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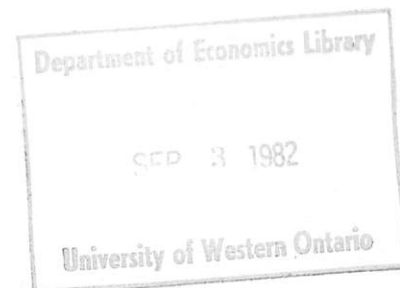# THE PREFERENCE FOUNDATIONS OF
SPATIAL MODELS OF BRAND CREATION

by

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and

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I.   Introduction

In his path-breaking work on firm location theory, "Stability in Competition," Harold Hotelling [7] suggested that the firm location model developed therein could be applied quite easily to the firm's choice of product variety as well. He states that:

> Distance, as we have used it for illustration, is only
> a figurative term for a great congeries of qualities.
> Instead of sellers of an identical commodity separated
> geographically we might have considered two competing
> cider merchants side by side, one selling a sweeter
> liquid than the other... . The measure of sourness now
> replaces distance, while instead of transportation costs
> there are the degrees of disutility resulting from a
> consumer getting cider more or less different from what
> he wants. (p. 54)

In recent years a number of authors have followed Hotelling's suggestion and adopted this modified spatial location approach when modelling markets in which product specification is a key firm decision variable. Schmalensee [18], for example, utilizes the notion of a circular product-location space to describe the process of brand proliferation in the ready-to-eat cereals market. Prescott and Visscher [14] and Eaton and Lipsey [4] employ straight line product spaces to analyze firm entry deterrence strategies based on choice of product specification (or design). Gabszewicz and Thisse [5] use the straight line product space to model the process of innovation in product specification (design). In all of these authors' works (and in others' work as well) the justification remains essentially that of Hotelling's; namely, that firm location is simply a special case of a more general process of product differentiation and therefore the spatial location model should be equally useful in analyzing these more general issues of product variety.

This product differentiation interpretation is not only intuitively appealing, but more importantly, it allows for the application of a set

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of models which is relatively tractable. However, the transition from
the simple location model to the general differentiated product model
gives rise to certain issues with which the economics profession has
heretofore not dealt. Moreover, the continued failure to do so takes
on ever increasing significance as these spatial models become not
only a more widely employed theoretical tool, but a tool of policy
analysis as well.[1] Put very simply, the source of the difficulty is
the fact that the spatial model assumption that each consumer can rank
alternative commodities in the product set according to their "nearness"
to his most preferred variety implies that any specification of the
structure of the set of goods-cum-locations must impose restrictions
on the structure of consumer preferences over the set. The broad
purpose of this paper is to deal with this problem through the
construction of a framework within which these restrictions can be
made explicit. More specifically, in this paper we will derive the
precise restrictions on consumer preferences which are equivalent to
the two most frequently employed location models: the line segment and
the circle. This will have the effect of not only making clear the
assumptions which underlie these two models, but it also will allow us
to shed some light on such related issues as: the identification of
the appropriate form for the product space in any market; the
appropriate markets to which location models may be applied; and the
nature of policy recommendations which can be made in any particular
differentiated product market.

Our analysis proceeds within the framework of the Lancaster model
in which consumer preferences are defined over bundles of characteristics
and there exists some technology which defines the way in which characteristics
are bundled into differentiated products.[2] We then ask what restrictions

must be placed on preferences and technology in order for this model
to be representable, in a well-defined sense, as a linear or circular
location model. This analysis is presented in Sections II and III and
is in many ways analogous to Rosen's [15] development of a mechanism by
which hedonic price models could be generated. Section IV presents an
illustration of the nature of the restrictions on preferences imposed
by the linear and circular location models in the context of a two
characteristic model. Section V discusses the implications of our
analysis for the application of spatial models. It focuses on
the way in which our analysis can be used to determine the shape of
the product space and the role that this plays in the analysis of
equilibrium and the question of optimal product variety. This section
also provides a short example of the way in which our analysis could
be applied to policy issues by reexamining the FTC's case against the
ready-to-eat cereal manufacturers. Concluding remarks are then contained
in Section VI.

II.   Analytic Framework and Preliminary Results

Implicit in any spatial model of product differentiation is the
assumption that a consumer's ranking of any particular point must be
related to the bundle of characteristics, be it actual location or
something less concrete, which the good at that point embodies. One
might reasonably conclude from this that the shape into which these
points (goods) are arranged (i.e., a circle, line, etc.) and the metric
which is used to define distance, as measured by utility loss, between
points should have implications not only for consumer preferences over
goods, but over characteristics as well (and conversely). The aim of

the present analysis is to define quite specifically what these latter implications are.

To accomplish this, we take as given the existence of a set, Z, of potentially producible characteristics bundles and a set of consumers, each of whom have preferences defined over Z. The exact nature of the set Z is given by the following:

Assumption 1. There exists a set $Z \subset \mathfrak{R}^n$, denoted as the characteristics possibility set, which is a compact, connected, and separable topological space. It is such that every point $z = (z_1, z_2, \ldots, z_n) \epsilon Z$, representing a bundle of characteristics, corresponds to some possible good.

This assumption corresponds to the assumed existence of what Lancaster calls alternatively a product differentiation curve [10] or the product spectrum [9].[3] In what follows, it will be assumed that each $z \epsilon Z$ corresponds to a bundle of characteristics available in one unit of the good.[4]

Consumers in this model, as in both the location and Lancasterian models, are indexed by that good in Z which they most prefer. For ease of exposition, we define a set θ which will serve as an index set both for elements of Z and for consumers. That is, with each $\alpha \epsilon \theta$ there is associated both an element $z^\alpha \epsilon Z$ and a consumer α for whom z^α is most preferred in Z.[5] This association is formalized in our second assumption.

Assumption 2. Associated with each $z^\alpha \epsilon Z$ is a consumer possessing a continuous weak ordering R_α over Z with the property:

for all $z \epsilon Z \setminus \{z^\alpha\}$, $z^\alpha P_\alpha z$.[6]

For simplicity, we will refer to the set of preference orderings over Z as $R(Z) = \{R_\alpha | z^\alpha \epsilon Z\} = \{R_\alpha | \alpha \epsilon \theta\}$. Any pair $(Z, R(Z))$ which

satisfies Assumptions 1 and 2 will be termed a <u>preference structure</u>.
In these terms, then, the issue under investigation is the delineation
of the conditions under which a preference structure can be represented
as a certain type of location model. Put more simply, the issue
with which we seek to deal is the conditions under which a Lancaster-
type model can be represented as either a circular or linear location
model. To address this problem, we require a formalization of the
notion of a location-model representation of our model. This is given
by the following definition:

<u>Definition</u>: The set $X \subset \Re^2$ is a <u>location-model representation</u> of

the preference structure $(Z, R(Z))$ if and only if there exists

a continuous, one-to-one function $f: Z \to \Re^2$ such that:

    (i) $f(z) = X$ and

    (ii) for all $\alpha \in \theta$ and all $z, \bar{z} \in Z$:

$$zR_\alpha \bar{z} \Leftrightarrow d[f(z^\alpha), f(z)] \leqq d[f(z^\alpha), f(\bar{z})],$$

where $d: \Re^2 \to \Re$ is the euclidean distance function.

The problem of the representability of a preference structure
in the above sense can be divided into two parts. The first concerns
the restrictions on the structure which are necessary or sufficient
for such a structure to exist. The second takes the existence of a
representation as given, and deals with conditions on the structure
which are necessary or sufficient for the representation to have a
specific form. Our analysis will focus primarily on the second question
in the particular cases in which the representation, X, is a line
segment or a circle. This is because this is the issue which has
always confronted authors utilizing spatial models to analyze product
differentiation problems but has not been addressed. In addition, it
is the one which involves most of the economically interesting results

in this area.

As for the first issue, while the derivation of sufficient conditions for the representability of a structure is quite beyond the scope of this paper, certain of the necessary conditions are of some interest, and we shall comment briefly on those. First, and most importantly, it is not the case that the results of the analysis to follow hold only vacuously. That is, it is not true that no $(Z,R(Z))$ exists having a linear or circular location-model representation, as at least two examples of each can be given. One is the case in which Z is a subset of a two-dimensional characteristic space, and is presented in Section IV below. Another is the case in which $f: X \to X$ is the identity map. In this case, our results are merely a delineation of the conditions on "preferences over locations" which are equivalent to the two location models.

Next, (ii) of the definition of a representation implies that for $(Z,R(Z))$ to be representable, each R_α must have a continuous, real-valued utility representation, $U_\alpha = -d[f(z), f(z^\alpha)]$. The representability of R_α is guaranteed, however, by our assumptions that each R_α is continuous on the connected and separable topological space, Z. Further, our assumption that Z is compact implies that there is an R_α-maximal (and minimal) element of Z for each α. Thus, the force of Assumption 2 is only to require the uniqueness of the maximal element.

Finally, a rather important result from the points of view of both the interpretation and application of spatial models is that if $(Z,R(Z))$ is representable, then Z and X must be of the same dimension.[7] This implies that if X is either a line segment or a circle, then while the number of characteristics in Z may be as large as one likes, at most

one characteristic in Z can be varied independently of all the others.
For all practical purposes, then, Z must be described by Lancaster's
[10] product differentiation curve.

This result has some interesting implications, particularly
as regards the sorts of markets to which one-dimensional spatial models
may be applied. It might make one skeptical, for instance, of the use
of these models in markets characterized by a high degree of product
heterogeneity.[8] In these instances, it is unlikely that products vary
in essentially only one characteristic dimension, and therefore, the
one-dimensional spatial model, strictly speaking, is not an accurate
characterization of this market. Moreover, given that the equilibria
in spatial models are not robust with respect to changes in dimensions,
the one-dimensional model is not even arguably a reasonable approximation
to some higher dimensional spatial model. On the other hand, one might
consider the one-dimensional spatial model a useful analytic tool in
those instances in which the market is characterized by a high degree
of product homogeneity. In particular these models would be appropriate
in those instances in which either the good possessed essentially a
single characteristic, or if possessing more than one characteristic,
each variety possessed the characteristics in very similar ratios (but
not in the same absolute amounts).[9] Interestingly enough, these latter
cases are generally markets in which products are distinguished by
their perceived quality or reliability and are markets in which trademarks
and brand names play a key role.[10]

III. Characterization of the Linear and Circular Representations

We now turn our attention to the second aspect of the representability question. That is, given that a structure is representable, what further conditions on it are equivalent to the representation, X, being either linear or circular? In both cases, it turns out that two conditions characterize the representation in question. The first of the conditions characterizing the linearly representable structure is:

Condition L.1. For any distinct $z, \bar{z} \in Z$, there exists a unique $a \in \theta$

such that $z \ I_a \ \bar{z}$.

Since any representation, X, must be connected--being the image of the connected set Z--(ii) of the definition of representability and Assumption 2 imply that there is always some consumer indifferent between any pair $z, \bar{z}$. L.1 imposes the further restriction that this consumer be unique. It therefore serves (as Lemmas A.1-A.4 of Appendix A make clear) to restrict the admissible representations to a subset of the class of simple arcs.[11]

The second condition characterizing the linearly representable structure is:

Condition L.2. For any a, b, c, d, e $\in \theta$ such that $z^a I_b z^c$, $z^a I_d z^b$

and $z^c I_e z^b$, it must be that $z^d I_b z^e$.

Although this condition has little intuitive appeal, its role is simply that of increasing the tractability of the linearly representable structure. It guarantees that the representations have the property that the fraction of consumers contained in any firm's market depends only on the (euclidean) length of that market in the representation. Thus all of the information relevant to the firm's product choice decision is captured by the single statistic, market length.[12]

It is clear that a linearly representable structure satisfies L.1 and L.2 but so would any structure representable by the arc of a circle up to a semi-circle. In fact, these two representations are completely equivalent, as stated in our first result.

Theorem 1. If $X \subset \Re^2$ is a location model representation of a preference structure $(Z, R(Z))$, then X is either a line segment or the arc of a circle (up to a semi-circle) if and only if $(Z, R(Z))$ satisfies L.1 and L.2.

Proof. See Appendix A.

There are also two conditions which characterize a structure with a circular representation. Contrasting them with the two conditions above proves to be an illuminating way of distinguishing between the two location models. The first condition is:

Condition C.1. For every $a \in \theta$, there exists a unique $b \in \theta$ such that:

$$\text{for all } z, \bar{z} \in Z: \quad zR_a\bar{z} \Leftrightarrow \bar{z}R_bz . \qquad (*)$$

As noted before, if a structure is representable there must be some consumer who is indifferent between any pair $z, \bar{z} \in Z$. In contrast to L.1, however, C.1 implies that there are always two such consumers. Further, it requires that this pair of consumers' preferences be opposites in the sense given by $(*)$ above.

As a matter of notational convenience, if $a, b \in \theta$ are such that R_a, R_b are opposites in the sense of $(*)$, we will write $b = \psi(a)$ (and $a = \psi(b)$). Given this, our second condition is as follows:

Condition C.2. For any $a, b, c, \in \theta$, if $\psi(a) = d$, $\psi(c) = e$ and $z^a I_b z^c$ then $z^d I_b z^e$.

This condition is analogous to L.2 in that it guarantees that the

representation of the structure $(Z,R(Z))$ will have the above-mentioned market-length property.

The second characterization can now be given.

Theorem 2. If $X \subset \Re^2$ is a location model representation of the preference structure $(Z,R(Z))$, then X is a circle if and only if $(Z,R(Z))$ satisfies C.1 and C.2.

Proof. See Appendix B.

This result and Theorem 1 above provide complete characterizations of the assumptions concerning consumers which are typically aggregated into a statement that consumers are distributed over a line or circle. As such, they make clear the fact that it is nothing more or less than variations in assumption on the distribution of consumers (and their preferences) which distinguish the two models. More specifically, Conditions L.2 and C.2 capture the extent to which the two structures are similar, in some sense. This similarity is evidenced by the fact that arcs of circles represent the same preference structures as do line segments. Conditions L.1 and C.1, on the other hand, define the difference between the two structures. This difference hinges on the existence or nonexistence of opposite consumers in the sense of (*). The significance of this will become apparent in what follows.

IV. The Difference Between the Linear and Circular Representations - An Example

One can obtain a better feeling for the nature of the restrictions imposed on the preference structure $(Z,R(Z))$ in the above two sections by considering two simple Lancasterian characteristic models which give rise respectively to the linear and circular representations. The purpose of this present section is to pursue such a consideration. In keeping with

the restrictions imposed by the necessary conditions for the existence

of a representation (Section II), we assume that the differentiated

product embodies two characteristics z_1 and z_2; and that the set Z

(Assumption 1) can be described by a continuous, monotone decreasing

and concave function of the form:

$$z_2 = \zeta(z_1) \ . \tag{1}$$

In addition, we assume that preferences (Assumption 2) are continuous

and monotonic on $\Re^2$, and that the set θ can be represented by a

bounded interval in $\Re$.

For the case of the linear representation, if one assumes that

each R_a is strictly convex and homothetic; and assumes that the slope

of an indifference curve for R_a along the ray determined by the origin

and the point $z^a \in Z$ is equal to ζ' at z^a, one obtains the result that

each consumer a's most preferred point in Z is z^a (Assumption 2). If,

in addition, one assumes that an indifference curve for exactly one

R_a intersects the graph of Z at each pair of points $z, \bar{z}$, then Condition L.1

is fulfilled as well (see Figure 1). The restrictions needed to satisfy

Condition L.2 are rather more complicated, but are illustrated in

Figure 2 in which, since $z^c I_a z^b$ and both $z^a I_d z^b$ and $z^a I_e z^c$, it must be

that $z^e I_a z^d$.

The illustration of the conditions for a circular X, not

surprisingly is more complex. Since θ in this case must be half

open, the set Z must exclude one of the points $z^\alpha = (0, \zeta(0))$ or $z^\omega = (\zeta^{-1}(0), 0)$

(in what follows we assume z^ω is excluded). Further, Condition C.2 implies

that the set θ can be partitioned into two disjoint sets such that

each consumer in one set possesses an opposite consumer, in the sense of

(*), in the other set. This implies that if z^β is defined by the condition

that $\psi(\alpha) = \beta$, then, by letting γ and δ be such that $z^\alpha I_\gamma z^\beta$ and

$\psi(\gamma) = \delta$, one can define a partition on the remainder of θ based

upon consumers' relative preferences for z^γ or z^δ; and this

partition will have the property that if a is in one set $\psi(a)$ must

be in the other. Thus, one also can partition the set of indifference

curves for $R(\theta') = \{R_a | a \in \theta \setminus \{\alpha, \beta\}\}$ into two sets representing the

preferences of the γ-preferring and δ-preferring consumers respectively;

and do so such that the indifference curves for opposite consumers of

each set "match-up" in the appropriate fashion.

   If this partition is to form a circular X, then $f(Z)$ must be

such that x^α and x^β form the endpoints of a diameter of X while x^γ and

x^δ form the endpoints of the diameter at right angles. Further, the

preference orderings in the partition must be such that as a (the index

of consumers) moves continuously from α through $\theta = [\alpha, \omega [$, $f(Z)$ moves

continuously around the circle from x^α to x^γ to x^β to x^δ and back to x^α.

They also must be such that as an individual consumer, a, moves

continuously through z from z^a to $z^{\psi(a)}$ utility decreases continuously,

while it increases continuously as a moves through the remainder of Z

from $z^{\psi(a)}$ to z^a. The sets of indifference curves for a representative

consumer γ and his opposite $\psi(\gamma)$ which generate this sort of ordering are

given in Figures 3 and 4 below.

   If it is assumed that the indifference curves of R_a are homothetic,

then the indifference curves for the other members of the γ-preferring

group can be envisioned as being obtained through a continuous distortion

of γ's indifference curves. This distortion involves one's rotating the

ray determined by the origin and the intersection with the graph of Z of

the indifference curve containing z^α and z^ω continuously through the

positive orthant. A similar construction can be used to obtain the indifference curves of other members of the z^{δ}-preferring group, one needing only to be careful that the indifference curves of each member of the group intersect with the graph of Z at the same points as the indifference curves of their respective opposites in the z^{γ}-preferring group. This procedure also generates the indifference curves for the consumers α and β (the consumers indifferent between z^{γ} and z^{δ}). These indifference curves are illustrated in Figure 5, and have the property that the intersection with the graph of Z of the indifference curve containing z^{α} and z^{ω} occurs at z^{α}, so that the indifference curves for α and β are everywhere concave and convex respectively. Finally, Figure 6 illustrates the nature of the restrictions imposed on an arbitrary consumer a's indifference curves (and implicitly, those of b and c) by Condition C.2.

While the above preference orderings are admittedly rather unorothodox, it is nevertheless true that one can provide a rather intuitive interpretation of them. In particular, the preference orderings over Z in the case of a circular product space and two characteristics can be interpreted as being such that each consumer other than α and β has a minimum characteristics level requirement over the set Z. Thus, consumer γ, for example, has a preference ordering that is such that the level of characteristic z_2 which any good he consumes embodies must be at least as great as z_2^{β} (see Figure 3) in order for him to be at least as well off as in the situation in which he consumes a good containing no z_2 at all. A similar property holds for all members of the γ-preferring group, with the only difference being in the value of the minimum level of z_2 required (being different for each member of the group). The members of the δ-preferring group likewise have a minimum characteristic level requirement, but over z_1.[13] The two

consumers not included in either group, α and β, possess preferences that are such that α always prefers goods containing all of one or all of the other characteristic while β always prefers goods containing a mixture of the two characteristics.

V.   Some Implications

1. General Results

A brief survey of the literature on product-location models makes clear the wide variety of results to be found there. Much of this variety is due to either differences in assumed firm behavior or to assumptions of elastic vs. inelastic demand. However, one source of variation in results is clearly the form of the spatial model used, and to date there has been offered no satisfactory economic explanation for this. Most often, differences in results between a circular, as opposed to a linear, spatial model are attributed to the existence or nonexistence of endpoints of the product space; and, in fact, this is the technical cause. However, it is not a very useful explanation for what is an important phenomenon, nor does it provide any guidance as to what the appropriate model might be for any given problem. Indeed, one would be hard-pressed to provide any information concerning whether or not the product space in some particular market possessed endpoints and where exactly these endpoints occur.

On the other hand, our results make it clear that the existence or nonexistence of endpoints in the product space stems from alternative specifications of the distribution of preferences over the set of products. That is, one can take a $(Z, R(Z))$ having a linear representation and alter $R(Z)$ in an obvious way to obtain a structure with a circular representation, and vice versa. Thus, we have provided an explanation for the difference in the two models in terms of one of the standard primitives of economic theory--preference orderings. This has the obvious value of providing an

interpretable economic criterion for the determination of the appropriate product space for any particular market. However, it also allows us to demonstrate the importance of one's choosing the correct product space by allowing us to show the extent to which results concerning the nature of equilibrium and the optimal level of product variety within these models depend upon the specification of the preference structure. This is done below.

Regardless of the assumptions made on producer behavior or the elasticity of demand, there seems to be one persistent difference in equilibria between the linear and circular models. That is, for the circular model, the equilibria are (or can be) symmetric, while for the linear model they are not (and cannot be). In fact, there is a tendency for firms in a linearly representable market to produce goods near the center of the market. We would maintain that these differences are due to the fact that in the circular case every consumer has an opposite, while on the line the set of consumers with opposites has measure zero (i.e. the endpoint consumers only). We can illustrate this by considering the simplest location model, in which each consumer purchases one unit and price is identical and exogenous for each firm.[14]

With only two firms in this model _any_ pair of product choices for the circular case results in a Nash equilibrium. The reason is straightforward. Consider any pair of firm product choices, z^1, z^2. If firm 1 instead were to choose $\bar{z}^1$ this earns him a _new_ customer $a \in \theta$ iff $\bar{z}^1 P_a z^2$ and $z^2 P_a z^1$. By C.1, however, there always exists $b = \Psi(a)$ for which $z^2 P_b \bar{z}^1$ and $z^1 P_b z^2$, so that b is lost as a customer by firm 1. No change can gain consumers (and thus

profits) for any firm. A similar result holds in the n-firm symmetric equilibrium on the circle. Here, however, while some consumers will switch to a firm which varies its product, there exists an equal number of consumers (not in the firm's market) for whom the new product is closer to their most preferred ones and who have opposites which the firm loses as customers.

On the line, Hotelling's minimum differentiation result holds, of course. This follows from the nonexistence of opposites. If z^1 and z^2 are the original, nonequilibrium product choices, a change to z^{-1} earns a new consumer $a \in \theta$ for firm 1 iff $z^{-1} P_a z^2$ and $z^2 P_a z^1$. Since no such consumer a exists who has an opposite, firm 1 can always gain customers without losing any by some "small" variation in its product. This is true so long as the new product z^{-1} is such that the consumer who most prefers z^2 prefers it to z^1. Thus, the only exception would be the case in which z^1 and z^2 are essentially identical. The minimum differentiation result thus follows immediately. With arbitrary numbers of firms greater than three, a similar result holds locally (i.e. at the market boundary) in that two firms will bunch together at the periphery of the market.

The above analysis makes it tempting to conclude that if some set of consumers having a positive measure, but less than the entire set, were to have opposites, one would obtain results which are intermediate in some sense. Surprisingly this is not the case. We have shown in Theorem 1 that a semi-circle and a line represent identical structures. If one were to start then with a semi-circle and add consumers and goods to complete the circle, this would define a path in the space of models leading from the

linear to the circular such that the set of consumers having opposites goes from none to all. In doing this, however, the sets of equilibria which one obtains from all of these intermediate models are identical to those obtained on the line. The only exception to this is the limiting model, the circle itself. Thus the important characteristic of the circular model (i.e. the one which distinguishes it from the linear model) is the fact that every consumer has an opposite. In this sense, it is a particularly special model.

This indication of the rather special nature of the circularly representable structure is reinforced when one considers the issue of optimal product differentiation in the two cases. For, consider the above simple models once again, with the added feature that each producing firm incurs a fixed cost F > 0 and zero variable costs.[15] In an analysis of the optimal level of product variety, an obvious welfare measure for this model would be the sum of fixed costs plus the utility losses due to consumers not being able to obtain their most preferred good in Z. One can then ask whether, in either market, this indicator increases or decreases in equilibrium with the addition of another firm (product).

Since in both markets the equilibria are not unique, for consistency we will compute the welfare measure at the welfare-maximizing (cost-minimizing) equilibria for each. Clearly, the answer to the question of the optimal level of product variety will depend on parameter values. However, the surprising fact is that for any initial number of firms N ≥ 4 and population density K > 0, if K and F satisfy

$$K|4N(N+1) < F < K|4(N-2)(N-1)$$

the addition of another firm will reduce welfare in the linear case and

increase it for the circular market. Further, there is no set of parameter values such that an additional firm has the reverse effect in each market.[16] The explanation for this lies in the nature of the equilibria in the two models. With the circular product space and a given N, the welfare-maximizing equilibrium configuration is also welfare maximizing over all possible con- figurations. Thus the addition of a new firm (product) improves welfare only in that it reduces the consumers' average utility loss. With the linear product space, the welfare-maximizing equilibrium configuration is not welfare-maximizing over all possible configurations. Thus the addition of a new firm (product) involves not only the utility loss reduction property of the circular product space, but also a movement to an equilibrium closer to the overall welfare-maximizing configuration. This additional welfare gain, then, produces the above results. Thus, it is ultimately because of the difference in preference structures (and the differences in equilibria which they produce) that allows cases in which an increase in variety can have an opposite effect on welfare in the two models.

2. Specific Results - The RTE Cereals Case

Perhaps the best known instance of the use of a spatial model as a tool for policy analysis is that by Schmalensee [18] in the U.S. Federal Trade Commission's suit against the ready-to-eat cereal manufacturers. In this case, the FTC argued that the three major U.S. cereal producers sought to reduce competition through the proliferation of different brands of cereal. Schmalensee argued this case in the context of a circular product market in which existing producers sought to "fill the product space" through

the introduction of new cereal brands so as to reduce competition from new firms. Our analysis allows us to make a number of comments concerning the interpretation of Schmalensee's results.

i) Schmalensee, in discussing his assumption of localized rivalry argues that cereals possess "at least four different attributes relevant to consumers" (p. 309). Our analysis indicates that unless only one of these attributes varies independently of the other three, which seems unlikely, there exists no circular location model representation of this market. Moreover, as was mentioned earlier, since equilibria vary so wildly (and often do not even exist) as the dimension of the product space changes, it is questionable whether this model can even reasonably be claimed as an approximation to a less tractable, higher dimensional problem. Therefore, given the model specific nature of the results which Schmalensee obtains, one must exercise some caution in interpreting them or employing them for the purpose of making policy prescriptions.

ii) Even if one is willing to argue that the only relevant characteristic in the market is brand name, so that the product space is essentially one dimensional, our analysis also makes clear the rather special nature of the circular product space. In particular, in order for Schmalensee to obtain his results he must assume that _every_ consumer in the market has an opposite in the sense of (*) in Section III (or Figures 3 and 4 in Section IV). Anything less than this, while not generating a linear product space, generates equilibria which are identical to the equilibrium for

a linear product space. Thus, while seeming to be rather innocuous, particularly given the difficulty one would have in defining an endpoint of a product space, Schmalensee's assumption of a circular product space is, in fact, quite strong; and his results, in this sense, are rather special.

iii) Schmalensee does conclude that the amount of product variety resulting from firm brand proliferation may have exceeded the socially optimal amount. Our analysis above demonstrates that, due to the fact that these sorts of results depend upon the equilibrium configuration which, in turn, depends upon the shape of the product space, Schmalensee might have arrived at a different conclusion had he employed a linear product space. In particular, it is possible that with a linear product space firm brand proliferation would have unequivocally resulted in too little (less than socially optimal) product variety.[17]

We cannot conclude this section without making some qualifications to the above comments. First, it would be wrong for anyone to conclude from our analysis that Schmalensee's results are necessarily in error. Our intention is only to point out that, given the difficulties one faces in constructing a compelling case for the application of a circular location model to the cereal market, one should proceed cautiously when attempting to formulate policy measures based upon an analysis which utilizes this model. Neither, however, should it be concluded from this that spatial models are an inappropriate tool for policy analysis. As was mentioned in

Section II above, one can conceive of a number of markets, many of which have brand names and trademarks as key elements, to which these models could be reasonably applied. In some of these instances[18] our analysis actually makes the application of these models more straightforward, in that it provides guidelines, based on standard economic concepts, for the determination of the proper shape of the product space. One can proceed on the basis of assumptions concerning consumer preferences rather than on some nebulous notion about the existence or nonexistence of product space endpoints.

VI.   Concluding Comments

In this paper we have sought to provide a preference foundation for the two most frequently employed spatial models of product differentiation: the line segment and the circle. In doing so, we have been able to define precisely what it means for a product space to have endpoints and the important role that this assumption plays in the analysis of equilibrium and issues of optimal product variety. We have also shown a way in which our analysis could be usefully brought to bear on policy questions, particularly those dealing with markets in which brand names and trademarks play a key role.

It might be argued that our analysis is very special in one respect in that it requires an assumption of completely inelastic demand. This, however, is not the case. For, since the inelastic demand model is simply a special case of a more general model of consumer behavior, the important characteristics of the inelastic demand analysis (i.e. Conditions L.1-L.2 and C.1-C.2) must be characteristics of the general model as well. Research which we are currently undertaking has established this point; and we are now considering a method by which elastic demand behavior can be sensibly incorporated into a spatial model.
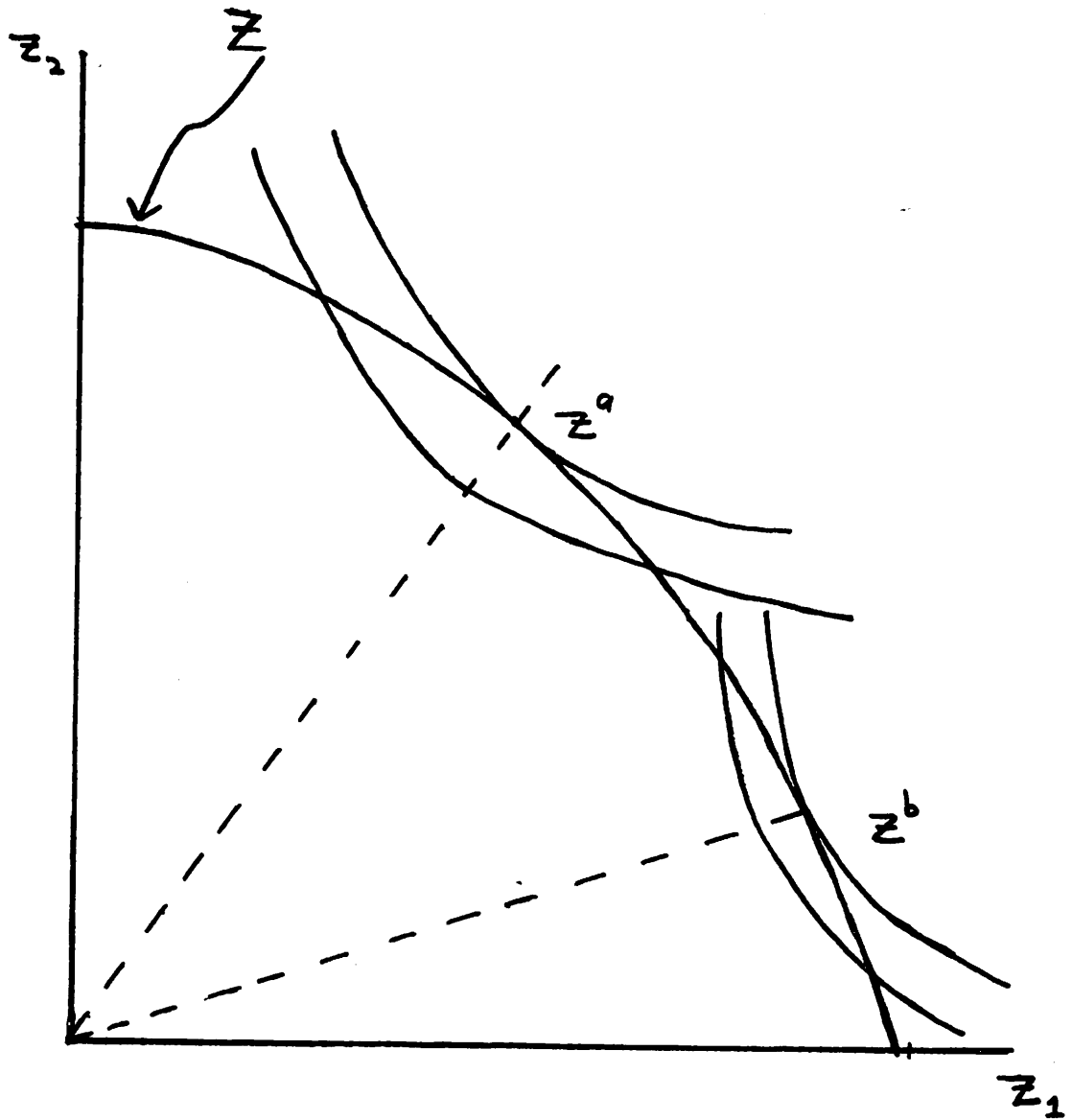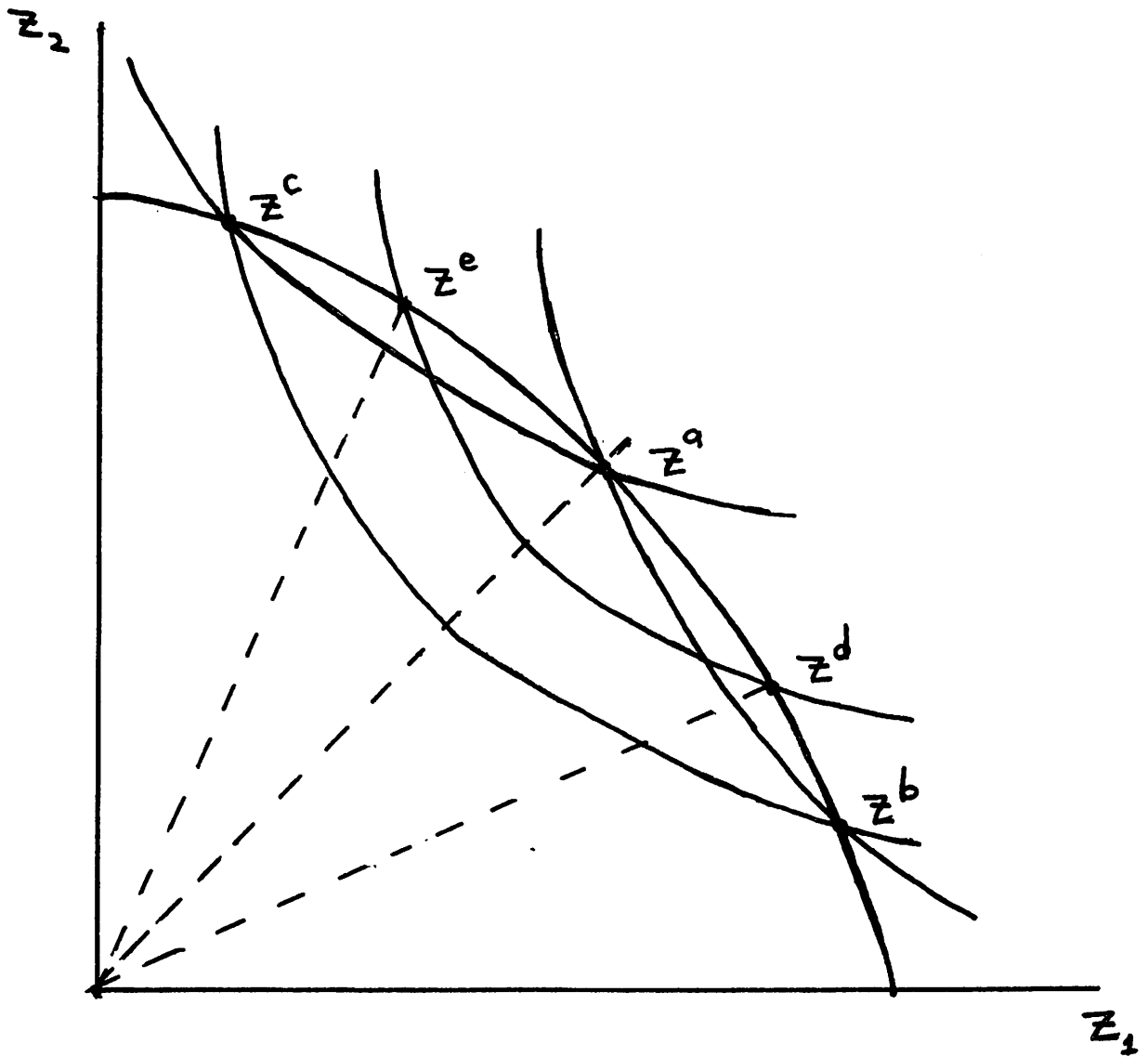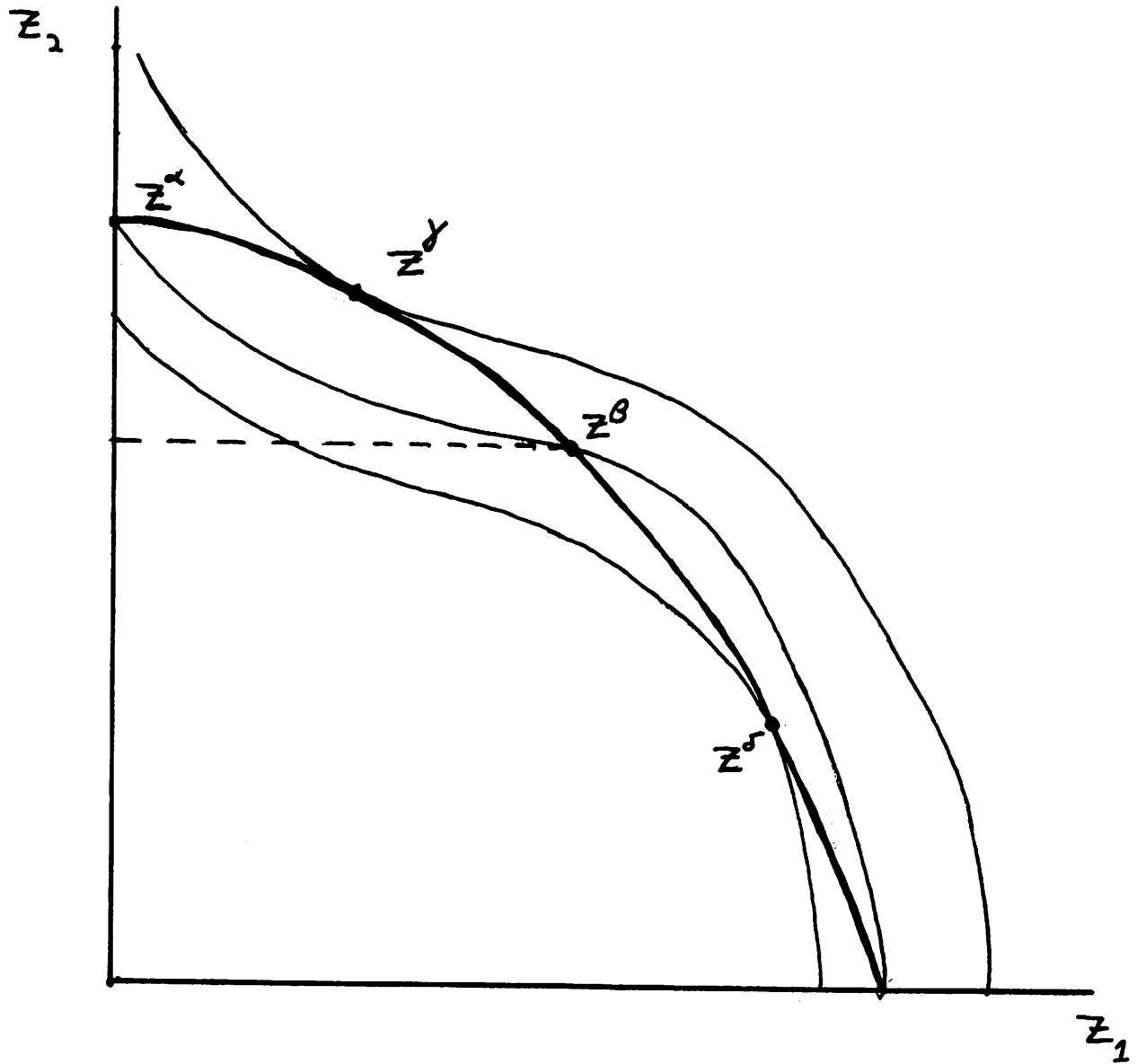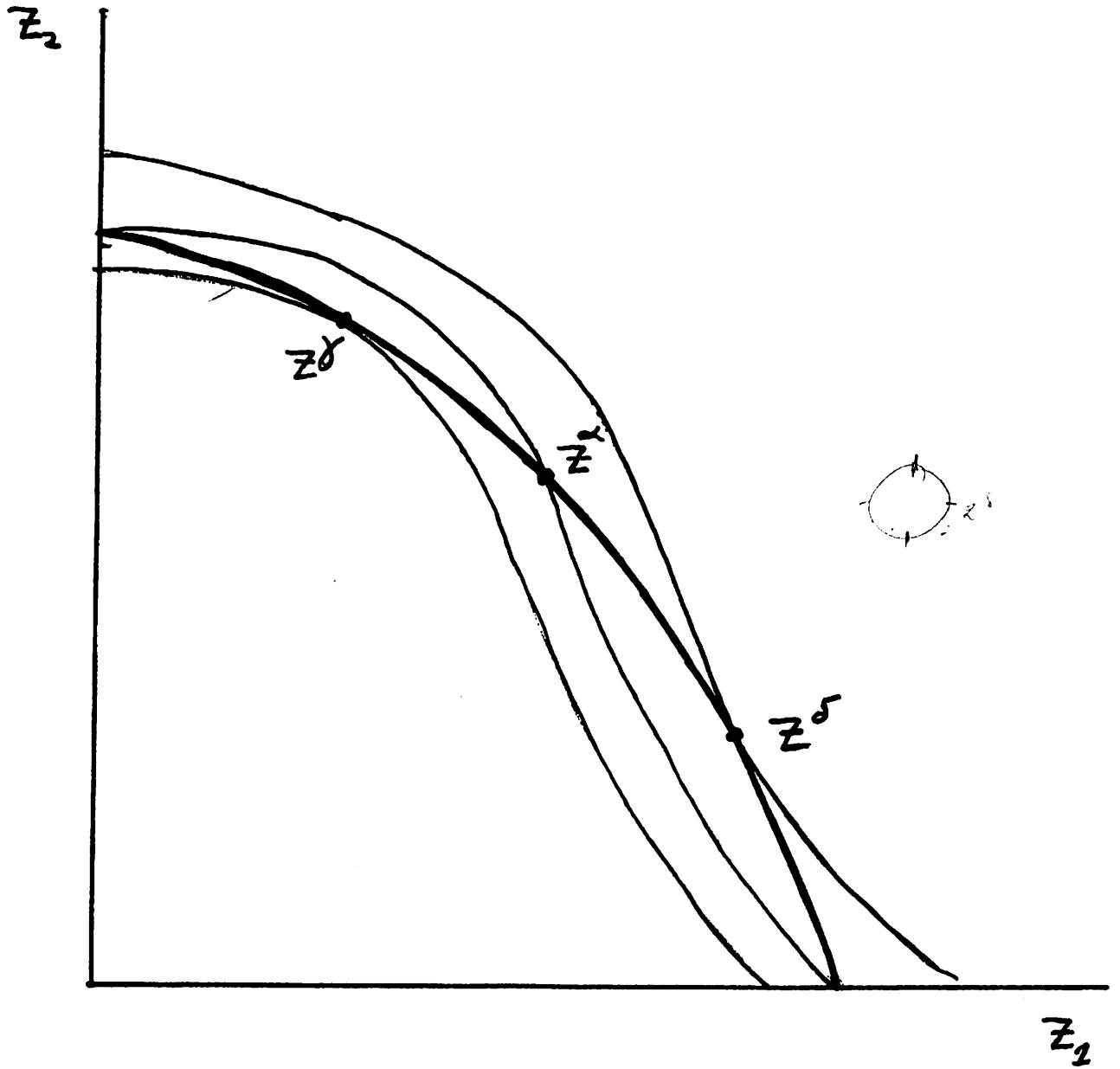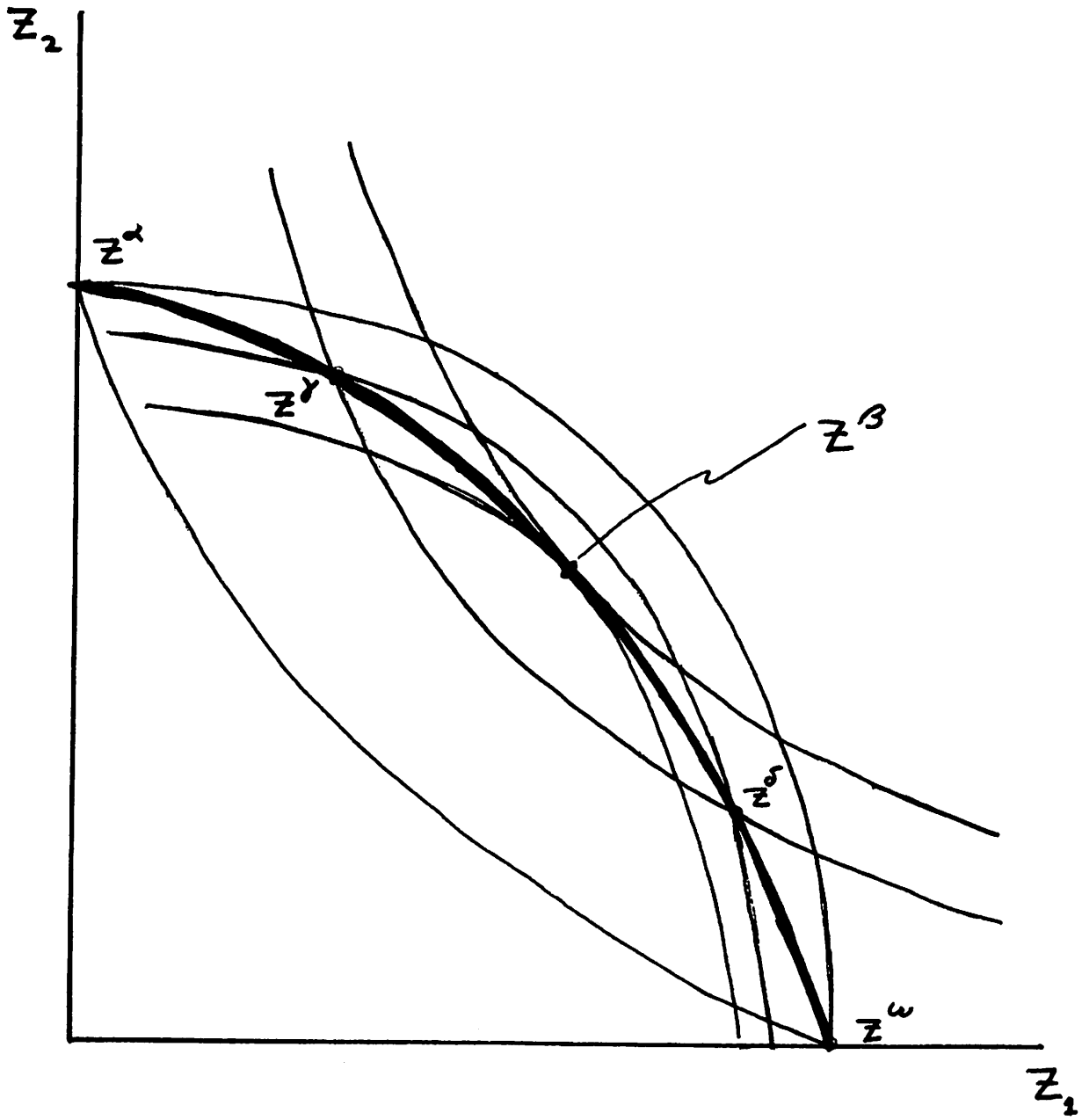
FIGURE 1

FIGURE 2

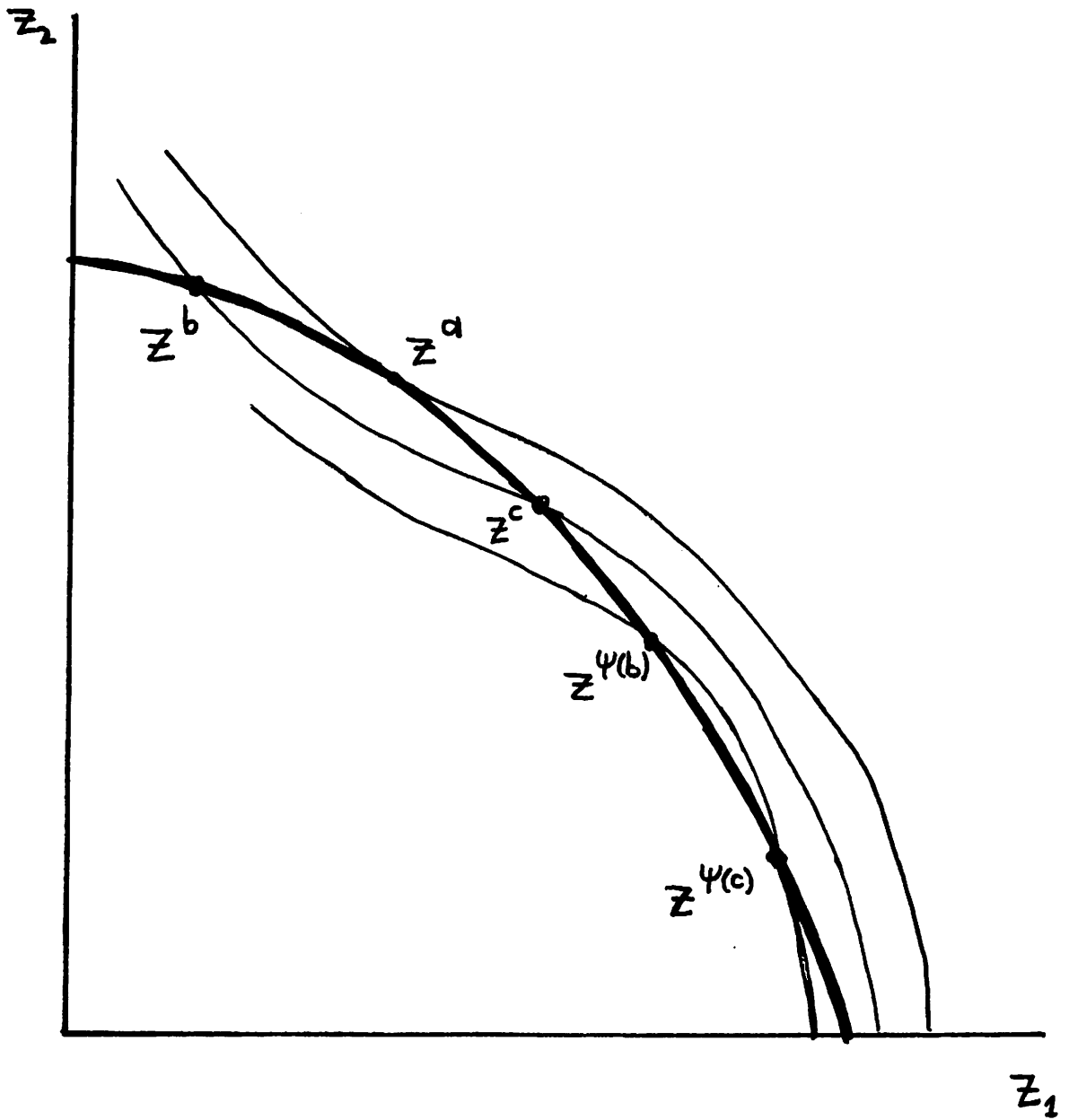FIGURE 3

FIGURE 4

FIGURE 5

FIGURE 6

APPENDICES

The two appendices that follow contain the proofs of Theorems 1 and 2, as well as statements and proofs of necessary preliminary results in each case. In addition to the notation already introduced in the text of the paper, we will make use of the following:

$$A \backslash B = \{x \in A \mid x \notin B\}$$

$$\bar{C} = \text{the closure of C.}$$

Further, for any $x, \bar{x} \in \mathfrak{R}^2$, we define:

$$[x, \bar{x}] = \{x' \in \mathfrak{R}^2 \mid (\exists t \in [0,1]): x' = tx + (1-t)\bar{x}\},$$

which is just the line segment having x and $\bar{x}$ as endpoints, and

$$L(x, \bar{x}) = \{x' \in \mathfrak{R}^2 \mid d(x'x) = d(x', \bar{x})\},$$

which is the perpendicular bisector of $[x, \bar{x}]$.

In appendix A, it is established in the first lemma that for any $x, \bar{x} \in X$, there is a unique path connecting them; and this path is denoted as $P(x, \bar{x})$. That is, there is a continuous function p defined on $[0,1]$ such that $p(0) = x$ and $p(1) = \bar{x}$, and we let $p([0,1]) \equiv P(x, \bar{x})$.

Finally, as a notational convenience, we will write $f(z^\alpha) = x^\alpha$, for any $\alpha \in \theta$.

Appendix A: Characterization of the Linear Production Set

Lemmas A.1 - A.4 all have as hypotheses that X is a representation of a structure satisfying Assumptions 1 and 2 and Condition L.1.

Lemma A.1 - For any $x^a, x^b \in X$, there exists a unique path $P \subset X$ connecting them.

Proof: Suppose, by way of contradiction, that $P_1, P_2 \subset X$ are distinct paths connecting x^b and x^a. Then there exist sub-paths $P_1' \subset P_1$, $P_2' \subset P_2$ having only endpoints in common, so that $[P_1' \cup P_2'] \subset X$ is a closed loop, contradicting L.1.

□

Lemma A.2 - If $d(x^a, x^b) = d(x^c, x^b)$, then $x^b \in P(x^a, x^c)$.

Proof: Immediate from lemma A.1 and condition L.1.

$\square$

Lemma A.3 - For all $a, b, c \in \theta$, if $d(x^a, x^c) = d(x^b, x^c) = \beta > 0$, then $d(x^d, x^c) = \beta$ implies $d = a$ or b.

Proof: Suppose such a distinct x^d exists. Then lemmas A.1 and A.2 imply that all three of x^a, x^b, x^d are located on a circle of radius β with center x^c, and that each pair of these 3 points is connected by a unique path in X containing x^c.

Let $\gamma = \max[d(x^d, x^c), d(x^d, x^b)] > 0$.

Then $d(x^d, x^c) = d(x^d, x^b)$ contradicts L.1 immediately, so suppose that,

$$\gamma = d(x^d, x^a) > d(x^d, x^b) = \delta > 0.$$

Then there exists an $\bar{x} \in P(x^a, x^d)$ such that $d(\bar{x}, x^d) = \delta$, so $\bar{z} I_d z^b$. But then there exists $x^f \in P(\bar{x}, x^b)$ such that $d(x^f, \bar{x}) = d(x^f, x^b) \Leftrightarrow \bar{z} I_f z^b$. Since $x^d \notin P(\bar{x}, x^b)$, this contradicts L.1.

$\square$

Lemma A.4 - If $x^a, x^b \in X$ are not endpoints of X, and $d(x^a, x^c) = d(x^b, x^c) = \beta$, then for any $\bar{x} \in X$ distinct from x^a, x^b, x^c;

$$\{x^a, x^b\} \cap P(x^c, \bar{x}) \neq \emptyset \Rightarrow d(\bar{x}, x^c) > \beta.$$

Proof: Suppose the hypotheses hold, with $x^a \in P(x^c, \bar{x})$, but $d(\bar{x}, x^c) = \beta' < \beta$. (Note that lemma A.3 rules out equality.) Then there must exist $\tilde{x} \in P(x^a, x^c)$ and $\hat{x} \in P(x^b, x^c)$ such that $d(x^c, \tilde{x}) = d(x^c, \hat{x}) = \beta'$, since these are connected sets. This, however contradicts lemma A.3.

$\square$

<u>Proof of Theorem 1</u>: The necessity part of the theorem is immediate. To prove sufficiency, one need only show that the curvature of X at two arbitrary points x^α, x^β (assumed not to be endpoints) must be the same.

Lemma A.2 implies there exists $\bar{x} \in X$ with $d(x^\alpha, \bar{x}) = d(x^\beta, \bar{x}) = \gamma$; and lemma A.4 implies there exists $x^a, x^b \in X$ such that

$$d(x^a, \bar{x}) = d(x^b, \bar{x}) = \gamma' > \gamma.$$

Further, $L(x^a, x^b) \cap X = \{\bar{x}\}$.

We will now show that the sets

$$C = \{x \in X \mid d(x, \bar{x}) \leq \gamma'\}$$

$$C_a = \{x \in C \mid d(x, x^a) \leq d(x, x^b)\}$$

$$C_b = \{x \in C \mid d(x, x^b) \leq d(x, x^a)\}$$

are such that there exists a function $\sigma : \mathfrak{R}^2 \to \mathfrak{R}^2$ which is a reflection or axial symmetry with respect to $L(x^a, x^b)$. That is (see Choquet (1969), p. 59),

(i) $x \in L(x^a, x^b) \Rightarrow \sigma(x) = x$

(ii) $x \notin L(x^a, x^b) \Rightarrow \sigma(x) = \{\hat{x} \in C \mid L(x, \hat{x}) = L(x^a, x^b)\}$.

Then, since $\sigma(x^\alpha) = x^\beta$ by construction, we will have the desired result.

We define $\sigma(\bar{x}) = \bar{x}$, and construct now a dense subset of C, such that a function $\sigma(\cdot)$ satisfying (ii) exists, from which it follows that its extension to C also satisfies (ii).

Let x^c, x^d be the unique points in X such that (from condition L.1),

$$d(\bar{x},x^c) = d(x^a,x^c) \text{ and } d(\bar{x},x^d) = d(x^b,x^d).$$

Condition L.2 implies then that all four of these distances are equal, so in particular, $\bar{x} \in L(x^c,x^d)$. There are two cases to consider now.

(a) $\bar{x}, x^a$ and x^b are collinear. Then if either of x^c, x^d are not also, then X must have an inflection point, in the sense that some line L in $\mathfrak{R}^2$ intersects X in at least three disconnected sets. However, then L.2 is violated, so x^c and x^d are collinear with the original points, and $L(x^c,x^d) = L(x^a,x^b)$.

(b) $\bar{x}, x^a$ and x^b are not collinear. Now let $L(\bar{x},x^a) \cap [\bar{x},x^a] = y$ and $L(\bar{x},x^a) \cap [x^a,x^b] = \bar{y}$. Then it must be the case that $d(x^c,\bar{y}) > d(x^c,y)$ and $d(\bar{y},x^c) > d(\bar{y},y)$. Otherwise, X must have an inflection point as in (a), violating L.2. Since the same argument applies to x^a, we have that $L(x^c,x^d) = L(x^a,x^b)$.

Continuing then, we construct the four points x^e, x^f, x^g, x^h such that:

$$d(x^a,x^e) = d(x^c,x^e) \quad , \quad d(x^c,x^f) = d(\bar{x},x^f)$$
$$d(\bar{x},x^g) = d(x^d,x^g) \quad , \quad d(x^d,x^h) = d(x^b,x^h).$$

Note that lemma A.4 implies that all of the points being constructed are in C. Condition L.2 implies that all eight distances above are in fact equal. An argument dealing with inflection points as above further implies that the four new points must be oriented such that

$$L(x^e,x^h) = L(x^g,x^f) = L(x^a,x^b).$$

This process results in the construction of a dense subset $B \subset C$ such that $B \cap C_a$ is a reflection of $B \cap C_b$, as above. By continuity this holds for $\bar{B} \cap C_a = C_a$ and $\bar{B} \cap C_b = C_b$ also, and since $d(\bar{x},x^\alpha) = d(\bar{x},x^\beta)$, we must have $\sigma(x^\alpha) = x^\beta$. Thus, the curvature of X at x^α and x^β must be the same, as was to be shown. Since x^α and x^β were arbitrary, X must be an arc of constant curvature. $\square$

Appendix B: Characterization of the Circular Product Set

Lemmas B.1 and B.2 assume that X is a representation of a structure satisfying Assumptions 1 and 2 and Condition C.1.

Lemma B.1 - If $b = \Psi(a)$, then x^b is the unique furthest point from x^a in X.

Proof: If $b = \Psi(a)$, then for all other $x \in X$,

$$d(x^a,x) > d(x^a,x^a) = 0,$$

so $\quad d(x^b,x) < d(x^b,x^a) \quad$ from condition C.1. $\qquad \square$

Lemma B.2 - For any $x, \bar{x} \in X$, there exists a unique $a \in \theta$ such that, for

$b = \Psi(a)$; $d(x^a,x) = d(x^a,\bar{x})$ and $d(x^b,\bar{x}) = d(x^b,x)$.

Proof: Since $X = f(x)$ is connected, there exists $a \in \theta$ such that $d(x^a,x)$

$= d(x^a,\bar{x})$, or else $L(x,\bar{x})$ would separate X. If $b = \Psi(a)$, then from C.1 it

follows that $d(x^b,\bar{x}) = d(x^b,x)$ also.

Suppose now that some third point $x^c \in L(x,\bar{x})$ also, and then necessarily

$x^d \in L(x,\bar{x})$, where $d = \Psi(b)$.

Then Lemma B.1 implies

$$d(x^a,x^c) < d(x^a,x^b),$$

and $\quad d(x^a,x^d) < d(x^a,x^b)$.

Thus, x^c and x^d both lie between x^a and x^b on $L(\bar{x},x)$, which implies

that

$$d(x^c,x^d) < d(x^c,\hat{x}) \text{ for } \hat{x} = x^a \text{ or } x^b,$$

and this contradicts lemma B.1. $\qquad \square$

Proof of Theorem 2: Again, the necessity part is trivial.

For sufficiency, let $a_1 \in \theta$ be arbitrary, with $b_1 = \Psi(a_1)$. From lemma B.2

there exists $a_2, b_2 \in \theta$ such that

$$d(x^{a_2}, x^{a_1}) = d(x^{a_2}, x^{b_1})$$

and $\quad d(x^{b_2}, x^{a_1}) = d(x^{b_2}, x^{b_1}).$

But since $b_1 = \Psi(a_1)$, condition C.1 implies

$$d(x^{a_1}, x^{a_2}) \geqq d(x^{a_1}, x^{b_2}) \Leftrightarrow d(x^{b_2}, x^{a_2}) \leqq d(x^{b_1}, x^{b_2})$$

and therefore,

$$d(x^{a_1}, x^{a_2}) = d(x^{a_1}, x^{b_2}) = d(x^{b_1}, x^{a_2}) = d(x^{b_1}, x^{b_2}).$$

These four points form the vertices of a square then, and we let $\bar{x}$ be the center of the circle C containing them, noting that by lemma B.2, $\bar{x} \notin X$. Using condition C.1 again, we construct four points $x^{a_3}, x^{a_4}, x^{b_3}, x^{b_4}$ such that

$$d(x^{a_3}, x^{a_1}) = d(x^{a_3}, x^{a_2}) \; , \; d(x^{b_3}, x^{b_1}) = d(x^{b_3}, x^{b_2})$$
$$d(x^{a_4}, x^{a_1}) = d(x^{a_4}, x^{a_2}) \; , \; d(x^{b_4}, x^{b_1}) = d(x^{b_4}, x^{a_2})$$

so that $\Psi(a_3) = b_3$ and $\Psi(a_4) = b_4$.

Again, it must be that these four points are the vertices of a square, and lie on a circle with center $\bar{x}$. It remains to show that they lie on the same circle C, as the first set of four points.

To show this let $x^{a_5}, x^{b_5} \in X$ be such that (using lemma B.2):

$$d(x^{a_5}, x^{a_1}) = d(x^{a_5}, x^{a_3}) \text{ and } d(x^{b_5}, x^{a_1}) = d(x^{b_5}, x^{a_3})$$

so $\Psi(a_5) = b_5$.

Then condition C.2 implies that:

$$d(x^{a_5}, x^{b_1}) = d(x^{a_5}, x^{b_3}) \text{ and } d(x^{b_5}, x^{b_1}) = d(x^{b_5}, x^{b_3}),$$

since $\Psi(a_1) = b_1$ and $\Psi(a_3) = b_3$. But then $x^{a_5}, x^{b_5} \in [L(x^{a_1}, x^{a_3}) \cap L(x^{b_1}, x^{b_2})]$, which can happen only of $L(x^{a_1}, x^{a_3}) = L(x^{b_1}, x^{b_3})$. This in turn implies that

$$d(\bar{x}, x^{a_3}) = d(\bar{x}, x^{a_1}) = d(\bar{x}, x^{b_1}) = d(\bar{x}, x^{b_3}).$$

Therefore, this argument establishes that all eight points constructed so far are equidistant from $\bar{x}$, and the procedure can be repeated to construct a dense subset C' of the circle C. Continuity of f implies then that $\overline{C'} = C \subset X$.

We must also have $X \subset C$, since if there exists $\hat{x} \in X \backslash C$, then letting L be the line determined by $\hat{x}$ and $\bar{x}$, we have <u>three</u> points ($\hat{x}$ plus $L \cap C$) which are equidistant from an infinity of pairs of points in C, violating lemma B.2. So, $X = C$. $\qquad\qquad \square$

FOOTNOTES

[1] The U.S. Federal Trade Commission, for instance, employed a spatial model in its case against the ready-to-eat cereal manufacturers. For discussions of this case see Schmalensee [18] and Scherer [17].

[2] For a discussion of the basics of the characteristic approach see Lancaster [10].

[3] In fact, it is a much less restrictive assumption than those of Lancaster. In [9], Lancaster assumes that the product spectrum is an n-dimensional, linear and convex subspace of an n+1-dimensional characteristics space.

[4] This is done not only for simplicity but also to capture the usual assumption in spatial models that each consumer purchases one unit of the good in question. The issue of what defines one unit of a good in the present context is left open. For one possible solution, see Drèze and Hagen [2].

[5] This captures the usual assumption of a uniform customer density. This assumption is not essential, however, and any density having Z as its support can be dealt with. Our discussion focuses on the uniform case because this is the model almost universally employed in the literature.

[6] P_α is the strict preference ordering derived from R_α, while I_α is the indifference relation.

[7] This follows from the fact that since $f(\cdot)$ is a continuous bijection from the compact set Z to $X \subset R^2$, a Hausdorff space, Z and X must be homeomorphic. (Cf Munkres [12], Theorem 5.6.)

[8]Heterogeneous goods, in this context, refers to goods which vary over a number of characteristic dimensions. Automobiles would be just one example. We will argue below that ready-to-eat cereals are another.

[9]While there are almost no products one can imagine which contain a single characteristic--and so essentially no markets to which these models can strictly speaking, be applied--we would argue that there are certain products for which one characteristic plays a dominant role in consumer decision-making and which can arguably be approximated therefore by a spatial model. Examples of the sorts of products we have in mind are household chlorine bleaches, or specific grocery items such as frozen peas, canned corn, etc.

[10]In a number of these markets, however, the model discussed in this paper is not strictly applicable. For unless one is willing to assume that quality (or reliability) becomes a bad beyond some point, the assumption that each consumer possesses a different most-preferred good no longer holds. For a way to modify this model to deal with the quality issue see Gabszewicz and Thisse [5].

[11]The subset is defined by those simple arcs which have the property that exactly one consumer is indifferent between any two points $f(Z), f(\bar{Z})$. An example of a simple arc which violates this condition is the arc of a circle beyond a semi-circle.

[12]This is strictly true only for a uniform distribution of consumers. It is still true for non-uniform distributions that firm markets of a given length differ only to the extent that the distribution of consumers within them does--the location of the market is irrelevant beyond that.

[13]In the case of a single characteristic, the δ-preferring group would have a maximum characteristic requirement. That is, each member of the group would have a level of $z_2 = \bar{z}_2$ such that he would prefer a good embodying the maximum amount of z_2 to one embodying any amount between $\bar{z}_2$

and this maximum; but would prefer any good embodying a level of $z_2 < \bar{z}_2$ to the maximum z_2 good.

[14]This is purely for simplicity. A more sophisticated model would illustrate the point no better while making the analysis significantly more complex.

[15]Any fixed marginal cost will yield the same result.

[16]There are values of K and F which either reduce or increase welfare in both markets.

[17]This result applies equally well to Scherer's [17] analysis.

[18]I.e. those in which the independently varying characteristic is not quality.

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