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On Production Functions and Elasticity of Substitution

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1. Introduction

It is not an overstatement to say that research in empirical

economics has tremendously increased both in quality and quantity ever

since Cobb and Douglas proposed the Cobb-Douglas production function

for finding relationships among economic variables. Two important

reasons for this are the simplicity of the Cobb-Douglas production

function from estimation point of view and that it explains most

economic data quite adequately. Even though the Cobb-Douglas production

function describes most economic data to a very good accuracy, it has

some serious drawbacks. The main ones being that it requires all

inputs to be positively employed and that it has a unitary elasticity

of substitution for all levels of factors. The generalized constant

elasticity of substitution (CES) production functions introduced by

Arrow, Chenay, Minhas and Solow (ACMS) [2] took care of the first of

the above criticisms. But if the elasticity of substitution were to

be an index for judging a production function as a means of explaining

relationships among economic variables the CES production function has

the same drawback as the Cobb-Douglas production function. Intuitively

speaking, one would expect that as one moves along a given isoquant the

elasticity of substitution to increase (decrease) as the relevant input

ratio goes to infinity as well as to zero. As Allen [1, p. 342] puts

it "the larger is the value of σ,[1] the flatter is the constant product curve and the more slowly does the marginal rate of substitution increase as B is substituted for A. The magnitude of σ is thus an indication of the ease with which product can be maintained by substituting B for A." Obviously the CES (and hence the Cobb-Douglas) production functions do not possess this desirable property. Recently some attempts have been made to find production functions which could meet this criticism. In this connection special reference[2] should be made to the works of Lu and Fletcher [3], Revanker [4] and Sato and Hoffman [6]. The general approach of these studies has been to assume that the elasticity of substitution is a linear function of the ratio of the two inputs and integrate the resulting differential equation to arrive at the implied production function. The resulting production functions, therefore do possess the property of exhibiting variable elasticity of substitution and, in a way, are a generalization of the CES production functions. But by the very nature of the assumptions made these production functions have the property that the elasticity of substitution is either a monotone increasing function of the input ratio or a monotone decreasing function of the input ratio. This means that the elasticity of substitution either reaches a maximum or a minimum as the input ratio increases (decreases). This property is against one's intuition because the elasticity of substitution of the input X_1 for X_2 is the same as the elasticity of substitution of X_2 for X_1. Therefore one would expect the

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[1] Here σ stands for the elasticity of substitution and A and B stand for the two inputs.

[2] Here no attempt is made to give a complete list of studies that have been done on this topic nor is any attention paid to the chronology.

elasticity of substitution to increase (decrease) as the input ratio increases as well as it decreases when the input ratio gets sufficiently small. In this paper a simple production function is presented. This production function includes as special cases the CES production function, the Lu-Fletcher [3] production function, the Revanker [4] and Sato-Hoffman [6] production functions and also is quite simple from estimation point of view. The production function is presented in section two. Section three presents some of the properties of the production function. Some generalizations and conclusions are presented in Section 4.

2. The Production Function

We assume the output Y is a function two[3] inputs, X_1 and X_2 the production function that is being proposed is given by

$$Y = F(X_1, X_2) = E(t) \, (\omega_{11} X_1^{2\rho} + 2\omega_{12} X_1^{\rho 1} X_2^{\rho 2} + \omega_{22} X_2^{2\rho})^{1/2\rho}. \qquad (1)$$

Here $E(t)$ stands (using ACMS terminology) for the efficiency parameter which also absorbs the neutral technical progress. ω_{11}, ω_{12} and ω_{22} are assumed to be non-negative. If the technical progress is neutral they will be independent of time, otherwise they are also functions of time (i.e., we have some kind of chronological ordering of the production process). We assume, without loss of generality, that $\omega_{11} + 2\omega_{12} + \omega_{22} = 1$. It is clear that F is homogeneous of degree one[4]

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[3] The case of more than two inputs is briefly discussed in Section 4.

[4] Increasing (decreasing) returns to scale can be covered by trivially extending (1) to $F(X_1, X_2) = E(t) \, (\omega_{11} X_1^{2\rho} + 2\omega_{12} X_1^{\rho 1} X_2^{\rho 2} + \omega_{22} X_2^{2\rho})^{\nu/2\rho}$ where $\nu > 1$ $(0 < \nu < 1)$.

in the inputs X_1 and X_2 (i.e., constant returns to scale). It is

assumed that $\rho_1 + \rho_2 = 2\rho$ and that ρ_1 and ρ_2 have the same sign as ρ.

This assumption is necessary to insure that the marginal products are

non-negative. This production function is a generalization of the

CES production function, the Lu-Fletcher production function and the

Revanker and Sato-Hoffman production function. For, by letting

$\omega_{12} = 0$; $\omega_{22} = 0$; and $\omega_{11} = 0$ (1) reduces to the CES, the Lu-Fletcher

and the Revanker-Sato-Hoffman production functions, respectively. We

will not pursue the production function (1) in that form. We will make

a slight simplifying assumption and then study its properties. Therefore,

from here on we will assume that $\rho_1 = \rho_2 = \rho$. This assumption does

restrict the form (1) to the degree that it has one parameter less.

But from an empirical research worker's point of view it is much

more easily adoptable and still maintains its essential characteristics.

With this assumption (1) can be written as

$$Y = F(X_1, X_2) = E(t) \left(\omega_{11} X_1^{2\rho} + 2\omega_{12} X_1^{\rho} X_2^{\rho} + \omega_{22} X_2^{2\rho} \right)^{1/2\rho} \tag{2}$$

It is clear that (2) and CES production functions are the same in the

following sense. The CES production functions and (2) are homogeneous

of degree 1/2 in the variables $X_{ij} = X_i X_j$; $i, j = 1, 2$. The only

difference being that the CES production functions assume the cross

term, ω_{12} to be zero. The parameter ω_{12} can be interpretted as a reaction

parameter between the inputs X_1 and X_2. Because of this slight modifi-

cation a great deal of variability is achieved by using the form (2)

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[5]It should be noted that this is only one of several ways of
looking at (1) as a generalization of the CES production function.

as opposed to the CES production function or its generalizations [2, 4, 6]
as will be seen from the following discussion.

3. Properties of the Production Function

The production function (2) is homogeneous of degree one in the
inputs X_1 and X_2 and the variables $Y^{1/\rho}$; X_1^{ρ} and $X_2^{1/\rho}$ are related by
a quadratic form. More specifically (2) can also be written as

$$(y \ x_1 \ x_2) A (y \ x_1 \ x_2)' = 0 \tag{3}$$

where A is the 3 x 3 matrix given by

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & E^{2\rho} \omega_{11} & E^{2\rho} \omega_{12} \\ 0 & E^{2\rho} \omega_{12} & E^{2\rho} \omega_{22} \end{pmatrix} \tag{4}$$

and $y = Y^{1/\rho}$, $x_1 = X_1^{1/\rho}$ and $x_2 = X_2^{1/\rho}$. It can be quickly verified
that the marginal products are non-negative. In the admissable space
of the parameters the second direct partial derivatives are negative
and the second cross partial derivative is positive as will be seen
below (equation (5)). The factor elasticity of substitution, σ is
given by

$$\sigma(x) = \frac{1}{1 - \rho + R} \tag{5}$$

where

$$R = \frac{-\rho(\omega_{11} \ \omega_{22} - \omega_{12}^{2})}{(\omega_{11} \ x^{-\rho} + \omega_{12}) (\omega_{12} + \omega_{22} \ x^{\rho})}$$

and $x = X_2/X_1$. Since

$$\frac{\partial R}{\partial x} = \frac{\rho^2(\omega_{11}\omega_{22} - \omega_{12}^2)\,\omega_{12}\,x^{-\rho-1}\,(\omega_{22}x^{2\rho} - \omega_{11})}{(\omega_{11}x^{-\rho} + \omega_{12})^2\,(\omega_{12} + \omega_{22}x^\rho)^2}$$

assuming[6] $\omega_{11}\omega_{22} - \omega_{12}^2 > 0$, R increases (decreases) with x as long as $x \gtrless (\omega_{11}/\omega_{22})^{1/2\rho}$, $(x < (\omega_{11}/\omega_{22})^{1/2\rho}$ and increases for $x > (\omega_{11}/\omega_{22})^{1/2\rho}$. When $\rho < 0$, σ decreases up to the point $x = (\omega_{11}/\omega_{22})^{1/2\rho}$ and increases for $x > (\omega_{11}/\omega_{22})^{1/2\rho}$. Therefore when $\rho > 0$ the maximum elasticity is attained at $x = (\omega_{11}/\omega_{22})^{1/2\rho}$ and equals σ_U^+ where

$$\sigma_U^+ = \frac{1}{1 - \rho - \rho\,\dfrac{\omega_{11}\omega_{22} - \omega_{12}^2}{(\omega_{11}^{1/2}\omega_{22}^{1/2} + \omega_{12})^2}} \tag{6}$$

and the minimum elasticity of substitution is attained as x tends to its two end points, 0 and ∞ and equals σ_L^+ where

$$\sigma_L^+ = \frac{1}{1 - \rho} \tag{7}$$

This implies that $\rho \leq \dfrac{(\omega_{11}^{1/2}\omega_{22}^{1/2} + \omega_{12})^2}{2(\omega_{11}\omega_{22} + \omega_{11}^{1/2}\omega_{22}^{1/2}\omega_{12})}$.

When $\rho < 0$ the maximum elasticity is attained as x approaches 0 and ∞ and equals σ_U^- where

$$\sigma_U^- = 1/1 - \rho \tag{8}$$

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[6]If $\omega_{11}\omega_{22} - \omega_{12}^2 < 0$ the words increase and decrease interchange their roles. But $\omega_{11}\omega_{22} - \omega_{12}^2 > 0$ is likely to be the case on intuitive grounds.

The minimum elasticity is attained at $x = (\omega_{11}/\omega_{22})^{1/2\rho}$ and equals σ_L^-, where

$$\sigma_L^- = \cfrac{1}{1 - \rho - \rho \cfrac{\omega_{11}\,\omega_{22} - \omega_{12}^2}{(\omega_{11}^{1/2}\,\omega_{22}^{1/2} + \omega_{12})^2}} \qquad (9)$$

It can be easily verified that (2) tends to (i) a Cobb-Douglas production function as $\rho \to 0$ (ii) a Leontief production function as $\rho \to -\infty$ and (iii) a linear[7] production function as $\omega_{12} \to 0$ and $\rho \to 1/2$. The ratio of the factor earnings is given by $X_1 F_1 / X_2 F_2$ and equals

$$\frac{X_1 F_1}{X_2 F_2} = \frac{\omega_{11} + (1 - \omega_{11} - \omega_{22})x^\rho}{(1 - \omega_{11} - \omega_{22})x^\rho + \omega_{22}\, x^{2\rho}}$$

4. Summary and Conclusions

In this paper we have extended the CES production functions to a class of production functions which are homogeneous and which possess the property that the elasticity of substitution varies with the input ratio as one moves along an isoquant and which is symmetric with respect to the two end points, 0 and ∞ of the input ratio. Even though the discussion of the paper has confined to two inputs the generalization to more than two inputs is immediate. One generalization to more than two inputs is clear. For, (2) can be written, when k inputs are employed as

$$Y = F(X_1, \ldots, X_k) = E(t)\left(\sum_{i=1}^{k} \omega_{ii}\, X_i^{2\rho} + 2\sum_{i<j} \omega_{ij}\, X_i^\rho\, X_j^\rho\right)^{1/2\rho} \qquad (10)$$

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[7]Of course, this is only one of several ways the way (2) approaches this form.

where $\omega_{ij} \geq 0$, i, j = 1,..., k and $\sum_{i=1}^{k} \omega_{ii} + \sum_{i<j} \omega_{ij} = 1$. One can also

compute the Allen or Uzawa-McFadden partial elasticities of substitution.

But, when more than two inputs are involved both the Allen elasticities

of substitution and the Uzawa-McFadden elasticities of substitution will,

in general, involve all the input ratios and they will not be as simple

as (5). If there are reasons to believe that there are internal economics

of scale (or the economies of scale are a function of the output) one

can make a transformation of (2) like, say Revanker and Zellner [5] to

get a production function which exhibits internal economies and still

maintains the properties (5), (6), (7), (8) and (9). To characterize

the production function (2), in general, is not possible. For, for

every $\sigma(x)$ there corresponds a production function and in most cases

it is not possible to solve for the specific form of the production

function for a given elasticity of substitution function $\sigma(x)$. The

next step is to see how good the production function explains

relationships among economic variables, particularly in those cases

where the CES and Cobb-Douglas production functions fail to give a

reasonable fit. It should be noted that the number of parameters in (2)

exceed that of the CES production function just by one. Therefore any

substantial gain that is achieved by fitting (2) rather than the CES

or the Cobb-Douglas production functions could be attributed to the form

of the production function and not due to the many additional parameters

introduced. The author is presently doing a study comparing these three

production functions using data for which the correlation coefficient

obtained by using CES or Cobb-Douglas production function is relatively

low. These results will be reported in a later study.

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