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TWO CHARACTERIZATIONS OF CONSISTENCY

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Abstract. This paper offers two characterizations of the Kreps-Wilson concept of consistent beliefs. One is primarily of applied interest: beliefs are consistent iff they can be constructed by multiplying together vectors of monomials which induce the strategies. The other is primarily of conceptual interest: beliefs are consistent iff they can be induced by a "product dispersion" whose marginal dispersions induce the strategies (a "dispersion" is defined as a relative probability system, and a "product" dispersion is defined as a joint dispersion whose marginal dispersions are independent). Both these characterizations are derived with linear algebra.

1. Introduction

1.1. Example

Figure 1.1 depicts the start of an extensive-form game. Xavier chooses between L and R, and simultaneously Yolanda chooses between ℓ, m, and r. Six nodes result, and the four nodes Lm, Lr, $R\ell$, and Rm constitute Helen's information set.

Suppose (for whatever reason) that both Xavier and Yolanda play to the right. What does this behaviour imply about Helen's belief? In other words, if Helen found herself with the move, which of the four nodes in her information set would she consider most likely?

This question is important even though there is zero probability that Helen will actually move. To appreciate its importance, consider Figure 1.2, which incorporates Figure 1.1 into an extensive-form game. Notice that the outcome Rr results from the sequential equilibrium consisting of the strategy profile $(p_L, p_R) = (0, 1)$, $(p_\ell, p_m, p_r) = (0, 0, 1)$, $(p_1, p_2) = (0, 1)$, and the belief $(p_{Lm}, p_{Lr}, p_{R\ell}, p_{Rm}) = (0, 1, 0, 0)$. This equilibrium outcome would vanish if Helen believed that Lm (or $R\ell$

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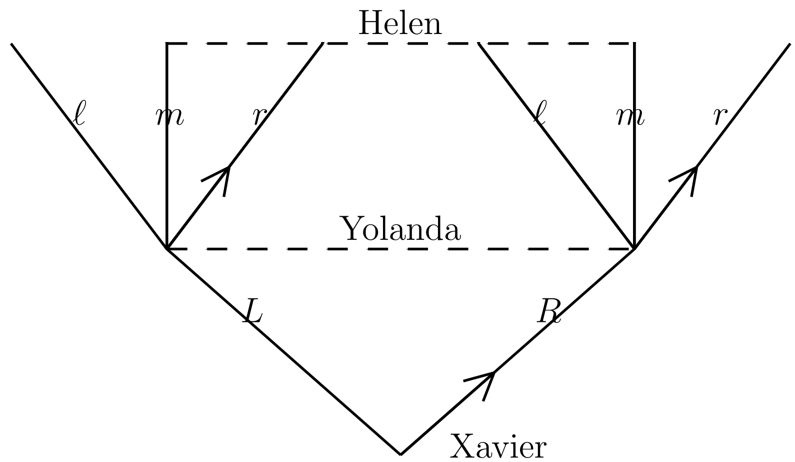

FIGURE 1.1. The Question of Helen's Belief

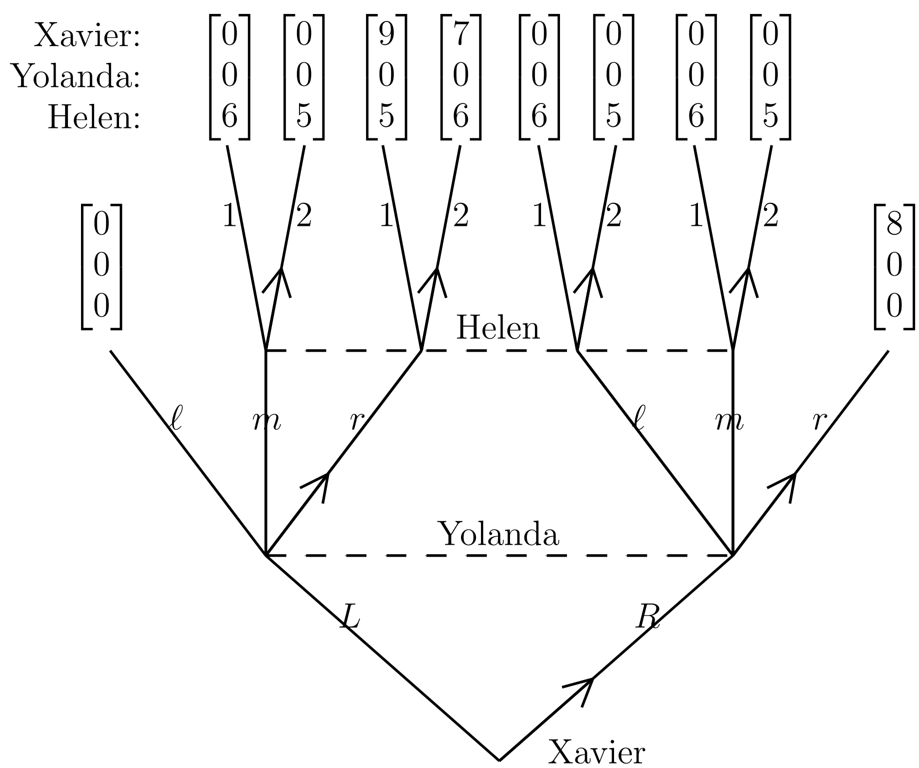

FIGURE 1.2. Motivation for the Question of Helen's Belief

or Rm) was more likely than Lr: she would then choose 1 over 2 and thereby induce Xavier to choose L over R. Hence, Helen's belief over her zero-probability information set is a critical component of the equilibrium.

One answer to the question of Helen's belief would be provided by ordinary probability theory. There Xavier's strategy $(p_L, p_R) = (0,1)$ can be multiplied by Yolanda's strategy $(p_\ell, p_m, p_r) = (0,0,1)$ to obtain the product distribution

$$\begin{array}{c|cc|} \cline{2-3} r & 0 & 1 \\ m & 0 & 0 \\ \ell & 0 & 0 \\ \cline{2-3} \multicolumn{1}{c}{} & L & \multicolumn{1}{c}{R} \end{array} \;.$$

This product distribution assigns zero probability to Lm, Lr, $R\ell$, and Rm, and thus imposes no restrictions on Helen's belief. Although this answer is logically coherent, it is unsatisfactory in the sense that Lm seems markedly less likely than the other three nodes because Lm is the only node in which *both* Xavier and Yolanda have failed to play right.

A better answer to the question is provided by Kreps and Wilson (1982)'s concept of consistent beliefs, and the purpose of this paper is to provide two characterizations of their concept. One is primarily of applied interest, the other primarily of theoretical interest.

1.2. A Characterization of Applied Interest

This subsection is more than introductory. Its four pages fully explain and illustrate the characterization of applied interest, and consequently, readers interested in consistent beliefs for application purposes may wish to stop at the end of this subsection.

In mathematics, a "monomial in the variable x" is an expression of the form cx^e, where the coefficient c is a real number and the exponent e is a nonnegative integer. In this paper, the word *monomial* will refer to an expression of the form cn^e, where c is a positive real number and e is any real number (allowing e to be real is only a matter of clarity and convenience: see Note 5).

Then consider a finite set Z such as Xavier's strategy set $\{L, R\}$ or Yolanda's strategy set $\{\ell, m, r\}$. To set up an analogy, recall that a distribution $[p_z]$ assigns a real number p_z to each element of Z. Obvious examples are Xavier's strategy $(p_L, p_R) = (0,1)$ and Yolanda's strategy $(p_\ell, p_m, p_r) = (0,0,1)$. Analogously, a vector of the form $[c_z n^{e_z}]$ assigns a monomial $c_z n^{e_z}$ to each element of Z. Examples are

(1) $$(c_L n^{e_L}, c_R n^{e_R}) = (n^{-1}, 1) \text{ and}$$

(2) $$(c_l n^{e_\ell}, c_m n^{e_m}, c_r n^{e_r}) = (n^{-2}, 5n^{-2}, 1) \;.$$

A monomial vector $[c_z n^{e_z}]$ is said to *induce* the distribution $[p_z]$ defined by

$$(3)\quad (\forall z)\ p_z = \lim_{n\to\infty} \frac{c_z n^{e_z}}{\Sigma_{z''\in Z}\, c_{z''} n^{e_{z''}}} = \frac{c_z 1(e_z{=}\max\{e_{z'}|z'\})}{\Sigma_{z''\in Z}\, c_{z''} 1(e_{z''}{=}\max\{e_{z'}|z'\})}$$

(the first equation is the definition, the second equation is an obvious identity, and the symbol $1(\cdot)$ denotes the indicator function assuming a value of 1 when its argument is true and a value of 0 when its argument is false). Note that the right-hand formula is simple: it says to use the exponents $[e_z]$ to find the support of the distribution, and then to normalize the coefficients $[c_z]$ to assign positive probabilities across that support. For example, (1) induces Xavier's strategy and (2) induces Yolanda's strategy.

It is important that one distribution can be induced by many monomial vectors. For example, Xavier's strategy can be induced by (1),

$$(4)\qquad (c_L n^{e_L}, c_R n^{e_R}) = (4n^{-2}, 1)\ ,\ \text{or}$$

$$(5)\qquad (c_L n^{e_L}, c_R n^{e_R}) = (n^{-3}, 1)\ .$$

In general, if $[p_z]$ is to be induced by $[c_z n^{e_z}]$, then the support of $[p_z]$ determines $\text{argmax}\{e_z|z\}$, and over this set, the ratios of the probabilities p_z determine the ratios of the coefficients c_z. However, outside of the support of $[p_z]$, there are no restrictions imposed on either the exponents e_z or the coefficients c_z (except that every such exponent be less than $\max\{e_z|z\}$). To put this very casually in terms of game theory, "off-equilibrium" monomials can be defined arbitrarily (almost).

Two monomial vectors can be multiplied together much like two distributions can be multiplied together. For example, the product of Xavier's (1) with Yolanda's (2) is

(6)

	L	R
r	n^{-1}	1
m	$5n^{-3}$	$5n^{-2}$
ℓ	n^{-3}	n^{-2}

.

This product assigns a monomial to every node in $\{L\ell, Lm, Lr, R\ell, Rm, Rr\}$ and thus assigns the vector

$$(c_{Lm} n^{e_{Lm}}, c_{Lr} n^{e_{Lr}}, c_{R\ell} n^{e_{R\ell}}, c_{Rm} n^{e_{Rm}}) = (5n^{-3}, n^{-1}, n^{-2}, 5n^{-2})$$

to Helen's information set. This monomial vector then induces the belief $(p_{Lm}, p_{Lr}, p_{R\ell}, p_{Rm}) = (0,1,0,0)$. In this sense the product (6) induces the belief $(p_{Lm}, p_{Lr}, p_{R\ell}, p_{Rm}) = (0,1,0,0)$.

Further examples are instructive. The product of Xavier's (4) with Yolanda's (2) is

$$(7) \qquad \begin{array}{c|cc|} r & 4n^{-2} & 1 \\ m & 20n^{-4} & 5n^{-2} \\ \ell & 4n^{-4} & n^{-2} \\ \hline & L & R \end{array} \ ,$$

which induces $(p_{Lm}, p_{Lr}, p_{R\ell}, p_{Rm}) = (\ 0,\ 4/10,\ 1/10,\ 5/10\ )$. Similarly, the product of Xavier's (5) with Yolanda's (2) is

$$(8) \qquad \begin{array}{c|cc|} r & n^{-3} & 1 \\ m & 5n^{-5} & 5n^{-2} \\ \ell & n^{-5} & n^{-2} \\ \hline & L & R \end{array} \ ,$$

which induces $(p_{Lm}, p_{Lr}, p_{R\ell}, p_{Rm}) = (\ 0,\ 0,\ 1/6,\ 5/6\ )$.

Theorem 6.2($a{\Rightarrow}a'$) shows that any of these three beliefs is consistent (Kreps and Wilson (1982)) with Xavier's and Yolanda's strategies. Admittedly this is not that surprising because every product $[c_x n^{e_x} c_y n^{e_y}]$ like (6), (7), or (8) corresponds to the full-support-product-distribution sequence

$$\langle [\pi_x^n \pi_y^n] \rangle_n = \Big\langle \Big[\frac{c_x n^{e_x}}{\Sigma_{x'} c_{x'} n^{e_{x'}}} \frac{c_y n^{e_y}}{\Sigma_{y'} c_{y'} n^{e_{y'}}} \Big] \Big\rangle_n \ ,$$

and this sequence establishes consistency precisely when $[c_x n^{e_x}]$ induces Xavier's strategy, $[c_y n^{e_y}]$ induces Yolanda's strategy, and $[c_x n^{e_x} c_y n^{e_y}]$ induces Helen's belief.

But Theorem 6.2($a{\Leftarrow}a'$) derives the converse, namely, that *every* consistent belief is induced by the product of two monomial vectors which induce the two strategies. In other words, the relatively small set of full-support-product-distribution sequences that correspond to products of monomial vectors is large enough to support the entire concept of consistency. Accordingly, Theorem 6.2(a$\Leftrightarrow$a′) provides an intuitive, finite-dimensional characterization of consistency.

We can use this result to derive the set of beliefs for Helen that are consistent with both Xavier and Yolanda playing right. Within this derivation, let the function P map a monomial vector to the distribution it induces (for example, $P(n^{-2},3) = (0,1)$, where $P((n^{-2},3))$ is written without double parentheses for readability). We will contend that

$$\{\ (p_{Lm}, p_{Lr}, p_{R\ell}, p_{Rm}) \text{ that are consistent with } (p_L, p_R){=}(0,1) \text{ and } (p_\ell, p_m, p_r){=}(0,0,1)\ \}$$

$$\begin{aligned}
&= \{\ P(c_Lc_mn^{e_L+e_m}, c_Lc_rn^{e_L+e_r}, c_Rc_\ell n^{e_R+e_\ell}, c_Rc_mn^{e_R+e_m})\ | \\
&\quad P(c_Ln^{e_L}, c_Rn^{e_R}){=}(0,1) \text{ and } P(c_\ell n^{e_\ell}, c_mn^{e_m}, c_rn^{e_r}){=}(0,0,1)\ \} \\
&= \{\ P(c_Lc_mn^{e_L+e_m}, c_Ln^{e_L}, c_\ell n^{e_\ell}, c_mn^{e_m})\ | \\
&\quad P(c_Ln^{e_L}, 1){=}(0,1) \text{ and } P(c_\ell n^{e_\ell}, c_mn^{e_m}, 1){=}(0,0,1)\ \} \\
&= \{\ P(c_Lc_mn^{e_L+e_m}, c_Ln^{e_L}, c_\ell n^{e_\ell}, c_mn^{e_m})\ | \\
&\quad e_L{<}0,\ e_\ell{<}0,\ e_m{<}0\ \} \\
&= \{\ (p_{Lm}, p_{Lr}, p_{R\ell}, p_{Rm})\ |\ p_{Lm}{=}0\ \}\ .
\end{aligned} \tag{9}$$

The first equality is Theorem 6.2(a⇔a′), the second equality holds by choosing the numeraires R and r (the appendix discusses numeraires and Remark A.2 is applied here), the third equality holds because $(c_Ln^{e_L}, 1)$ induces $(0,1)$ iff e_L is negative and because $(c_\ell n^{e_\ell}, c_mn^{e_m}, 1)$ induces $(0,0,1)$ iff both e_ℓ and e_m are negative, and the last equality holds because $e_L{+}e_m$ must be less than e_L because e_m is negative. This result accords with our earlier intuition that Helen should regard Lm as particularly unlikely and also shows that this is the only restriction imposed by consistency within this example.

If the reader is interested in consistent beliefs for application purposes, now may be a good time to stop. The characterization of applied interest has been stated and illustrated, and there are no technical assumptions to learn because the entire paper is algebraic.

1.3. A Characterization of Conceptual Interest

The remainder of the paper proves the previous characterization, and simultaneously, defines and proves a second characterization which is primarily of conceptual interest. Since the reader may find these developments unexpected in three ways, the paper is structured in terms of three steps. To put it poetically, Sections 1 through 3 invite the reader to take three steps into the dark woods (the first step is yet in this section). Then Sections 4 through 6 bring the reader back to familiar ground, first by a key mathematical result (Theorem 4.1 of Section 4) and then by comparatively simple reversals of the first two steps (Sections 5 and 6).

The first step into the woods is to suspend, until Section 6, all interest in the sequences with which Kreps and Wilson (1982) defined consistency. This effectively divorces the results in Sections 2 through 5 from the economics literature (footnotes will mark equivalent concepts whenever possible).

Section 2 will take a second step into the woods. Before taking that step, the section will define a "dispersion" to be a system of

(a) monomial vectors
|\
| \
| (a′) sequences of distributions (Kreps-Wilson)
| /
|/
(b) product dispersions

FIGURE 1.3. Kreps-Wilson consistency (a′) can be characterized by either (a) monomial vectors or (b) product dispersions. The downhill conditionals a⇒a′, a′ ⇒b and a⇒b are easy. The key result in this paper is a⇐b.

relative probabilities, and will note the simple one-to-many relationship between distributions and dispersions (in contrast to distributions, dispersions specify the relative probability between two distinct zero-probability events). Section 2 will then ask the reader to suspend any interest in distributions (such as strategies and beliefs) and to focus on dispersions alone. This is the second step into the woods.

Next, Section 3 will take the third and final step into the woods by defining the "products" of two dispersions. (Or, in words which are not used formally in the paper, Section 3 will define "joint" dispersions over two variables such that the two variables are dispersed "independently.") This producthood will be specified in terms of numerous cancellation laws which embody the notion that cancellation can proceed in each of the two dimensions independently. The reader is apt to find the concept conceptually appealing but hopelessly intractable.

However, Section 4 will provide Theorem 4.1, which is the mathematical foundation of the paper. It states that the products of monomial vectors characterize the products of dispersions. Its proof can be reduced to the solution of matrix equations, and accordingly, this entire paper is based on nothing more than undergraduate linear algebra. To be more specific, the second half of Theorem 4.1's proof directly concerns the solution of a certain matrix equation, while the first half of the proof employs Scott (1964). Although that paper is unfamiliar to economists, it is familiar to mathematical psychologists and Krantz, Luce, Suppes, and Tversky (1971) explain in detail how Scott's Theorem can be proven by solving a number of matrix equations.

Section 5 will reintroduce distributions such as strategies and beliefs. Its Theorem 5.1 states that (a) beliefs can be constructed by multiplying together monomial vectors inducing the strategies iff (b) beliefs can be induced by a product dispersion whose marginals induce the strategies (see Figure 1.3). Note that (a) is the characterization of the previous subsection. Meanwhile, (b) will become the second characterization of consistency. This second characterization is conceptually interesting to the extent that the reader is interested in the cancellation laws defining product dispersions in Section 3.

Finally, Section 6 reintroduces sequences and relates the paper to the economics literature. Its Theorem 6.2 states that (a) and (b) are both equivalent to (a′) Kreps-Wilson consistency. The equivalence of (a′) and (b) is closely related to a result in Kohlberg and Reny (1997) (see the paragraph following Theorem 6.1).

To put this all in perspective, one might want to weigh the relative merits of the three equivalent concepts of Figure 1.3. Subjectively, one might be most comfortable with (a′). At the other extreme, one might come to feel that (a) is the most tractable in applications and that (b) is the most clean conceptually. If so, (a′) is the least interesting of the three concepts, and Theorem 5.1's equivalence between (a) and (b) is the most important result in this paper. Further, it then becomes interesting that this Theorem 5.1 was derived in Section 5 without any reference to the sequences of Section 6. In this fashion one could conclude that the concept of consistency can be well understood without any reference to the sequences of Kreps and Wilson (1982)'s original definition. From this perspective, consistency is an algebraic rather than a topological concept. Such a perspective, however, remains a matter of personal taste.

2. Focus on Dispersions and not Distributions

2.1. Informally

The second step into the woods requires some new terminology. To set up an analogy, recall that a distribution $[p_z]$ (e.g., a strategy or a belief) is a vector of ordinary (i.e., absolute) probabilities. Analogously, Subsection 2.2 will formally define a "dispersion" $[q_{z/z'}]$ to be a matrix of relative probabilities (that obeys two elementary properties to be discussed there).

In the example, Xavier's strategy ($p_L = 0,\ p_R = 1$) is equivalent to the dispersion

$$\begin{bmatrix} q_{L/R} = 0 & q_{R/R} = 1 \\ q_{L/L} = 1 & q_{R/L} = \infty \end{bmatrix}. \tag{10}$$

The equation $q_{R/L} = \infty$ means that the probability of R relative to L is infinite. This information appears redundantly as $q_{L/R} = 0$, which means that the probability of L relative to R is zero. In general, every diagonal element of a dispersion is one, and every off-diagonal element is the reciprocal of the corresponding element on the other side of the diagonal.

Further, for the sake of illustration, suppose that Yolanda's dispersion is

$$\begin{bmatrix} q_{\ell/r}=0 & q_{m/r}=0 & q_{r/r}=1 \\ q_{\ell/m}=1/5 & q_{m/m}=1 & q_{r/m}=\infty \\ q_{\ell/\ell}=1 & q_{m/\ell}=5 & q_{r/\ell}=\infty \end{bmatrix}. \tag{11}$$

Note that the relative probabilities $q_{r/m}=\infty$ and $q_{r/\ell}=\infty$ are determined by the strategy $(p_\ell,p_m,p_r)=(0,0,1)$, but that there is nothing in this distribution which would determine $q_{m/\ell}$. That additional information is specified by the dispersion alone (and $q_{m/\ell}=5$ has been arbitrarily selected for the sake of illustration).

Section 2.2 will use the word "induce" to describe the relationship between distributions and dispersions. Because a distribution does not specify the relative probability between two distinct zero-probability events, one distribution can be induced by many dispersions. On the other hand, it will be seen that exactly one distribution is induced by each dispersion. This one-to-many relationship between distributions and dispersions is important.

However, there is a further issue regarding dispersions which is best studied in isolation from distributions. Accordingly, the second step into the woods is to suspend (until Section 5) all interest in distributions (e.g., strategies and beliefs). Instead, focus your attention on dispersions.

2.2. Formally

Consider a finite set Z. A *table* over Z is a $[q_{z/z'}]\in[0,\infty]^{Z^2}$ which lists a relative probability $q_{z/z'}\in[0,\infty]$ for every pair of elements z and z' from Z. A *dispersion*[1] over Z is a table $[q_{z/z'}]$ that satisfies *unit diagonality*

$$(\forall z)\ q_{z/z}=1 \tag{12}$$

and the *basic cancellation law*

$$(\forall z,z',z'')\ q_{z/z''}\in q_{z/z'}\odot q_{z'/z''}\ , \tag{13}$$

in which the correspondence $\odot$ maps $[0,\infty]^2$ into subsets of $[0,\infty]$ according to

$$a\odot b=\begin{pmatrix} [0,\infty] & \text{if } (a,b) \text{ equals } (0,\infty) \text{ or } (\infty,0) \\ \{ab\} & \text{otherwise} \end{pmatrix}.$$

Examples (10) and (11) satisfy these two properties, and therefore, they are dispersions.

Note that every dispersion satisfies *reciprocity*

$$(14) \qquad (\forall z, z')\ q_{z/z'} = 1/q_{z'/z}\ .$$

This holds because unit diagonality (12) and the basic cancellation law (13) at $(z, z', z'') = (z, z', z)$ yield that $(\forall z, z')\ 1 = q_{z/z} \in q_{z/z'} \odot q_{z'/z}$, and thus, it must be the case that either $1 = q_{z/z'} q_{z'/z}$ for some real numbers $q_{z/z'}$ and $q_{z'/z}$ or that one of $q_{z/z'}$ and $q_{z'/z}$ is 0 and the other ∞.

Finally, say that a dispersion $[q_{z/z'}]$ *induces* the distribution $[p_z]$ satisfying

$$(15) \qquad (\forall z \in Z)\ p_z = \frac{q_{z/z^*}}{\Sigma_{z' \in Z}\, q_{z'/z^*}}\ ,$$

for some $z^* \in Z$ satisfying $(\forall z' \in Z)\ q_{z'/z^*} < \infty$. In other words, a dispersion induces the distribution that is derived by normalizing any row that contains only finite relative probabilities (the induced distribution is invariant to the row chosen by the following remark).

Remark 2.1. *Every dispersion induces exactly one distribution.*

Proof. The existence of the induced distribution follows from two observations. (a) There is a z^* satisfying $(\forall z' \in Z)\ q_{z'/z^*} < \infty$ because otherwise we would have that $(\forall z)(\exists z')\ q_{z'/z} = \infty$, and thus, by the finiteness of Z, we would have a cycle $\langle z^m \rangle_{m=1}^M$ such that $(\forall m < M-1)\ q_{z^{m+1}/z^m} = \infty$ and $q_{z^1/z^M} = \infty$. The first condition together with $M-1$ applications of the basic cancellation law (13) would yield that $q_{z^M/z^1} = \infty$, while the second condition together with reciprocity

[1] Three other papers have defined equivalent concepts using properties similar to unit diagonality and the basic cancellation law. The most closely related concept is a matrix of extended real numbers ("log-likelihoods") which satisfies McLennan (1989b, page 146, equation (2.5)). Such a matrix is the logarithm of a dispersion by the brief discussion at Streufert (2003, page 28). The second concept is Myerson (1986, page 337)'s "conditional probability system," which is equivalent to McLennan's concept as explained in detail by Hammond (1994, Subsection 4.1, $\Delta_C \approx \Delta_M$). Finally, the third concept is Kohlberg and Reny (1997, p. 282-283)'s "random variable defined on a relative probability space," which is equivalent to a dispersion by the lengthy discussion at Streufert (2003, Remark B.5).

Many other equivalent concepts have been defined without reference to properties like unit diagonality and the basic cancellation law. The sentence with Note 4 will discuss the concepts nearest to monomial vectors, the sentence with Note 7 will discuss the concepts defined via sequences of probability distributions, and finally, Note 9 will discuss the concepts defined via nonstandard probability distributions.

(14) would yield that $q_{z^M/z^1} = 0$. (b) The denominator in (15) must be positive because $q_{z^*/z^*} = 1$ by unit diagonality (12).

Uniqueness also follows from two observations. (a) Equation (15) uniquely determines $[p_z]$ for a given z^* satisfying $(\forall z' {\in} Z)\ q_{z'/z^*} < \infty$. (b) Suppose that a first z^* satisfied $(\forall z' {\in} Z)\ q_{z'/z^*} < \infty$ and that a second z^{**} satisfied $(\forall z' {\in} Z)\ q_{z'/z^{**}} < \infty$. Then $q_{z^{**}/z^*} < \infty$ and $q_{z^*/z^{**}} < \infty$, and consequently by reciprocity (14), $q_{z^*/z^{**}} \in (0, \infty)$. Thus the basic cancellation law (13) yields that $(\forall z)\ q_{z/z^{**}} = q_{z/z^*} q_{z^*/z^{**}}$. The last sentence and the sentence before that yield

$$(\forall z)\ \frac{q_{z/z^{**}}}{\sum_{z' \in Z} q_{z'/z^{**}}} = \frac{q_{z/z^*} q_{z^*/z^{**}}}{\sum_{z' \in Z} q_{z'/z^*} q_{z^*/z^{**}}} = \frac{q_{z/z^*}}{\sum_{z' \in Z} q_{z'/z^*}} \ ,$$

and thus, by (15), the induced distribution is invariant to the choice of z^* or z^{**}. □

Remark 2.1 shows a dispersion uniquely determines a distribution. But conversely, a distribution does not uniquely determine a dispersion because a distribution does not specify relative probabilities between distinct zero-probability events. Section 5 will return to this one-to-many relationship between distributions and dispersions. In the meantime, forget about distributions and focus on dispersions.

3. Define the Products of Two Dispersions

3.1. Informally

The third step into the woods is to consider taking the product of two dispersions. There is precious little motivation for this other than academic curiosity and the vague promise that it eventually leads to two characterizations of consistency.

Begin by returning to the example and noting that a product of Xavier's dispersion (10) and Yolanda's dispersion (11) would have to contain 36=6^2 relative probabilities because there are 6 nodes in the Cartesian product $\{L, R\} {\times} \{\ell, m, r\} = \{\ L\ell,\ Lm,\ Lr,\ R\ell,\ Rm,\ Rr\ \}$.

Or, from another angle, a product has to contain 36=4×9 relative probabilities because each of Xavier's 4 relative probabilities must be multiplied by each of Yolanda's 9 relative probabilities. For instance,

$$\begin{aligned} q_{Lr/Lm} &= q_{L/L} q_{r/m} = 1 {\times} \infty = \infty \\ q_{R\ell/Lm} &= q_{R/L} q_{\ell/m} = \infty {\times} (1/5) = \infty \\ q_{Rm/Lm} &= q_{R/L} q_{m/m} = \infty {\times} 1 = \infty \ , \end{aligned} \tag{16}$$

which accords with our earlier intuition that Helen should regard Lm as particularly unlikely.

In fact, 32 of the 36 relative probabilities in a product can be determined in this fashion. These 32 appear in

$$
(17)\quad [q_{xy/x'y'}] \;=\;
\begin{array}{c|ccc|ccc}
x'y' & & & & & & \\
Rr & 0 & 0 & 0 & 0 & 0 & 1 \\
Rm & 0 & 0 & \square & 1/5 & 1 & \infty \\
R\ell & 0 & 0 & \square & 1 & 5 & \infty \\
\hline
Lr & 0 & 0 & 1 & \square & \square & \infty \\
Lm & 1/5 & 1 & \infty & \infty & \infty & \infty \\
L\ell & 1 & 5 & \infty & \infty & \infty & \infty \\
\hline
 & L\ell & Lm & Lr & R\ell & Rm & Rr \quad xy
\end{array} .
$$

To see all these multiplications quickly, note that the southwest quadrant is Yolanda's dispersion multiplied by $q_{L/L} = 1$, the northeast quadrant is her dispersion times $q_{R/R} = 1$, the northwest quadrant is her dispersion times $q_{L/R} = 0$, and the southeast quadrant is her dispersion times $q_{R/L} = \infty$. The three relative probabilities calculated in (16) appear as the first three ∞'s in the row Lm.

However, 4 of the 36 relative probabilities are not determined by $q_{xy/x'y'} = q_{x/x'} q_{y/y'}$. These 4 appear as empty boxes in (17). In particular, the two relative probabilities

$$q_{R\ell/Lr} = q_{R/L} q_{\ell/r} = \infty \times 0 \text{ and}$$
$$q_{Rm/Lr} = q_{R/L} q_{m/r} = \infty \times 0 \;,$$

as well as their reciprocals, are undetermined. This accords with the fact that there is nothing in Xavier's and Yolanda's dispersions which would tell Helen which of the two is more likely to fail to play right. Accordingly, the product of Xavier's dispersion with Yolanda's dispersion is not unique. Rather, there will be many products corresponding to the many values that might be assigned to these two undetermined relative probabilities.

One response to this indeterminateness would be to assign $q_{R\ell/Lr}$ and $q_{Rm/Lr}$ arbitrarily. But this would admit unattractive assignments like $q_{R\ell/Lr} = \infty$ and $q_{Rm/Lr} = 0$. Here the first equality states that $R\ell$ is infinitely more likely than Lr, the second equality states that Lr is infinitely more likely than Rm, and thus (by some sort of transitivity), it seems that $R\ell$ should be infinitely more likely than Rm. Unfortunately, this contradicts $q_{R\ell/Rm} = q_{R/R} q_{\ell/m} = 1/5$.

A better response to the indeterminateness is to assign $q_{R\ell/Lr}$ and $q_{Rm/Lr}$ while imposing the cancellation laws defined formally in Subsection 3.2. One such cancellation law is

$$q_{R\ell/Lr} = q_{R\ell/Rm} q_{Rm/Lr}$$

(informally, this law says that the cancellation $q_{R\ell/\cancel{Rm}}\, q_{\cancel{Rm}/Lr}$ is legal). Since $q_{R\ell/Rm} = q_{R/R} q_{\ell/m} = 1/5$, this cancellation law requires

$$q_{R\ell/Lr} = (1/5) q_{Rm/Lr} \ .$$

Hence, cancellation laws impose one restriction on the two relative probabilities $q_{R\ell/Lr}$ and $q_{Rm/Lr}$, and accordingly, the set of products is one- rather than two-dimensional (full details for this example appear in Subsection 4.6). That single dimension corresponds to the extent to which either Xavier or Yolanda is more likely to fail to play right.

In general, finding the products of two dispersions will entail many undetermined relative probabilities and a great many cancellation laws. In particular, there can be an arbitrary number of terms on the right-hand side of a cancellation law, and much more subtly, there are a great many laws like

$$q_{R\ell/Lr} = q_{Rm/Mr} q_{M\ell/Lm} \tag{18}$$

when each dispersion has at least three elements in its domain (the M appearing in this law is a third option for Xavier; and informally, the law says that the two cancellations $q_{R\cancel{m}/\cancel{M}r}\, q_{\cancel{M}\ell/L\cancel{m}}$ are legal). Such cancellation laws operate on the individual coordinates "independently." Indeed, such independent cancellation is the way in which the definition of producthood will specify the notion that x and y are "dispersed" "independently."

At this point, the reader has patiently taken three steps into the dark woods. The familiar terminology of consistency and game theory has completely disappeared, and the definition of a producthood may appear impractical.

3.2. Formally

Consider a nonempty finite Cartesian product $X{\times}Y$ and denote an element of $X{\times}Y$ by xy rather (x, y). The definitions of Subsection 2.2 apply at $Z = X{\times}Y$. In particular, a table over $X{\times}Y$ is a $[q_{xy/x'y'}] \in [0, \infty]^{(X\times Y)^2}$ listing a relative probability $q_{xy/x'y'} \in [0, \infty]$ for every pair of elements xy and $x'y'$ taken from $X{\times}Y$. Further, a dispersion over $X{\times}Y$ is a table $[q_{xy/x'y'}]$ that satisfies unit diagonality

$$(\forall xy)\ q_{xy/x'y'} = 1 \tag{19}$$

and the basic cancellation law

$$(20)\qquad (\forall xy, x'y', x''y'')\ q_{xy/x''y''} \in q_{xy/x'y'} \odot q_{x'y'/x''y''}\ .$$

A *product*[2] over $X \times Y$ is a table $[q_{xy/x'y'}]$ such that for every $m \geq 0$, and for every pair of permutations σ and τ of the set $\{0, 1, 2, ...m\}$,

$$(21)\qquad (\forall \langle x^i y^i \rangle_{i=0}^m)\ \ q_{x^0y^0/x^{\sigma(0)}y^{\tau(0)}} \in \odot_{i=1}^m\ q_{x^{\sigma(i)}y^{\tau(i)}/x^iy^i}$$

where the product on the right-hand side is defined by

$$\odot_{i=1}^m a_i = \begin{pmatrix} [0,\infty] & \text{if } (\exists i)\, a_i{=}0 \text{ and } (\exists i)\, a_i{=}\infty \\ \{\Pi_{i=1}^m a_i\} & \text{otherwise} \end{pmatrix}$$

for $m \geq 1$, and by $\odot_{i=1}^m a_i = \{1\}$ for $m = 0$. Each instance of (21) at a particular m and (σ, τ) is called a *cancellation law*, and the number m is called the *order* of the cancellation law.[3]

There are many cancellation laws. To be precise, there are $((1{+}m)!)^2$ cancellation laws of order m since there are $(1{+}m)!$ permutations of $\{0, 1, 2, ...\, m\}$. The $1{=}(1!)^2$ zero-order law is

$$(22)\qquad (\forall x^0y^0)\ q_{x^0y^0/x^0y^0} = 1\ ,$$

which coincides with (19). Thus products and dispersions are similar in that they both have unit diagonals (in fact we will see shortly that products are a special kind of dispersion).

The $4{=}(2!)^2$ first-order laws are

$$(23a)\qquad (\forall x^0y^0, x^1y^1)\ q_{x^0y^0/x^0y^0} = q_{x^1y^1/x^1y^1}$$

$$(23b)\qquad (\forall x^0y^0, x^1y^1)\ q_{x^0y^0/x^1y^0} = q_{x^0y^1/x^1y^1}$$

$$(23c)\qquad (\forall x^0y^0, x^1y^1)\ q_{x^0y^0/x^0y^1} = q_{x^1y^0/x^1y^1}$$

$$(23d)\qquad (\forall x^0y^0, x^1y^1)\ q_{x^0y^0/x^1y^1} = q_{x^0y^0/x^1y^1}\ .$$

While (23d) is vacuous and (23a) is implied by (22), laws (23b) and (23c) are substantial. They might be called *separation laws*, and they

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[2] These cancellation laws are equivalent to the acyclicity appearing in Kohlberg and Reny (1997, Theorem 2.10) [details in Streufert (2003, Remark B.6(a⇔a^{KR}))]. While they treat the idea comparatively lightly, this paper adopts it as its fundamental concept. Dissimilar but nonetheless equivalent concepts of producthood have also been defined via full-support-distribution sequences (see the clause containing Note 8) and nonstandard distributions (see Note 9). All these concepts define what it means for x to be "independent" of y.

[3] The terms "cancellation law" and "order" are taken from analogous concepts for binary relations in Krantz, Luce, Suppes, and Tversky (1971, page 427). (Their page 431 places Scott (1964)'s theorem in a broader mathematical context, and this theorem of Scott is fundamental to Subsection 4.3's proof of Theorem 4.1.)

will soon be used by Remark 3.1 to establish the existence of a product's marginals.

The 36=$(3!)^2$ second-order laws are too numerous to list. However, one of the second-order laws is

$$(24)\qquad (\forall x^0y^0, x^1y^1, x^2y^2)\ q_{x^0y^0/x^2y^2} \in q_{x^0y^0/x^1y^1} \odot q_{x^1y^1/x^2y^2}\ ,$$

which coincides with the basic cancellation law (20). This and (22)'s unit diagonality yield that a product over $X \times Y$ must be a dispersion over $X \times Y$ (hence "product" and "product dispersion" are synonymous). Another of the second-order laws is

$$(\forall x^0y^0, x^1y^1, x^2y^2)\ q_{x^0y^0/x^2y^1} \in q_{x^0y^2/x^1y^1} \odot q_{x^1y^0/x^2y^2}$$

which is equivalent to the *cross cancellation law*

$$(25)\qquad (\forall xy, x'y', x''y'')\ q_{xy/x''y''} \in q_{xy'/x'y''} \odot q_{x'y/x''y'}$$

(set $x^0y^0{=}xy$, $x^1y^1{=}x'y''$, $x^2y^2{=}x''y'$). Equation (18) was an instance of this law.

By the way, there are many redundancies within the cancellation laws that define producthood, and these redundancies lead to some interesting observations which are tangential to the paper. For example, it can be shown that unit diagonality (22), the two separability laws (23b&c), the basic cancellation law (24), and the cross cancellation law (25) are together sufficient for all zero-, first-, and second-order laws. Also, it can be shown that if a table satisfies unit diagonality (22) and all cancellation laws of order m, then it must satisfy all cancellation laws of order less than m. However, it seems tricky to determine when lower-order laws imply upper-order laws.

The *marginals* of a product $[q_{xy/x'y'}]$ are the dispersions $[q_{x/x'}]$ and $[q_{y/y'}]$ which satisfy

$$(26)\qquad (\forall xy, x'y')\ q_{xy/x'y'} \in q_{x/x'} \odot q_{y/y'}\ .$$

Note that marginals are defined to be dispersions, and consequently, each marginal must itself satisfy unit diagonality (12) and the basic cancellation law (13) (hence "marginal" and "marginal dispersion" are synonymous).

Remark 3.1. *Every product $[q_{xy/x'y'}]$ has unique marginals $[q_{x/x'}]$ and $[q_{y/y'}]$. Further, $[q_{x/x'}] = [q_{xy^\circ/x'y^\circ}]$ for any y°, and $[q_{y/y'}] = [q_{x^\circ y/x^\circ y'}]$ for any x°.*

Proof. Take any product $[q_{xy/x'y'}]$. First we show that $[q_{xy/x'y'}]$ has at least one pair of marginals. Choose some x° and y°, and define

$[q_{x/x'}] = [q_{xy^\circ/x'y^\circ}]$ and $[q_{y/y'}] = [q_{x^\circ y/x^\circ y'}]$. The table $[q_{x/x'}]$ is a dispersion because

$$(\forall x)\ q_{x/x} = q_{xy^\circ/xy^\circ} = 1$$

by the definition of $[q_{x/x'}]$ and by the unit diagonality (22) of $[q_{xy/x'y'}]$; and because

$$(\forall x, x', x'')\ q_{x/x''} = q_{xy^\circ/x''y^\circ} \in q_{xy^\circ/x'y^\circ} \odot q_{x'y^\circ/x''y^\circ} = q_{x/x'} \odot q_{x'/x''}$$

by the definition of $[q_{x/x'}]$, by the basic cancellation law (24) of $[q_{xy/x'y'}]$, and by two applications of the definition of $[q_{x/x'}]$. A symmetric argument shows that $[q_{y/y'}]$ is also a dispersion. Finally, (26) holds by an unnamed second-order law (cancel terms to check its validity), by the separability laws (23b&c), and by the definitions of $[q_{x/x'}]$ and $[q_{y/y'}]$:

$$(\forall xy, x'y')\ q_{xy/x'y'} \in q_{xy/x'y} \odot q_{xy/xy'} = q_{xy^\circ/x'y^\circ} \odot q_{x^\circ y/x^\circ y'} = q_{x/x'} \odot q_{y/y'}\ .$$

Second, suppose that $[q_{x/x'}]$ and $[q_{y/y'}]$ are marginals of $[q_{xy/x'y'}]$. By (26) and by the unit diagonality (12) of $[q_{y/y'}]$ we have

$$(\forall x, x', y^\circ)\ q_{xy^\circ/x'y^\circ} \in q_{x/x'} \odot q_{y^\circ/y^\circ} = \{q_{x/x'}\}$$

and thus the marginal $[q_{x/x'}]$ must equal $[q_{xy^\circ/x'y^\circ}]$ for any value of y°. A symmetric argument shows that $[q_{y/y'}]$ must equal $[q_{x^\circ y/x^\circ y'}]$ for any value of x°. □

As expected, we define a *product of* $[q_{x/x'}]$ *and* $[q_{y/y'}]$ to be a product whose marginals are $[q_{x/x'}]$ and $[q_{y/y'}]$. Keep in mind that the marginals of a product dispersion are unique (Remark 3.1) but the product of two dispersions might not be unique (recall (17)). That can be surprisingly tricky to digest because *distributions* are fundamentally different: the marginals of a product distribution are unique, *and* the product of two distributions is unique. Thus, a product distribution is equivalent to its two marginal distributions. Unfortunately, we don't have that luxury here: marginal dispersions are ambiguous.

4. Monomial Vectors

4.1. Representing Dispersions

As introduced in Section 1, let $[c_z n^{e_z}]$ denote a vector of monomials in which each monomial $c_z n^{e_z}$ has a positive coefficient c_z and a real exponent e_z. Such a monomial vector $[c_z n^{e_z}]$ is said to *represent* the table $[q_{z/z'}]$ if

$$(27) \qquad (\forall z, z')\ q_{z/z'} = \lim_{n\to\infty} \frac{c_z n^{e_z}}{c_{z'} n^{e_{z'}}} = \begin{pmatrix} \infty & \text{if } e_z > e_{z'} \\ c_z/c_{z'} & \text{if } e_z = e_{z'} \\ 0 & \text{if } e_z < e_{z'} \end{pmatrix}$$

(the first equality is the definition while the second is an obvious fact). For example, the monomial vector

$$(c_L n^{e_L}, c_R n^{e_R}) = (n^{-1}, 1) \tag{28}$$

represents Xavier's dispersion (10). Further, the set of all monomial vectors which represent Xavier's dispersion is characterized by $e_L < e_R$ (the coefficients c_L and c_R are irrelevant).

For another example, the monomial vector

$$(c_\ell n^{e_\ell}, c_m n^{e_m}, c_r n^{e_r}) = (n^{-2}, 5n^{-2}, 1) \tag{29}$$

represents Yolanda's dispersion (11). To exercise the definition of representation (27), note that $e_r = 0$ exceeds $e_m = -2$, and that this accords with $q_{r/m} = \infty$. Also note that $e_m = -2$ equals $e_\ell = -2$, and that this accords with $q_{m/\ell} = 5$ since $c_m = 5$ and $c_\ell = 1$. Further, the monomial vectors which represent Yolanda's dispersion are characterized by $e_\ell = e_m < e_r$ and $c_\ell = c_m/5$ (the coefficient c_r is irrelevant). In every such monomial vector, the exponents (e_ℓ, e_m, e_r) represent (in the economist's usual sense) an ordering of $\{\ell, m, r\}$ which partitions $\{\ell, m, r\}$ into a lower equivalence class $\{\ell, m\}$ and an upper equivalence class $\{r\}$. Then the coefficients (c_ℓ, c_m, c_r) serve to establish nonzero finite relative probabilities within each equivalence class (the coefficient c_r is irrelevant because $\{r\}$ is a singleton equivalence class).

In general, it is well understood that a dispersion over Z is equivalent to (1) an ordering of Z such that z is in a higher equivalence class than z' if and only if z is infinitely more likely than z', and (2) a full-support probability distribution within each equivalence class.[4] Accordingly, a table is a dispersion iff it can be represented by a monomial vector.

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[4] This equivalence appeared first in McLennan (1989b, page 147), and is formulated conveniently for our purposes by Hammond (1994, Subsection 4.1, $\Delta_M \approx \Delta_L$)): There Δ_M consists of McLennan's log-likelihood matrices, each of which is equivalent to a dispersion by Note 1; and Δ_L consists of Blume, Brandenburger, and Dekel (1991, Definition 5.2)'s lexicographic conditional probability systems, each of which is an ordered list of probability distributions whose supports partition Z. For example, Yolanda's dispersion (11) is equivalent to the lexicographic conditional probability system $\rho = (p_1, p_2)$ in which $p_1 = (0, 0, 1)$ and $p_2 = (\frac{1}{6}, \frac{5}{6}, 0)$. By inspection, such a ρ is equivalent to (1) an ordering of Z such that z' is in a higher equivalence class than z iff z' is infinitely more likely than z, and (2) a full-support probability distribution within each equivalence class.

Similar concepts also appear elsewhere. McLennan (1989a, page 127) and Monderer, Samet, and Shapley (1992, page 31) would formulate Yolanda's dispersion via the ordered partition $\{\{\ell, m\}, \{r\}\}$ and the corresponding within-class distributions $\{(\frac{1}{6}, \frac{5}{6}), (1)\}$. Vieille (1996, page 209) would formulate it by listing the elements of $\{\ell, m, r\}$ in order of descending probability as $\sigma^{-1} = (r, m, \ell)$, and then listing

4.2. Representing Products

As introduced in Section 1, two monomial vectors can be multiplied together. For example, the product of Xavier's vector (28) and Yolanda's vector (29) is

$$\begin{array}{c} r \\ m \\ \ell \end{array} \begin{array}{|cc|} \hline n^{-1} & 1 \\ 5n^{-3} & 5n^{-2} \\ n^{-3} & n^{-2} \\ \hline \end{array} \;. \qquad \begin{array}{cc} L & R \end{array}$$

The 6 monomials in this product represent a 6×6 dispersion over the 6-element set $\{L\ell, Lm, Lr, R\ell, Rm, Rr\}$. Two of the 36 relative probabilities in that dispersion are

$$q_{R\ell/Lr} = \lim_{n\to\infty} n^{-2}/n^{-1} = 0 \text{ and}$$
$$q_{Rm/Lr} = \lim_{n\to\infty} 5n^{-2}/n^{-1} = 0 \;,$$

two similar calculations would show that both $q_{Lr/R\ell}$ and $q_{Lr/Rm}$ are ∞, and 32 more calculations would show that the remainder of the 6×6 dispersion coincides with the 32 numbers in (17).

Theorem 4.1(a⇒b) demonstrates that this dispersion is a product of Xavier's dispersion (10) and Yolanda's dispersion (11). And more subtlely, Theorem 4.1(a⇐b) demonstrates that *every* product of Xavier's and Yolanda's dispersions can be represented by some such product of monomial vectors. This theorem is the key to the whole paper (Subsection 4.6 continues the example).

Theorem 4.1. *For any table $[q_{xy/x'y'}]$, (a) $[q_{xy/x'y'}]$ is represented by some $[c_x c_y n^{e_x+e_y}]$ iff (b) $[q_{xy/x'y'}]$ is a product. Furthermore, the marginals of the product represented by $[c_x c_y n^{e_x+e_y}]$ are represented by $[c_x n^{e_x}]$ and $[c_y n^{e_y}]$. (Proof in Subsections 4.3 through 4.5.)*

Theorem 4.1 reflects the fact that a product over $X\times Y$ is a special kind of dispersion over $X\times Y$ (recall the sentence following (24)). In particular, dispersionhood requires (by the previous subsection) that there be an ordering of $X\times Y$ such that xy is in a higher equivalence class than $x'y'$ if and only if xy is infinitely more likely than $x'y'$. Producthood then requires something more: it requires that this ordering has an additive representation of the form $[e_x+e_y]$. Further, dispersionhood requires that there be nonzero finite relative probabilities within each equivalence class. But producthood requires something more: it

the relative probabilities between adjacent elements as $\alpha = (q_{m/r}, q_{\ell/m}) = (0, 1/5)$ (here a zero relative probability signals a break between equivalence classes). All these are very similar to a monomial vector within the context of representing a single dispersion (monomial vectors are embellished with nontrivial exponents, and these become important in the context of representing products).

requires that these relative probabilities can be established with coefficients of the multiplicative form $[c_x c_y]$.

4.3. Proof of Theorem 4.1(a⇐b)

4.3.1. Overview. This subsection's proof is the key to the whole paper: it derives a representation of the form $[c_x c_y n^{e_x+e_y}]$ for every product $[q_{xy/x'y'}]$. In accord with the previous paragraph, this proof will first derive exponents $[e_x]$ and $[e_y]$ and then derive coefficients $[c_x]$ and $[c_y]$.

As discussed two paragraphs ago, the exponents must be chosen so that $[e_x+e_y]$ is an additive representation of the ordering $\succeq$ that is defined by $xy \succ x'y'$ iff xy is infinitely more likely than $x'y'$. The derivation of this additive representation comes from an unfamiliar source. To set up an analogy, recall that much economics literature has developed around Gorman (1968), who showed that an additive representation is implied by separability. Similarly, it appears that much mathematical psychology literature has developed around Scott (1964), who showed that an additive representation is implied by cancellation laws. The following proof will use Scott's result to derive the additive representation $[e_x+e_y]$ from the cancellation laws defining producthood.

Krantz, Luce, Suppes, and Tversky (1971) place Scott's result in a broader context. In particular, their Subsection 9.2 notes that Scott's result is one of several results that can be derived from the rank condition characterizing the existence of a solution to a system of linear inequalities. Further, their Theorem 2.7 derives this characterization from the undergraduate rank condition for a system of linear *equalities.*

The second half of this subsection's proof derives coefficients $[c_x]$ and $[c_y]$. As discussed in the last paragraph of the previous subsection, these coefficients must be chosen so that $[c_x c_y]$ determines the relative probabilities within all the equivalence classes of $\succeq$. It happens that this half of the proof is also based on the undergraduate rank condition for a system of linear equalities. Accordingly, the mathematical basis of this entire paper is nothing more than elementary linear algebra.

4.3.2. The Exponents $[e_x]$ and $[e_y]$. Take any product $[q_{xy/x'y'}]$. Then define the ordering $\succeq$ of $X{\times}Y$ by $xy \succ x'y'$ iff $q_{xy/x'y'} = \infty$. The well-definition of $\succeq$ follows from the fact that producthood implies dispersionhood (by the sentence following (24)) and from the fact that a dispersion over $Z = X{\times}Y$ is equivalent to (1) the ordering $\succeq$ and (2) a full-support probability distribution within each equivalence class of $\succeq$ (by the sentence with Note 4). As with any dispersion, reciprocity

(14) implies that $\succeq$ satisfies

(30a) $$q_{xy/x'y'} = \infty \text{ iff } xy \succ x'y'$$

(30b) $$q_{xy/x'y'} \in (0,\infty) \text{ iff } xy \approx x'y'$$

(30c) $$q_{xy/x'y'} = 0 \text{ iff } xy \prec x'y' \ .$$

for every xy and $x'y'$. The project here is to find an additive representation $[e_x{+}e_y]$ for $\succeq$.

This paragraph establishes Scott (1964, page 243, conditions (1_V) and (2_V), at $(A,A^*) = (X,Y)$, $xx^* = xy$, $V = \succeq$, $n = m{+}1$, and $(\pi,\sigma) = (\sigma^{-1},\tau^{-1})$). The first of these two conditions is the completeness of $\succeq$, which follows from the fact that $\succeq$ is an ordering. To prove the second condition, consider any $m \geq 1$, any permutations σ and τ of $\{0,1,2,...\,m\}$, and any $\langle x^i y^i \rangle_{i=0}^m$ such that

$$(\forall i{\geq}1)\ x^{\sigma(i)} y^{\tau(i)} \succeq x^i y^i \ .$$

Since (30a&b) yield that $(\forall i{\geq}1)\ q_{x^{\sigma(i)}y^{\tau(i)}/x^i y^i} > 0$, it must be that

$$0 \notin \odot_{i=1}^m q_{x^{\sigma(i)}y^{\tau(i)}/x^i y^i} \ .$$

Thus, since the producthood of $[q_{xy/x'y'}]$ implies the cancellation law

$$q_{x^0y^0/x^{\sigma(0)}y^{\tau(0)}} \in \odot_{i=1}^m q_{x^{\sigma(i)}y^{\tau(i)}/x^i y^i} \ ,$$

it must be that $q_{x^0y^0/x^{\sigma(0)}y^{\tau(0)}} > 0$. This together with (30a&b) yields that $x^0y^0 \succeq x^{\sigma(0)}y^{\tau(0)}$.

The previous paragraph and Scott (1964, Theorem 3.1, with "utility functions" set to $[e_x]$ and $[e_y]$) yields the existence of $[e_x] \in \mathbb{R}^X$ and $[e_y] \in \mathbb{R}^Y$ such that $xy \succeq x'y'$ iff $e_x{+}e_y \geq e_{x'}{+}e_{y'}$.[5] Thus, by (30) we have for any xy and $x'y'$ that

(31a) $$q_{xy/x'y'} = \infty \text{ iff } xy \succ x'y' \text{ iff } e_x{+}e_y > e_{x'}{+}e_{y'}$$

(31b) $$q_{xy/x'y'} \in (0,\infty) \text{ iff } xy \approx x'y' \text{ iff } e_x{+}e_y = e_{x'}{+}e_{y'}$$

(31c) $$q_{xy/x'y'} = 0 \text{ iff } xy \prec x'y' \text{ iff } e_x{+}e_y < e_{x'}{+}e_{y'} \ .$$

[5] This and the preceding two paragraphs have used Scott (1964, Theorem 3.1) to derive *real* vectors $[e_x]$ and $[e_y]$ such that $(\forall xy, x'y')\ xy \succeq x'y'$ iff $e_x{+}e_y \geq e_{x'}{+}e_{y'}$. An omitted five-page appendix uses Krantz, Luce, Suppes, and Tversky (1971, Theorem 2.7) to derive *nonnegative integer* vectors with the same property. If those five pages were inserted here, all the paper's proofs could be trivially modified to show that all the paper's results hold for monomials with nonnegative integer exponents. I decided to use real exponents because of their clarity (the integerness of exponents has nothing to do with the representation of dispersions) and convenience (finding integer exponents is an unnecessary tedium when working examples).

4.3.3. The Coefficients $[c_x]$ *and* $[c_y]$. It remains to find coefficient vectors $([c_x],[c_y])$ such that

$$(\forall\ (xy,x'y') \in \approx)\ c_x c_y (c_{x'} c_{y'})^{-1} = q_{xy/x'y'}$$

(recall that $(xy,x'y') \in \approx$ iff $xy \approx x'y'$). Since (31b) yields the critical fact that $q_{xy/x'y'} \in (0,\infty)$ for every $(xy,x'y')$ in $\approx$, this is equivalent to finding real numbers $([d_x],[d_y])$ such that

$$(\forall\ (xy,x'y') \in \approx)\ d_x + d_y - d_{x'} - d_{y'} = \ln(q_{xy/x'y'})\ .$$

Index the set $\approx$ as $\langle (x_j^0 y_j^0, x_j^1 y_j^1) \rangle_{j=1}^{|\approx|}$ (the set $\approx$ is nonempty since it must include diagonal elements). Our task is then to find real numbers $([d_x],[d_y]) \in \mathbb{R}^{X \cup Y}$ such that

$$(\forall j)\ d_{x_j} + d_{y_j} - d_{x'_j} - d_{y'_j} = \ln(q_{x_j y_j / x'_j y'_j})\ . \tag{32}$$

First notice that for any $xy \in X \times Y$, we can define a row vector $1_x 1_y \in \{0,1\}^{X \cup Y}$, in which $1_x \in \{0,1\}^X$ is the unit vector of x and $1_y \in \{0,1\}^Y$ is the unit vector of y. For example, if $X = \{L,R\}$ and $Y = \{\ell,m,r\}$, then $1_L 1_m = [1\ 0\ 0\ 1\ 0]$ because $1_L = [1\ 0]$ and $1_m = [0\ 1\ 0]$. Using this notation, the system (32) can be rewritten as

$$(\forall j)\ (1_{x_j} 1_{y_j} - 1_{x'_j} 1_{y'_j}) \cdot ([d_x],[d_y]) = \ln(q_{x_j y_j / x'_j y'_j}).$$

This is a matrix equation of the form $Ad = b$, in which row j of the coefficient matrix A is $1_{x_j} 1_{y_j} - 1_{x'_j} 1_{y'_j}$ and element j of the vector b is $\ln(q_{x_j y_j / x'_j y'_j})$.

Recall from elementary linear algebra that Gaussian elimination is equivalent to premultiplying the augmented matrix $[A\,b]$ with a certain square matrix E which replicates the elementary row operations and row permutations. Further recall that back substitution then reveals a solution to $Ad = b$ provided that $E[A\,b]$ does not contain a row which is zero in all but the last column (see for example Strang (1980, Chapter 1)). In the present circumstance, E has only rational elements because the coefficient matrix A has only rational elements. As a result, each row of $E[A\,b]$ can be written as

$$[\ \Sigma_j \lambda_j (1_{x_j} 1_{y_j} - 1_{x'_j} 1_{y'_j}) \quad \Sigma_j \lambda_j \ln(q_{x_j y_j / x'_j y'_j})\ ]$$

for some rational numbers $\langle \lambda_j \rangle_{j=1}^{|\approx|}$ equal to a row of E. Thus $E[A\,b]$ does not have a row in which all but the last column is zero if

$$\Sigma_j \lambda_j (1_{x_j} 1_{y_j} - 1_{x'_j} 1_{y'_j}) = 0 \text{ implies } \Sigma_j \lambda_j \ln(q_{x_j y_j / x'_j y'_j}) = 0. \tag{33}$$

for all rational numbers $\langle \lambda_j \rangle_{j=1}^{|\approx|}$. This we will establish to complete the proof.

Equation (33) holds if

$$\Sigma_j \ell_j (1_{x_j} 1_{y_j} - 1_{x'_j} 1_{y'_j}) = 0 \text{ implies } \Sigma_j \ell_j \ln(q_{x_j y_j / x'_j y'_j}) = 0. \tag{34}$$

for all integers $\langle \ell_j \rangle_{j=1}^{|\approx|}$ (to see the contrapositive of this claim, note that if $\langle \lambda_j \rangle_j$ violates (33) then some multiple of $\langle \lambda_j \rangle_j$ violates (34)). Accordingly, assume that

$$\Sigma_j \ell_j (1_{x_j} 1_{y_j} - 1_{x'_j} 1_{y'_j}) = 0 \tag{35}$$

holds for some integers $\langle \ell_j \rangle_{j=1}^{|\approx|}$. The remainder of this proof will establish that $\Sigma_j \ell_j \ln(q_{x_j y_j / x'_j y'_j}) = 0$.

First, define $\langle (\theta_j, \theta'_j) \rangle_{j=1}^{|\approx|}$ by

$$(\theta_j, \theta'_j) = \begin{pmatrix} (0,1) & \text{if } \ell_j \geq 0 \\ (1,0) & \text{if } \ell_j < 0 \end{pmatrix} .$$

Note that (35) is equivalent to

$$\Sigma_j \; |\ell_j| \; (1_{x_j^{\theta_j}} 1_{y_j^{\theta_j}} - 1_{x_j^{\theta'_j}} 1_{y_j^{\theta'_j}}) = 0 \; , \tag{36}$$

and that the symmetry of $\approx$ implies

$$(\forall j) \; (x_j^{\theta_j} y_j^{\theta_j}, x_j^{\theta'_j} y_j^{\theta'_j}) \in \approx \; . \tag{37}$$

Next, define $m^\star = \Sigma_j |\ell_j|$ and define $\langle (x_i y_i, x_i^\star y_i^\star) \rangle_{i=1}^{m^\star}$ by

$$(x_i y_i, x_i^\star y_i^\star) = (x_j^{\theta_j} y_j^{\theta_j}, x_j^{\theta'_j} y_j^{\theta'_j})$$

$$\text{for } i \in \{ \; \Sigma_{k=1}^{j-1} |\ell_k| + 1, \; \Sigma_{k=1}^{j-1} |\ell_k| + 2, ... \; \Sigma_{k=1}^{j-1} |\ell_k| + |\ell_j| \; \}$$

(at $j = 1$, $\Sigma_{k=1}^{j-1} |\ell_k| = 0$; at any j, the set is empty if $\ell_j = 0$; and at $j = |\approx|$, $\Sigma_{k=1}^{j-1} |\ell_k| + |\ell_j| = \Sigma_{j=1}^{|\approx|} |\ell_j| = \Sigma_j |\ell_j| = m^\star$). Note that (36) is equivalent to

$$\Sigma_{i=1}^{m^\star} \; (1_{x_i} 1_{y_i} - 1_{x_i^\star} 1_{y_i^\star}) = 0 \; , \tag{38}$$

and that (37) yields

$$(\forall i) \; (x_i y_i, x_i^\star y_i^\star) \in \approx \; . \tag{39}$$

Finally, note that (38) is equivalent to

$$\Sigma_{i=1}^{m^\star} \; 1_{x_i} = \Sigma_{i=1}^{m^\star} \; 1_{x_i^\star} \; \text{ and } \; \Sigma_{i=1}^{m^\star} \; 1_{y_i} = \Sigma_{i=1}^{m^\star} \; 1_{y_i^\star} \; ,$$

which is in turn equivalent to the existence of permutations $\sigma^\star$ and $\tau^\star$ of $\{1, 2, ... \, m^\star\}$ such that

$$(\forall i) \; x_i = x^\star_{\sigma^\star(i)} \text{ and } y_i = y^\star_{\tau^\star(i)} \; .$$

The producthood of $[q_{xy/x'y'}]$ implies

$$(\forall \langle x^i y^i \rangle_{i=1}^m)\ 1 \in \odot_{i=1}^m q_{x^{\sigma(i)} y^{\tau(i)}/x^i y^i}$$

for any $m \geq 1$ and any permutations σ and τ of $\{1, 2, ...m\}$ (this follows from (21) by defining $\sigma(0) = 0$ and $\tau(0) = 0$). By applying this at $m^\star$, $\sigma^\star$, $\tau^\star$, and $\langle x_i^\star y_i^\star \rangle_{i=1}^{m^\star}$, one obtains

$$1 \in \odot_{i=1}^{m^\star} q_{x^\star_{\sigma^\star(i)} y^\star_{\tau^\star(i)}/x_i^\star y_i^\star}$$

which by the definition of $\sigma^\star$ and $\tau^\star$ is equivalent to

$$1 \in \odot_{i=1}^{m^\star} q_{x_i y_i/x_i^\star y_i^\star}\ .$$

Since every $q_{x_i y_i/x_i^\star y_i^\star} \in (0, \infty)$ by (31b) and (39), this is equivalent to

$$\Pi_{i=1}^{m^\star} q_{x_i y_i/x_i^\star y_i^\star}\ =\ 1$$

and also to

$$\Sigma_{i=1}^{m^\star} \ln q_{x_i y_i/x_i^\star y_i^\star}\ =\ 0\ .$$

By the definitions of $m^\star$ and $\langle (x_i y_i, x_i^\star y_i^\star) \rangle_{i=1}^{m^\star}$, this is equivalent to

$$\Sigma_j |\ell_j| \ln q_{x_j^{\theta_j} y_j^{\theta_j}/x_j^{\theta'_j} y_j^{\theta'_j}} = 0\ ,$$

which is equivalent to

$$\Sigma_{j|\ell_j<0}\ (-\ell_j) \ln q_{x_j^{\theta_j} y_j^{\theta_j}/x_j^{\theta'_j} y_j^{\theta'_j}} + \Sigma_{j|\ell_j \geq 0}\ \ell_j \ln q_{x_j^{\theta_j} y_j^{\theta_j}/x_j^{\theta'_j} y_j^{\theta'_j}} = 0\ .$$

By the definition of $\langle (\theta_j, \theta'_j) \rangle_{j=1}^{|\approx|}$, this is equivalent to

$$\Sigma_{j|\ell_j<0}\ (-\ell_j) \ln q_{x_j^1 y_j^1/x_j^0 y_j^0} + \Sigma_{j|\ell_j \geq 0}\ \ell_j \ln q_{x_j^0 y_j^0/x_j^1 y_j^1} = 0\ ,$$

which by reciprocity (14) is equivalent to

$$\Sigma_{j|\ell_j<0}\ \ell_j \ln q_{x_j^0 y_j^0/x_j^1 y_j^1} + \Sigma_{j|\ell_j \geq 0}\ \ell_j \ln q_{x_j^0 y_j^0/x_j^1 y_j^1} = 0$$

and also to

$$\Sigma_j\ \ell_j \ln q_{x_j^0 y_j^0/x_j^1 y_j^1} = 0\ .$$

4.4. Proof of Theorem 4.1(a⇒b)

It is straightforward to show that the special structure of $[c_x c_y e^{e_x+e_y}]$ implies every cancellation law in the definition of producthood.[6] We do this immediately after the following lemma.

Lemma 4.2. *Suppose that* $\{\langle a_j^n\rangle_n\}_j$ *is a finite set of sequences in* $(0,\infty)$*, that each* $\lim_n a_j^n$ *exists in* $[0,\infty]$*, and that* $\lim_n \Pi_j a_j^n$ *exists in* $[0,\infty]$*. Then* $\lim_n \Pi_j a_j^n \in \odot_j \lim_n a_j^n$*.*

Proof. If each $\lim_n a_j^n < \infty$, then $\lim_n \Pi_j a_j^n = \Pi_j \lim_n a_j^n$ [Rudin (1976, page 49, Theorem 3.3)] and $\odot_j \lim_n a_j^n = \{\Pi_j \lim_n a_j^n\}$. If some $\lim_n a_j^n = \infty$ and every $\lim_n a_j^n > 0$, then $\lim_n \Pi_j a_j^n = \infty$ and $\odot_j \lim_n a_j^n = \{\infty\}$. Finally, if some $\lim_n a_j^n = \infty$ and some other $\lim_n a_j^n = 0$, the conclusion $\lim_n \Pi_j a_j^n \in \odot_j \lim_n a_j^n$ holds vacuously because $\odot_j \lim_n a_j^n = [0,\infty]$. □

We now demonstrate Theorem 4.1(a⇒b) Suppose that $[q_{xy/x'y'}]$ is represented by some $[c_x c_y n^{e_x+e_y}]$. $[q_{xy/x'y'}]$ satisfies the zero-order cancellation law because

$$(\forall xy)\ \ q_{xy/xy} = \lim_n \frac{c_x c_y n^{e_x+e_y}}{c_x c_y n^{e_x+e_y}} = 1\ ,$$

by the definition of representation and by algebra. Further, for any $m \geq 1$ and any permutations σ and τ, $[q_{xy/x'y'}]$ satisfies

$$(\forall \langle x^i y^i\rangle_{i=0}^m)\ \ q_{x^0y^0/x^{\sigma(0)}y^{\tau(0)}} = \lim_n \frac{c_{x^0} c_{y^0} n^{e_{x^0}+e_{y^0}}}{c_{x^{\sigma(0)}} c_{y^{\tau(0)}} n^{e_{x^{\sigma(0)}}+e_{y^{\tau(0)}}}} =$$

$$\lim_n \Pi_{i=1}^m \frac{c_{x^{\sigma(i)}} c_{y^{\tau(i)}} n^{e_{x^{\sigma(i)}}+e_{y^{\tau(i)}}}}{c_{x^i} c_{y^i} n^{e_{x^i}+e_{y^i}}} \in \odot_{i=1}^m \lim_n \frac{c_{x^{\sigma(i)}} c_{y^{\tau(i)}} n^{e_{x^{\sigma(i)}}+e_{y^{\tau(i)}}}}{c_{x^i} c_{y^i} n^{e_{x^i}+e_{y^i}}}$$

$$= \odot_{i=1}^m q_{x^{\sigma(i)}y^{\tau(i)}/x^iy^i}$$

by the definition of representation, by algebra, by Lemma 4.2, and by m applications of the definition of representation. Thus $[q_{xy/x'y'}]$ satisfies every cancellation law in the definition of producthood.

4.5. Proof of Theorem 4.1's Sentence About Marginals

Let $[q_{xy/x'y'}]$ be the product represented by $[c_x c_y n^{e_x+e_y}]$. Fix any y°. By Remark 3.1, the marginal with respect to x equals $[q_{xy^\circ/x'y^\circ}]$. Since this $[q_{xy^\circ/x'y^\circ}]$ is a restriction of $[q_{xy/x'y'}]$ and since all of $[q_{xy/x'y'}]$ is represented by $[c_x c_y n^{e_x+e_y}]$, we have that the marginal with respect to x is represented by $[c_x c_{y^\circ} n^{e_x+e_{y^\circ}}]$, which equals $c_{y^\circ} n^{e_{y^\circ}}[c_x n^{e_x}]$. This yields that the marginal with respect to x is represented by $[c_x n^{e_x}]$ because the definition of representation depends only on the ratios between

[6] This reformulates the limiting argument of Kohlberg and Reny (1997, page 305, Proof of Theorem 2.10, first paragraph).

monomials (the appendix places this last step in a broader context). A symmetric argument holds for the marginal with respect to y.

4.6. The Products of Xavier and Yolanda

This subsection uses Theorem 4.1 to derive all of the products of Xavier's dispersion (10) with Yolanda's dispersion (11). Within this derivation, let the function Q map a monomial vector to the dispersion it represents, and let (10), (11) and (17) denote the tables appearing at those equation numbers (for instance, one could write $Q(n^{-2},3) = \left[\begin{smallmatrix} 0 & 1 \\ 1 & \infty \end{smallmatrix}\right] = (10)$ to show that the monomial vector $(n^{-2},3)$ represents Xavier's dispersion (10)).

The first equality below follows from Theorem 4.1. The second follows from choosing R and r to be the two numeraires (the appendix discusses numeraires and Remark A.1 can be applied here). The third follows from Subsection 4.1's discussion of which monomial vectors represent Xavier's and Yolanda's dispersions (recall the text after (28) and (29)).

$$\{ \text{ products of (10) and (11) } \} \tag{40}$$

$$= \left\{ Q \begin{pmatrix} c_L c_r n^{e_L+e_r} & c_R c_r n^{e_R+e_r} \\ c_L c_m n^{e_L+e_m} & c_R c_m n^{e_R+e_m} \\ c_L c_\ell n^{e_L+e_\ell} & c_R c_\ell n^{e_R+e_\ell} \end{pmatrix} \right| \tag{41}$$

$$Q(c_L n^{e_L}, c_R n^{e_R}) = (10) \text{ and } Q(c_\ell n^{e_\ell}, c_m n^{e_m}, c_r n^{e_r}) = (11) \left. \vphantom{\int} \right\}$$

$$= \left\{ Q \begin{pmatrix} c_L n^{e_L} & 1 \\ c_L c_m n^{e_L+e_m} & c_m n^{e_m} \\ c_L c_\ell n^{e_L+e_\ell} & c_\ell n^{e_\ell} \end{pmatrix} \right| \tag{42}$$

$$Q(c_L n^{e_L}, 1) = (10) \text{ and } Q(c_\ell n^{e_\ell}, c_m n^{e_m}, 1) = (11) \left. \vphantom{\int} \right\}$$

$$= \left\{ Q \begin{pmatrix} c_L n^{e_L} & 1 \\ c_L c_m n^{e_L+e_m} & c_m n^{e_m} \\ c_L c_\ell n^{e_L+e_\ell} & c_\ell n^{e_\ell} \end{pmatrix} \right| \tag{43}$$

$$e_L < 0,\ c_L \text{ free},\ e_\ell = e_m < 0,\ 5c_\ell = c_m \ \}$$

The next step is to evaluate $Q(\cdot)$ within (43) by evaluating

$$q_{xy/x'y'} = \lim_{n\to\infty} \frac{c_x c_y n^{e_x+e_y}}{c_{x'} c_{y'} n^{e_{x'}+e_{y'}}}$$

at each of the 36 $xy/x'y'$ in $X{\times}Y$. Fortunately, 32 of the 36 equal the 32 scalars in the 6×6 table at (17): one example is that

$$q_{Rm/Lm} = \lim_{n\to\infty}\frac{c_m n^{e_m}}{c_L c_m n^{e_L+e_m}} = \lim_{n\to\infty}\frac{1}{c_L n^{e_L}} = \infty$$

since $e_L < 0$ by the last line of (43); and the remaining 31 are surprisingly easy because patterns such as unit diagonality and reciprocity readily emerge. However, 4 of the 36 are more difficult and they appear with little simplification in (44).

$$\begin{aligned} \{\ (17) \mid\ & q_{R\ell/Lr} = \lim_{n\to\infty}(c_\ell/c_L)n^{e_\ell-e_L}, \\ & q_{Lr/R\ell} = \lim_{n\to\infty}(c_L/c_\ell)n^{e_L-e_\ell}, \\ & q_{Rm/Lr} = \lim_{n\to\infty}(c_m/c_L)n^{e_m-e_L}, \\ & q_{Lr/Rm} = \lim_{n\to\infty}(c_L/c_m)n^{e_L-e_m}, \\ & e_L<0,\ c_L \text{ free},\ e_\ell = e_m < 0,\ 5c_\ell = c_m\ \} \end{aligned} \tag{44}$$

Then, (45) is reached because there are two pairs of reciprocal limits, (46) is reached by substituting out c_m and e_m, (47) is reached by factoring out the 5 in the limit defining $q_{Rm/Lr}$, and finally, (48) is reached by noting that $q_{R\ell/Lr}$ can be made any value while keeping both e_ℓ and e_L negative ($q_{R\ell/Lr}$ can be made infinite by setting $e_\ell > e_L$, made zero by setting $e_\ell < e_L$, and made equal to any finite nonzero a by setting $e_\ell = e_L$ and $c_\ell/c_L = a$).

$$\begin{aligned} \{\ (17) \mid\ & 1/q_{Lr/R\ell} = q_{R\ell/Lr} = \lim_{n\to\infty}(c_\ell/c_L)n^{e_\ell-e_L}, \\ & 1/q_{Lr/Rm} = q_{Rm/Lr} = \lim_{n\to\infty}(c_m/c_L)n^{e_m-e_L}, \\ & e_L<0,\ c_L \text{ free},\ e_\ell = e_m < 0,\ 5c_\ell = c_m\ \} \end{aligned} \tag{45}$$

$$\begin{aligned} = \{\ (17) \mid\ & 1/q_{Lr/R\ell} = q_{R\ell/Lr} = \lim_{n\to\infty}(c_\ell/c_L)n^{e_\ell-e_L}, \\ & 1/q_{Lr/Rm} = q_{Rm/Lr} = \lim_{n\to\infty}(5c_\ell/c_L)n^{e_\ell-e_L}, \\ & e_L<0,\ c_L \text{ free},\ e_\ell < 0,\ c_\ell \text{ free}\ \} \end{aligned} \tag{46}$$

$$\begin{aligned} = \{\ (17) \mid\ & 1/q_{Lr/R\ell} = q_{R\ell/Lr} = \lim_{n\to\infty}(c_\ell/c_L)n^{e_\ell-e_L}, \\ & 1/q_{Lr/Rm} = q_{Rm/Lr} = 5q_{R\ell/Lr}, \\ & e_L<0, c_L \text{ free}, e_\ell < 0,\ c_\ell \text{ free}\ \} \end{aligned} \tag{47}$$

$$\begin{aligned} = \{\ (17) \mid\ & 1/q_{Lr/R\ell} = q_{R\ell/Lr},\ q_{R\ell/Lr} \text{ free}, \\ & 1/q_{Lr/Rm} = q_{Rm/Lr} = 5q_{R\ell/Lr}\ \} \end{aligned} \tag{48}$$

The equality of (40) with (48) shows that the set of all products of Xavier's dispersion (10) with Yolanda's dispersion (11) is a one-dimensional set in a 36-dimensional space. Subsection 4.2's example is the point in this set where $q_{R\ell/Lr} = 0$, and all of this accords with Subsection 3.1's intuition concerning the definition of producthood.

5. Reintroduce Distributions

5.1. Strategies and Beliefs

This section reverses the second step into the woods by reintroducing distributions such as strategies and beliefs. In particular, this section considers a strategy $[p_x]$ over X, a strategy $[p_y]$ over Y, and a belief $[p_{xy}]_{xy \in H}$ over some information set $H \subseteq X{\times}Y$. This section's only result is the following theorem, which is derived from Theorem 4.1, the key result in the paper.

Theorem 5.1. *The following are equivalent for any strategy $[p_x]$, any strategy $[p_y]$, and any belief $[p_{xy}]_{xy \in H}$. (a) There are monomial vectors $[c_x n^{e_x}]$ and $[c_y n^{e_y}]$ such that $[c_x n^{e_x}]$ induces $[p_x]$, $[c_y n^{e_y}]$ induces $[p_y]$, and the restriction of $[c_x c_y n^{e_x + e_y}]$ to H induces $[p_{xy}]_{xy \in H}$. (b) There is a product dispersion $[q_{xy/x'y'}]$ whose marginal dispersions induce $[p_x]$ and $[p_y]$ and whose restriction to H^2 induces $[p_{xy}]_{xy \in H}$.*

Informally, Theorem 5.1 states that strategies and beliefs are induced by a pair of monomial vectors iff they are induced by a product dispersion. The theorem's formal statement is more cumbersome because it uses the word "induce" precisely, as defined for monomial vectors at (3) in Section 1 and as defined for dispersions at (15) in Section 2.

Theorem 5.1's proof employs the following lemma, which concerns the two avenues by which a monomial vector can determine a distribution. The first is the direct route: (3) in Section 1 shows how a monomial vector induces a distribution. The second is the indirect route: (27) in Section 4 shows how a monomial vector represents a dispersion, and then (15) in Section 2 shows how a dispersion induces a distribution. Unsurprisingly, the following lemma shows that these two avenues are equivalent.

Lemma 5.2. *Suppose that $[c_z n^{e_z}]$ induces $[p_z^A]$, and that the same $[c_z n^{e_z}]$ represents $[q_{z/z'}]$ which induces $[p_z^B]$. Then $[p_z^A] = [p_z^B]$.*

Proof. By the definitions at (3), (27), and (15), we are given that $[c_z n^{e_z}]$ and $[p_z^A]$ satisfy

$$(\forall z)\ \ p_z^A = \frac{c_z 1(e_z{=}\max\{e_{z'}|z'\})}{\Sigma_{z'' \in Z}\ c_{z''} 1(e_{z''}{=}\max\{e_{z'}|z'\})}\ , \tag{49}$$

that $[c_z n^{e_z}]$ and $[q_{z/z'}]$ satisfy

$$(50)\qquad (\forall z,z')\;\; q_{z/z'} = \begin{pmatrix} \infty & \text{if } e_z > e_{z'} \\ c_z/c_{z'} & \text{if } e_z = e_{z'} \\ 0 & \text{if } e_z < e_{z'} \end{pmatrix} ,$$

and that $[q_{z/z'}]$, $[p^B_z]$, and z^* satisfy

$$(51)\qquad (\forall z)\;\; p^B_z = \frac{q_{z/z^*}}{\Sigma_{z'\in Z}\; q_{z'/z^*}}$$

$$(52)\qquad \text{and } (\forall z')\; q_{z'/z^*} \;<\; \infty \;.$$

(52) and (50) together yield that $z^* = \max\{e_{z^0}|z^0\}$. Hence by (50),

$$(\forall z)\;\; q_{z/z^*} = \begin{pmatrix} \infty & \text{if } e_z > e_{z^*} \\ c_z/c_{z^*} & \text{if } e_z = e_{z^*} \\ 0 & \text{if } e_z < e_{z^*} \end{pmatrix} = (c_z/c_{z^*})1(e_z{=}\max\{e_{z^0}|z^0\}) \;.$$

Hence by (51), by algebra, and by (49)

$$\begin{aligned}(\forall z)\;\; p^B_z &= \frac{(c_z/c_{z^*})1(e_z{=}\max\{e_{z^0}|z^0\})}{\Sigma_{z'\in Z}\;(c_{z'}/c_{z^*})1(e_{z'}{=}\max\{e_{z^0}|z^0\})} \\ &= \frac{c_z 1(e_z{=}\max\{e_{z^0}|z^0\})}{\Sigma_{z'\in Z}\; c_{z'}1(e_{z'}{=}\max\{e_{z^0}|z^0\})} = p^A_z \;.\end{aligned}$$

□

Proof of Theorem 5.1. Assume (a). By Theorem 4.1(a⇒b) together with that theorem's sentence about marginals, $[c_x c_y n^{e_x+e_y}]$ represents a product dispersion $[q_{xy/x'y'}]$ whose marginal dispersions are represented by $[c_x n^{e_x}]$ and $[c_y n^{e_y}]$. Let $[q_{x/x'}]$ and $[q_{y/y'}]$ denote these marginals. Since $[c_x n^{e_x}]$ induces $[p_x]$ (by assumption), since $[c_x n^{e_x}]$ represents $[q_{x/x'}]$ (by the previous sentence), and since $[q_{x/x'}]$ induces some distribution (by Remark 2.1), Lemma 5.2 shows that $[q_{x/x'}]$ induces $[p_x]$. A similar argument shows that $[q_{y/y'}]$ induces $[p_y]$. Further, since $[c_x c_y n^{e_x+e_y}]$ represents all of $[q_{xy/x'y'}]$ (by the second sentence of this paragraph), it must be that $[c_x c_y n^{e_x+e_y}]_{xy\in H}$ represents $[q_{xy/x'y'}]_{(xy,x'y')\in H^2}$. Since $[c_x c_y n^{e_x+e_y}]_{xy\in H}$ induces $[p_{xy}]_{xy\in H}$ (by assumption), since this same $[c_x c_y n^{e_x+e_y}]_{xy\in H}$ also represents $[q_{xy/x'y'}]_{(xy,x'y')\in H^2}$ (by the previous sentence), and since $[q_{xy/x'y'}]_{(xy,x'y')\in H^2}$ induces some distribution (by Remark 2.1), Lemma 5.2 shows that $[q_{xy/x'y'}]_{(xy,x'y')\in H^2}$ induces $[p_{xy}]_{xy\in H}$.

Assume (b), and let $[q_{x/x'}]$ and $[q_{y/y'}]$ denote the marginals assumed there. By Theorem 4.1(a⇐b) together with that theorem's sentence about marginals, there exists $[c_x n^{e_x}]$ and $[c_y n^{e_y}]$ such that $[c_x c_y n^{e_x+e_y}]$ represents $[q_{xy/x'y'}]$, $[c_x n^{e_x}]$ represents $[q_{x/x'}]$, and $[c_y n^{e_y}]$ represents its $[q_{y/y'}]$. Since $[c_x n^{e_x}]$ induces some dispersion (by the definition at (3)), since $[c_x n^{e_x}]$ represents $[q_{x/x'}]$ (by the previous sentence), and since

$[q_{x/x'}]$ induces $[p_x]$ (by assumption), Lemma 5.2 shows that $[c_x n^{e_x}]$ induces $[p_x]$. A similar argument shows that $[c_y n^{e_y}]$ represents $[p_y]$. Further, since $[c_x c_y n^{e_x+e_y}]$ represents all of $[q_{xy/x'y'}]$ (by this paragraph's second sentence), $[c_x c_y n^{e_x+e_y}]_{xy \in H}$ represents $[q_{xy/x'y'}]_{(xy,x'y') \in H^2}$. Since $[c_x c_y n^{e_x+e_y}]_{xy \in H}$ induces some distribution (by the definition at (3)), since $[c_x c_y n^{e_x+e_y}]_{xy \in H}$ represents $[q_{xy/x'y'}]_{(xy,x'y') \in H^2}$ (by the previous sentence), and since $[q_{xy/x'y'}]_{(xy,x'y') \in H^2}$ induces $[p_{xy}]_{xy \in H}$ (by assumption), Lemma 5.2 shows that $[c_x n^{e_x} c_y n^{e_y}]_{xy \in H}$ induces $[p_{xy}]_{xy \in H}$. $\square$

5.2. The End of Xavier, Yolanda, and Helen

The example of Xavier, Yolanda, and Helen has appeared on several occasions, three of which are important here. First recall that Subsection 2.1 showed that Yolanda's strategy can be induced by any dispersion in a one-dimensional set (her strategy does not determine the relative probability $q_{m/\ell}$). Accordingly, that subsection fixed one such dispersion for the purposes of illustration (that dispersion has $q_{m/\ell} = 5$). Second recall that Subsection 4.6 showed that the product of Xavier's dispersion with Yolanda's dispersion is a one-dimensional set (that degree of freedom concerns which of Xavier and Yolanda is more likely to fail to play right). Finally, recall that Subsection 1.2 showed that the set of consistent beliefs has two dimensions (Helen could have any belief with support $\{Lr, R\ell, Rm\}$).

This example illustrates that the multiplicity of consistent beliefs stems from two distinct sources. The first is that a player's strategy does not specify the relative probabilities between her own zero-probability actions. This first degree of freedom appears when choosing one of the many dispersions that induce her strategy. The second source of multiplicity is that two dispersions do not specify the relative probability between one player's zero-probability actions and the other player's zero-probability actions. This second degree of freedom appears when choosing one of the many products of the two players' dispersions. In this simple example, both of these sources contribute one dimension so that the set of consistent beliefs has two dimensions.

5.3. Taking Stock

As explained at the outset in Section 1, this paper is concerned with the question of how Helen should assign probabilities to the nodes in her zero-probability information set. A good answer to that question would appeal both computationally and conceptually.

Saying that Helen's belief should be induced by a pair of monomial vectors is an answer which appeals computationally (at least in comparison with the alternatives). Saying that Helen's belief should be induced by a product dispersion is an answer which appeals conceptually (dispersions and their products seem to be the appropriate "fundamentals"). Hence Theorem 5.1's equivalence suggests that we have a single answer which appeals both computationally and conceptually. (And in addition, each cancellation law is a necessary condition which has some computational appeal of its own.)

We have found this answer without any reference to the sequences with which Kreps and Wilson defined consistency, and virtually without reference to any results in the economics literature. To put this poetically, we have found this answer while yet remaining one step in the woods. Subjectively, one might find this a comfortable place to pause or even to stop.

6. Reintroduce Sequences

6.1. Approximation

This final section reverses the first step into the woods by reintroducing the sequences with which Kreps and Wilson defined consistency. It thereby connects this paper with the economics literature.

This subsection begins the task by studying how dispersions and products can be approximated by sequences of full-support distributions. Subsection 6.2 will then discuss consistency (which concerns how distributions such as strategies and beliefs can be induced by sequences of full-support distributions).

A full-support-distribution sequence $\langle[\pi^n_z]\rangle_n$ is said to *approximate* a table $[q_{z/z'}]$ if

$$(53) \qquad (\forall z, z')\; q_{z/z'} = \lim_{n\to\infty} \pi^n_z/\pi^n_{z'} \;.$$

It is well understood that (a) a table over Z can be represented by a monomial vector iff (a′) it can be approximated by a full-support-distribution sequence iff (b) it is a dispersion.[7]

Analogously (in two dimensions rather than one), Theorem 6.1 will show that (a) a table over $X{\times}Y$ is represented by the product of two

[7] The equivalence of (a) representation and (b) dispersionhood was discussed in Note 4. The equivalence of (a′) approximation and (b) dispersionhood has been noted twice in the literature. First, recall from Note 1 that a dispersion is equivalent to Myerson (1986)'s conditional probability system. Thus his Theorem 1 (page 337) can be regarded as the equivalence between dispersionhood and approximation. Second, recall from Note 1 that a dispersion is also equivalent to a matrix of likelihood ratios that satisfy McLennan (1989b)'s condition (2.5). Thus his Lemma 2.1 (page 147) can also be regarded as the equivalence between dispersionhood and approximation.

monomial vectors iff (a′) it is approximated by the product of two full-support-distribution sequences iff (b) it is a product. This theorem is a straightforward extension of Theorem 4.1, and its intuition is as follows.

First, note that any representation can be regarded as a special sort of approximation: given $[c_z n^{e_z}]$, define $\langle[\pi_z^n]\rangle_n = \langle[c_z n^{e_z}(\Sigma_{z'} c_{z'} n^{e_{z'}})^{-1}]\rangle_n$ and note that the definition of representation then coincides with the definition of approximation. Accordingly, a table is represented by the product of two monomial vectors only if it is approximated by the product of two full-support-distribution sequences. In brief, (a) representation by two monomial vectors implies (a′) approximation by two distribution sequences.

Second, note that the table of relative probabilities derived from the product of two full-support distributions must satisfy all the cancellation laws defining producthood: given $[\pi_x \pi_y]$, define $[q_{xy/x'y'}] = [\pi_x \pi_y (\pi_{x'} \pi_{y'})^{-1}]$, and observe by elementary algebra with positive reals that any such table satisfies every cancellation law. Hence, the limit of a sequence of such tables must also satisfy every cancellation law. Accordingly, (a′) representation by two distribution sequences implies (b) producthood.

Third, recall Theorem 4.1(a⇐b), which showed that (a) representation by two monomial vectors is implied by (b) producthood (this is the paper's key result). Hence, the last three paragraphs together show the equivalence of (a) representation by two monomial vectors, (a′) approximation by two distribution sequences, and (b) producthood. The following theorem states this formally.

Theorem 6.1. *For any table $[q_{xy/x'y'}]$, (a) $[q_{xy/x'y'}]$ is represented by some $[c_x c_y n^{e_x+e_y}]$ iff (a′) $[q_{xy/x'y'}]$ is approximated by some $\langle \pi_x^n \pi_y^n \rangle_n$ iff (b) $[q_{xy/x'y'}]$ is a product. Further, the marginals of the product represented by $[c_x c_y n^{e_x+e_y}]$ are represented by $[c_x n^{e_x}]$ and $[c_y n^{e_y}]$. Similarly, the marginals of the product approximated by $\langle \pi_x^n \pi_y^n \rangle_n$ are approximated by $\langle \pi_x^n \rangle_n$ and $\langle \pi_y^n \rangle_n$.*

Before proving the entire theorem, we note that the equivalence of (a′) and (b) reformulates a result in Kohlberg and Reny (1997). In particular, their strong independence concept (page 286) is equivalent and similar to (a′),[8] and their Theorem 2.10 (page 288) shows that this strong independence is equivalent to a kind of acyclicity which is equivalent and similar to (b) [the details are sorted out in Streufert (2003, Appendix B.1)]. Further, and perhaps more importantly, the present paper fundamentally reflects their focus on relative probability as a means of understanding consistency.

Proof. (a⇒a′) Suppose $[q_{xy/x'y'}]$ is represented by some $[c_x c_y n^{e_x+e_y}]$. Define

$$\langle[\pi_x^n]\rangle_n = \langle[c_x n^{e_x}](\Sigma_{x''} c_{x''} n^{e_{x''}})^{-1}\rangle_n \text{ and}$$
$$\langle[\pi_y^n]\rangle_n = \langle[c_y n^{e_y}](\Sigma_{y''} c_{y''} n^{e_{y''}})^{-1}\rangle_n \ .$$

Then by the definition of representation (27), by algebra, and by the definitions of $\langle[\pi_x^n]\rangle_n$ and $\langle[\pi_y^n]\rangle_n$,

$$(\forall xy, x'y')\ q_{xy/x'y'} = \lim_{n\to\infty} \frac{c_x n^{e_x} \cdot c_y n^{e_y}}{c_{x'} n^{e_{x'}} \cdot c_{y'} n^{e_{y'}}} =$$
$$\lim_{n\to\infty} \frac{c_x n^{e_x} (\Sigma_{x''} c_{x''} n^{e_{x''}})^{-1} \cdot c_y n^{e_y} (\Sigma_{y''} c_{y''} n^{e_{y''}})^{-1}}{c_{x'} n^{e_{x'}} (\Sigma_{x''} c_{x''} n^{e_{x''}})^{-1} \cdot c_{y'} n^{e_{y'}} (\Sigma_{y''} c_{y''} n^{e_{y''}})^{-1}} = \lim_{n\to\infty} \frac{\pi_x^n \pi_y^n}{\pi_{x'}^n \pi_{y'}^n} \ .$$

This entire equality is the definition of approximation (53).

(a′⇒b) Suppose $[q_{xy/x'y'}]$ is approximated by some $\langle \pi_x^n \pi_y^n \rangle_n$. Take any order $m \geq 0$ and any permutations σ and τ of $\{0, 1, ...m\}$. By $2m$ cancellations of identical pairs of positive reals in the numerator and the denominator of the right-hand side, we have that

$$(54) \qquad (\forall n)\ \frac{\pi_{x^0}^n \pi_{y^0}^n}{\pi_{x^{\sigma(0)}}^n \pi_{x^{\tau(0)}}^n} = \Pi_{i=1}^m \frac{\pi_{x^{\sigma(i)}}^n \pi_{y^{\tau(i)}}^n}{\pi_{x^i}^n \pi_{y^i}^n} \ .$$

By approximation, all m terms on the right-hand side have limits. Similarly, by approximation, the term on the left-hand side has a limit, and thus by (54) itself, the product of the m terms on the right-hand side has a limit. The last two sentences and Lemma 4.2 (in Subsection 4.4) yield that

$$\lim_{n\to\infty} \Pi_{i=1}^m \frac{\pi_{x^{\sigma(i)}}^n \pi_{y^{\tau(i)}}^n}{\pi_{x^i}^n \pi_{y^i}^n} \in \odot_{i=1}^m \lim_{n\to\infty} \frac{\pi_{x^{\sigma(i)}}^n \pi_{y^{\tau(i)}}^n}{\pi_{x^i}^n \pi_{y^i}^n} \ .$$

Hence by (54)

$$\lim_{n\to\infty} \frac{\pi_{x^0}^n \pi_{y^0}^n}{\pi_{x^{\sigma(0)}}^n \pi_{x^{\tau(0)}}^n} \in \odot_{i=1}^m \lim_{n\to\infty} \frac{\pi_{x^{\sigma(i)}}^n \pi_{y^{\tau(i)}}^n}{\pi_{x^i}^n \pi_{y^i}^n} \ ,$$

which by the definition of approximation (53) is equivalent to

$$q_{x^0 y^0 / x^{\sigma(0)} y^{\tau(0)}} = \odot_{i=1}^m q_{x^{\sigma(i)} y^{\tau(i)} / x^i y^i} \ .$$

Therefore, the mth-order cancellation law with permutations σ and τ is satisfied. Since this holds for all m, σ, and τ, $[q_{xy/x'y'}]$ is a product.

(a⇐b) Theorem 4.1(a⇐b). (This is the key step.)

[8] Also equivalent and similar to (a′) is the construction of the set Ψ within the equilibrium existence proof of McLennan (1989b, page 170) [details at Streufert (2003, page 32)].

The Representation of Marginals. Theorem 4.1's sentence about marginals.

The Approximation of Marginals. Let $[q_{xy/x'y'}]$ be the product approximated by $\langle[\pi_x^n \pi_y^n]\rangle_n$. Fix any y°. By Remark 3.1, the marginal of $[q_{xy/x'y'}]$ with respect to x equals $[q_{xy^\circ/x'y^\circ}]$. Since this $[q_{xy^\circ/x'y^\circ}]$ is a restriction of $[q_{xy/x'y'}]$ and since all of $[q_{xy/x'y'}]$ is approximated by $\langle[\pi_x^n \pi_y^n]\rangle_n$, we have that the marginal with respect to x is approximated by $\langle[\pi_x^n \pi_{y^\circ}^n]\rangle_n$. Thus, by noting the common $\pi_{y^\circ}^n$ in conjunction with the definition of approximation (53), we have that the marginal with respect to x is also approximated by $\langle[\pi_x^n]\rangle_n$. A symmetric argument holds for the marginal with respect to y. □

This section has studied how dispersions and products can be approximated by sequences of full-support distributions. Analogously, other papers have studied how dispersions and products can be expressed by nonstandard vectors.[9]

6.2. Consistency

A distribution $[p_z]$ is said to be *induced* by a full-support-distribution sequence $\langle[\pi_z^n]\rangle_n$ if

$$(\forall z)\ p_z = \lim_{n\to\infty}\pi_z^n\ . \tag{55}$$

As in Kreps and Wilson (1982, page 872 with their $\pi = ([p_x],[p_y])$ and their $\mu = [p_{xy}]_{xy\in H}$), the belief $[p_{xy}]_{xy\in H}$ is said to be *consistent* with the strategies $[p_x]$ and $[p_y]$ if there exist full-support-distribution sequences $\langle[\pi_x^n]\rangle_n$ and $\langle[\pi_y^n]\rangle_n$ such that $\langle[\pi_x^n]\rangle_n$ induces $[p_x]$, $\langle[\pi_y^n]\rangle_n$ induces $[p_y]$, and $\langle[\pi_x^n\pi_y^n]_{xy\in H}(\Sigma_{x'y'\in H}\pi_{x'}^n\pi_{y'}^n)^{-1}\rangle_n$ induces $[p_{xy}]_{xy\in H}$.

Theorem 6.2. *The following are equivalent for any strategies $[p_x]$ and $[p_y]$ and any belief $[p_{xy}]_{xy\in H}$. (a) There are monomial vectors $[c_x n^{e_x}]$ and $[c_y n^{e_y}]$ such that $[c_x n^{e_x}]$ induces $[p_x]$, $[c_y n^{e_y}]$ induces $[p_y]$, and $[c_x c_y n^{e_x+e_y}]_{xy\in H}$ induces $[p_{xy}]_{xy\in H}$. (a′) $[p_{xy}]_{xy\in H}$ is consistent with $[p_x]$ and $[p_y]$. (b) There is a product dispersion $[q_{xy/x'y'}]$ whose marginal distributions induce $[p_x]$ and $[p_y]$ and whose restriction to H^2 induces $[p_{xy}]_{xy\in H}$.*

[9] Here is a brief summary. Say that a nonstandard vector $[a_z]$ *expresses* $[q_{z/z'}]$ if each $q_{z/z'}$ equals ∞ whenever $a_z/a_{z'}$ is infinite, and equals the standard part of $a_z/a_{z'}$ whenever $a_z/a_{z'}$ is finite. Then, dispersionhood is equivalent to expression by a nonstandard vector $[a_z]$, and further, producthood is equivalent to expression by a nonstandard product $[a_x a_y]$. Thus, if one views nonstandard vectors solely as a means of expressing dispersions, this paper's concept of producthood is equivalent to the producthood appearing in Blume, Brandenburger, and Dekel (1991, Definition 7.1) and Hammond (1994, Subsection 6.5). See Streufert (2003, Appendix B.2) for details.

Casually, Theorem 6.2 says that (a) strategies and beliefs are induced by some $[c_x c_y n^{e_x+e_y}]$ iff (a′) they are consistent iff (b) they are induced by some product dispersion $[q_{xy/x'y'}]$. This (a) is the characterization of applied interest that was discussed in Subsection 1.2, while (b) is the characterization of theoretical interest that was introduced in Subsection 1.3. The entire theorem is a straightforward extension of Theorem 5.1, which is restated here as the equivalence of (a) and (b). Note that the equivalence of (a′) and (b) is closely related to a result in Kohlberg and Reny (1997) (see the text following Theorem 6.1).[10]

The theorem's proof occupies the remainder of this section. It begins with the following lemma, which is analogous to Lemma 5.2. It concludes with a critical one-line reference to Theorem 5.1(a⇐b), which was in turn derived from Theorem 4.1(a⇐b), the fundamental theorem in this paper.

Lemma 6.3. *Suppose that $\langle[\pi_z^n]\rangle_n$ induces $[p_z^A]$ and that the same $\langle[\pi_z^n]\rangle_n$ approximates $[q_{z/z'}]$ which induces $[p_z^B]$. Then $[p_z^A] = [p_z^B]$.*

Proof. By the definitions at (55), (53), and (15), we are given that $\langle[\pi_z^n]\rangle_n$ and $[p_z^A]$ satisfy

$$(\forall z)\ \ p_z^A = \lim_{n\to\infty}\pi_z^n\ , \tag{56}$$

that $\langle[\pi_z^n]\rangle_n$ and $[q_{z/z'}]$ satisfy

$$(\forall z, z')\ \ q_{z/z'} = \lim_{n\to\infty}\pi_z^n/\pi_{z'}^n\ , \tag{57}$$

and that $[q_{z/z'}]$, $[p_z^B]$, and z^* satisfy

$$(\forall z)\ \ p_z^B = \frac{q_{z/z^*}}{\Sigma_{z'\in Z}\ q_{z'/z^*}} \tag{58}$$

$$\text{and } (\forall z')\ q_{z'/z^*}\ <\ \infty\ . \tag{59}$$

By (58), by (57), by Rudin (1976, page 49, Theorem 3.3) with (59) and the fact that the denominator is positive because $\lim_{n\to\infty}\pi_{z^*}^n/\pi_{z^*}^n = 1$, by algebra, and by (56), we arrive at the desired equality:

$$\begin{aligned}(\forall z)\ p_z^B &= \frac{q_{z/z^*}}{\Sigma_{z'\in Z}\, q_{z'/z^*}} = \frac{\lim_{n\to\infty}\pi_z^n/\pi_{z^*}^n}{\Sigma_{z'\in Z}\lim_{n\to\infty}\pi_{z'}^n/\pi_{z^*}^n} \\ &= \lim_{n\to\infty}\frac{\pi_z^n/\pi_{z^*}^n}{\Sigma_{z'\in Z}\ \pi_{z'}^n/\pi_{z^*}^n} = \lim_{n\to\infty}\pi_z^n = p_n^A\ .\end{aligned}$$

□

[10] Also note that the equivalence of (a) and (a′) bears some resemblance to Govindan and Klumpp (2002, Theorem 2.4)'s use of polynomials to characterize perfection. Indeed, this whole paper reflects their paper's use of algebra to clarify difficult concepts. Yet the mathematics of the two papers are entirely different: this paper uses linear algebra while their paper uses algebraic topology.

Proof of Theorem 6.2. (a⇒a′) Assume (a). Define

$$\langle[\pi^n_x]\rangle_n = \langle[c_x n^{e_x}](\Sigma_{x'} c_{x'} n^{e_{x'}})^{-1}\rangle_n \text{ and}$$
$$\langle[\pi^n_y]\rangle_n = \langle[c_y n^{e_y}](\Sigma_{y'} c_{y'} n^{e_{y'}})^{-1}\rangle_n \ .$$

By the assumption that $[c_x n^{e_x}]$ induces $[p_x]$, and by the definition of $\langle[\pi^n_x]\rangle_n$, we have

$$(\forall x)\ p_x = \lim_{n\to\infty} c_x n^{e_x}(\Sigma_{x'} c_{x'} n^{e_{x'}})^{-1} = \lim_{n\to\infty} \pi^n_x \ .$$

Thus $\langle[\pi^n_x]\rangle_n$ induces $[p_x]$. Similarly, $\langle[\pi^n_y]\rangle_n$ induces $[p_y]$. Further, by the assumption that $[c_x c_y n^{e_x+e_y}]$ induces $[p_{xy}]_{xy\in H}$, by algebra, and by the definitions of $\langle[\pi^n_x]_x\rangle_n$ and $\langle[\pi^n_y]_y\rangle_n$, we have

$$(\forall xy{\in}H)\ p_{xy} = \lim_{n\to\infty} \frac{c_x c_y n^{e_x+e_y}}{\Sigma_{x''y''\in H} c_{x''} c_{y''} n^{e_{x''}+e_{y''}}}$$
$$= \lim_{n\to\infty} \frac{c_x n^{e_x}(\Sigma_{x'} c_{x'} n^{e_{x'}})^{-1} \cdot c_y n^{e_y}(\Sigma_{y'} c_{y'} n^{e_{y'}})^{-1}}{\Sigma_{x''y''\in H} c_{x''} n^{e_{x''}}(\Sigma_{x'} c_{x'} n^{e_{x'}})^{-1} \cdot c_{y''} n^{e_{y''}}(\Sigma_{y'} c_{y'} n^{e_{y'}})^{-1}}$$
$$= \lim_{n\to\infty} \frac{\pi^n_x \pi^n_y}{\Sigma_{x''y''\in H} \pi^n_{x''} \pi^n_{y''}} \ .$$

Hence $\langle[\pi^n_{xy}]_{xy\in H}\rangle_n$ induces $[p_{xy}]_{xy\in H}$. By the previous sentence and by the earlier sentences about $[p_x]$ and $[p_y]$, we have that $[p_{xy}]_{xy\in H}$ is consistent with $[p_x]$ and $[p_y]$.

(a′⇒b) Assume (a′). By the definition of consistency, there exist full-support-distribution sequences $\langle[\pi^n_x]\rangle_n$ and $\langle[\pi^n_y]\rangle_n$ such that $\langle[\pi^n_x]\rangle_n$ induces $[p_x]$, $\langle[\pi^n_y]\rangle_n$ induces $[p_y]$, and $\langle[\pi^n_x \pi^n_y]_{xy\in H}\rangle_n$ induces $[p_{xy}]_{xy\in H}$. By Theorem 6.1(a′⇒b) together with that theorem's second sentence about marginals, $\langle[\pi^n_x \pi^n_y]\rangle_n$ approximates a product dispersion $[q_{xy/x'y'}]$ whose marginals are approximated by $\langle[\pi^n_x]\rangle_n$ and $\langle[\pi^n_y]\rangle_n$. Let $[q_{x/x'}]$ and $[q_{y/y'}]$ denote these marginals. Since $\langle[\pi^n_x]_x\rangle_n$ induces $[p_x]$ (by assumption), since $\langle[\pi^n_x]\rangle_n$ approximates $[q_{x/x'}]$ (by the previous two sentences), and since $[q_{x/x'}]$ induces some distribution (by Remark 2.1), Lemma 6.3 shows that $[q_{x/x'}]$ induces $[p_x]$. A symmetric argument shows that $[q_{y/y'}]$ induces $[p_y]$. Finally, since $\langle[\pi^n_x \pi^n_y]\rangle_n$ approximates all of $[q_{xy/x'y'}]$ (by this paragraph's third sentence), it must be that $\langle[\pi^n_x \pi^n_y]_{xy\in H}\rangle_n$ approximates $[q_{xy/x'y'}]_{(xy,x'y')\in H^2}$. Since $\langle[\pi^n_x \pi^n_y]_{xy\in H}\rangle_n$ induces $[p_{xy}]_{xy\in H}$ (by assumption), since this same $\langle[\pi^n_x \pi^n_y]_{xy\in H}\rangle_n$ approximates $[q_{xy/x'y'}]_{(xy,x'y')\in H^2}$ (by the previous sentence), and since $[q_{xy/x'y'}]_{(xy,x'y')\in H^2}$ induces some distribution (by Remark 2.1), Lemma 6.3 shows that $[q_{xy/x'y'}]_{(xy,x'y')\in H^2}$ induces $[p_{xy}]_{xy\in H}$.

(a⇐b) Theorem 5.1(a⇐b). □

Appendix A. Numeraires

This appendix shows how calculations using monomial vectors can be simplified by choosing one numeraire for X and another numeraire for Y (indeed, our calculations for Xavier and Yolanda used Remark A.1 to justify (42) and Remark A.2 to justify (9)).

First consider a set Z. Recall how a monomial vector $[c_z n^{e_z}]$ induces a distribution (3) and represents a dispersion (27). Both definitions concern only the *ratios* among the monomials listed in $[c_z n^{e_z}]$. Thus both constructions are invariant to multiplying $[c_z n^{e_z}]$ by a monomial cn^e, and consequently, both are unaffected by specifying an element of Z to be the "numeraire" (that is, by choosing some $z^\star$ and maintaining that $c_{z^\star} n^{e_{z^\star}} = 1$).

Next consider a Cartesian product $X \times Y$. Intuitively, one should be able to choose one numeraire for X and another for Y. The following remarks confirm this intuition. The first remark is for Theorems 4.1 and 6.1 (which concern dispersions) while the second is for Theorems 5.1 and 6.2 (which concern distributions).

Remark A.1. *Choose any $x^\star$ and $y^\star$. A table $[q_{xy/x'y'}]$ is represented by some $[c_x c_y n^{e_x+e_y}]$ iff it is represented by some $[\xi_x \xi_y n^{\varepsilon_x+\varepsilon_y}]$ for which $\xi_{x^\star} n^{\varepsilon_{x^\star}} = 1$ and $\xi_{y^\star} n^{\varepsilon_{y^\star}} = 1$.*

Proof. The "if" part is obvious. To prove the "only if" part, take any $x^\star$ and any $y^\star$, assume that $[q_{xy/x'y'}]$ is represented by $[c_x c_y n^{e_x+e_y}]$, and define

$$[\xi_x n^{\varepsilon_x}] = c_{x^\star}^{-1} n^{-\varepsilon_{x^\star}} [c_x n^{e_x}]$$
$$[\xi_y n^{\varepsilon_y}] = c_{y^\star}^{-1} n^{-\varepsilon_{y^\star}} [c_y n^{e_y}] \ .$$

Then

$$[\xi_x \xi_y n^{\varepsilon_x+\varepsilon_y}] = (c_{x^\star} c_{y^\star})^{-1} n^{-e^{x^\star}-e^{y^\star}} [c_x c_y n^{e_x+e_y}]$$

represents $[q_{xy/x'y'}]$ because it is a monomial multiple of $[c_x c_y n^{e_x+e_y}]$ which represents $[q_{xy/x'y'}]$ by assumption. □

Remark A.2. *Choose any $x^\star$ and $y^\star$. The following are equivalent for any strategy $[p_x]$, any strategy $[p_y]$, and any belief $[p_{xy}]_{xy \in H}$. (a) There are $[c_x n^{e_x}]$ and $[c_y n^{e_y}]$ such that $[c_x n^{e_x}]$ induces $[p_x]$, $[c_y n^{e_y}]$ induces $[p_y]$, and $[c_x c_y n^{e_x+e_y}]_{xy \in H}$ induces $[p_{xy}]_{xy \in H}$. (a*) There are $[\xi_x n^{\varepsilon_x}]$ and $[\xi_y n^{\varepsilon_y}]$ such that $[\xi_x n^{\varepsilon_x}]$ induces $[p_x]$, $[\xi_y n^{\varepsilon_y}]$ induces $[p_y]$, $[\xi_x \xi_y n^{\varepsilon_x+\varepsilon_y}]_{xy \in H}$ induces $[p_{xy}]_{xy \in H}$, $\xi_{x^\star} n^{\varepsilon_{x^\star}} = 1$, and $\xi_{y^\star} n^{\varepsilon_{y^\star}} = 1$.*

Proof. It is obvious that (a)⇐(a*). To show that (a)⇒(a*), take any $x^\star$ and any $y^\star$, and assume that $[c_x n^{e_x}]$ and $[c_y n^{e_y}]$ are such that $[c_x n^{e_x}]$ induces $[p_x]$, $[c_y n^{e_y}]$ induces $[p_y]$, and $[c_x c_y n^{e_x+e_y}]_{xy \in H}$ induces

$[p_{xy}]_{xy \in H}$. Then define

$$[\xi_x n^{\varepsilon_x}] = c_{x^\star}^{-1} n^{-\varepsilon_{x^\star}} [c_x n^{e_x}] \text{ and}$$
$$[\xi_y n^{\varepsilon_y}] = c_{y^\star}^{-1} n^{-\varepsilon_{y^\star}} [c_y n^{e_y}] \ .$$

$[\xi_x n^{\varepsilon_x}]$ induces $[p_x]$ because it is a monomial multiple of $[c_x n^{e_x}]$ which induces $[p_x]$ by assumption; $[\xi_y n^{\varepsilon_y}]$ induces $[p_y]$ because it is a monomial multiple of $[c_y n^{e_y}]$ which induces $[p_y]$ by assumption; and

$$[\xi_x \xi_y n^{\varepsilon_x + \varepsilon_y}]_{xy \in H} = (c_{x^\star} c_{y^\star})^{-1} n^{-\varepsilon^{x^\star} - \varepsilon^{y^\star}} [c_x c_y n^{e_x + e_y}]_{xy \in H}$$

induces $[p_{xy}]_{xy \in H}$ because it is a monomial multiple of $[c_x c_y n^{e_x + e_y}]_{xy \in H}$ which induces $[p_{xy}]_{xy \in H}$ by assumption. □

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