_i is the cost of the i^{th} search measured in utility units. Since W is twice differentiable and $W_Y > 0$ and $W_{YY} < 0$ we have that $c_i > 0$ and $c_i < c_{i+1}$, $i=1,2,\ldots$. From now on whenever we mention the cost of search we shall be referring to its utility measure c_i.

Now, it happens that in most of the search literature, either the search costs are considered non-existent (Phelps and Winter [36], Mortensen [37]) or constant (McCall [31] and [32], Rothschild [40], Rothschild and Arrow [42], Telser [46]). But, on the light of our previous results, if searching carries with it a constant depletion of the given endowments of some good (leisure, for example) or a constant expenditure of the given income, then the hypothesis of a constant cost of search (measured in utility terms) is contradictory.[16] This is so since for both a constant depletion (alt. expenditure) and a constant cost of search to be true, the (indirect) marginal utility of the price of the good (alt. income) has to be independent of the prices and income (alt. the prices), which requires an indirect utility function additive in the price of the good (with respect to both the other prices and income) (alt. an indirect utility function additive in the income). But this, as we saw above, implies that the indirect utility function has to be the sum of one term, independent of the price of the good (alt. income), plus the log of the price of the good (alt. income), contradicting therefore the assumption of constant costs.

Nevertheless, due to its wide use and to the neat simplifications that it allows, I will not completely refrain from using the assumption of constant cost of search.

As a rationale for it I shall suppose, in that case, that the consumer receives a fixed amount of income every period and completely spends it during it. Unfortunately, for most reasonable additive indirect utility functions, this implies that the consumer buys at every period a positive amount of the good that he is searching since, otherwise, his utility would become $-\infty$. The decision of the consumer, of course, will be about searching or not searching during the period.

Finally, it will be convenient to cast our discussion of the search problem in terms of the probability space defined by the triple $(S, \mathfrak{B}, Pr)$ where $\mathfrak{B}$ is a σ-field of sets in S and Pr a probability measure on $\mathfrak{B}$. Then it is advantageous to interpret P as a random variable, $P: S \to R$ and make the following assumption.

7. There exists a cdf $G(p) = Pr(P \leqq p)$ which describes the consumer's probability assessment about the prices quoted by the stores.

Now, the cfd itself may not be known with certainty. In that case the observation of prices in the sequential search will be a learning experience to the consumer, who will update his beliefs accordingly.

III. Consumer Behavior with Knowledge of the Price Distribution

We shall suppose that the consumer is aware that stores may be charging different prices for the m^{th} commodity and believes he knows the distribution of those prices. We shall denote by P_i the variable price-observed-at-period-i (i=1,2,...)[17] and assume that $P_1, P_2, \ldots$ are independently[18] and identically distributed random variables. We can view the consumer, therefore, as sequentially sampling from the price population. Or equivalently, since we have postulated that to each price there corresponds a numerical utility determined by a strictly decreasing function V(P), we can describe the consumer as sampling from a population of utilities with cdf F[V(P)]. This way of looking at the problem, although less appealing to common sense, simplifies the exposition.

In addition to the assumptions made in the last section we make the following:

8. $E[V(P)]^2 < \infty$.

If the consumer decides to buy after a number n of searches, i.e., after observing $P_n = p_n$ and recall is not allowed, his utility if buying at p_n will be $V(p_n) + W(y - n\bar{e}) = V(p_n) + W(y) - \sum_{i=1}^{n} c_i$, provided that $y - n\bar{e} > 0$. Now since W(y) is a parameter we can dispense with it in any maximization problem. Making use of the expected utility hypothesis, what we want to find is, therefore, a stopping rule that maximizes

$$E[V(P_{N_T}) - \sum_{i=1}^{N_T} c_i] \tag{1}$$

over Δ, where N_T is, for every T, a random variable indicating the number of searches required to enter the stopping set T for the first time.[19] Observe that expression (1) can also be written as

$$E_{N_T}\{E[V(P_{N_T})|N_T] - \sum_{i=1}^{N_T} c_i\} ,$$

where first, the conditional expected value of $V(P_{N_T})$ given that the consumer takes the N_T th price observed is obtained, and then, N_T is expected out.

Similarly, if recall is allowed, calling $m_n = \max\{V(P_1), V(P_2),\ldots, V(P_n)\}$, we want the stopping rule that maximizes

$$E[m_{N_T} - \sum_{i=1}^{N_T} c_i] \tag{2}$$

over Δ.[20] Let us consider in turn the cases of no recall and recall.

1. <u>No Recall</u>

We assume that given the consumer's income y, and the cost of search at period j being $c_j > 0$, j=1,2,..., an optimal stopping rule exists and the expected utility of following the optimal rule is finite.[21] Let us call the expected utility of following the optimal rule prior to search $v_o^* + W(y)$, where v_o^* is the maximum of (1). Now we can describe the consumer's search in the following way: After observing the first price $P_1 = p_1$, the consumer either stops or continues searching. If he stops, his utility will be $V(p_1) + W(y) - c_1$.[22] If he decides to continue searching, then he is in a position similar to the one he was in before starting the search, since his prior beliefs are still the same.[23] But there are differences. First of all, the sequence of costs that he has to pay starting his search from his new position is the following:

c_2 for the first search, c_3 for the second and c_i for the $i-1^{th}$. Therefore, the expected utility of following the optimal rule, even supposing that he has not incurred any cost, will be different from $v_o^* + W(y)$. We could call it $v^*(p_1) + W(y)$. But he has already spent $\bar{e}$. Therefore, the expected utility of following the optimal rule when one price is observed is $v^*(p_1) + W(y) - c_1$. Note that this utility depends also on the price observed, because the optimal rule might require to stop after $P_1 = p_1$ is observed, in which case $v^*(p_1) = V(p_1)$. In a similar vein we will call the expected utility of following the optimal rule when the i^{th} price has been observed $v^*(p_i) + W(y) - \sum_{j=1}^{i} c_j$.[24]

Since $v^*(p_1)$ is the expected utility of following the optimal rule after $P_1 = p_1$ has been observed, exclusive of the cost of previous search, it has to be equal either to $V(p_1)$ or to $v^*(p_2) - c_2$, the expected utility of following the optimal rule once the second price has been observed (c_2, of course, is the cost of observing the second price). Now, since at this first stage the second price has not yet been observed, the expected value of following the optimal rule after observing the second price should be written $E[v^*(P_2) - c_2]$, where P_2 is a random variable. And so, in general, we can state

$$v^*(p_i) = \max \{V(p_i), E[v^*(P_{i+1}) - c_{i+1}]\} \quad i=1,2,\ldots, \qquad [3]$$

Clearly, then, if we call $E[v^*(P_{i+1}) - c_{i+1}] = \alpha_i, \quad i=1,2,\ldots, \qquad [4]$

the optimal rule is to stop searching whenever $V(p_i) \geq \alpha_i$ and to continue searching otherwise. Or equivalently, to stop the search if $p_i \leq p_i^*$, where

$$p_i^* = V^{-1}(\alpha_i) \; . \qquad [5]$$

Note that α_i is the expected utility at stage i of taking one more observation and following the optimal rule afterwards.

Combining [3] and [4] we obtain

$$\alpha_i = E[\max\{V(P_{i+1}), \alpha_{i+1}\} - c_{i+1}].$$

Therefore, since $V(P_1)$, $V(P_2)$,...are iid, by lemma 1,[25]

$$\alpha_i = \int_{\alpha_{i+1}}^{\infty} (V(P) - \alpha_{i+1})\, dF[V(P)] + \qquad [5^*]$$

$$+ \alpha_{i+1} - c_{i+1}, \quad i=1,2,\ldots$$

or calling $g(\alpha_{i+1}) = \int_{\alpha_{i+1}}^{\infty} (V(P) - \alpha_{i+1})\, dF[V(P)]$, by lemma 2,

the sequence (α_i) is determined once one of its elements is known.

$$\alpha_i = g(\alpha_{i+1}) + \alpha_{i+1} - c_{i+1} \quad i=1,2\ldots \qquad [6]$$

Note that if the cost of search is constant, then at the beginning of each period the situation is exactly the same except for the costs already incurred. Therefore calling α^* the expected utility of following the optimal rule, exclusive of the search cost incurred in previous search, we have

$$\alpha^* = \int_{\alpha^*}^{\infty} (V(P) - \alpha^*)\, dF[V(P)] - c + \alpha^*,$$

that is, α^* is the (unique, for $c>0$, by lemma 2) solution of

$$g(\alpha^*) = c, \qquad [7]$$

and the optimal rule is to stop the search as soon as the price observed p_i, $i=1,2,\ldots$, satisfies $p_i \leq p^*$, where

$$p^* = V^{-1}(\alpha^*) \qquad [8]$$

2. Recall

This case is easier to discuss than the previous one since we are now in what Chow and Robbins[26] called the monotone case, which is defined for a stochastic sequence (u_i) of utilities as the one satisfying

$$A_1 \subset A_2 \subset \ldots , \bigcup_{i=1}^{\infty} A_i = \Xi$$

where $A_i = \{E(U_{i+1}) \leq u_i\}$ i=1,2... and Ξ is, the set of all possible utilities in R, the set of real numbers, and the expectation is taken over the distribution of utilities at period i + 1.

In this case, if some additional conditions are satisfied (see, Chow, Robbins, Siegmund (1971), Ch. 3, Sect. 5),[27] the optimal procedure is a myopic one in which at any stage the consumer is allowed to search at most one more time. In this case, the optimal rule will be to stop at stage N, where N = first $i \geq 1$ such that $u_i \geq E(U_{i+1})$.[28]

Now, as we shall presently show, the case in which the consumer is allowed the recall privilege falls into the monotone case. Let us call $m_i = \max(V(p_1),\ldots V(p_i))$ and $u_i = m_i - \sum_{j=1}^{i} c_j$, i=1,2,..., where u_i is the utility of stopping at the ith search. Then $U_{i+1} - u_i = \max[V(P_{i+1}), - m_i - c_{i+1}$; that is $E(U_{i+1}) \leq u_i \Leftrightarrow E\{\max[V(P_{i+1}), m_i]\} - m_i \leq c_{i+1}$.

If we define α_i as the (unique by lemmas 1 and 2) solution of

$$E\{\max[V(P_{i+1}), \alpha_i]\} - \alpha_i = c_{i+1} \qquad [9]$$

or, equivalently, $g(\alpha_i) = c_{i+1}$, we have $E(U_{i+1}) \leq u_i \Leftrightarrow m_i \geq \alpha_i$ by lemma 2.

Note that [9] can be written as $E\{\max\ [V(P_{i+1}) - \alpha_i, 0]\} = c_{i+1}$. Therefore, since $\{V(P_i)\}$ is a sequence of i.i.d. random variables, $c_{i+1} \geq c_i \Rightarrow \alpha_i \leq \alpha_{i-1}$ and since by definition $m_i \geq m_{i-1}$, $[(m_{i-1} \geq \alpha_{i-1}) \Rightarrow (m_i \geq \alpha_i)] \Leftrightarrow [(E(U_i) \leq u_{i-1}) \Rightarrow [E(U_{i+1}) \leq u_i)]$ and we are in the monotone case. For a verification of the additional conditions that guarantee that the optimal stopping rule is to stop at N, N = first $i \geq 1$ such that $m_i \geq \alpha_i$, see Chow, Robbins, Siegmund [12], Ch. 3, Sect. 6. Therefore, the optimal rule is to stop at the first i at which $p_i < p_i^*$, $i=1,2...$ where $p_i^* = V^{-1}(\alpha_i)$, i.e.,

$$g[V(p_i^*)] = c_{i+1} \quad . \qquad [10]$$

In particular, if the search cost is constant, the optimal rule is to stop whenever $m_i \geq \alpha^*$, where α^* is the unique solution of

$$E\{\max[V(P), \alpha^*]\} - \alpha^* = c.$$

From this it is clear that the rule that says to stop at N, N being the first $i \geq 1$ such that $m_i \geq \alpha^*$, is equivalent to the optimal rule, obtained with no recall and constant cost (namely to stop at N, N being the first $i \geq 1$ such that $V(p_i) \geq \alpha^*$) because $M_N = \max(V(p_1)...V(p_N)) = V(p_N)$, since if $V(p_j) \geq V(p_N)$ for $j \in \{1,2,..., N-1\}$ the search would have stopped at stage j.

Definition:

A price p^* (alt. p_i^*, if the cost of search is not constant) is called the reservation price, relative to the optimal stopping rule, if for prices above p^* (alt. p_i^*, $i=1,2,...$, at stage i) the consumer continues the search and stops it for prices less or equal to p^* (alt. for prices less or equal to p_i^*, $i=1,2,...$, at stage i). Note that p^* has to satisfy [8] (alt. p_i^* has to satisfy $p_i^* = V^{-1}(\alpha_i)$ where α_i is defined in [4] or [9] depending on the problem).

A. Constant Cost of Search

When the cost of search is constant from period to period the expected utility of following the optimal rule is the same, as we have seen, with or without recall. In this case we can obtain the following results.

Proposition 1.1

Let c_1 and c_2 be two different costs of search such that $c_1 > c_2$, and let p_1^* and p_2^* be the reservation prices corresponding to the cost c_1 and to the cost c_2 respectively, the price distribution being the same. Then $p_1^* > p_2^*$. Let α_1^* and α_2^* be respectively the expected utility of following the optimal rule with cost c_1 and with cost c_2. Then $\alpha_1^* < \alpha_2^*$. This can be read as implying that the larger the cost of search the smaller is the expected amount of search.

<u>Proof</u>:

That $p_1^* > p_2^*$ and $\alpha_1^* < \alpha_2^*$ is clear from [7], [8] and lemma 2. Let N be the number of searches under the optimal stopping rule. N is therefore the random number of trials for a price lower or equal than the reservation price to be found for the first time. Therefore N has a geometric distribution with parameter $\nu = \int_{\alpha^*}^{\infty} dF(V(P))$ and $E(N) = \nu^{-1}$.[29] Note that ν is the probability of V(P) being greater than α^* or, equivalently, of p being smaller than p^*. Now $\nu_1 = \int_{\alpha_1^*}^{\infty} dF(V(P))$ and $\nu_2 = \int_{\alpha_2^*}^{\infty} dF(V(P))$. Since $\alpha_1^* < \alpha_2^*$, $\nu_1 > \nu_2$ and the expected number of searches when the cost is c_1 is smaller than when the cost is c_2.

Proposition 1.2

Suppose that there exists a minimum price $\underline{p}$ and a maximum price $\bar{p}$ in the sense that $Pr(P < \underline{p}) = 0$, $Pr(P > \bar{p}) = 0$, $Pr(\underline{p} < P \leq \underline{p} + \epsilon) > 0$ and $Pr(\bar{p} - \epsilon \leq P < \bar{p}) > 0$ for any $\epsilon > 0$. Then

a) $\underline{p}$ will be the reservation price if and only if $c=0$.

b) $\bar{p}$ will be the reservation price if and only if $c = E(V(P)) - V(\bar{p}) > 0$.

Proof

a) Suppose that $\underline{p}$ is the reservation price.

Since $Pr(P < \underline{p}) = 0$, $Pr(V(P) > V(\underline{p})) = 0$ and therefore $g(V(\underline{p})) = 0$.

$\underline{p}$ being the reservation price requires that $g(V(\underline{p})) = c$. Therefore $c=0$. Conversely if $c=0$, the reservation price p^* has to satisfy $g(V(p^*)) = 0$. But $g(V(p^*)) = 0 \Leftrightarrow Pr(V(P) > V(p^*)) = 0 \Leftrightarrow Pr[P > p^*] = 0$. Therefore $p^* \in \{P \leq \underline{p}\}$. But of all the elements of this set only $\underline{p}$ can be the reservation price (see the definition). Therefore $p^* = \underline{p}$.

b) Suppose that $\bar{p}$ is the reservation price.

Since $Pr(P > \bar{p}) = 0$, $Pr(V(P) < V(\bar{p})) = 0$ and therefore $g(V(\bar{p})) = E(V(P)) - V(\bar{p})$. (Note that $\lim_{p \to \infty} [g(V(p)) - \{E(V(P)) - V(p)\}] = 0$). $\bar{p}$ being the reservation price requires that $g(V(\bar{p})) = c$. Therefore $c = E(V(P)) - V(\bar{p})$. Conversely, if $c = E(V(\bar{p}) > 0$, the reservation price p^* has to satisfy $g(V(p^*)) = E(V(P)) - V(\bar{p})$. Now, $p^* = \bar{p}$ is a solution of this equation, and since $g(V(p^*))$ is strictly decreasing in $V(p^*)$, it is the only solution.

Proposition 1.3

If $c \geq E[V(P) - V(\bar{p})]$, then the consumer will take the first price observed $(\underline{p} \leq p \leq \bar{p})$.

Proof

Since $E[V(P) - V(\bar{p})] \geq g(V(\bar{p})$, then $c \geq g(V(\bar{p}))$ and $p^* \geq \bar{p}$, where $g(V(p^*)) = c$.

Proposition 1.4

If $\nexists$ a $\underline{p}$ such that $Pr[V(P) > V(\underline{p})] = 0$, and $c=0$, the search will never end.

Proof: Trivial

Proposition 1.5

If $\exists$ a $\underline{p}$ such that $Pr[V(P) > V(\underline{p})] = 0$ and $Pr[\underline{p} < P \leq \underline{p} + \epsilon] > 0$ for any $\epsilon > 0$, and $c = 0$, the search will never end with probability 1.

Proof: Trivial

Proposition 1.6

If $\nexists$ $\bar{p} < \infty$ such that $Pr[V(P) < V(\bar{p})] = 0$ and $c \geq \lim_{\bar{p} \to \infty} E[V(P) - V(\bar{p})]$, the consumer will take the first price observed.

Proof: Trivial

B. Variable Cost of Search

a. with recall.

Using equation [10] and lemmas 1 and 2, it is clear that, with the obvious modifications, the results obtained in the previous section hold when the cost of search is not constant.

b. without recall.

Proposition 2.1

$$\alpha_{i-1} \geq \alpha_i \quad , \quad i=2,3,\ldots .$$

Proof:

$$\alpha_{i-1} \geq \alpha_i \Leftrightarrow \int_{\alpha_i}^{\infty} (V(P) - \alpha_i)\, dF[V(P)] \geq c_i, \text{ from [5*]}.$$

Let us call $\alpha_i^* = g^{-1}(c_i)$ i.e., $g(\alpha_i^*) = c_i > 0$.

Then $\alpha_{i-1} \geq \alpha_i \Leftrightarrow g(\alpha_i) \geq g(\alpha_i^*) > 0 \Leftrightarrow \alpha_i^* \geq \alpha_i$ by lemma 2.

α_i is the expected utility of , at stage i-1, paying c_i, observing P_i and following the optimal rule afterwards. α_i^* is the expected utility of, at stage i-1, paying c_i, observing P_i and following the optimal rule afterwards when all the future costs of search are c_i.

Let us consider the following No. 1 auxiliary optimal stopping problem. At stage i-1 the consumer has to pay c_i and observe P_i and then follow the optimal rule when the cost of search is thereafter c_{i+1}.

No. 1: (i-1) (i) (i+1) (i+2) ...
c_i c_{i+1} c_{i+1} c_{i+2}

Now, the expected utility of continuing the search optimally after P_i has been observed is α_{i+1}^*. Let us call, in the auxiliary problem, the optimal expected utility at stage i-1 $^1(\alpha_i)$. Then we have

$$^1(\alpha_i) = E[\max\{V(P_i), \alpha_{i+1}^*\}] - c_i.$$

Since $\alpha_i^* = E[\max\{V(P_i), \alpha_i^*\}] - c_i$ and by Proposition 1.1 $\alpha_i^* > \alpha_{i+1}^*$, then $\alpha_i^* \geq {}^1(\alpha_i)$, with strict inequality if the probability of not stopping at stage i is greater than zero.

Similarly, let us construct another auxiliary optimal stopping problem. At stage i-1 the consumer has to pay c_i and observe P_i. If the search continues after observing P_i, the cost of observing P_{i+1} is c_{i+1}. After stage i+1, the cost of observing any other price is constant at c_{i+2}.

No. 2: (i-1)	(i)	(i+1)	(i+2)	...	
	c_i	c_{i+1}	c_{i+2}	c_{i+2}	

So, if price P_{i+1} is ever observed the expected utility of <u>continuing</u> the search optimally is α^*_{i+2}, while in the auxiliary problem no. 1, after stage i+1 the expected utility of continuing the search optimally is α^*_{i+1}. Only if the probability of reaching this stage is zero, the auxiliary problems No. 1 and No. 2 have the same optimal rule with probability one. Therefore, if we call $^2(\alpha_i)$ the maximum expected utility in the auxiliary problem No. 2, then $^1(\alpha_i) \geq {}^2(\alpha_i)$, since $\alpha^*_{i+2} < \alpha^*_{i+1}$ by Proposition 1.1. The inequality, as before, is strict if the probability of reaching stage i+1 is positive.

In the same way we can define a sequence $^k(\alpha_i)$, k=1,2,... such that

$$^\ell(\alpha_i) \geq {}^m(\alpha_i), \quad \text{for any } \ell < m, \quad \ell, m = 1,2,\ldots$$

Now, observing that $\lim_{k\to\infty} {}^k(\alpha_i) = \alpha_i$, we have

$$\alpha^*_i \geq {}^1(\alpha_i) \geq {}^2(\alpha_i) \geq \ldots \geq \alpha_i.$$

Proposition 2.2

a. If $\exists$ a $\bar{p}$, as defined above, and $0 < c_{i+1} < g(V(\bar{p}))$, then $\alpha_{i-1} > \alpha_i$ (i=2,3,...) with probability one.

b. If $\nexists$ a $\bar{p}$, as defined above, and $0 < c_{i+1} < \lim_{p\to\infty} g(V(p))$, then $\alpha_{i-1} > \alpha_i$ (i=2,3,...) with probability one.

Proof

a. $c_{i+1} = g(\alpha^*_{i+1}) < g(V(\bar{p})) \Leftrightarrow \alpha^*_{i+1} > V(\bar{p})$, where $P^* = V^{-1}(\alpha^*_{i+1})$

is the reservation price for the optimal stopping problem with constant cost of search c_{i+1}. Therefore, by the definition of $\bar{p}$, $Pr[p^* < p_i \le \bar{p}] > 0$. So, there is, at stage i, a positive probability of observing a price below p^* and therefore continuing the search. Then, by the argument developed in the previous proof, $\alpha^*_i > {}^1(\alpha_i)$, with probability one. And since

$${}^1(\alpha_i) \ge {}^2(\alpha_i) \ge \ldots \ge \alpha_i,$$

we have that $\alpha_{i-1} > \alpha_i$ with probability 1.

b. The proof follows the same steps.

Corollary 2.1. No recall

If the consumer knows the price distribution and the cost of search increases from period to period, $0 < c_1 < c_2 < c_3 \ldots$, the reservation prices for each period form a nondecreasing sequence (p^*_i), $p^*_i \ge p^*_{i-1}$, $i=2,3,\ldots$

Proof

By Proposition 2.1, $\alpha_{i-1} \ge \alpha_i$, and from [4] and lemma 2 the result follows.

Corollary 2.2. No recall

If the consumer knows the price distribution and the cost of search increases from period to period, $0 < c_1 < c_2 < c_3 \ldots$, then the reservation prices for each period form an increasing sequence (p^*_i), $p^*_i > p^*_{i-1}$, $i=2,3,\ldots$, with probability one if c_{i+1}, $i=2,3,\ldots$, satisfies either inequality a) or inequality b) in Proposition 2.2.

Proof:

By Proposition 2.2, $\alpha_{i-1} < \alpha_i$ with probability one, and from [4] and lemma 2 the result follows.

Corollary 2.3. Recall

If the consumer knows the price distribution and the cost of search increases from period to period, $0 < c_1 < c_2 < c_3 \ldots$, the reservation prices for each period form an increasing sequence (p_i^*), $p_i^* > p_{i-1}^*$, $i=2,3,\ldots$

Proof:

Immediate from equation [10] and lemma 2.

So far, these results are encouraging. There is a discernible evolutive pattern of the reservation prices which might be crucial in a description of the dynamics of the market prices when both consumers and stores are supposed to behave optimally. Unfortunately, we will see later on that when the price distribution is not known with certainty there is no way to guarantee that the reservation prices are monotonically non-decreasing.

Finally it is easy to verify that an increase in the cost of search reduces the expected utility of optimal search.

A SPECIAL CASE

To obtain further results we will assume that prices are distributed lognormally $P \sim \Lambda(\mu^*, \tau^*)$, where μ^* and τ^* are respectively the mean and precision of the random variable log P which is normally distributed. We will also assume that V(P) is a Bernouilli utility function $V(P) = \log a - b \log P, b > 0, P > 0$, the random variable V(P) is distributed $N(\mu, \tau)$, where $\mu = a - b\mu^*$ and $\tau = \frac{\tau^*}{b^2}$.[30,31]

a) Constant cost of search

We know that if the cost of search is constant, the optimal stopping rule will be defined by a price p^* such that $g[V(p^*)] = c$. But now, since V(P) is normally distributed with mean μ and precision τ, using Lemma 3 we obtain

$$g[V(p^*)] = \tau^{-\frac{1}{2}} \Psi[\tau^{\frac{1}{2}} (V(p^*) - \mu)], \text{ where}$$

$$\Psi(s) = \int_s^\infty (z-s)\ \psi(z)\ dz\ , \quad \text{and}$$

$\psi(z)$ is the pdf of a standard normal distribution. Now, $\Psi(\cdot)$ is, by Lemma 2, a strictly decreasing function for $\Psi(\cdot) \neq 0$. And if $p^* > 0$, $V(p^*) < \infty$, and

$$\Psi[\tau^{\frac{1}{2}}(V(p^*) - \mu)] \neq 0 .$$

Therefore Ψ^{-1} exists everywhere and since $\Psi[\tau^{\frac{1}{2}}(V(p^*) - \mu)] = \tau^{\frac{1}{2}}c$,

$$\alpha^* = V(p^*) = \mu + \tau^{-\frac{1}{2}}\Psi^{-1}[c\,\tau^{\frac{1}{2}}] . \qquad [11]$$

Proposition 3.1

A mean preserving increase in the price dispersion,[32] that is, a decrease in β with α constant (where β and α are respectively the precision and the mean of the price distribution) leads to an increase in the expected utility of optimal search α^* and to a decrease in the reservation price p^*.

Proof

Since[33] $\beta = \frac{1}{\alpha^2\eta^2}$, where $\eta^2 = e^{1/\tau^*} - 1$, a decrease in β with α constant can take place if and only if τ^* decreases. Since, on the other hand, $\alpha = e^{\mu^* + \frac{1}{2\tau^*}}$, a decrease in τ^* with α constant requires a decrease in μ^*. Since, moreover, $\frac{d\mu}{d\mu^*} = -b < 0$, it follows that μ increases. Therefore from [11], $\tau = \tau^*/b^2$, and Ψ^{-1} being a strictly decreasing function by Lemma 2, the results follow.

Proposition 3.2

A utility mean preserving[34] increase in the price dispersion, that is, a decrease in β with μ constant leads to an increase in the expected utility of optimal search α^* and to a decrease in the reservation price p^*.

Proof

Since, μ^* being constant, a change in α can only be the result of a change in τ^*, a decrease in β can take place if and only if τ^* decreases. Therefore the result follows.

Note that so far we have postulated that the consumer had certain beliefs about the price distribution but, since we had assumed that there was no learning in the process of observing the prices quoted and, therefore, no adaptation of beliefs, for the purpose of determining the maximum expected utility of further search and the optimal stopping rule, there was no need to relate the consumer's subjective beliefs about the price distribution with the "true" price distribution. No need for specifying the "true" distribution exists, as well, when we are concerned with the consumer's expected number of searches. It is a completely different matter when we want to determine the "objective" expected number of searches on the part of the consumers. By this we mean the expected number of searches computed on the basis of the "true" price distribution. It is trivial to show that if two consumers have the same utility functions and confront the same price distribution but differ in their estimation of the price dispersion, the consumer that believes that the distribution is more dispersed will search more (the "objective" expected number of searches will be larger) than the other, since his reservation price will be lower. But more interesting than this is the question of whether a consumer who knows the price distribution will search more when the price becomes more dispersed. As Rothschild [40, p. 692] has pointed out, this rule, if observed, will result in some sort of stability in any reasonable and complete model of the market. Rothschild, unfortunately for us, believes it is obvious that if as a result of an increase in price dispersion the reservation price decreases then, the expected number of searches increases, and, consequently, does not provide a proof. Since it can be shown (see below Chapter IV) that this is not true in the adaptive case and, alas, it is not obvious to me in the non-adaptive case, I prefer to present a proof under my present assumptions.

Proposition 3.3

An increase in the price dispersion leads to an increase in the expected amount of search.

Proof

Observe that $V(P) \sim N(\mu, \tau)$, and if we call $\nu = Pr[V(P) \geq \alpha^*]$, then the expected number of searches $E(N) = \nu^{-1}$. To make the notation simpler, I shall add to the relevant variables and parameters and the index A when the precision of the price distribution is β_A, and the index B when the precision of the price distribution is β_B.

Let me assume that $\beta_A > \beta_B$. Then, $\tau_A > \tau_B$. From [11], $\nu_A = \Pr[V(P_A) \geq \mu_A + \tau_A^{-\frac{1}{2}} \Psi^{-1} [c\tau_A^{\frac{1}{2}}]]$. Now $V(P_B) = (V(P_A) - \mu_A)(\frac{\tau_A}{\tau_B})^{\frac{1}{2}} + \mu_B$. From this and [11], $\nu_B = \Pr[V(P_A) \geq \mu_A + \tau_A^{-\frac{1}{2}} \Psi^{-1}[c\tau_B^{\frac{1}{2}}]]$.

Therefore, since Ψ^{-1} is a decreasing function, $\nu_A > \nu_B$ and $E(N_A) < E(N_B)$.

Proposition 3.4

A utility mean preserving increase in the price dispersion increases the expected number of searches.

<u>Proof</u>

Exactly as above, with $\mu_A = \mu_B$.

b) Non-constant cost of search

If the recall privilege is allowed, one would expect that the results obtained with a constant cost of search carry over to this case. In intuitive terms the higher is the price dispersion the more likely it is to observe a low price and, although the likelihood of observing a high price is also increased, the consumer is protected against this eventuality by being able to go back to a price previously observed[35]. This can be made rigorous by rewriting equation [10] in the following form:

$$\alpha_i^* = V(p_i^*) = \mu + \tau^{-\frac{1}{2}} \Psi^{-1} [\tau^{\frac{1}{2}} \cdot c_{i+1}] . \qquad [11^*]$$

It is clear then that by using the same procedure as above, the following two propositions can be proved.

Proposition 3.5 Recall.

A mean preserving increase in the price dispersion leads at any search stage i, $i=1,2,\ldots,$ to an increase in the expected utility of optimal search α_i, to a decrease of the reservation price p_i^*, and to an increase in the expected amount of search.

Proposition 3.6 Recall.

A utility mean preserving increase in the price dispersion leads at any search stage i, $i=1,2,\ldots,$ to an increase in the expected utility of optimal search α_i, to a decrease of the reservation price p_i^*, and to an increase in the expected amount of search.

Although less intuitive, it is also true that when no recall is allowed the two previous propositions hold. To show it, observe that if all future and present utilities are reduced by μ, we have

$$v(p_i)^* - \mu = \max \left\{ V(p_i) - \mu \, , \, E\left[V(P_{i+1})^* - \mu - c_{i+1} \right] \right\} .$$

Let $V(P_i) - \mu = Z_i$, then $Z_i \sim N(o,\tau)$. Calling $E\left[Z^*_{i+1} - c_{i+1}\right] = \hat{\gamma}_i$, then, clearly the optimal rule after observing $P_i = p_i$ is to stop whenever $z_i \geq \hat{\gamma}_i$ or equivalently $V(p_i) \geqq \mu + \hat{\gamma}_i$.

Now, proceeding as for [6] we have that $\hat{\gamma}_i = \hat{\gamma}_{i+1} + g(\hat{\gamma}_{i+1}) - c_{i+1}$, and using Lemma 3, we obtain

$$\hat{\gamma}_i = \hat{\gamma}_{i+1} + \tau^{-\frac{1}{2}} \Psi \left[\tau^{\frac{1}{2}} \hat{\gamma}_{i+1}\right] - c_{i+1}.$$

By Lemma 7

$$\hat{\gamma}_i = \tau^{-\frac{1}{2}} \Psi \left[\tau^{\frac{1}{2}} (-\hat{\gamma}_{i+1})\right] - c_{i+1} . \qquad [11^{**}]$$

We want to prove now the following

Lemma

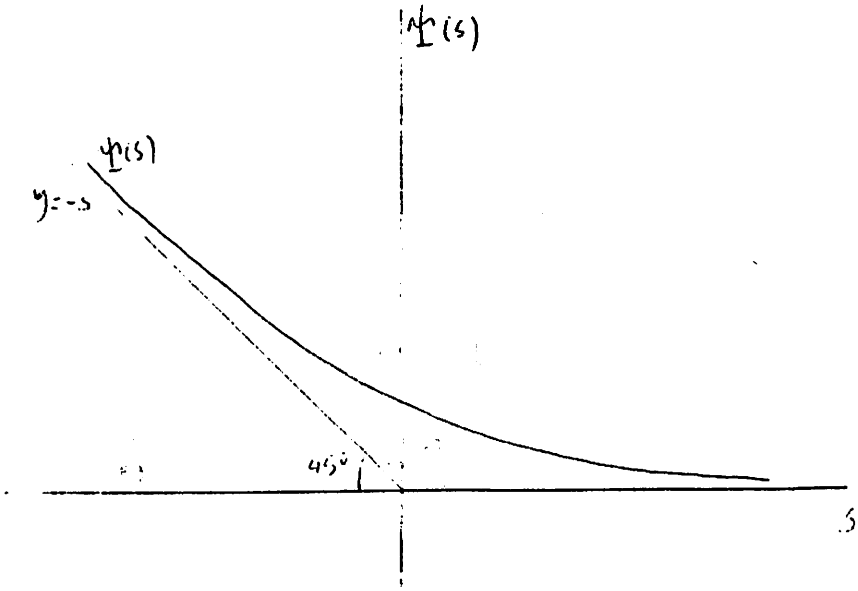
$$\frac{d\hat{\gamma}_i}{d\tau} > 0 .$$

Proof

Consider an auxiliary search problem equal to ours except for the fact that, after the i-th stage, all future costs of search are constant and equal to c_{i+1}. $\overline{\gamma}_i = \tau^{-\frac{1}{2}}\Psi\left[\tau^{\frac{1}{2}} (-\hat{\gamma}_i)\right] - c_{i+1}$ is the optimal expected utility at period i when the consumer must search at least once more. Clearly $\overline{\gamma}_i(\tau) \geq \hat{\gamma}_i(\tau)$, that is, the function $\hat{\gamma}_i(\tau)$ is bounded from above by $\overline{\gamma}_i(\tau)$. We want to show now that $\frac{d\,\overline{\gamma}_i(\tau)}{d\,\tau} < 0$. $\frac{d\overline{\gamma}_i}{d\tau} = -\frac{1}{2}\,\tau^{-3/2}\,\Psi(\cdot) + \tau^{-\frac{1}{2}}\,\Psi'(\cdot)\left[-\frac{1}{2}\tau^{-\frac{1}{2}}\overline{\gamma}_i - \tau^{\frac{1}{2}}\frac{d\overline{\gamma}_i}{d\tau}\right]$,

and so $$\frac{d\overline{\gamma}_i}{d\tau} = \frac{-\frac{1}{2}\tau^{-1}\left[\tau^{-\frac{1}{2}}\,\Psi(\cdot) + \hat{\gamma}_i\,\Psi'(\cdot)\right]}{1 + \Psi'(\cdot)}$$

It should be observed that the function $\Psi(s) \equiv \int_s^\infty (x - s)\,\varphi(x)\,dx$ --where, we must remember, $\varphi(x)$ is the density function of $x \sim N(o,\tau)$--is a positive function, $-1 < \Psi'(s) < 0$ for $-\infty < s < \infty$, and when $s < 0$, $\Psi(s) > |s|$. A generic representation of $\Psi(s)$ is given in the following graph.



Therefore,

$$\text{sign}\left[\frac{d\overline{\gamma}_i}{d\tau}\right] = \text{sign}\left\{-\left[\tau^{-\frac{1}{2}}\,\Psi(\cdot) + \hat{\gamma}_i\,\Psi'(\cdot)\right]\right\}.$$

Since the functions Ψ and Ψ' are evaluated at $-\tau^{\frac{1}{2}}\,\overline{\gamma}_i$, we are interested in the sign of

$\tau^{-\frac{1}{2}}\,\Psi(-\tau^{\frac{1}{2}}\,\overline{\gamma}_i) + \hat{\gamma}_i\,\Psi'(-\tau^{\frac{1}{2}}\,\overline{\gamma}_i)$, where the first term is positive and the second negative. Now, from the properties of Ψ it follows that

$$\Psi(-\tau^{\frac{1}{2}}\,\overline{\gamma}_i) > \tau^{\frac{1}{2}}\,\overline{\gamma}_i\ ,$$

and

$$\tau^{\frac{1}{2}}\,\hat{\gamma}_i > \left|\Psi'(-\tau^{\frac{1}{2}}\,\overline{\gamma}_i)\,\tau^{\frac{1}{2}}\,\overline{\gamma}_i\right|.$$

Therefore,

$$\tau^{-\frac{1}{2}}\,\Psi(-\tau^{\frac{1}{2}}\,\overline{\gamma}_i) > \left|\Psi'(-\tau^{\frac{1}{2}}\,\overline{\gamma}_i)\,\overline{\gamma}_i\right|,$$

and

$$\frac{d\overline{\gamma}_i}{d\tau} < 0\ .$$

Now, $\hat{\gamma}_i$ being a monotone function of a monotone function, ad infinitum, it is a monotone function of τ bounded from above by a strictly decreasing function which tends to zero as τ becomes very large. Therefore $\hat{\gamma}_i$ is itself a decreasing function of τ.

We can therefore state the following.

Proposition 3.7 No recall

A mean preserving increase in the price dispersion leads at any search stage i, i=1,2,..., to an increase in the expected utility of optimal search α_i and to a decrease of the reservation price p_i^*.

Proposition 3.8 No recall

A utility mean preserving increase in the price dispersion leads at any search stage i, i=1,2,..., to an increase in the expected utility of optimal search α_i and to a decrease of the reservation price p_i^*.

Next we want to see if it is still true, for the case of increasing cost of search, that an increase in the price dispersion leads to an increase in the expected number of searches.

Proposition 3.9 No recall

An increase in the price dispersion leads to an increase in the expected number of searches.

Proof

Suppose that $\beta_A > \beta_B$. Then $\tau_A > \tau_B$. Let us call

$$\nu_i^A = \Pr\ [V(P_A) \geq \mu_A + \hat{Y}_i^A]$$

$$\nu_i^B = \Pr\ [V(P_B) \geq \mu_B + \hat{Y}_i^B].$$

Then, from [11**] and following the steps of the proof of Proposition 3.5,

$$\nu_i^A = \Pr\left[V(P_A) \geq \mu_A + \tau_A^{-\frac{1}{2}}\ \Psi\ [\tau_A^{\frac{1}{2}}\ (-\hat{Y}_{i+1}^A)] - c_{i+1}\right]$$

$$\nu_i^B = \Pr\left[V(P_B) \geq \mu_A + \tau_B^{-\frac{1}{2}}\ \Psi\ [\tau_B^{\frac{1}{2}}\ (-\hat{Y}_{i+1}^B)] - c_{i+1}\right]$$

From Proposition 3.7, $\hat{Y}_{i+1}^B > \hat{Y}_{i+1}^A$ and since $\tau_A > \tau_B$, and Ψ is a decreasing function, it follows that $\nu_A > \nu_B$ and $E(N_A) < E(N_B)$, establishing the claim.

Proposition 3.10 No recall

A utility mean preserving increase in the price dispersion leads to an increase in the expected amount of search.

Proof

Analogous to the previous one with $\mu_A = \mu_B$.

In conclusion, we observe that, under the particular assumptions about the price distribution and the utility function that we have made, and when the consumer does not learn from searching, he shows a preference for price dispersion, under both a mean-preserving and a utility mean preserving definition of dispersion.

Is this so because the consumer is a risk-taker?

The utility function was defined in chapter II as an indirect additive utility function.

$$\bar{V}\ (P',Y,) = V(P) + W(Y)\ .$$

Afterwards, for notational convenience, we dropped the second term and concentrated on V(P). In a sense we could say that if V(P) is strictly concave the consumer is a risk averter and, conversely, a risk lover if V(P) is strictly convex, meaning by that, that a risk averse consumer will choose a definite E(P) which is certain in preference to a gamble which yields a random price P. This follows simply from Jensen's inequality. But this definition of a risk averse consumer refers to a once and for all bet, not to a sequence of experiments. In addition it is not equivalent to the customary one (see, e.g., Pratt [37], Arrow [4]) which says that a consumer is an absolute risk averter whenever $r(Y) = -u''(Y)/u'(Y)$ is positive, x being the amount of wealth or income and $\mu(Y)$ the utility of wealth or income. With our notation, we can define absolute risk aversion $r(Y) = -\frac{\bar{V}_{YY}}{\bar{V}_Y}$.

Since $\bar{V}(P',Y)$ is homogeneous of degree zero, $\frac{\bar{V}_{YY}}{\bar{V}_Y}$ is homogenous of degree minus one, and therefore $r(Y) = -\frac{k}{Y}$, k being a parameter. If $k > 0$ the consumer is a risk taker, if $k < 0$ the consumer is a risk averter.

But, since $V(p',y) = \log a - \sum_{i=1}^{n} b_i p_i + d \log y$, where $d = \sum_{i=1}^{n} b_i$, $r(y) = \frac{1}{y}$ and the consumer is a risk averter.[36] Therefore, the "preference for price dispersion" that consumers show is not the result of the consumers being risk lovers.

Note, finally, that once the consumer has decided to accept a price and, therefore, to stop searching, the determination of the demand for the m goods is straightforward. All the prices are now known, and so is the income available to the consumer (being the initial income minus the searching expenses).[37]

Lemmata

Lemma 1

Let X be a random variable with a d.f. F(x) for which the mean exists.

Then, for any $s \in (-\infty, \infty)$,

$$E[\max(X, s)] = s + \int_s^\infty (x - s)\, dF(x) .$$

Proof

$$E[\max(X, s)] = s \int_{-\infty}^{s} dF(x) + \int_s^\infty x\, dF(x) = s[1 - \int_s^\infty dF(x)]$$

$$+ \int_s^\infty x\, dF(x) = s + \int_s^\infty (x - s)\, dF(x) .$$

Lemma 2

Let F be a distribution function on the real line for which a mean exists.

Then $g(s) = \int_s^\infty (x - s)\, dF(x)$ $-\infty < s < \infty$, is a positive strictly decreasing function of s, for any value of s such that $g(s) \neq 0$.

Proof

Using Leibniz' formula (see, e.g., Bartle [5] p. 307),

$$g'(s) = -\int_s^\infty dF(x) = -[F(x)]_s^\infty$$

which is negative as long as $\Pr(s \le x < \infty) \neq 0$, that is, as long as $g(s) \neq 0$.

Lemma 3

Let F be the d.f.of a random variable x which is normally distributed with mean μ and precision τ.

Then $\int_{s}^{\infty} (x - s)\ dF(x) = \tau^{-\frac{1}{2}} \int_{\tau^{\frac{1}{2}}(s-\mu)}^{\infty} (z - \tau^{\frac{1}{2}}(s - \mu))\ \psi(z)\, dz$, where z is a standard normal random variable and $\psi(z)$ is its p.d.f.

Proof

Applying the change of variable theorem for integral equations (see, e.g., Bartle [5] p.305) and the change of variable theorem for random variables (see, e.g., Hoel [24], App. B) and calling $z = (x - \mu)\tau^{\frac{1}{2}}$, the result is immediate.

Footnotes

[1]See [3] p. 43, [27] p. 179, or [10] p. 49.

[2]See [23] p. 1.

[3]For some work on disequilibrium models with maximizing agents see [39].

[4]For a survey of this field see Rothschild [39].

[5]This follows the traditional methodology of economic theory which basically views the concept of rational behaviour as the rational choice of an isolated individual which maximizes a well defined function.

[6]The commodities being homogeneous, the consumer will search, in Nelson's [34] terminology, for a "pure search good" as opposed to an "experience good".

[7]As it is customary we shall denote variables by capital letters and the values taken by the variables by the same low case letters.

[8]This is not an innocent assumption. It appears natural when we make abstraction of locational and product differences, goodwill, etc., and it is more justified when we have in mind pure search good than when we think of experience goods. And yet it creates problems. Because if search is random, how do we explain that stores do not divide themselves in very small units in order to increase the number of the consumer's visits?

[9]See, for example, Telser [46].

[10]By "knows with or without certainty" I mean that the consumer might be uncertain about the parameters that characterize the m-th commodity price distribution. E.g., he might only know the probability distribution of the parameters, one of them being non-degenerate.

[11]Although the additive (direct) utility function has been thoroughly studied (even when defined over probability measures, see, e.g., P. C. Fishburn [20] ch. 11 or Debreu [14]) only partial results are available specifying the characteristics of the family of (direct) utility functions that yield additive indirect utility functions (see, e.g., Lau [28] and the references there mentioned).

[12]Only in this example we shall denote the price of the i-th commodity (i=1,2,...,m) P_i. In the rest of the paper we shall call P_i the price of the m-th commodity observed at the i-th search.

[13]If there is no recall, the only relevant price is the last price observed since either the previous prices have been incorporated into shaping the consumer's beliefs (in the case in which the search itself is a learning experience, as will be discussed in chapter IV) or they are water under the bridge, and cannot affect the consumer's decision (as will be discussed in chapter III).

[14]This assumption is not needed as long as $c_i < c_{i+1}$, $i=1,2,\ldots$ (c_j $(j=1,2,\ldots)$ is defined below). But everything looks more neat if it is made.

[15]This constraint will be always satisfied by assuming $\lim_{y \to 0} W(y) = -\infty$.

[16]If the depletion of the given endowments or the expenditure of the given income are not constant, of course the assumption of constant cost of search need not be contradictory but then it will be utterly implausible.

[17]See note 12.

[18]Strictly speaking this is a strong assumption. Still, with a large number of undifferentiated stores, it can be taken as a fairly good approximation.

[19]N_T: $S \to I^+$, I^+ being the set of natural numbers. Let $i,j \in I^+$, then for all i,

$$N_T(x_i) = i \quad \text{if } (\forall x_j)(j<i)\ x_j \notin T \text{ and } x_i \in T$$

$$N_T(x_i) < i \quad \text{if } (\exists x_j)(j<i)\ x_j \in T$$

$$N_T(x_i) > i \quad \text{if } (\forall x_j)(j \leq i)\ x_j \notin T.$$

Note that for every stopping rule the search can be interpreted as a Bernouilli experiment in which there are only two outcomes of interest--that the price observed belongs to T or that it does not. Every visit to a store is an independent Bernouilli trial. If we call $\pi = \Pr(P \in T)$ and therefore $1-\pi = \Pr(P \notin T)$, $\Pr(N=n) = (1-\pi)^{n-1}\cdot\pi$. The distribution defined by these probabilities is said to be geometric, and the mean of N is $1/\pi$.

[20]Having introduced the symbol N_T, Δ can be defined more compactly as the set of all $T \subseteq S$ such that $\Pr(N_T < \infty) = \lim_{n\to\infty} \Pr(N<n) = 1$. Note that it need not be assumed that there is some finite upper bound n such that $\Pr(N \leq n) = 1$.

[21]In the case of a constant cost of search, DeGroot [15] Chapter 13, section 10, shows that $E[V(P)] < \infty$ and $E[V(P)]^2 < \infty$ are sufficient conditions for the existence of an optimal rule and an expected utility of following the optimal rule which is finite, both with or without recall. By a simple modification of the proof it can be shown that they are still sufficient for non-constant cost of search.

[22]H. Raiffa and R. Schlaifer [38] part II and ff. use a utility function which is the sum of two terms. One is called the terminal utility, and the other the sampling utility. Our additive indirect utility function could be similarly described.

[23]Note that in this chapter we suppose that the consumer's beliefs do not change.

[24]From now on, for notational simplicity, we will drop the term $W(y)$ that appears in all the expressions that have to be compared.

[25]The lemmas appear at the end, in a special section.

[26]See Chow and Robbins [11].

[27]These conditions are:

(a) $\lim_{n\to\infty} \int_{\{N>n\}} x_n \, dF(X_n) = 0$, where N is the stopping time

(b) There exists a r.v. $w \geq 0$ with finite expectation and non-negative costs c_n such that

$$-\min\{X_n, 0\} \leq w + \sum_{i=1}^{n} c_i \qquad n=1,2,\ldots$$

It can be shown that in a monotone case in which those additional conditions are satisfied, the sequence of utilities (x_i) form a regular supermartingale. For a definition of a regular supermartingale and related properties see DeGroot [15], Chapter 13, section 14.

[28]Note that in a problem of optimal stopping in which the random variables $Y_1, Y_2, \ldots$ are observed and where $X_i = x_i(Y_1, Y_2, \ldots, Y_i)$, $i=1,2,\ldots$ are the corresponding utilities, if at any stage i of the sampling $Y_1 = y_1$, $Y_2 = y_2, \ldots, Y_i = y_i$ have been observed and

$$E(X_{i+1} \mid y_1, y_2, \ldots, y_i) > x_i(y_1, y_2, \ldots, y_i),$$

then it is clear that it is preferable to take at least one more observation. But if, instead,

$$E(X_{i+1} \mid y_1, y_2, \ldots, y_i) \leq x_i(y_1, y_2, \ldots, y_i), \qquad [*]$$

then it is not clear whether it is preferable to stop or to continue since, even if the expected utility of taking <u>just</u> one more observation is not greater than the utility of terminating the sampling, there might be a continuation with higher expected utility than $x_i(y_1, y_2, \ldots, y_i)$. But if the sequence of utilities form a regular supermartingale, the optimal stopping rule is to stop as soon as [*] happens.

[29]See, for example, Lindgren [29], p. 151, and footnote 19.

[30]See, for example, Aitchison and Brown [1] p.

[31]For a justification of those assumptions see the Part II of this paper.

[32]On this concept see Rothschild and Stiglitz [37].

[33]For the relation between the mean and precision of a random variable X, lognormally distributed, and the mean and precision of a random variable $Y = \log X$, see, for example, Aitchison and Brown [1].

[34]On this concept see Diamond and Stiglitz [18].

[35]Actually our intuition is more easily convinced that the results hold for the recall case than for the no recall case with constant costs. Only the understanding that the recall privilege is irrelevant when the cost of search is constant, may convince us that the results are also acceptable in the no recall, constant cost case.

[36]By the way, R. Deschamps [16] claims that it is not possible to find an indirect utility function (and therefore homogeneous of degree zero) with relative risk aversion equal to one. Well this is one.

[37]See Roy, R., [44].

References

[1] Aitchison, J., and Brown, J. A. C., The Lognormal Distribution, Cambridge University Press, 1957.

[2] Alexander, S. S., "Price Movements in Speculative Markets: Trends or Random Works," Industrial Management Review, May 1961, pp. 7-26.

[3] Arrow, K. J., "Toward a Theory of Price Adjustment" in Abramovitz et al., The Allocation of Economic Resources, Stanford University Press, 1959.

[4] Arrow, K. J., Aspects of the Theory of Risk-Bearing, Yrjö Jahnssonin Säätiö, Helsinki, 1965.

[5] Bartle, R. G., The Elements of Real Analysis, Wiley, New York, 1964.

[6] Belth, J. M., The Retail Price Structure in American Life Insurance, Bureau of Business Research, Indiana University, Bloomington, 1966.

[7] Bosch-Domènech, A., Learning by Searching: An Analysis of Consumer Decision-Making, U.W.O. Research Report, March 1975.

[8] Box, G. E. P. and Tiao, G. C., Bayesian Inference in Statistical Analysis, Addison-Wesley, Reading, Mass., 1973.

[9] Breiman, Leo, "Stopping Rules Problems," in E. F. Beckenback, ed., Applied Combinatorial Mathematics, Wiley, New York, 1964.

[10] Chipman, J. S., "The Nature and Meaning of Equilibrium in Economic Theory," in Functionalism in the Social Sciences, Monograph No. 5 of the American Academy of Political and Social Science, Philadelphia, February 1965.

[11] Chow, Y. S. and Robbins H., "A Martingale System Theorem and Application, Proc. 4th Berkeley Symp. Math. Stat. and Prob., 1961.

[12] Chow, Y. S., Robbins, H., and Siegmund, D., Great Expectations: The Theory of Optimal Stopping, Houghton Mifflin Co., Boston, 1971.

[13] Clark, P. K., "A Subordinated Stochastic Process Model with Finite Variance for Speculative Prices," Econometrica, January 1973, pp. 135-155.

[13*] Courant, R., Differential and Integral Calculus, Vol. II, London, 1936.

[14] Debreu, G., "Topological Methods in Cardinal Utility Theory," in Arrow, Karlin, Suppes, eds., Mathematical Methods in the Social Sciences, 1959, Stanford University Press.

[15] DeGroot, M. H., Optimal Statistical Decisions, McGraw-Hill, New York, 1970.

[16] Deschamps, R., "Risk Aversion and Demand Functions," Econometrica, May 1973, pp. 455-465.

[17] Diamond, P. A., "A Model of Price Adjustment," Journal of Economic Theory, 3, 1971, pp. 156-168.

[18] Diamond, P. A. and Stiglitz, J. E., "Increases in Risk and in Risk Aversion," Working Paper No. 97, MIT, 1973.

[19] Edwards, W., Lindman, H. and Savage, L. J., "Bayesian Statistical Inference for Psychological Research," Psychological Review, 70, 1963, pp. 193-242.

[20] Fishburn, P. C., Utility Theory for Decision Making, Wiley, New York, 1970.

[21] Gastwirth, J. L., "On Probabilistic Models of Consumers Search for Information," Mimeographed, Dpt. of Statistic, Johns Hopkins University, 1971.

[22] Gould, J. P., "Risk, Stochastic Preference, and the Value of Information," Mimeo, University of Chicago, Business School, working paper 7330, 1973.

[23] Hahn, F. H., "Some Adjustment Problems," Econometrica, 38-1, January 1970.

[24] Hoel, P. G., Introduction to Mathematical Statistics, 2nd ed., Wiley, New York, 1954.

[25] Jung, A. F., "Price Variations Among Automobile Dealers in Metropolitan Chicago," Journal of Business, 1960, pp. 31-42.

[26] Jung, A. F., "Price Policy and Discounts in the Medium and High-Priced Car Markets," Journal of Business, 1960, pp. 342-47.

[27] Koopmans, T. C., Three Essays on the State of Economic Science, New York, McGraw-Hill, 1957.

[28] Lau, L. J., "Duality and the Structure of Utility Functions," Journal of Economic Theory 1, pp. 374-396, 1970.

[29] Lindgren, B. W., Statistical Theory, 2nd ed., Macmillan, New York, 1968.

[30] Lindley, D. V., Bayesian Statistics, A Review, Society for Industrial and Applied Mathematics, Philadelphia, 1971.

[31] McCall, J. J., "Economics of Information and Job Search," Quarterly Journal of Economics, 84, pp. 113-126, 1970.

[32] McCall, J. J., "Probabilistic Economics," Bell Journal of Economics and Management Science, 2, pp. 403-433, 1971.

[33] Mortensen, "Search Equilibrium in a Simple Multi-Market Economy," mimeo, Northwestern University, July 1973.

[34] Nelson, P., "Information and Consumer Behaviour," Journal of Political Economy, 78, 1970, pp. 311-329.

[35] Osborne, M. F. M., "Periodic Structure in the Browman Motions of Stock Prices," Operations Research, Vol. 10, May-June 1962, pp. 345-379.

[36] Phelps, E. S., and Winter, S. G., "Optimal Price Policy under Atomistic Competition," in Microeconomic Foundations of Inflation and Employment Theory, by E. S. Phelps et al., New York, 1970.

[37] Pratt, J. W., "Risk Aversion in the Small and in the Large," Econometrica, 32, 1964, pp. 122-136.

[38] Raiffa, H. and Schlaifer, R., Applied Statistical Decision Theory, Division of Research, Graduate School of Business Administration, Harvard University, 1961.

[39] Rothschild, M., "Models of Market Organization with Imperfect Information: A Survey," Journal of Political Economy, December 1973.

[40] Rothschild, M., "Searching for the Lowest Price when the Distribution of Prices is Unknown," Journal of Political Economy, July 1974, pp. 689-712.

[41] Rothschild, M., "A Two-Armed Bandit Theory of Market Pricing," Working Paper No. 10, MSSBW, Berkeley, Summer 1971.

[42] Rothschild, M., and Arrow, L., "Preliminary Notes on Equilibrium Price Distribution with Limited Information," paper presented at the American Meetings of the Econometric Society, New York, Dec. 1973.

[43] Rothschild, M. and Stiglitz, J. E., "Increasing Risk: I. A Definition," Journal of Economic Theory 2, 1970, pp. 225-243.

[44] Roy, R., "La Distribution du Revenu entre les Divers Biens," Econometrica 15, 1947, pp. 205-225.

[45] Savage, L. J., The Foundations of Statistics, 2nd ed., Wiley, New York, 1954.

[46] Telser, L. G., "Searching for the Lowest Price," American Economic Review, Proceedings 63, 1973, pp. 41-49.