1970

Intermediate Products and the Two-Sector Growth Model

Raveendra Batra

Ranjit Singh

RESEARCH REPORT 7027

INTERMEDIATE PRODUCTS AND THE TWO-SECTOR GROWTH MODEL

by

Raveendra Batra*

and

Ranjit Singh

*The authors are respectively Assistant Professors of Economics at the University of Western Ontario and the University of Atlanta. This paper is to be presented at the forthcoming meetings of the Econometric Society, Detroit, December 1970 in the session on "Growth Models."

Intermediate Products and the Two-Sector Growth Model

I

A rich literature has recently grown on the conditions for relative stability in a two-sector model of economic growth. It has been shown by Uzawa [12] [13] that a sufficient condition for the economy to be relatively stable is that the consumption-goods sector is capital-intensive relative to the capital-goods sector. In the literature on two-sector growth models, this condition has been called the <u>capital-intensity</u> condition. Shinkai [9] has shown that the capital-intensity condition is a necessary and sufficient condition for relative stability in the case of fixed coefficients of production. Takayama [10] [11], Amano [1], Drandakis [4], Sato [7], Shell [8], Batra [2] [3] and others have derived still weaker conditions for relative stability. For example, Drandakis [4] has shown that the two-sector model is relatively stable if the elasticity of factor substitution in each sector is at-least equal to unity. This condition is now commonly known as the "elasticity of substitution" or simply "elasticity" condition. Yet a weaker elasticity condition has been derived recently by Batra [3]. He shows that if the elasticity of substitution in the capital-goods sector exceeds or equals relative share of labor in the consumption-goods sector, or, if the elasticity of substitution in the consumption-goods sector is at least equal to the relative share of capital in the capital-goods sector, then the two-sector growth model is relatively stable. It is, therefore, evident that the sufficient conditions for the relative stability of the two-sector model are not very restricted, and there is a general agreement among growth theorists that the model is most likely to be stable. This optimism is well reflected by Sato in his conclusion that "the essential point is that the two-sector-growth model is most likely stable, unless the model has extreme assumptions concerning its parameters" [7, p. 117] [Sato's italics].

Can then the two-sector growth model be successfully applied to explain the growth behavior of an actual economy? The answer is no, because of the presence of intermediate goods which constitute a large proportion of an economy's production activity and which are produced solely to be utilized as material inputs for final goods. Even if it was possible to divide an economy roughly into two sectors, because of the presence of a large intermediate goods sector, the economy may still not follow the growth path prescribed by the traditional two-sector growth model.

The purpose of this paper is to fill this gap by introducing intermediate goods into the two-sector model of economic growth. It is shown that the growth model is still stable, provided, the capital-intensity condition is satisfied. That is, if the capital-goods sector is labor-intensive relative to the intermediate-goods sector and the consumption-goods sector, and if the consumption-goods sector is capital-intensive relative to the intermediate goods sector and the capital-goods sector, the two-sector growth model is relatively stable.[1] It is further shown that the "weak" elasticity conditions derived by Drandakis, Sato and Batra may no longer be sufficient for relative stability when intermediate products are introduced. However, under certain restrictions, it is possible to derive some "strong" elasticity conditions.

II. Assumptions and the Model

It is assumed that an economy consists of three sectors of production, Y_1, the capital-goods sector, Y_2, the consumption-goods sector and; Y_3, the intermediate-goods sector,[2] and two primary factors of production, capital (K) and labor (L). There is perfect competition; production functions are linear homogeneous; factors of production are fully employed, in inelastic supply, and completely mobile among the three sectors; finally capital and labor are both homogeneous and can be used in either sector.

The three production functions are:

(1) $Y_1 = F_1(K_1, L_1, Y_{31}) = L_1 f_1(k_1, y_{31})$

(2) $Y_2 = F_2(K_2, L_2, Y_{32}) = L_2 f_2(k_2, y_{32})$

(3) $Y_3 = F_3(K_3, L_3) = L_3 f_3(k_3)$

where K_i and L_i are respectively the capital and labor inputs and $k_i = K_i/L_i$ is the capital/labor ratio in the ith sector (i = 1, 2, 3), and Y_{3j} is the amount of Y_3 utilized as material input in the jth final product (j = 1, 2) and $y_{3j} = Y_{3j}/L_j$. Let a_j $(=Y_{3j}/Y_j)$ denote the requirement of Y_3, the intermediate product, per unit of the jth final product (j = 1, 2). Following Samuelson [6], Vanek [14] and others who have worked with the input-output models, a_j is assumed to be constant. Then

(4) $Y_3 = a_1 Y_1 + a_2 Y_2$.

Let v_i = marginal productivity of capital in the ith sector and u_i = marginal productivity of labor in the ith sector (i = 1, 2, 3). Then

$$v_i = f_i', \text{ and}$$

$$u_i = f_i - k_i f_i'.$$

We assume Inada's [5] derivative conditions: $f_i > 0$. $f_i' > 0$, $f_i'' < 0$, $f_i(\infty) = \infty$, $f_i'(0) = \infty$, $f_i'(\infty) = 0$.

With perfect competition in both product and factor markets, the price of each factor equals its marginal value-added product and is the same in all three industries. Let w stand for the wage-rate, r for the rental of capital, p_i for the price of the ith commodity and ω for the w/r ratio.

Factor prices can then be expressed as:

$$(5)\quad r = f_1'(p_1 - p_3a_1) = f_2'(p_2 - p_3a_2) = p_3f_3'$$

$$(6)\quad w = (f_1 - k_1f_1')(p_1 - p_3a_1) = (f_2 - k_2f_2')(p_2 - p_3a_2) = p_3(f_3 - k_3f_3')$$

$$(7)\quad \omega = \frac{f_i}{f_i'} - k_i.$$

With full employment:

$$(8)\quad \rho_1 + \rho_2 + \rho_3 = 1$$

$$(9)\quad \rho_1k_1 + \rho_2k_2 + \rho_3k_3 = k$$

where $\rho_i = L_i/L$.

The introduction of the intermediate-goods sector has already complicated the model. Since our main purpose is to show how the presence of such goods modify the stability conditions, we employ, for the sake of simplicity, the classical savings-function in which the owners of capital save everything and laborers consume everything. The total amount of savings, therefore, equals rK. The level of investment in the next period, however, is given by the value of output in the capital-goods sector. With savings and investment remaining always in equilibrium, $p_1Y_1 = rK$, or

$$(10)\quad Y_1 = (1 - \frac{p_3}{p_1}a_1)f_1'K.$$

With this last equation, our description of the two-sector model with intermediate products is complete. There are 10 variables, Y_1, Y_2, Y_3, ρ_1, ρ_2, ρ_3, k_1, k_2, k_3 and ω in 10 equations. Hence, for any aggregate capital labor ratio k, the model is determinate. Moreover, while counting the number of equations, it may be observed that equation (7) represents three independent relations in k_i obtained from equations (5) and (6). Thus equations (5) and (6) are not counted in these 10 independent equations. On the other hand, (7) represents three independent equations.

III. Existence and Uniqueness of the Short Run Equilibrium

This section deals with the derivation of certain sufficient conditions which will ensure the existence and the uniqueness of the short-run equilibrium characterized by, among other things, the absence of capital accumulation.

Let

$$(11)\quad \sigma_i = \frac{\omega}{k_i} \cdot \frac{dk_i}{d\omega}$$

be the elasticity of substitution between capital and labor in the ith sector. Differentiating 7 with respect to ω, we obtain:

$$(12)\quad \frac{dk_i}{d\omega} = - \frac{f_i'^2}{f_i f_i''} ,$$

which, in view of $f_i'' < 0$, and $f_i > o$, is positive. Therefore, σ_i from (11) is also positive. Let

$$(13)\quad \beta_i = \frac{f_i - k_i f_i'}{f_i} = \frac{f_i'(f_i - k_i f_i')}{f_i' f_i} = \frac{\omega}{\omega + k_i}$$

be the relative share of labor in the ith sector. From (4)

$$L3f_3 = a_1 L_1 f_1 + a_2 L_2 f_2,$$

so that, by dividing through L,

$$(14)\quad \rho_3 = (a_1 \rho_1 f_1 + a_2 \rho_2 f_2)/f_3.$$

From (10), we can write

$$\frac{L_1 f_1}{L} = (1 - \frac{p_3}{p_1} a_1) f_1' \frac{K}{L}$$

or substituting from (5) and (7)

$$(15)\quad \rho_1 = \left[\frac{k}{\omega + k_1}\right]\left[\frac{f_3'}{f_3' + a_1 f_1'}\right].$$

By substituting ρ_3 from (14) in (8) and (9), we obtain:

$$(16)\quad \rho_1(f_3 + a_1 f_1) + \rho_2(f_3 + a_2 f_2) = f_3$$

$$(17)\quad \rho_1(k_1 f_3 + k_3 a_1 f_1) + \rho_2(k_2 f_3 + k_3 a_2 f_2) = kf_3.$$

Solving for ρ_1 from (16) and (17) yields:

$$(18)\quad \rho_1 = \frac{k_2f_3 + k_3a_2f_2 - k(f_3 + a_2f_2)}{f_3(k_2 - k_1) + a_1f_1(k_2 - k_3) + a_2f_2(k_3 - k_1)}$$

From (15) and (18) we have:

$$(19)\quad k = \frac{[f_3k_2 + a_2f_2k_3][f_3'(\omega + k_1) + a_1f_1]}{(a_1f_1 + f_3)\ [a_2f_2 + f_3'(\omega + k_2)]} = \frac{a}{b}$$

Equation (19) describes a relationship between k, k_i, f_i, a_i and ω, but since a_i is assumed to be constant and k_i and f_i are functions of ω, (19) shows a functional relationship between k and ω only. A unique solution for ω is ensured if k is strictly a monotonic function of ω. Differentiating (19) logarithmically with respect to ω, we have:

$$(20)\quad \frac{1}{k}\frac{dk}{d\omega} = \frac{\sigma_1c_1 + \sigma_2c_2 + \sigma_3c_3 + e}{ab\omega} = \frac{\sigma}{\omega}\ .$$

where

$$c_1 = k_1f_3[f_3k_2 + k_3a_2f_2][f_3' + a_1f_1'][a_2f_2 + f_3'(\omega + k_2)]$$
$$c_2 = k_2f_3\omega[f_3 + a_1f_1][f_3' + a_2f_2'][a_1f_1 + f_3'(\omega + k_1)]$$
$$c_3 = k_3[a_1f_1+f_3'(\omega+k_1)][a_2f_2+f_3'(\omega+k_2)][a_1f_1k_2f_3'+a_2f_2(a_1f_1+\omega f_3')]$$
$$e = \frac{\omega f_3'}{\omega + k_3}[f_3 + a_1f_1][f_3k_2 + a_2f_2k_3][f_3(k_2-k_1) + a_1f_1(k_2-k_3)+a_2f_2(k_3-k$$

As stated before, a unique solution for ω requires that k be a monotonic function of ω. Furthermore, the relationship between k and ω must be positive, if the model is to have a determinate solution. The reason for this lies in the fact that the relationship between k_i and ω from (12) is positive, and since k and k_i must be directly related, k must be an increasing function of ω. In other words, for the system to be uniquely determined at every moment of time,

$$\frac{1}{k}\frac{dk}{d\omega} = \sigma > 0,$$

where σ may be called the aggregate elasticity of substitution.

A. Absence of Intermediate Goods: There are several conditions under which σ can be shown to be positive. To begin with, we can show that the short run stability conditions obtained from the traditional two sector growth models without intermediate products are special cases of the general solution given by (20). In the absence of intermediate products $a_i = 0 (i=1,2)$ so that

$$c_1 = k_1 f_3^2 [k_2(\omega + k_2) f'^2_3], \; c_2 = k_2 f_3^2 \omega(\omega + k_1) f_3'^2, \; c_3 = 0, \text{ and}$$
$$e = \omega f_3'^2 f_3^2 k_2 (k_2 - k_1).$$

It is clear that c_1, c_2 and c_3 are positive where e may be positive or negative, depending on the relative magnitude of k_2 and k_1. From (20) $\sigma > 0$ if (i) $e \geq 0$, or (ii) $\sigma_1 c_1 + e \geq 0$ or (iii) $\sigma_2 c_2 + e \geq 0$, or (iv) $\sigma_1 c_1 + \sigma_2 c_2 + e > 0$. Now $e \geq 0$ if $k_2 \geq k_1$. In other words, if the consumption-goods sector is at least as capital-intensive as the capital-goods sector, the short run equilibrium will be uniquely determined. This is Uzawa's theorem [12]. Secondly, $\sigma_1 c_1 + e \geq$ if

$$f_3^2 f_3'^2 [\sigma_1 k_1 k_2 (\omega + k_2) + \omega k (k_2 - k_1)] \geq 0$$

which is necessarily satisfied if

$$\sigma_1 \geq \frac{\omega}{\omega + k_2} = \beta_2.$$

In other words, if the elasticity of substitution in the capital-goods sector exceeds or equals the relative share of labor in the consumption-goods sector, $\sigma > 0$. This is the "weak" elasticity condition derived by Sato [7] and Batra [3]. Thirdly, $\sigma_2 c_2 + e \geq 0$, if

$$f_3^2 f_3'^2 [\sigma_2 k_2 \omega(\omega + k_1) + \omega k_2 (k_2 - k_1)] \geq 0,$$

which is necessarily satisfied if

$$\sigma_2 \geq \frac{k_1}{\omega + k_1} = 1 - \beta_1.$$

In other words, if the elasticity of substitution in the consumption goods sector is at least equal to the relative share of capital in the capital-goods sector, the short run equilibrium is unique. This is also Batra's theorem.

Finally, $\sigma_1 c_1 + \sigma_2 c_2 + e \geq 0$ if

$$f_3{}^2 f_3{}'^2[\sigma_1 k_1 k_2(\omega + k_2) + \sigma_2 k_2 \omega(\omega + k_1) + \omega k_2(k_2 - k_1)] > 0,$$

which is necessarily satisfied if

$$k_1 k_2 \omega(\sigma_1 + \sigma_2 - 1) \geq 0, \text{ or } \sigma_1 + \sigma_2 \geq 1.$$

In other words, if the two elasticities of substitution add up to at least unity, the uniqueness of the short-run equilibrium is ensured. This is the condition obtained by Drandakis [4]. Note that if $\sigma_1 \geq 1$, a condition obtained by Takayama [10], the Drandakis condition is necessarily fulfilled.

Thus the traditional conditions ensuring the uniqueness of the short-run equilibrium turn out to be the special cases of the general solution given by (20).

B. Intermediate Goods in the Capital-Goods Sector:

Consider now the case where the intermediate product is used in the capital-goods sector alone, so that a_2 is still zero but a_1 is positive. Evidently, c's and e have have different values now. However, c's are still positive, whereas e may be negative or positive. The following theorems may now be derived.

Theorem 1: The short-run equilibrium in the two-sector model where intermediate goods are used in capital-goods alone is unique if the consumption-goods sector is capital-intensive relative to the capital-goods sector and the intermediate-goods sector.

This theorem requires $e \geq 0$, which now equals

$$\frac{\omega f_3{}'}{\omega + k_3}[f_3 + a_1 f_1][f_3(k_2 - k_1) + a_1 f_1(k_2 - k_3)]f_3 k_2,$$

so that $e \geq 0$ if $k_2 \geq k_1$ and $k_2 \geq k_3$.

Theorem 2: If $a_2 = 0$, but $a_1 > 0$, $\sigma > 0$ if the elasticity of substitution in the consumption-goods sector exceeds or equals the relative share of capital in the capital-goods sector and the intermediate-goods sector.

This theorem can be proved by seeing that

$$\sigma_2 c_2 + e = \omega f_3 f_3' k_2 (f_3 + a_1 f_1)[\sigma_2 a_1 f_1 + \frac{k_2(f_3+a_1f_1) - k_3 a_1 f_1}{\omega + k_3} + \sigma_2 f_3'(\omega + k_1) - \frac{f_3 k_1}{\omega + k_3}]$$

so that $\sigma_2 c_2 + e \geq 0$ if

$$\sigma_2 \geq \frac{k_3}{\omega + k_3} = 1 - \beta_3$$

and

$$\sigma_2 \geq \frac{k_1}{\omega + k_1} = 1 - \beta_1.$$

A sufficient condition for theorem 2 to hold is that $\sigma_2 \geq 1$.

C. Intermediate Goods in the Consumption-Goods Sector:

Consider now the case where the intermediate good is used in the production of the consumption-goods alone. Here $a_1 = 0$, but $a_2 > 0$. The following theorems may now be derived:

Theorem 3: The short run equilibrium is uniquely determined in a two-sector model where intermediate products are used in the consumption-goods sector alone, if the capital-goods sector is labor-intensive relative to the consumption-goods sector and the intermediate-goods sector.

The proof of this theorem follows from the fact that

$$e = \frac{\omega f_3'}{\omega + k_3} [f_3(f_3 k_2 + k_3 a_2 f_3)\{f_3(k_2 - k_1) + a_2 f_2(k_3 - k_1)\}] \geq 0$$

if $k_2 \geq k_1$ and $k_3 \geq k_1$.

Theorem 4: If $a_1 = 0$, and $a_2 > 0$, a sufficient condition for $\sigma > 0$ is that the elasticity of substitution in the capital-goods sector be greater than or equal to the relative share of labor in the intermediate-goods sector and the consumption-goods sector.

This theorem is derived by showing that

$$\sigma_1 c_1 + e = f_3 f_3'(f_3 k_2 + a_2 f_2 k_3)[k_1 a_2 f_2(\sigma_1 - \frac{\omega}{\omega + k_3}) + k_1 f_3'\{\sigma_1(\omega + k_2) - \omega\}] + f_3'^2 \omega (f_3 k_2 + a_2 f_2 k_3)^2 \geq$$

if $\sigma_1 \geq \frac{\omega}{\omega + k_3} = \beta_3$

and

$$\sigma_1 \gtreqless \frac{\omega}{\omega + k_2} = \beta_2.$$

Evidently, a sufficient condition for theorem 4 to hold is that $\sigma_1 \gtreqless 1$.

D. Intermediate Goods in Both Sectors: We now turn to the case where the intermediate good is utilized as a material input in the consumption-goods sector as well as in the capital-goods sector. Here both a_1 and a_2 are positive. Bearing this in mind, we derive the following theorem:

Theorem 5: The short-run equilibrium is uniquely determined in a two-sector model where intermediate products are used in both sectors if the consumption-goods sector is the most capital-intensive of the other two sectors and the capital-goods sector is the most labor-intensive of the other two sectors.

This theorem can be proved by showing that $e \gtreqless 0$ if

$$k_2 \geq k_1, \; k_2 \geq k_3, \text{ and } k_3 \gtreqless k_1.$$

Thus a sufficient condition for the short run equilibrium to be unique in the two-sector model with intermediate goods is that the capital-goods sector be labor-intensive not only relative to the consumption-goods sector, but also to the intermediate-goods sector. Thus the capital-intensity condition in the model with intermediate products is given by theorem 5. One can see that if this condition is satisfied, the short-run equilibrium is unique whether the intermediate good is used in the capital-goods sector or the consumption-goods sector or both.[2]

IV. Existence and Stability of the Long Run Equilibrium

The long run growth path of an economy is determined by the rate of capital accumulation and the exogenously determined rate of growth of labor (η) which is assumed to be constant throughout the accumulation process.

Let μ be the constant rate of depreciation. The rate of capital accumulation is given by the following differential equation:

$$(21)\quad \frac{dK}{dt} = Y_1 - \mu K = [(Y_1/K) - \mu]K.$$

Substituting (5) and (10) in (21), we have:

$$(22)\quad \frac{1}{K}\frac{dK}{dt} = G = f_1' \left[\frac{f_3'}{a_1 f_1' + f_3'}\right] - \mu.$$

It is evident from (22) that the rate of growth of capital, G, is determined by the rate of depreciation, the marginal productivity of capital in the capital-goods and the intermediate-goods sector as well as by the amount of intermediate goods required to produce a unit of output in the capital-goods sector. The following theorem is immediate:

Theorem 6: The introduction of intermediate goods results in a decline in the rate of growth of capital. However, if the intermediate product is used in the consumption-goods sector alone, the growth rate of capital remains unaltered.

In the absence of intermediate goods, or if intermediate products are used in Y_2 alone, $a_1 = 0$, so that $G = (f_1' - \mu)$. Since

$$\frac{f_3'}{a_1 f_1' + f_3'} < 1 \text{ when } a_1 > 0,$$

the introduction of intermediate goods leads to a decline in G.

Differentiating (22) logarithmically with respect to k, we obtain:

$$(23)\quad \frac{1}{G}\frac{dG}{dk} = G^* = \frac{-d\omega/dk}{a_1 f_1' + f_3'} \left[\frac{a_1 f_1'}{\omega + k_3} + \frac{f_3'}{\omega + k_1}\right].$$

Now the existence of the long run equilibrium is defined by the equality of G and η, and the capital/labor ratio prevailing at that equilibrium is called the balanced capital/labor ratio. This balanced capital/labor ratio is unique if, whenever there is a divergence between G and η, G comes back to η, such that the balanced capital/labor ratio remains the same. This means that whenever k rises above the balanced capital/labor ratio, the adjustment mechanism of the economic system should be such as to lower G, and conversely. Therefore the condition for the existence and stability of a unique long run equilibrium is that $G^* < 0$.

It is clear from (23) that $G^* < 0$ if $d\omega/dk > 0$. It may be recalled that in our discussion of the existence of the short run equilibrium in Section III, we have obtained sufficient conditions for $d\omega/dk > 0$. Hence the conditions for the stability of the long run equilibrium are given by theorems 1 - 5. The following theorem may now be derived:

Theorem 7: If intermediate goods are used in both sectors, the two-sector growth model is globally stable if theorem 5 or the capital-intensity condition is satisfied.

If intermediate goods are used only in the capital-goods sector, stability of the growth model is assured if theorem 1 or theorem 2 holds.

If intermediate goods are used in the consumption-goods sector alone, the two-sector growth model is stable if theorem 3 or theorem 4 is satisfied.

V. Concluding Remarks.

The general conclusion that emerges from the foregoing discussion of our two-sector growth model with intermediate products is that the growth model is stable if the consumption-goods sector is the most capital-intensive and the capital-goods sector is the most labor-intensive of all the sectors including the intermediate-goods sector. This conclusion holds whether intermediate goods are used in one sector or both. If, however, intermediate goods are used only in one sector, then a unit or greater than unity elasticity of substitution in the other sector is a sufficient condition for global stability.

Finally, it may be observed that Shinkai's theorem that the capital-intensity condition is a necessary and sufficient condition for stability of the two-sector growth model with fixed coefficients no longer holds when intermediate goods are introduced. In this case, the elasticities of substitution approach zero and from (20)

$$\sigma = e = f_3(k_2-k_1) + a_1f_1(k_2-k_3) + a_2f_2(k_3-k_1),$$

so that $\sigma > 0$ if

$$(24)\quad k_2(f_3+a_1f_1) + k_3a_2f_2 > k_1(f_3+a_2f_2) + k_3a_1f_1,$$

which may be satisfied even if $k_2 < k_1$. Thus, the capital-intensity condition becomes a sufficient, but not necessary, condition for the stability of a two-sector growth model with intermediate goods even if production coefficients are fixed. A numerical example will perhaps add further to the exposition. Let $f_3 = 10$, $a_1f_1 = 4$, $a_2f_2 = 8$, $k_2 = 4$, and $k_1 = 5$, then by substituting these values in (24), we find that if $k_3 > \frac{17}{2}$, condition (24) will be satisfied, even though k_2 (=4) is less than k_1 (=5).

Footnotes

1. This condition is thus a variant of the so called capital-intensity condition for the stability of the two-sector growth model.

2. The reader may feel that we are deploying a three-sector model. However, sector 3 is just an intermediate-goods sector and its output is utilized solely in the production of the output in the first two sectors. Therefore, as far as the final goods are concerned, we still have a two-sector model.

References

1. Amano, A., "A Further Note on Professor Uzawa's Two-Sector Model of Economic Growth," Review of Economic Studies, 31 (1965), 97 - 102.

2. Batra, R., "Hicks and Harrod-Neutral Technical Progress and the Relative Stability of a Two-Sector Growth Model with Fixed Coefficients," Journal of Political Economy, Jan. 1970.

3. __________, "The Saving-Hypothesis, Factor Substitution and Relative Stability in a Two-Sector Model of Economic Growth," (mimeographed).

4. Drandakis, E. M., "Factor Substitution in the Two-Sector Growth Model," Review of Economic Studies, 30 (1963) 217 - 28.

5. Inada, K., "On a Two-Sector Model of Economic Growth: Comments and a Generalization," Review of Economic Studies, Vol. 30, 1963, pp. 119-27.

6. Samuelson, P. A., "Prices of Factors and Goods in General Equilibrium," Review of Economic Studies, Vol. 21 (1953), pp. 1-20.

7. Sato, R., "Stability Conditions in Two-Sector Model of Economic Growth," Journal of Economic Theory, Volume 1, 1969, p. 107 - 117.

8. Shell, K., "Comparative Statics for Two-Sector Model," Metroeconomica, Vol. 18, May - August 1966, pp. 118 - 124.

9. Shinkai, Y., "On Equilibrium Growth of Capital and Labor," International Economic Review, 1 (1960), 107 - 111.

10. Takayama, A., "On a Two-Sector Model of Economic Growth: A Comparative Statics Analysis," Review of Economic Studies, 30 (1963), 95 - 104.

11. __________, "On a Two-Sector Model of Economic Growth with Technical Progress: A Comparative Statics Analysis," Review of Economic Studies, 32 (July 1965), 251 - 62.

12. Uzawa, H., "On a Two-Sector Model of Economic Growth," Review of Economic Studies, 29 (1961), 40 - 47.

13. __________, "On a Two-Sector Model of Economic Growth, II," Review of Economic Studies, 30 (1963), 105 - 118.

14. Vanek, Jaroslav, "Variable Factor Proportions and Interindustry Flows in the Theory of International Trade," Quarterly Journal of Economics, Vol. 77 (1963), 129 - 42.

MATHEMATICAL APPENDIX

A: Derivation of k as a Function of ω

In equation (10) we substitute for $\frac{p_3}{p_1} \left(= \frac{f_1'}{f_3' + f_1' a_1} \right)$ from equation (5) and for $\frac{f_1'}{f_1} \left(= \frac{1}{\omega + k_1} \right)$ from equation (7) and get equation (15)

$$\rho_1 = \left[\frac{k}{\omega + k_1} \right] \left[\frac{f_3'}{f_3' + a_1 f_1'} \right]. \qquad (15)$$

Full employment equations (8) and (9) and full utilization of intermediate product, equation (4) can be written as:

$$\rho_1 + \rho_2 + \rho_3 = 1 \qquad (8)$$

$$\rho_1 k_1 + \rho_2 k_2 + \rho_3 k_3 = k \qquad (9)$$

$$\rho_1 a_1 f_1 + \rho_2 a_2 f_2 = \rho_3 f_3 \qquad (4')$$

Substituting the value of ρ_3 from equation (4') into (8) and (9) we get

$$(1 + a_1 \frac{f_1}{f_3})\rho_1 + (1 + a_2 \frac{f_2}{f_3})\rho_2 = 1 \qquad (8')$$

$$(k_1 + a_1 k_3 \frac{f_1}{f_3})\rho_1 + (k_2 + a_2 k_3 \frac{f_2}{f_3})\rho_2 = k \qquad (9')$$

eliminating ρ_2 from (8') and (9') we get

$$[(1 + a_1 \frac{f_1}{f_3})(k_2 + a_2 k_3 \frac{f_2}{f_3}) - (k_1 + a_1 k_3 \frac{f_1}{f_3})(1 + a_2 \frac{f_2}{f_3})]\rho_1$$

$$= (k_2 + a_2 k_3 \frac{f_2}{f_3}) - k(1 + a_2 \frac{f_2}{f_3}).$$

Simplifying this last equation we get equation (18)

$$\rho_1 = \frac{k_2 f_3 + k_3 a_2 f_2 - k(f_3 + a_2 f_2)}{f_3(k_2 - k_1) + a_1 f_1 (k_2 - k_3) + a_2 f_2 (k_3 - k_1)} \qquad (18)$$

From (15) and (18) we have

$$k = \frac{(f_3 k_2 + a_2 f_2 k_3)(\omega + k_1)(f_3' + a_1 f_1')}{f_3'[f_3(\omega + k_2) + a_2 f_2(\omega + k_3) + a_1 f_1 (k_2 - k_3)] + a_1 f_1'(\omega + k_1)(f_3 + a_2 f_2)}$$

Using relation $f_i = f_i'(\omega + k_i)$ from equation (7) the numerator of k becomes

$$[f_3 k_2 + a_2 f_2 k_3][f_3'(\omega + k_1) + a_1 f_1]$$

and the denominator of k can be written as

$$a_1f_1[k_2f_3' + a_2f_2 + (f_3 - k_3f_3')] + f_3'f_3(\omega + k_2) + a_2f_2f_3$$

$$= a_1f_1[k_2f_3' + a_2f_2 + \omega f_3'] + f_3'f_3(\omega + k_2) + a_2f_2f_3$$

$$= f_3'[a_1f_1(\omega + k_2) + f_3(\omega + k_2)] + a_2f_2(a_1f_1 + f_3)$$

$$= [a_1f_1 + f_3]\ [a_2f_2 + f_3'(\omega + k_2)].$$

B. Logarithmic Differentiation of k with respect to ω

$$k = \frac{[f_3k_2 + a_2f_2k_3]\ [f_3'(\omega + k_1) + a_1f_1]}{[a_1f_1 + f_3]\ [a_2f_2 + f_3'(\omega + k_2)]} = \frac{a}{b}$$

$$\frac{1}{k}\frac{dk}{d\omega} = \frac{1}{a}\frac{da}{d\omega} - \frac{1}{b}\frac{db}{d\omega},$$

f_i, f_i' and k_1 are all functions of ω.

We shall use the relations:

$$\frac{df_i}{d\omega} = \frac{df_i}{dk_i}\frac{dk_i}{d\omega} = f_i'\frac{dk_i}{d\omega}$$

and

$$\frac{df_i'}{d\omega} = f_i''\frac{dk_i}{d\omega} = -\frac{f_i'}{\omega + k_i}$$

[by differentiating $\omega = \frac{f_i}{f_i'} - k_i$ with respect to k_i we get $f_i''\frac{dk_i}{d\omega} = -\frac{f_i'}{\omega + k_i}$].

Differentiating 'a' with respect to ω and collecting the coefficients of $\frac{dk_i}{d\omega}$, we get:

$$\frac{da}{d\omega} = [f_3k_2 + a_2f_2k_3]\ [f_3' + a_1f_1']\frac{dk_1}{d\omega}$$

$$+ [f_3'(\omega + k_1) + a_1f_1]\ [f_3 + a_2f_2'k_3]\frac{dk_2}{d\omega}$$

$$+ [f_3'(\omega + k_1) + a_1f_1]\ [k_2f_3' + a_2f_2]\frac{dk_3}{d\omega}$$

$$+ f_3'\frac{k_3 - k_1}{\omega + k_3}[f_3k_2 + a_2f_2k_3]$$

Similarly:

$$\frac{db}{d\omega} = a_1f_1'[f_3'(\omega + k_2) + a_2f_2]\,\frac{dk_1}{d\omega}$$

$$+ [f_3 + a_1f_1]\,[f_3' + a_2f_2']\,\frac{dk_2}{d\omega}$$

$$+ f_3'[f_3'(\omega + k_2) + a_2f_2]\,\frac{dk_3}{d\omega}$$

$$+ f_3'\,\frac{k_3 - k_2}{\omega + k_3}\,[f_3 + a_1f_1]$$

$$\therefore \quad \frac{1}{a}\frac{da}{d\omega} - \frac{1}{b}\frac{db}{d\omega} =$$

$$= \frac{1}{ab}\Big\{[b.\ (f_3k_2 + a_2f_2k_3)(f_3' + a_1f_1') - a.\ a_1f_1'\{f_3'(\omega + k_2) + a_2f_2\}]\,\frac{dk_1}{d\omega}$$

$$+ [b.\ \{f_3'(\omega + k_1) + f_1\}(f_3 + a_2f_2'k_3) - a.\ (f_3 + a_1f_1)(f_3' + a_2f_2')]\,\frac{dk_2}{d\omega}$$

$$+ [b.\ \{f_3'(\omega + k_1) + a_1f_1\}(k_2f_3' + a_2f_2) - a.\ f_3'\{f_3'(\omega + k_2) + a_2f_2\}]\,\frac{dk_3}{d\omega}$$

$$+ [b.\ f_3'.\ \frac{k_3 - k_1}{\omega + k_3}\,(f_3k_2 + a_2f_2k_3) - a.\ f_3'.\ \frac{k_3 - k_2}{\omega + k_3}\,(f_3 + a_1f_1)]\Big\}$$

Substituting the expressions for a and b and from equation (11) writing $\frac{dk_i}{d\omega} = \frac{k_i\sigma_i}{\omega}$ and taking $\frac{1}{\omega}$ common and simplify the coefficients of σ_i we get

$$\frac{1}{k}\frac{dk}{d\omega} = \frac{c_1\sigma_1 + c_2\sigma_2 + c_3\sigma_c + e}{ab\omega}$$

where c_1, c_2, c_3 and e are given expressions following equation (20).

[In simplifying the expressions for c_i's and e we make use of relations $f_i - k_if_i' = \omega f_i'$ and $f_i = f_i'(\omega + k_i)$].

C. Logarithmic Differentiation of G with respect to k

$$G = \frac{f_1'f_3'}{a_1f_1' + f_3'}$$

$$\frac{1}{G}\frac{dG}{dk} = \frac{1}{f_1'f_3'}\frac{d(f_1'f_3')}{dk} - \frac{1}{a_1f_1' + f_3'}\frac{d(a_1f_1' + f_3')}{dk}$$

$$= \frac{1}{f_1'f_3'(a_1f_1'+f_3')}\{(a_1f_1'+f_3')[f_1'f_3''\frac{dk_3}{d\omega}\frac{d\omega}{dk} + f_3'f_1''\frac{dk_1}{d\omega}\cdot\frac{d\omega}{dk}]$$

$$- f_1'f_3'[a_1f_1''\frac{dk_1}{d\omega}\frac{d\omega}{dk} + f_3''\frac{dk_3}{d\omega}\cdot\frac{d\omega}{dk}]\}$$

taking $\frac{d\omega}{dk}$ common and using the relation $f_i''\frac{dk_i}{d\omega} = -\frac{f_i'}{\omega+k_i}$ we have

$$\frac{1}{G}\frac{dG}{dk} = \frac{-\frac{d\omega}{dk}}{a_1f_1' + f_3'}\left[\frac{a_1f_1'}{\omega + k_3} + \frac{f_3'}{\omega + k_1}\right]$$