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In 1951 Samuelson [6] showed that in production models with intermediate inputs and a single primary factor (usually considered to be labour), even if the production functions allow substitution between factors, such substitution will never take place. In equilibrium there will be a single production coefficient used in each industry and furthermore the autarky production possibility surface will be linear.[1] The purpose of this note is to present a simple diagrammatic demonstration of this theorem and to show that for the two good case, when both functions are Cobb-Douglas, exact expressions for all the parameters of the model can be derived. Furthermore the derivation will illustrate another interesting feature of the model; namely that the two production functions can be written as functions of the labour inputs only, and that these functions will be homogeneous of degree one in labour if the original functions are homogeneous of degree one in labour and the intermediate input.[2]

Samuelson's theorem is illustrated graphically in Figure 1. We assume that two goods, X_1 and X_2, are each produced with homogeneous production functions using labour and the output of the other industry as inputs. Thus the production functions are

(1) $\quad X_1 = f_1(X_{21}, L_1)$

(2) $\quad X_2 = f_2(X_{12}, L_2)$,

where X_{ij} is the quantity of commodity i used to produce commodity j , and L_j is the amount of labour used in the production of commodity j . Labour is assumed to be in fixed supply so that $L_1 + L_2 = L$ where L is some positive constant. In Figure 1 outputs are measured in a positive direction and inputs in a negative direction.[3] The curve f_1^* is the total product curve for commodity 1 ; it shows the relation between the levels of output of X_1 and the levels of input of X_{21}

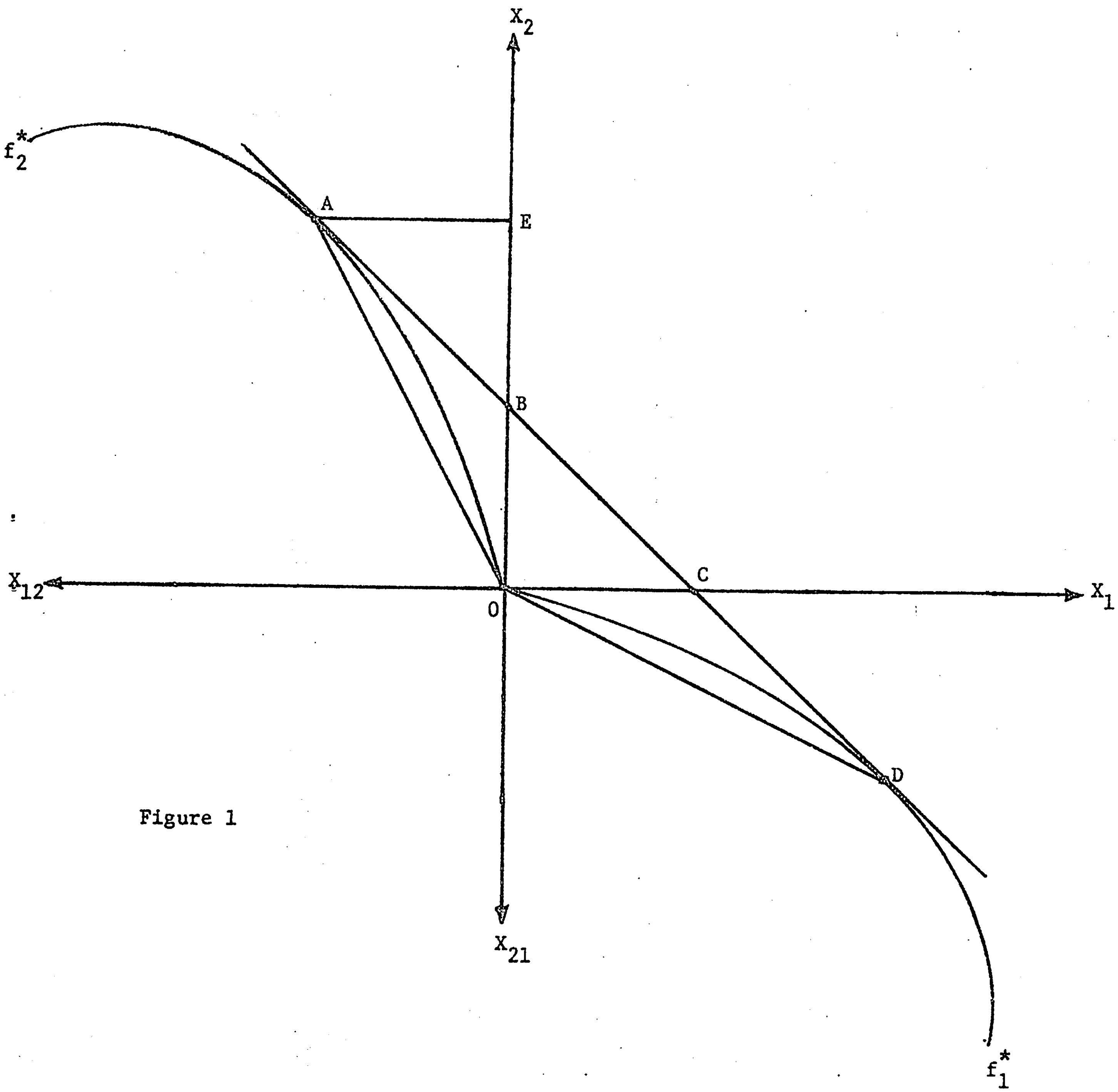

Figure 1

under the assumption that $L_1 = L$. Thus any point in f_1^* completely exhausts the labour supply. The curve f_2^* is the corresponding curve for commodity 2.

Since both f_1^* and f_2^* assume the use of the total labour supply, possible production points will be the convex hull of all points in the production sets defined by these two curves.[4] One set of possible production points is the line ABCD and it is clear that this line contains all the efficient production points. Of course in autarky only the points on BC are possible, for points on AB and CD imply negative net outputs of one or the other of the two commodities.[5] The Samuelson result is now clear. First, the autarky production possibility curve, BC, is linear, and second, although many combinations of the two inputs are possible, as represented by all the points along f_1^* and f_2^*, only one such point for each industry will ever be chosen; A in industry 2 and D in industry 1. All efficient production points are convex combinations of A and D. Observe that the reciprocals of the slopes of OA and OD give the production coefficients for the two industries.

In Figure 1 we diagrammatically solved the pair of equations (1) and (2) for the equilibrium commodity price ratio (the slope of AD) and for the two input-output coefficients (the reciprocals of the slopes of OA and OD). And of course the line segment BC represents all the possible efficient autarky equilibrium points. We now want to demonstrate how these solutions can be obtained algebraically for the Cobb-Douglas case. For Cobb-Douglas production functions equations (1) and (2) become:

$$(3) \qquad X_1 = X_{21}^{\alpha_1} L_1^{\beta_1}$$

$$(4) \qquad X_2 = X_{12}^{\alpha_2} L_2^{\beta_2}$$

where $\alpha_i + \beta_i = 1$, and $\alpha_i > 0$, $\beta_i > 0$, $(i = 1,2)$. For the first industry, maximizing profits and solving for the demand for the intermediate input we obtain:

$$(5) \qquad X_{21} = \left[\alpha_1 \cdot \frac{P_1}{P_2}\right]^{1/\beta_1} \cdot L_1 \ .$$

Observe that this expression is homogeneous of degree one in L_1. Now from (5) and the equation for the marginal product of labour we can derive:

$$(6) \qquad \frac{w_1}{P_2} = \beta_1 \alpha_1^{\alpha_1/\beta_1} \left[\frac{P_1}{P_2}\right]^{1/\beta_1} \ .$$

The same procedure for the second industry will yield

$$(7) \qquad \frac{w_2}{P_1} = \beta_2 \alpha_2^{\alpha_2/\beta_2} \left[\frac{P_2}{P_1}\right]^{1/\beta_2} \ .$$

Equations (6) and (7) give the wage rates in the two industries, normalized by the price of the intermediate input, as functions of the commodity price ratio. Since $0 < \beta_i < 1$, $i = 1,2$, it is clear that both equations describe monotonic increasing functions of their arguments and that both functions increase at an increasing rate. In Figure 2 equation (6) is plotted in the upper half of the diagram and equation (7) in the lower half, with 0 being the common origin for both graphs. Note that both equations are plotted as functions of P_1/P_2.

These two functions give, for the two industries, the ratio of input prices implied by any commodity price ratio. In equilibrium, however, the wages in the two industries must be equal, and so the problem we must solve is whether there exists any P_1/P_2 such that $w_1 = w_2$. Or in other words does there exist a commodity price ratio which will allow production by both industries. To answer this question we divide (6) by (7) to obtain

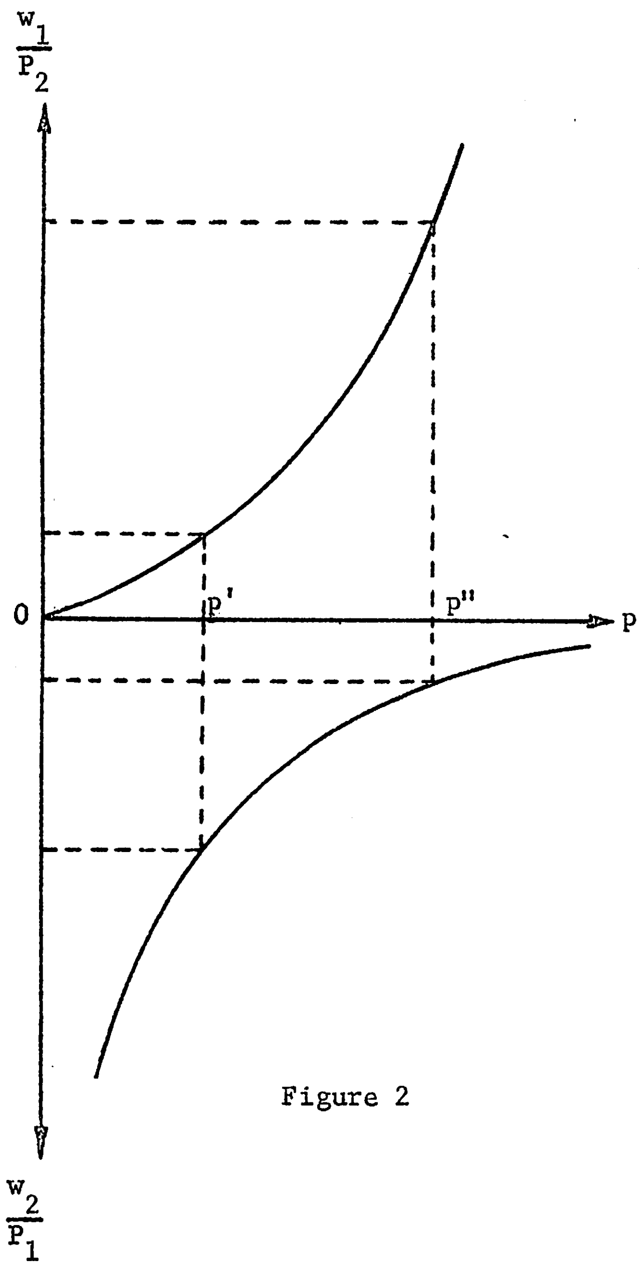

Figure 2

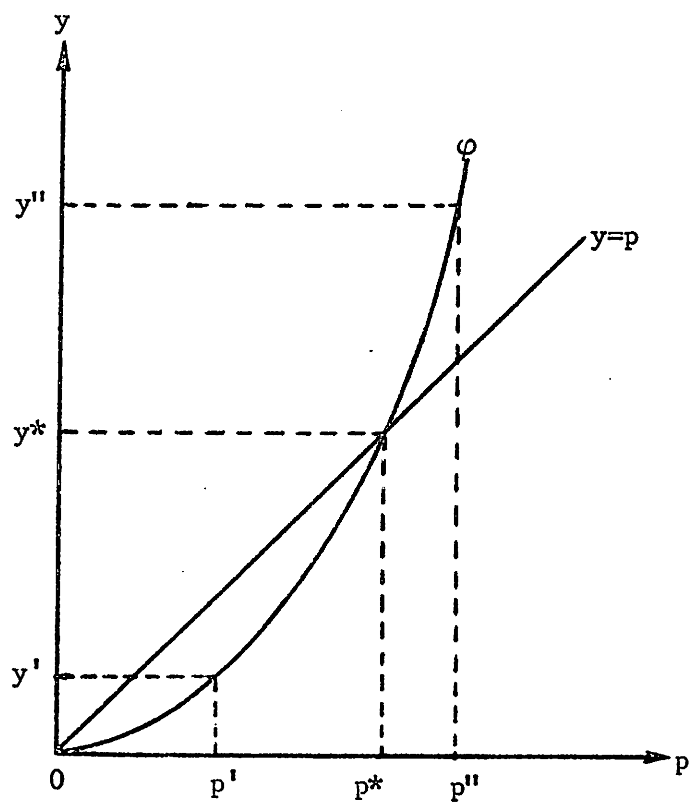

Figure 3

$$(8) \qquad \frac{w_1 P_1}{w_2 P_2} = \frac{\beta_1 \alpha_1^{\alpha_1/\beta_1}}{\beta_2 \alpha_2^{\alpha_2/\beta_2}} \cdot \left[\frac{P_1}{P_2}\right]^{1/\beta_1 + 1/\beta_2} .$$

which we can write as

$$(9) \qquad y = \varphi(p) ,$$

where $p = P_1/P_2$ and where y is some real number. Observe that when $w_1 = w_2$, $y = p$. Thus we want to know if there exists a p, p^* say, such that

$$y^* = \varphi(p^*)$$

where $p^* = y^*$. Diagrammatically we can divide the upper function of Figure 2 by the lower one to obtain φ of Figure 3. In Figure 3 we have drawn in the line $y = p$ for reference purposes, and the question of whether or not an equilibrium p exists becomes the question of whether or not the two functions $y = p$ and $y = \varphi(p)$ have a solution, or in other words, does φ cross the line $y = p$.[6] From (8) it is obvious that there are values of p such that $y > p$ and values such that $y < p$, and then from continuity the existence of p^* is established. Indeed, setting $w_1 = w_2$ in (8) and solving for p we obtain[7]

$$(10) \qquad p = \frac{P_1}{P_2} = \left[\frac{\beta_2 \alpha_2^{\alpha_2/\beta_2}}{\beta_1 \alpha_1^{\alpha_1/\beta_1}}\right]^{\frac{\beta_1\beta_2}{\beta_1 + \beta_2 - \beta_1\beta_2}}$$

where the exponent must be positive since $\beta_i < 1$, $(i = 1,2)$. It is clear that for any choice of the production parameters (10) exists, is positive and is unique. Observe that (10) depends only on the values of the α's and β's, and thus the equilibrium commodity price ratio is known as soon as the production functions are specified. In terms of Figure 1, p is the slope of AD.

Now that we have an exact expression for the output price ratio, all other parameters of the equilibrium are now exactly determined as well. For example it is easily shown that

(11) $X_1 = \gamma_1 L_1$

where $\gamma_1 = (\alpha_1 p)^{\alpha_1/\beta_1}$, and that

(12) $a_{21} = \alpha_1 p$

where $a_{21} = X_{21}/X_1$, the intermediate input production coefficient for industry 1. Equation (12) gives the reciprocal of the slope of OD in Figure 1, and (11) gives the gross output of X_1 as a linear function of the amount of labour used. Being linear, (11) is obviously homogeneous of degree 1 in labour, and it illustrates the fact that when output price ratios are known, even though the production functions may have intermediate inputs, they can be written as functions of the primary factors alone. Exactly analogous expressions can be derived for industry 2.

As a final step we want to derive an expression for the net output of one commodity in terms of the other, i.e., to derive an expression for BC in Figure 1.[8] This can be done by substitution using the various expressions shown above or more simply can be derived from the geometry of Figure 1. Defining $X_i^n = X_i - X_{ij}$ it is clear that we have

(13) $X_2^n = a_o - bX_1^n$

where a_o is the intercept (point B) and b is the slope of BC which we have shown to be equal to p. To find an expression for a_o observe that, from Figure 1, a_o = OE - BE. OE is equal to the maximum producible quantity of X_2 consistent with the price ratio p, which we know from (11) to be $\gamma_2 L$, where $L = L_1 + L_2$. Using the fact that AE/OE = a_{12}, and with the slope of AB = p, it can be shown that EB = $\alpha_2 \gamma_2 L$ so that $a_o = \beta_2 \gamma_2 L$. Thus (13) has the specific form

(14) $X_2^n = \beta_2 \gamma_2 L - pX_1^n$.

This simple expression for the transformation curve is interesting in the light of the difficulty which is encountered when one attempts to derive such a function for the case where the two inputs are capital and labour rather than labour and an intermediate input.[9]

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FOOTNOTES

1. By the autarky production possibility surface we mean the set of all efficient (net) production points, all of whose elements are non-negative.

2. For the general proof of this proposition see [5].

3. This kind of diagram was first used by Georgescu-Roegen [1]. For a more recent use see [3].

4. The convex hull of two sets A and B can be defined as the intersection of all the convex sets containing A and B. In this case the set A, for example, would be all the points on and below the curve f_2^*.

5. For a trading situation we would also want to include in the set of efficient points the points on f_2^* to the left of A (up to the point where f_2^* reaches a maximum) and the points on f_1^* below D (up to the point where f_1^* reaches a maximum). However, since we are not concerned with the possibility of trade, such points are not of interest.

6. Note that what we are asking here is whether the function (9) has a fixed point. In this simple case it is obvious that it has, as can be seen from equation (10). This is not always true, however, and this property depends on the fact that the functions are Cobb-Douglas. Even for the popular CES functions an output price ratio allowing the production of both goods cannot in general be assumed to exist, as is shown in [4].

7. Since hereafter we will refer only to this equilibrium price ratio we will drop the asterisk.

8. It should perhaps be observed that while this model will always allow some positive net output of at least one of the goods, this depends on the fact that the production functions are Cobb-Douglas. For a mathematical treatment of this problem see [4].

9. For example, see [2]. Of course, in the case where the inputs are capital and labour the transformation curve is not linear.