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Most general economic theories assume that waste is freely disposed of, or that its accumulation has no cost to society. This paper will introduce waste disposal into a simple economy with one produced consumer good where waste accumulation provides flows of disutility over time.

Waste in this model may represent pollutants, sewage, or garbage, the residual of consumption. It is assumed that whenever production takes place, a proportion will ultimately result in waste. Examples are old newspapers, beer containers and sulphur dioxide. This waste may be biodegradable, or be eliminated (at a cost) by man, or accumulated.

The model will be a variation of the simple Ramsey model.[1] The goal will be a maximization of discounted utilities over an infinite horizon. Because of the publicness property of waste, disposal services will not be voluntarily provided in a competitive economy by a private sector. The problem is a control problem and central regulation is warranted.[2] It is hoped that there will be an optimal level of waste and that the theory will describe how to achieve this optimal.

The Model

Assume there is constant population. The labour force is a constant proportion of this population, and is fully employed. Assume a fixed amount L of a variable input (which may be a capital-efficiency unit of labor mix) is to be allocated between two sectors. Sector 1 consists of production of a consumer good in amount $Y_1(t)$

with a technology defined by $f(L_1(t))$ where $C_1(t)$ is consumed.

$$C_1(t) \leqq f(L_1(t)).$$

In period t, a fixed proportion γ of output accumulates as waste, and a constant proportion α of the existing stock of waste $G(t)$ biodecomposes. Moreover some variable input $L_2(t)$ is assigned to production of waste disposal services. Production in this sector is defined by $g(L_2(t))$.

Thus waste accumulates according to

$$\dot{G}(t) = \gamma(f\ L_1(t)) - g(L_2(t)) - \alpha\ G(t).$$ [3]

The waste stock $G(t)$ provides a flow of disservices $C_2(t)$ assumed proportional to the existing stock. The units are chosen so that the constant of proportionality is unity.

The problem a controller must solve is to maximize

$$\int_0^{\infty} U(C_1, C_2) e^{-\delta t}\, dt \qquad (1)$$

subject to

$$f(L_1) \geqq C_1 \qquad (2)$$

$$G = C_2 \qquad (3)$$

$$L = L_1 + L_2 \qquad (4)$$

and

$$\dot{G} = \gamma\ f(L_1) - g(L_2) - \alpha\ G \qquad (5)$$

where δ represents a social rate of discount, assumed positive.

Assume for the present time that

$$U(C_1, C_2) \equiv u(C_1 + v(C_2)$$

and that

$$u'(C_1) > 0 \qquad u''(C_1) < 0 \qquad u(0) = 0,\ u'(0) = \infty$$

$$v'(C_2) < 0 \qquad v''(C_2) < 0 \qquad v(0) = 0$$

Thus U is a concave function.

Assume $f', g' > 0$ and $f'', g'' < 0$ for all positive L_i.

The Maximum Principle[4] states that there exists an auxiliary variable p_t and Lagrangian multipliers λ_1, λ_2 and w such that (for an interior solution) the following conditions are necessary for an extremum:[5]

$$u'(C_1) = \lambda_1 \tag{6}$$

$$v'(C_2) = \lambda_2 \tag{7}$$

$$(p\gamma+\lambda_1)f'(L_1) = w = -p\ g'(L_2) \tag{8}$$

$$\dot{p} = p(\delta + \alpha) - \lambda_2 \tag{9}$$

Now $w, \gamma > 0$ and so $p < 0$ and $\lambda_1 > 0$ from (8).

Hence $C_1 = f(L_1)$

One may interpret p(t) as the imputed price of an addition to the stock of waste and w as a real wage.

$$\lambda_1 = u'(C_1) = u'(C_1(L_1))$$

so $$\frac{d\lambda_1}{dL_1} = u''f' < 0$$

Equation (8) defines L_1 implicitly as a function of p.

Define $M(p, L_1) \equiv (p\gamma+\lambda_1)f'(L_1) + pg'(L-L_1) = 0$

where $$M_2 = \frac{\partial M}{\partial L_1} = (p\gamma+\lambda_1)f'' + f'\frac{d\lambda_1}{dL_1} - pg''$$

$$= w\frac{f''}{f'} + f'(\frac{d\lambda_1}{dL_1}) + w\frac{g''}{g'} < 0$$

$$M_1 = \frac{\partial M}{\partial p} = \gamma f' + g' > 0$$

By the Implicit Function Theorem[6] one may write

$$L_1 = F(p) \quad \text{where} \quad f(p) \text{ is continuous at } p$$
$$f'(p) \text{ is continuous at } p$$

$$\text{and} \quad F' = -\frac{M_1}{M_2} > 0$$

So L_1 is an increasing function of p and

L_2 is a decreasing function of p .

Since p is negative these conditions state that as the imputed price of waste accumulation increase, for optimal allocation of L one should produce less waste disposal. Alternatively if p decreases, G increases and the rule says to produce more waste disposal.

Steady-State Solution

Consider next the steady-state defined by $\dot{p} = \dot{G} = 0$ for the autonomous system of differential equations:

$$\dot{p} = p(\delta+\alpha) - v'(G)$$
$$\dot{G} = \gamma\, f(L_1(p)) - g\, L_2(p) - \alpha\, G$$

Consider the graph of $\dot{p}=0$ in phase space. $\dot{p}=0$ implies $R(p,G) \equiv p(\delta+\alpha) - v'(G) = 0$

So $R_G = -\, v''(G)$ where $v''(G) < 0$

$R_p = (\delta+\alpha) > 0$

$$\text{So} \quad \left.\frac{dp}{dG}\right)_{\dot{p}=0} = -\frac{-v''(G)}{\delta+\alpha} = \frac{v''(G)}{\delta+\alpha} < 0$$

Consider next the graph of $\dot{G}=0$

$$\left.\frac{dp}{dG}\right)_{\dot{G}=0} = \frac{\alpha}{\gamma f' \frac{dL_1}{dp} - g' \frac{dL_2}{dp}} \gtreqless 0 \quad \text{as} \quad \alpha \gtreqless 0$$

Assume the curves $\dot{p}=0$ and $\dot{G}=0$ intersect at (G^*,p^*) as illustrated in Figure I below.

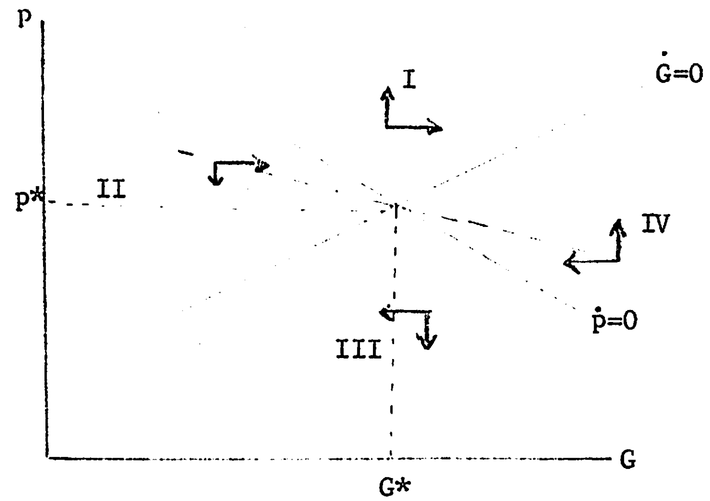

Figure 1

Regions are labelled with Roman numerals for identification.

Directions of movements of points in phase space are determined next. For fixed G, R(p,G) is an increasing function of p. So $\dot{p} < 0$ ($\dot{p} > 0$) for points below (above) the line $\dot{p}=0$.

For fixed G, $\gamma f(L_1(p)) - g(L_2(p)) - \alpha G$ is an increasing function of p. Hence $\dot{G} < 0$ below the line $\dot{G} = 0$ and $\dot{G} > 0$ above it. Directions are indicated in Figure 1.

In an appendix it is shown that (G^*,p^*) is a saddle-point. The optimal trajectory is indicated in Figure 1 by a jagged line.

Achieving Optimality through Taxation

A central planner has information, through the optimal trajectory of an optimal price p_t for each level of waste accumulation G_t. It can use this price to devise a taxing scheme on the producers in Sector 1. This is a generalization of the policy of fining firms who produce excesses of pollutants currently evidenced in urban areas, and for firms polluting lakes and rivers.

Suppose output Y_1 is produced by a single producer which maximizes discounted profits and treats input price w and output price λ_1 as parameters. This firm will maximize

$$\int_0^\infty (\lambda_1 f(L_1) - wL_1)e^{-\delta t}dt \text{ with respect to } L_1.$$

First order necessary conditions for this maximum are

$$\lambda_1 f' = w \quad (10)$$

[7]

which differs from the optimal allocation of L given in (8) by amount $p\,\gamma\, f'(L_1)$.

If the central authorities tax output (production) with per unit tax of amount $(-p\gamma)$ the firm will hire L up to the point where

$$\lambda_1 f' + p\gamma f' = w \quad (11)$$

which will provide a socially acceptable allocation.

At the same time the central authorities can undertake waste disposal at an imputed price of (-p) per unit, or contract out the job at this rate. In a private economy services will be provided according to

$$w = -pg'(L_2)$$

which is optimal. Budget deficits or surpluses can be financed (or redistributed) in a non-distorting way, such as by per head taxes or subsidies.

Comments and Conclusions

The basic model as presented here is a gross simplification. It seems reasonable to assume for some applications of the model, that waste accumulation makes production more costly. This cost may be internal or external, and could be treated in the above framework with some modification.

Another simplification, that $U_{C_1C_2} = 0$, was made for expository reasons. In many cases the existence of waste in fact affects the enjoyment of consumption and one would expect $U_{C_1C_2} < 0$. Examples are garbage in a picnic area, and swimming in a polluted lake.

A slight modification in the specification of $U(C_1,C_2)$ can be made as follows. Assume[8]

a) $U_{C_iC_i} < 0$ and $J = U_{C_1C_1} U_{C_2C_2} - U^2_{C_1C_2} > 0$

b) $\dfrac{U_{C_1C_1}}{U_{C_1}} - \dfrac{U_{C_1C_2}}{U_{C_2}} < 0, \quad \dfrac{U_{C_2C_2}}{U_{C_2}} - \dfrac{U_{C_1C_2}}{U_{C_1}} > 0$

Concavity of $U(C_1,C_2)$ still holds, and the above presentation is valid with minor alterations.

It is not surprising that the optimal level of G exceeds zero. It is interesting to note that if $v''(G) > 0$ no non trivial steady state exists. In this case, presumably no waste disposal is optimal, because as waste accumulates marginal units become less objectionable.

Although garbage, sewage and chemical pollution of air and water fit best into the model perhaps noise pollution and other psychic "bads" related to development and urbanization apply.

Extension of the model in two directions seems warranted. First, population increases should be considered. However, for an urban area it may not be reasonable to assume exponential growth.

Next the model should incorporate a theory of capital.[9] The accumulation of waste will change the optimal saving models. The existence of two or more state variables makes the mathematics difficult. Recent models by Uzawa [8], and Arrow and Kurz [2] which include public goods or public capital can be used where the public good is pollution control, or the public capital is used for pollution control.

Footnotes

[1]See for example Arrow [1].

[2]Urban governments already provide direct waste disposal services such as those related to garbage and sewage. Pollution control has recently become more common through systems of fines. Some areas have used garbage recently as a fill to construct ski slopes, and an artificial island.

[3]$\dot{X}$ will be used to denote $\frac{dX}{dt}$. Subscripts t will be dropped in the development except where their inclusion is instructive.

[4]See Arrow [1] p. 14, Theorem 6, Hestenes [4] or Pontryagin *et. al*. [5].

[5]If a steady state exists (p^*, G^*) the transversability conditions will be met since $\lim_{t \to \infty} e^{-\delta t} \tilde{p}^* G^* = 0$

and $\lim e^{-\delta}$

[6]See Taylor [7], p. 241.

[7]This condition holds if the producer assumes λ_1,w are constant, or if he knows the time paths $\lambda_1(t)$,w(t).

[8]For a comparison in the case of a public good instead of G, see Uzawa [8], p. 131.

[9]See Cass [3]

APPENDIX: To show (G^*,p^*) is a Saddle Point.

Evaluated at (G^*,p^*)

$$\frac{dp}{dG} = \frac{\frac{d}{dG}[p(\delta+\alpha) - v'(G)]}{\frac{d}{dG}[\gamma\ f(L_1(p)]-g(L_2(p)) - \alpha\ G]}$$

$$p' = \frac{(\alpha+\delta)p' - v''}{\gamma\ f'L_1'p' - g'L_2'p' - \alpha}$$

$$p'^2[\gamma\ f'L_1'-gL_2'] - (\delta+2\alpha)p' + v'' = 0 \qquad \text{(A1)}$$

Equation (A1) is quadratic in p'. The sign of the product of the roots is the same as the sign of $v''[\gamma\ f'L_1' - g\ L_2']$ and hence negative. Hence at (G^*,p^*), $\frac{dp}{dG}$ is real and has two values opposite in sign. So (G^*,p^*) is a saddle point. The optimal trajectory corresponds to the negative root.

The same result can be achieved by expanding $\dot{p}$ and $\dot{G}$ about (G^*,p^*) as a Taylor series and showing the characteristic roots are real and of opposite sign.

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