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EXPLOITATION OF COMMON PROPERTY REPLENISH-
ABLE RESOURCES: AN EXTENSION

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EXPLOITATION OF COMMON PROPERTY REPLENISHABLE NATURAL
RESOURCES: AN EXTENSION*

Recently articles have appeared illustrating the dynamic properties of natural resource exploitation.

Cummings and Burt [3] employ the principles of dynamic programming to find an efficient intertemporal production theory for a mining firm. They use a discrete time, finite horizon model to treat an exhaustible resource.¹ Their result incorporates the statical concept of 'user cost'² to distinguish their sole ownership model from the model of Vernon Smith³ where the resource is non-appropriated.

I have presented a simple model⁴ of replenishable⁵ resource allocation using the Pontryagin Maximum Principle⁶ with continuous time and an infinite horizon in a one good economy to analyze maximum sustained yield conservation programs. This model applies fairly well to fisheries which are non-appropriated. The conclusion of the analysis was that maximum sustained yield programs are non-optimal.

The present paper is meant to generalize my previous model by including costs of production in a two-good economy. Once again the validity of maximum sustained yield conservation programs will be questioned. The presentation will be more general than earlier works by Crutchfield and Zellner [2] and Smith [16] in that a welfare functional will be maximized whereas their models were concerned with efficiency in production.⁷ Important issues such as mesh size and overcrowding as treated by Smith will not be included in order

* This paper is taken in part from a Ph.D. dissertation written at the University of Minnesota. I wish to thank my adviser Professor Edward Foster, and Professors John Chipman and Thomas Muench for helpful comments. All errors are the responsibility of the author.

to more efficiently attack the primary problem.

This paper will include development of a model with several restrictive assumptions on tastes and technologies in order to achieve unambiguous results. Following this the problem of control will be addressed, and a method of taxing described which will allow the control solution to be achieved within a competitive structure. The tax will have the effect of appropriating the common-property resource, or internalizing the externality.

The Model

Assume there are two consumer goods Y_1 and Y_2 and one variable input of production L , fixed in total supply and fully employed.

Production of resource product Y_2 is summarized by the function $G(L_2, N)$ where N is the available natural resource mass and L_2 is the amount of L allocated to the resource sector. For convenience G will be specified as Cobb-Douglas. Hence G is continuous with continuous second partial derivatives, has positive marginal products for all positive amounts of inputs, and $G_{ii} < 0$. L_2 and N are indispensable.

Production in the other sector is summarized by specification of $F(L_1)$ where $F' > 0$, $F'' < 0$ and $F(0) = 0$.

Input L may be interpreted as labor, or a mixture of many variable inputs.

Assume human population is constant over time, and that there exists a social discount rate $\delta > 0$.

It is required that the welfare functional

$$\int_0^{\infty} U(C_1, C_2) e^{-\delta t} dt \quad (1)$$

be maximized subject to the constraints

$$G(L_2, N) - C_2 \cong 0 \quad (2)$$

$$F(L_1) - C_1 \cong 0 \quad (3)$$

$$L_1 + L_2 = \bar{L} \quad (4)$$

and

$$\dot{N} = \lambda N - \epsilon N^2 - G(L_2, N) \quad (5)$$

C_i is the amount of good Y_i consumed.

Equation (5) represents the biological growth rate of the resource mass after human extraction of amount $G(L_2, N)$ ⁸. Assume $U(C_1, C_2) \equiv U(C_1) + V(C_2)$.

See footnote 10.

$$U'(*) > 0, \quad U''(*) < 0, \quad U'(C_1) \rightarrow \infty \text{ as } C_1 \rightarrow 0$$

$$V'(*) > 0, \quad V''(*) < 0, \quad V'(C_2) \rightarrow \infty \text{ as } C_2 \rightarrow 0$$

The Maximum Principle states the necessary conditions for maximization subject to initial conditions

$N(0) = N_0 > 0$ are that there exist an auxiliary variable p_t and Lagrangian multipliers q_1, q_2, w ⁹ such that controls L_1, L_2, C_1, C_2 , instantaneously maximize H , the current-value Hamiltonian (for all values of N_t)

where

$$H = U(C_1) + V(C_2) + p[\lambda N - \epsilon N^2 - G(L_2, N)]$$

and

$$\begin{aligned} L = & U(C_1) + V(C_2) + p[\lambda N - \epsilon N^2 - G(L_2, N)] \\ & + q_1 [F(L_1) - C_1] + q_2 [G(L_2, N) - C_2] \\ & + w (\bar{L} - L_1 - L_2) \end{aligned}$$

and such that

$$\dot{p} = \delta p - p(\lambda - 2\epsilon N - \frac{\partial G}{\partial N}) - q_2 \frac{\partial G}{\partial N} \quad (6)$$

$$\dot{N} = \lambda N - \epsilon N^2 - G(L_2, N) \quad (5)$$

evaluated at the optimal values of the controls. Thus

$$a) \quad U'(C_1) = q_1$$

$$b) \quad V'(C_2) = q_2$$

$$c) \quad (q_2 - p)G_1 = w = q_1 F'$$

q_1 thus represents the price of output in Sector 1 and q_2 the price a consumer would pay for a unit of resource product.¹⁰

The auxiliary variable p represents the imputed demand price of investment in terms of present consumption foregone. For example, p represents the price placed upon a unit of fish in the ocean as an investment good.

A central planner would be interested in controlling variable input such that, from a social point of view, marginal revenue products are equal in both sectors. That equality is stated in (c) above.

In Sector 1 the value of the marginal product is simply $q_1 F'$, the implicit price of a unit of output. In Sector 2 it is $q_2 G_1$, the implicit price of a unit of current output, minus $p G_1$, the 'cost' of a unit of current output in terms of future consumption foregone.

Condition (c) will be referred to as a condition that input L be used "efficiently".

L_1 and L_2 as functions of N and p

For any given values (\bar{N}, \bar{p}) of the state and auxiliary variables, (c) states that

$$(q_2 - \bar{p})G_1(L_2, \bar{N}) = q_1 F'(L_1)$$

From (a) and (b)

$$\{V'(G(L_2, \bar{N})) - p\}G_1(L_2, N) = U'(F(L_1)) \cdot F'(L_1) \quad (7)$$

This relationship implicitly defines L_1 and L_2 as functions of N and p .

Define

$$M \equiv \{V'(G(L_2, N)) - p\}G_1(L_2, N) - U'(F(L-L_2)) \cdot F'(L-L_2) \quad (8)$$

Now

$$M(N, p, L_2) = 0$$

and

$$M_3 = D$$

where

$$D = V''G_1^2 + (V' - p)G_{11} + U'' \cdot F'F' + U' \cdot F''$$

But

$$V'', G_{11}, U'', \text{ and } F'' \text{ are negative.}$$

Moreover

$$G_1^2, V' - p, F', \text{ and } U' \text{ are non-negative.}$$

Unless no production takes place at all, $D < 0$.¹¹

So for (N, p) satisfying the necessary conditions for maximizing H one may write

$$L_2 = f(N, p)$$

where

- 1) f is continuous at (N, p)
- 2) f has continuous first partials given by

$$f_1 = - \frac{M_1}{D}$$

$$f_2 = - \frac{M_2}{D} \quad (\text{Taylor [18], p. 241})$$

$$f_1 = - \frac{V''G_1 G_2 + (V' - p) G_{12}}{D} \quad (9)$$

$$f_2 = - \frac{G_1}{D} \quad (10)$$

$$\frac{\partial L_2}{\partial p} = f_2 < 0$$

The sign of f_1 is troublesome. The question to be answered is: "What happens to L_2 as N is increased and the necessary conditions for optimality are met?" In one case, more L may be allocated to Sector 2, i.e.,

$\left\{ \frac{\partial L_2}{\partial N} > 0 \right\}$ because, combined with more N , L is more productive in this sector.

Or L might move out of Sector 2 i.e., $\left\{ \frac{\partial L_2}{\partial N} < 0 \right\}$ if, for example, no more of C_2

is wanted and N is substituted for L_2 in production.

Generally the sign of $\frac{\partial L_2}{\partial N}$ depends upon the substitutability of factors L and N , and upon the desire for the products Y_1 and Y_2 .

One may demonstrate this dependence as follows: Define

$$\bar{m} = - C_2 \frac{V''}{V'} \text{ the elasticity of marginal utility, }^{12}$$

and

$$e = \frac{G_1 G_2}{G \cdot G_{12}} \text{ the elasticity of substitution between factors } L_2 \text{ and } N.$$

In equilibrium

$$q_2 > 0$$

and

$$C_2 = G$$

so

$$m \bar{e} = - C_2 \frac{V''}{V'} \cdot \frac{G_1 G_2}{G \cdot G_{12}}$$

$$\text{Sign } f_1 = \text{sign} \left\{ V'' G_1 G_2 + V' G_{12} - p G_{12} \right\}$$

$$\begin{aligned} &\cong 0 \\ &< 0 \end{aligned}$$

as

$$1 \cong m \bar{e} + p/q_2 \quad \text{and hence as } 1 - em = \cong p/q_2$$

For $G(L_2, N)$ Cobb-Douglas $e=1$. Assume $G_{12} > 0$

Hence the sign of f_1 is the same as the sign of $(1-m-p/q_2)$

From condition (c) $(q_2-p) > 0$ and $(1-p/q_2) > 0$

Without prior knowledge, the cases will necessarily be considered individually. In this paper the case $\frac{\partial L_2}{\partial N} \cong 0$ will be treated in full. If

$\frac{\partial L_2}{\partial N} < 0$ a similar analysis applies but is somewhat more difficult.

$\frac{\partial L_2}{\partial N} > 0$ means that for efficiency in allocation of input L, if the stock of resource rises, more of the variable input should be allocated to the second sector.

$\frac{\partial L_2}{\partial p} < 0$ can be interpreted as a condition that the amount of input L allocated to Sector 2 should fall as the imputed demand price of investment rises for efficient and optimal allocation of L. That is, the more one wishes to 'save' the resource for the future, the less of L should be put into its production today.

Steady-State Equilibrium for Interior Solutions

For notational convenience call

$$G(L_2(N, p), N) = Q(N, p)$$

so

$$Q_N = G_1 \frac{\partial L_2}{\partial N} + G_2$$

$$Q_p = G_1 \frac{\partial L_2}{\partial p}$$

The differential equations which describe flows of points (N_t, p_t) in phase space are

$$\dot{N}_t = \lambda N_t - \epsilon N_t^2 - Q(N, p) \quad (5)$$

$$\dot{p}_t = p_t (\delta - \lambda + 2\epsilon N_t + G_2) - V' [Q(N, p)] G_2 \quad (6)$$

where

$$G_2 = \frac{\partial G}{\partial N} \quad 13$$

For this autonomous system an equilibrium is defined as (N^*, p^*) such that

$$\dot{N}_t = \dot{p}_t = 0 \text{ at } (N^*, p^*)$$

Slopes of $\dot{p}=0$ and $\dot{N}=0$ curves in Phase Space

The following analysis will consider the question of the slopes $\left. \frac{d p}{d N} \right|_{\dot{p}=0}$ and $\left. \frac{d p}{d N} \right|_{\dot{N}=0}$ in phase space.

Consider the graph of $\dot{N} = 0$.

$$\dot{N} = 0 \text{ if and only if } \lambda N - \epsilon N^2 = Q(N, p)$$

or

$$\gamma(N, p) = 0$$

where

$$\gamma = \lambda N - \epsilon N^2 - Q(N, p)$$

$$\gamma_2 = -Q_p = -G_1 \frac{\partial L_2}{\partial p} > 0 \text{ for all } (N, p) \text{ so implicit differentiation is valid.}$$

$$\therefore \left. \frac{d p}{d N} \right|_{\dot{N}=0} = \left. \frac{\frac{\partial \gamma}{\partial N}}{\frac{\partial \gamma}{\partial p}} \right|_{\dot{N}=0}$$

$$= \left. \frac{\lambda - 2\epsilon N - Q_N}{-G_1 \frac{\partial L_2}{\partial p}} \right|_{\dot{N}=0} \quad (11)$$

If $Q_N \cong \lambda - 2\epsilon N$ then the numerator of $\left. \frac{d p}{d N} \right|_{N=0} \cong 0$ and $\left. \frac{d p}{d N} \right|_{N=0} \cong 0$

If $Q_N < \lambda - 2\epsilon N$ then $\left. \frac{d p}{d N} \right|_{N=0} < 0$

Consider next the graph of $\dot{p} = 0$.

Define

$$R(N, p) = p[\delta - \lambda + 2\epsilon N + G_2] - q_2 G_2 = 0 \quad (12)$$

where

$$q_2 = q_2(N, p) = V'(Q(N, p))$$

and

$$G_2(L_2, N) = G_2(L_2(N, p), N)$$

$$\frac{\partial q_2}{\partial p} = V'' \cdot Q_p = V'' \cdot G_1 \cdot \frac{\partial L_2}{\partial p} = \frac{-V'' G_1 \cdot G_1}{D}$$

where

$$D < V'' G_1^2 < 0$$

Hence

$$0 < \frac{\partial q_2}{\partial p} < 1$$

This condition states that a rise in the imputed demand price of investment (in terms of the utility of present consumption foregone) will be accompanied by a rise (smaller in size) of the imputed demand price of resource product. (I.e., the more one desires to have his cake in the future, the higher will be his present imputed price of cake.)

Since $q_2 > 0$, then $R(N, p) = 0$ implies $p \neq 0$. Interpret p as the imputed demand price of the unharvested resource. This study is interested

only in commercially exploitable natural resources, not in such resource questions as Lamprey control, so only the case where $p > 0$ will be of interest.

So

$$\delta - \lambda + 2\epsilon N + G_2 = \frac{q_2}{p} G_2$$

and in order to satisfy $\dot{p} = 0$, the value of N will be defined by

$$N^{**} = \frac{\lambda - \delta + \frac{q_2 - p}{p} G_2}{2\epsilon} \quad (13)$$

Since

$$(q_2 - p)G_1 = w = q_1 F' > 0 \quad \text{and} \quad G_1 > 0$$

\therefore

$$(q_2 - p) > 0 \quad \text{and} \quad N^{**} > \frac{\lambda - \delta}{2\epsilon} \quad \text{for all } p^{**} \quad (14)$$

Heuristically, the condition $(q_2 - p) > 0$ states that the imputed demand price of the resource product exceeds the imputed demand price of the unharvested resource. In my earlier model¹⁴ these prices were equal. The difference is the existence of production costs in this model. The equilibrium value of N here must exceed the value $\frac{\lambda - \delta}{2\epsilon}$, which was the equilibrium value of the simplified model.

So, whenever there are costs of production, the optimal steady-state population will exceed the modified maximum sustained yield population.

This conclusion was reached earlier by Crutchfield and Zellner [2]. Now

$$\left. \frac{dp}{dN} \right]_{\dot{p}=0} = \frac{\frac{\partial R}{\partial N}}{\frac{\partial R}{\partial p}} \quad \text{if} \quad \frac{\partial R}{\partial p} \neq 0$$

$$\frac{\partial R}{\partial p} = \left[(p - q_2) G_{21} \frac{\partial L_2}{\partial p} + G_2 \left(1 - \frac{\partial q_2}{\partial p} \right) + \delta - \lambda + 2\epsilon N \right] \quad (15)$$

where

$$\left(1 - \frac{\partial q_2}{\partial p} \right) > 0$$

$$\frac{\partial L_2}{\partial p} < 0$$

and

$$N^{**} > \frac{\lambda - \delta}{2\epsilon} \quad (16)$$

\therefore

$$\delta - \lambda + 2\epsilon N > 0$$

Hence

$$\frac{\partial R}{\partial p} > 0 \text{ for all } (N^{**}, p^{**}) \text{ satisfying } \dot{p} = 0 .$$

The sign of $\frac{\partial R}{\partial N}$ will depend upon the sign of $\frac{\partial L_2}{\partial N}$.

$$\frac{\partial R}{\partial N} = 2\epsilon p + (p - q_2) \left[G_{22} + G_{21} \frac{\partial L_2}{\partial N} \right] - G_2 \frac{\partial q_2}{\partial N} \quad (17)$$

where

$$\frac{\partial q_2}{\partial N} = Q_N \cdot v'' = v'' \left[G_2 + G_1 \frac{\partial L_2}{\partial N} \right]$$

From Appendix

$$\frac{\partial L_2}{\partial N} \cong 0 \text{ implies } (G_{22} + G_{21} \frac{\partial L_2}{\partial N}) < 0$$

So

$$\frac{\partial R}{\partial N} > 0 \text{ for } \frac{\partial L_2}{\partial N} \cong 0 \text{ and} \quad (18)$$

$$\left. \frac{dp}{dN} \right|_{\dot{p}=0} < 0 \quad (19)$$

Phase Space

It was shown that the slope of the curve $\dot{p} = 0$ in phase space is negative. The $\dot{N} = 0$ curve will intersect the $\dot{p} = 0$ curve in zero, one or two points. Since it is the intention of this analysis to study properties of steady-state equilibria, attention will be directed at the last two cases.

If the number of intersections is one, it will be shown that the slope of the $\dot{N} = 0$ will be positive and the resulting equilibrium will be a saddle-point.

If there are two intersections (see Figure 2) then it will be shown that one is unstable, the other is a saddle-point.

Next consider

$$\left. \frac{dp}{dN} \right|_{\dot{N}=0} \quad \text{with} \quad \frac{\partial L_2}{\partial N} \cong 0 \quad \text{for all } N$$

For each $p > 0$ there may be two positive values of N satisfying $\dot{N} = 0$. Label then \tilde{N}_1 and \tilde{N}_2 for a cross-section of $Q(N,p)$ at $p = \tilde{p}$. Assume $\tilde{N}_1 < \tilde{N}_2$

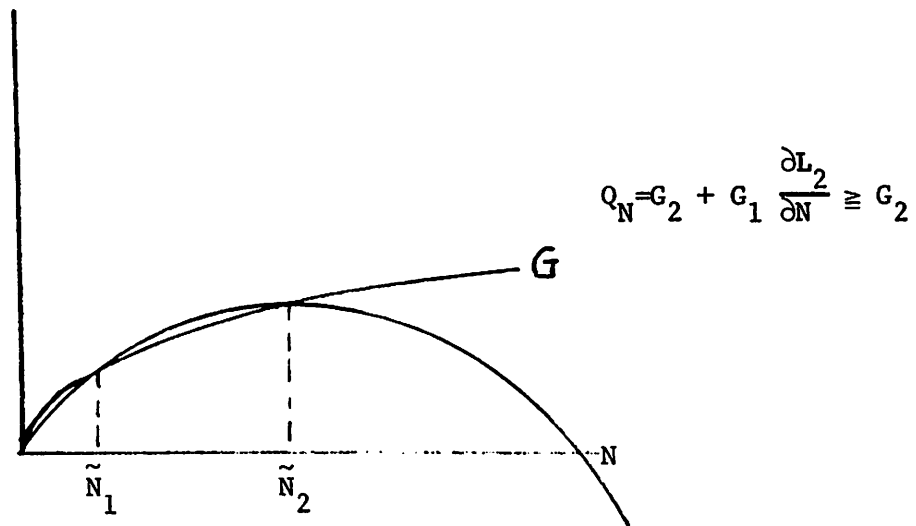


Figure 1

$$\text{At } (\tilde{N}_2, \tilde{p}), Q_N > \lambda - 2\epsilon N \quad \therefore \quad \left. \frac{dp}{dN} \right|_{\dot{N}=0} > 0$$

$$\text{and at } (\tilde{N}_1, \tilde{p}) \text{ assume } Q_N < \lambda - 2\epsilon N \quad \therefore \quad \left. \frac{dp}{dN} \right|_{\dot{N}=0} < 0$$

$$\text{If } \tilde{N}_2 = \tilde{N}_1 \text{ then } Q_N = \lambda - 2\epsilon N \quad \therefore \quad \left. \frac{dp}{dN} \right|_{\dot{N}=0} = 0$$

The corresponding diagram in phase space is as follows:

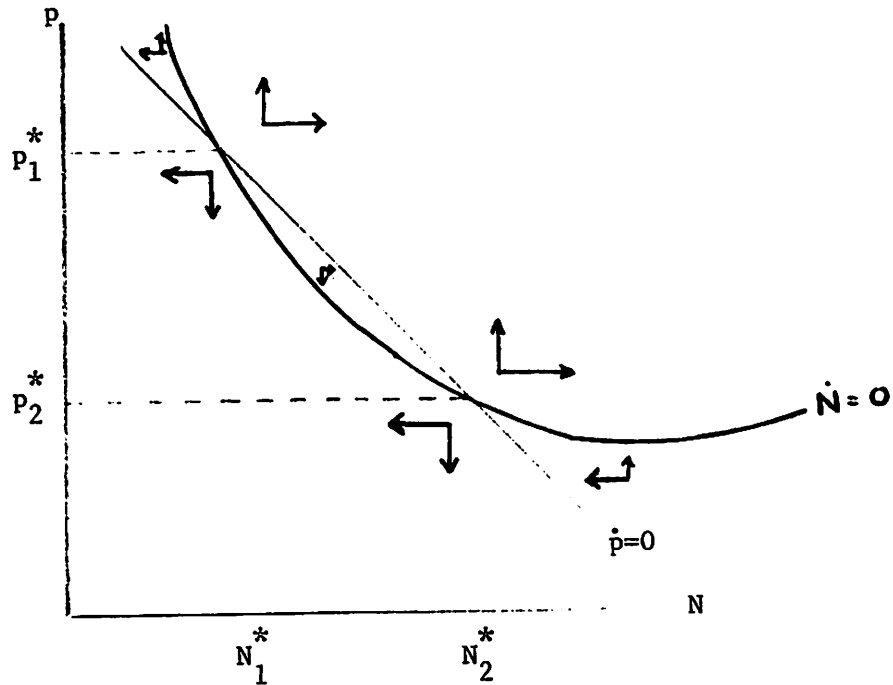


Figure 2

Equilibrium (N_1^*, p_1^*) will be analyzed later. Directions of arrows will be shown to be as indicated in Figure 2. Thus (N_1^*, p_1^*) will be unstable.

(N_2^*, p_2^*) is a saddle-point. There exists an optimal ¹⁵ trajectory, which will be described later. If $N_0 > N_1^*$ a controller will follow this trajectory.

If (N_1^*, p_1^*) is in fact an optimum, with the information assumed at his disposal, a controller can compare the welfare provided by (N_1^*, p_1^*) and (N_2^*, p_2^*) . If $N_0 = N_1^*$, the controller can leave the system unaltered, or force p downward and direct the economy toward (N_2^*, p_2^*) depending upon which state provides more welfare.

If $N_0 < N_1^*$ the resource mass should be reduced to zero.

Trajectories in Phase Space

$$\dot{N} = \lambda N - \epsilon N^2 - Q(N, P)$$

It is easily verified that above the line $\dot{N} = 0$ it is true that $\dot{N} > 0$ and below it $\dot{N} < 0$ since $G(L_2(\bar{N}, p), \bar{N})$ is a decreasing function of p .

Since $\dot{p} = p(\delta - \lambda + 2\epsilon N) - (q_2 - p) G_2$ is an increasing function of N from equation (18) therefore above and to the right of $\dot{p} = 0$ it follows that $\dot{p} > 0$ and to the left $\dot{p} < 0$.

Arrows in Figure 2 indicate directions of movements in Phase Space.

To Show (N_2^*, p^*) is a Saddle Point

Expand \dot{p} and \dot{N} about (N_2^*, p^*) as a Taylor series, taking only linear terms.

$$\begin{bmatrix} \dot{p} \\ \dot{N} \end{bmatrix} \approx A \begin{bmatrix} p - p^* \\ N - N_2^* \end{bmatrix}$$

where A is a constant 2×2 matrix.

It can be shown that the eigenvalues of A are real and of opposite sign.

Alternatively, using L'Hospital's Rule

$$\left. \frac{dp}{dN} \right]_{(N_2^*, p^*)} = \frac{\left. \frac{d}{dN}(\dot{p}) \right]_{(N_2^*, p^*)}}{\left. \frac{d}{dN}(\dot{N}) \right]_{(N_2^*, p^*)}} \quad (20)$$

This results in a quadratic equation in $\left. \frac{dp}{dN} \right]_{(N_2^*, p^*)}$.

The coefficient of $\left. \left(\frac{dp}{dN} \right) \right]_{(N_2^*, p^*)}^2$ is Q_p which is negative and the absolute term is

$\left. \frac{\partial R}{\partial N} \right]_{(N_2^*, p^*)}$ which is positive. Hence the roots are of opposite sign,

and real so (N_2^*, p^*) is a saddle-point. The optimal trajectory will correspond to the negative root.

A Sole-Ownership Producer

If a sole-ownership firm is given exogenous time paths of product price $q_2(t)$ and wages $w(t)$ and maximizes discounted profits, that is, maximizes

$$\int_0^{\infty} [q_2 G(L_2, N) - wL_2] e^{-\delta t} dt$$

subject to

$$\dot{N} - \lambda N + \epsilon N^2 + G(L_2, N) = 0$$

the Euler necessary conditions for an extremum are

$$w = (q_2 - p)G_1 \tag{21}$$

and

$$\dot{p} = \delta p - p(\lambda - 2\epsilon N - G_2) - q_2 G_2 \tag{22}$$

These conditions are identical to condition (c) and equation (6) for a controller. The competitive producer's allocation of L_2 is socially optimal. For him the resource use is an internal problem. No externality is present. It is the common-property nature of the resource which necessitates control in the general case.

The Case of n Identical Competitive Firms

Consider next the situation where there are n firms with identical technologies in resource product (fish) production represented by $g(L_{2i}, N)$ where L_{2i} is the amount of L_2 each will exploit, and N is the resource mass.

A controller's optimum would require L_2 to be allocated according to

$$w = (q_2 - p)g_1 \quad (23)$$

If the controller sets a tax per unit output of amount $p(t)$ and if each firm maximizes discounted profits, given prices $q_2(t)$ and $w(t)$ each firm will allocate variable input according to equation (23).¹⁶ The tax $p(t)$ which the controller determines from the optimal trajectory, has the effect of internalizing the common-property externality, or of appropriation.

Concluding Remarks

Replenishable common-property natural resources are not optimally exploited by untrammelled competitive firms. Central control is necessary.

One central control solution is to attempt, via quotas, to achieve a maximum sustained production, i.e., maintain the resource size $N = \frac{\lambda}{2e}$. This is generally non-optimal.

Optimal control can be achieved by imposition of a correctly determined per unit tax $p(t)$. The problem of determining $p(t)$ however is difficult because of the information that the controller must have at his disposal.

Appendix

To show

$$a) \quad G_{21} + G_{11} \frac{\partial L_2}{\partial N} > 0$$

$$b) \quad G_{22} + G_{21} \frac{\partial L_2}{\partial N} < 0$$

when

$$\frac{\partial L_2}{\partial N} > 0$$

Consider $G_{21} + G_{11} \frac{\partial L_2}{\partial N} > 0$ where $\frac{\partial L_2}{\partial N} > 0$ for $G(L_2, N)$ Cobb-Douglas

of the form $AN^\alpha L_2^\beta$ where $\alpha + \beta = \gamma < 1$.

Homogeneity of degree $\gamma < 1$ is not unreasonable in view of the fact that fixed capital is an input in production but is not represented explicitly in G .

Proof

$$NG_2 + L_2G_1 = \gamma G \quad (\text{Euler's Theorem})$$

Hence

$$N(G_{22} + G_{21} \frac{\partial L_2}{\partial N}) = (\gamma - 1) (G_1 \frac{\partial L_2}{\partial N} + G_2) - L_2(G_{12} + G_{11} \frac{\partial L_2}{\partial N})$$

Now

$$\begin{aligned} \frac{\partial L_2}{\partial N} &= - \frac{V''G_2G_1 + (V' - p)G_{12}}{V''G_1G_1 + (V' - p)G_{11} + U'' \cdot F' \cdot F' + U' \cdot F''} \\ &= - \frac{G_2 + \frac{V' - p}{V''} \frac{G_{12}}{G_1}}{G_1 + \frac{V' - p}{V''} \frac{G_{11}}{G_1} + P} \end{aligned}$$

where P represents the positive terms

$$\frac{U''}{V''} \cdot \frac{F'F'}{G_1} + \frac{U' \cdot F''}{V'' \cdot G_1}$$

∴

$$\frac{\partial L_2}{\partial N} \left[G_1 + \frac{(V' - p)G_{11}}{V''} + P \right] + G_2 + \frac{V' - p}{V''} \cdot \frac{G_{12}}{G_1} = 0$$

Subtract

$$\frac{\partial L_2}{\partial N} G_1 + G_2 > 0$$

∴

$$\frac{(V' - p)}{V''} \left[G_{11} \frac{\partial L_2}{\partial N} + G_{12} \right] + G_1 P < 0$$

Since

$$\frac{(V' - p)}{V''} < 0$$

∴

$$G_{12} + G_{11} \frac{\partial L_2}{\partial N} > 0$$

and also since

$$(\gamma - 1) < 0$$

∴

$$G_{22} + G_{21} \frac{\partial L_2}{\partial N} < 0$$

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¹'Exhaustible' in their use means that the supply of the resource is not naturally replenished within the horizon. 'Exhaustible' in this paper shall mean that the biological growth rate λ is zero.

²See Gordon, H.S. [4], and Scott, A. [14].

³Vernon Smith [16].

⁴In A.E.R., June 1970.

⁵By 'replenishable' is meant that the biological growth rate λ exceeds zero. In fact it is required in this model that $\lambda > \delta$ where δ is the social discount rate.

⁶See Pontryagin, et. al. [12].

⁷After the present model was essentially developed, a similar presentation by Quirk and Smith [13] was brought to my attention.

⁸See [11] for details. N is interpreted as the mass of resource, such as tonnage of fish.

⁹Subscripts of t will be dropped except where their inclusion is instructive.

¹⁰It is at this point that specification of a (separable) additive utility function is convenient. Since the interesting case where resource product is desired has $q_2 > 0$, therefore $C_2 = G(L_2, N)$. It is instructive to write q_2 as a function of L_2 and N , i.e.,

$$q_2 = v'(C_2) = v'[G(L_2, N)]$$

in order to show L_2 as a function of N and p . To see the complications that arise when the utility function is not separable, see Uzawa [19].

¹¹If $(q_2 - p)$ and q_1 were given market prices, and if condition (c) were the first-order condition for profit maximization for a competitive firm, then $D < 0$ would be the second-order condition that profits be maximized, or, equivalently, that variable inputs be used efficiently.

¹²For $V(C_2^2) = C_2^\alpha$, $\bar{m} = 1 - \alpha$. For $V(C_2)$ linear $\bar{m} = 0$, and quadratic $\bar{m} = 1$. A. K. Sen [15] shows that in a primitive, one factor, economy \bar{m} constant and less than unity is the only case that leads to intuitively appealing results.

Green [5] and Hicks [7], p. 252, conclude that if utility functions are to be stationary, homogeneous and independent over time, then $\bar{m} > 1$ and constant. Upon integrating one can show this condition requires there exist a Bliss state, as Ramsey employed.

¹³More properly one would write

$$\dot{p} = \delta p - \frac{\partial L}{\partial N} \quad \text{where } L_2^* = L_2^*(N, p)$$

$$N_t$$

$$p_t$$

$$L_2^*$$

and

$$\begin{aligned} L = & U(C_1) + V(C_2) + p(\lambda N - \epsilon N^2 - G(L_2, N)) \\ & + q_2(G(L_2, N) - C_2) + q_1(F(L_1) - C_1) \\ & + w(\bar{L} - L_1 - L_2) \end{aligned}$$

in which case, evaluated at (N_t, p_t, L_2^*)

$$\begin{aligned} \frac{\partial L}{\partial N} &= p(\lambda - 2\epsilon N - G_1 \frac{\partial L_2}{\partial N} - G_2) + q_2 G_1 \frac{\partial L_2}{\partial N} \\ &+ G_2 + q_1 F' \frac{\partial L_1}{\partial N} + w(-\frac{\partial L_1}{\partial N} - \frac{\partial L_2}{\partial N}) \\ &= p(\lambda - 2\epsilon N) + (q_2 - p)G_2 \\ &+ \frac{\partial L_2}{\partial N} [(q_2 - p)G_1 - q_1 F'] \\ &+ w[\frac{\partial L_2}{\partial N} - \frac{\partial L_2}{\partial N}] \\ &= p(\lambda - 2\epsilon N) + (q_2 - p)G_2 \text{ as above because} \end{aligned}$$

$$\frac{\partial L_1}{\partial N} = -\frac{\partial L_2}{\partial N} \quad \text{and condition (c)}$$

¹⁴A.E.R., June 1970.

¹⁵ Since $p_t e^{-\delta t} \rightarrow 0$ and $N_t \rightarrow N_i^*$ as $t \rightarrow \infty$ it follows that the transversality conditions are met. Sufficient conditions are not easily found. For $\frac{\partial L_2}{\partial N} = 0$, $H(N, p)$ is concave.

¹⁶ This results from maximizing

$$\int_0^{\infty} \{(q_2 - p)g(L_{2i}, N) - wL_{2i}\} e^{-\delta t} dt$$

with respect to L_{2i} .