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A SIMPLE PROOF OF THE EXISTENCE OF SUBGAME PERFECT EQUILIBRIA IN INFINITE-ACTION GAMES OF PERFECT INFORMATION*

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1. INTRODUCTION

Consider perhaps the prototypical extensive form game with perfect information. There are T players moving in strict sequence at times t=1,...,T. Player t has perfect information on all previous moves and must select from some choice set. A strategy for player t then is a function from histories before t into this choice set. Each strategy vector yields a unique choice vector and this choice vector can be assumed then to determine all payoffs.

If all choice sets are finite, it is easy to construct a subgame perfect equilibrium in pure strategies for such a game. If choice sets are infinite, however, existence of an SPE is less obvious. The apparent problem is that the strategies of players t+1,...,T seem to lack continuity properties to guarantee that player t's maximization problem has a solution.

Harris (1985), however, does prove the existence of an SPE in pure strategies for a class of extensive form games including that described in the first paragraph. Harris considers two substantial generalizations -- that the choice sets can depend on histories in a continuous fashion, and that the number of players can be infinite. Hellwig and Leininger (forthcoming) provide a somewhat more accessible existence result for essentially the same class of games as Harris. (Hellwig and Leininger also establish that the equilibrium strategies may be taken to be measurable functions of the histories.) Hellwig and Leininger (1986) also provide an existence proof for the case of the first paragraph. This is, however, incidental to the main focus of this second paper which is to investigate the relationship between an infinite-action game and approximating finite action games. Indeed,

this new proof, is to some extent, an adaption of the proof of existence in their first paper and is of a similar length and degree of sophistication.

The main contribution of the present paper is to provide a shorter and more elementary proof of existence which is also thus more intuitively compelling. A by-product of the proof is a demonstration that infinite-action games can be usefully approximated by appropriate finite-action games. The present paper assumes that choice-sets are history-independent and that there are a finite number of players. This is for the sake of brevity, as the current approach can be extended. (Robson, 1987, provides an example of an oligopoly model satisfying these restrictions. Harris, 1985, and Hellwig and Leininger, forthcoming, are motivated to allow greater generality by infinite-horizon planning models with, perhaps, capital. Hellwig and Leininger, 1986, make the simpler assumptions.) The topological assumptions in the present paper are, however, quite general in that choice sets are assumed to be sequentially compact, first countable and separable. (In both of their papers cited here Hellwig and Leininger make stronger assumptions--in the first, that choice sets are compact subsets of R^n--in the second, that they are compact metric spaces. Harris assumes choice sets are compact Hausdorff spaces which neither implies nor is implies by the present assumption. However, compact metric spaces satisfy both the present assumption and Harris'.)

The method of establishing existence here, as in Hellwig and Leininger (1986), is to approximate the choice sets by a finite but arbitrarily large "grid" of choices. Any such finite-action game has an SPE. The general idea behind each proof is to show that an SPE of the original infinite-action game can be obtained by a limiting construction based on these SPE's of finite-action approximating games. Each construction also establishes

a link between some SPE of the infinite action game and SPE's of the finite-action games. However, the present construction generates the equilibrium strategies in a much more direct and elementary fashion.

We intend to use the present techniques to construct equilibria for classes of infinite-action games without perfect information.

2. TWO EXAMPLES

The nature of the difficulty and the present resolution of it can best be illustrated by means of examples. With two players, backward induction, augmented by a simple tie-breaking rule, suffices to establish existence of an SPE. To illustrate this consider a game with two players--1 and 3. (Of course, player 2 waits in the wings.) Players 1 and 3 move in the obvious sequence and have payoffs and choice sets given by

$$U_1(c_1, c_3) = c_1 - c_3,\ c_1 \in [-1,1] = C_1$$

$$U_3(c_1, c_3) = c_1 \cdot c_3 \quad c_3 \in [-1,2] = C_3.$$

Player 3's equilibrium strategy must be a selection from the best reply correspondence

$$\varphi_3(c_1) = \begin{cases} \{-1\} & c_1 \in [-1,0) \\ [-1,2] & c_1 = 0 \\ \{2\} & c_1 \in (0,1] \end{cases}.$$

This has a closed graph and hence $U_1(c_1, c_3)$ reaches a maximum on this graph. Indeed this is at the point

$$(c_1, c_3) = (0, -1).$$

This can be supported as an equilibrium path by the SPE strategies

$$f_1 = c_1 = 0$$

$$f_3(c_1) = \begin{cases} -1 & c_1 \in [-1,0] \\ 2 & c_1 \in (0,1] \end{cases}.$$

Thus the tie-breaking rule for two player games is just that the second player breaks ties in favor of the first.

Consider an example with three players. The notationally overdue player 2 enters and moves in the obvious place in the sequence. Now payoffs and choice sets are given by

$$U_1(c_1, c_2, c_3) = c_1 - c_3,\ c_1 \in [-1,1] = C_1$$
$$U_2(c_1, c_2, c_3) = c_2 \cdot c_3,\ c_2 \in [-1,2] = C_2$$
$$U_3(c_1, c_2, c_3) = c_1 \cdot c_3,\ c_3 \in [-1,2] = C_3.$$

A natural generalization of the tie-breaking rule from the two player case would be to have any player break a tie in favor of his immediate predecessor. Indeed this procedure will work in some examples. But suppose here that player 3 breaks ties in favor of player 2, choosing strategy

$$f_3'(c_1, c_2) = \begin{cases} -1 & , \ c_1 \in [-1,0) \\ 2 & , \ c_1 \in [0,1] \end{cases}.$$

Then player 2 chooses the strategy

$$f_2'(c_1) = \begin{cases} -1 & c_1 \in [-1,0) \\ 2 & c_1 \in [0,1] \end{cases},$$

and player 1's maximization problem has no solution. In this example, it is clear why this naive tie-breaking rule fails, since player 2 has no substantive effect on the original game between players 1 and 3. The unique SPE is clearly

$$f_1 = c_1 = 0$$

$$f_2(c_1) = \begin{cases} -1 & c_1 \in [-1,0] \\ 2 & c_1 \in (0,1] \end{cases}$$

$$f_3(c_1,c_2) = \begin{cases} -1 & c_1 \in [-1,0] \\ 2 & c_1 \in (0,1] \end{cases}$$

where 3, as before, breaks ties in favor of player 1. What is not immediately obvious, however, how to carry out an analysis in terms of tie-breaking in the general case.

It should be noted that Harris (1985) presents a proof of existence without explicitly discussing this issue of tie-breaking. Hellwig and Leininger (forthcoming) do more explicitly discuss tie-breaking rules and are able to show that these can be devised so as to produce an SPE. Hellwig and Leininger, 1986, contains a proof of existence which is, to some extent, an adaptation of that in their first paper, but which does not explicitly discuss tie-breaking. The approach here is the most direct in terms of obtaining existence and also obviates the need to discuss tie-breaking explicitly.

To illustrate the present procedure, consider again the three player example above. Replace the infinite choice sets C_1, C_2, and C_3 by

$$C_1^k = \{-1, -1 + \frac{2}{k}, \ldots, 1\}$$

$$C_2^k = C_3^k = \{-1, -1 + \frac{3}{k}, \ldots, 2\}$$

where each of these finite sets has k+1 elements. The resultant finite action game must have an SPE. To construct any such SPE, ties can be broken arbitrarily. If $0 \in C_1^k$ and player 3 breaks the resulting tie by choosing

-1, then the equilibrium path of the original infinite-action game is attained exactly. If $0 \notin C_1^k$ or if player 3 breaks the tie in any other way, then player 1's equilibrium choice is

$$c_1^k = -\frac{1}{k} \text{ or } -\frac{2}{k} \in C_1^k$$

so that player 2 and 3 choose

$$c_2^k = c_3^k = -1 \in C_2^k = C_3^k.$$

The key feature here is that these finite-action game equilibrium <u>paths</u> converge to an equilibrium <u>path</u> of the infinite-action game. This is true even if player 3 breaks the tie in the "wrong" way in the finite-action game. However, note that it is <u>not</u> true that the equilibrium <u>strategies</u> of the finite-action games must converge to equilibrium <u>strategies</u> of the infinite-action game. Indeed, if player 3 breaks the tie "incorrectly" so that

$$f_3^k(0,c_2) = 2$$

then, of course,

$$\lim_k f_3^k(0, c_2^k) = 2$$

which has already been shown to be inconsistent with an SPE. Nonetheless, the complete equilibrium strategies of the infinite-action game can be derived by consideration of equilibrium paths of finite-action games and those of all subgames of these. To see how this can be done, set firstly

$$f_1 \equiv c_1 \equiv \lim c_1^k = 0$$
$$f_2(0) \equiv \lim f_2^k(c_1^k) = \lim c_2^k = -1$$
$$f_3(0,-1) \equiv \lim f_3^k(c_1^k, f_2^k(c_1^k)) = \lim c_3^k = -1.$$

Note here that the arguments of each strategy function are also subject to the limit. This completely defines f_1, but f_2 and f_3 are not prescribed off the equilibrium path. Consider then a subgame of the infinite-action game defined by any $c_1 \neq 0$. Find some sequence

$$\overset{\sim k}{c}_1 \to c_1, \text{ where } \overset{\sim k}{c}_1 \in C_1^k \ \forall k.$$

Define then

$$f_2(c_1) \equiv \lim f_2^k(\overset{\sim k}{c}_1) = \begin{cases} -1, & c_1 \in [-1,0). \\ 2, & c_1 \in (0,1] \end{cases}$$

$$f_3(c_1, f_2(c_1)) \equiv \lim f_3^k(\overset{\sim k}{c}_1, f_2^k(\overset{\sim k}{c}_1)) = \begin{cases} -1, & c_1 \in [-1,0) \\ 2, & c_1 \in (0,1] \end{cases}.$$

This completes the definition of f_2. In general it would be necessary to complete the description of f_3 by considering subgames defined by (c_1, c_2) where $c_2 \neq f_2(c_1)$. However, in this case c_2 is irrelevant to player 3's equilibrium choice, and so, for all $c_2 \in C_2$, define

$$f_3(c_1, c_2) = f_3(c_1, f_2(c_1))$$

where $f_3(c_1, f_2(c_1))$ is as above. That this method of finding SPE strategies is quite general is shown in the next two sections.

3. THE CLASS OF INFINITE ACTION GAMES

The structure here is derived essentially from Hellwig and Leininger. As noted above, choice sets are history-independent but have a more general topology. In addition, only the case with a finite number of players is considered.

Definition 1. Infinite Action Game, Γ

The set of players is $P = \{1,\ldots,T\}$ assumed to be finite. Each player t has a choice set C_t which is sequentially compact (meaning that all sequences have convergent subsequences), first countable (meaning each point has a countable local basis), and separable (meaning C_t possesses a countable dense subset.) The set of histories up to and including time (t-1) is

$$E_{t-1} = C_1 x \ldots x C_{t-1}.$$

Payoffs are continuous functions

$$U_t: \quad E_T \to R.$$

When player t moves at time t, it is with perfect information on

$$e_{t-1} \in E_{t-1}.$$

Hence, the set of strategies for player t is

$$S_1 = C_1, \; S_t = \{f_t: \quad E_{t-1} \to C_t\}, \; t=2,\ldots,T.$$

Clearly any strategy vector implies a uniquely determined choice vector and hence the payoff vector in R^T.

Definition 2. Subgames, Subgame Perfect Equilibrium

These are defined in the obvious fashion. Any history $e_{t-1} \in E_{t-1}$ determines a unique subgame played by $t,\ldots,T$ over strategies in $S_t x \ldots x S_T$. The complete sequence of choices $(e_{t-1}, c_t,\ldots,c_T)$ in this subgame determines a unique payoff vector in R^T.

A strategy vector $(f_1,\ldots,f_T)$ is a subgame perfect equilibrium iff, for all $t=1,\ldots,T$, for all $e_{t-1} \in E_{t-1}$, and for all $c_t' \in C_t$, $U_t(e_{t-1}, f_t, f_{t+1},\ldots,f_T) \geq U_t(e_{t-1}, c_t', f_{t+1},\ldots,f_T)$ where the arguments of $f_t,\ldots,f_T$ are omitted for compactness of notation. For example, on the left-hand side of the above inequality,

$f_t \equiv f_t(e_{t-1})$

$f_{t+1} \equiv f_{t+1}(e_{t-1}, f_t(e_{t-1}))$

$f_{t+2} \equiv f_{t+2}(e_{t-1}, f_t(e_{t-1}), f_{t+1}(e_{t-1}, f_t(e_{t-1})))$

and so on. Thus, particular choices are denoted by c_t, particular histories by e_{t-1}, but f_t, $f_{t+1},\ldots,f_T$ denote strategies, that is, functions of previous histories.

The choice vector determined uniquely by $(f_1,\ldots,f_T)$ is called the equilibrium path of the game Γ. The equilibrium path for any subgame is defined analogously.

4. THE EXISTENCE RESULT

Consider the following approximation to Γ.

Definition 3. Finite Action Game Γ^k

Since each C_t is separable, there exists a countable dense subset

$\{c_t^1, c_t^2,\ldots,c_t^k,\ldots\} \subset C_t$ t=1,...,T.

Define

$C_t^k = \{c_t^1,\ldots,c_t^k\}$ t=1,...,T, $k \in N$.

The game Γ^k is defined precisely as is Γ, where C_t^k replace the C_t, t=1,...,T. Now E_{t-1}^k denotes the set of histories at t and S_t^k the set of strategies for player t, t=1,...,T. An SPE for Γ^k is then as in Definition 2.

Lemma 1. Existence of an SPE for Γ^k

The game Γ^k always has an SPE.

Proof. See Selten (1975).

Therem 1. Existence of an SPE of Γ; The Approximation Result

A. The game Γ, as in Definition 1, has an SPE.

B. At least one of the SPE's of Γ has the property that its equilibrium path is the limit of a subsequence of equilibrium paths of SPEs of Γ^k. Furthermore, this SPE of Γ has the property that the equilibrium path for any subgame of Γ is the limit of a subsequence of the equilibrium paths for approximating subgames of Γ^k generated by SPE of Γ^k. (The subsequences involved may depend on the subgame considered.)

Proof. An SPE of Γ will be constructed to satisfy "B".

Given Lemma 1, denote any particular SPE of Γ^k by

$$(f_1^k,\ldots,f_T^k) \in S_1^k x \ldots x S_T^k,\ k=1,2,\ldots$$

The definition of the candidate for an SPE of Γ is carried out by the following step-by-step procedure. The idea is that stage "t" defines the equilibrium path for any subgame involving player "t" deviating from his equilibrium strategy and being the last to do so. If player "t" does not deviate, then stage "t" adds nothing to what has been already defined.

As a preliminary step, choose for all $t=1,\ldots,T$, for all $c_t \in C_t$, fixed sequences

$$\overset{\sim k}{c}_t(c_t) \to c_t,\ \overset{\sim k}{c}_t(c_t) \in C_t^k \text{ for all } k \in N$$

using the separability and first countability of C_t (see Dugundji, p. 218, 6.2).

Stage 0. This is the case in which no player deviates. Consider the sequence

$$\{f_1^k\} = \{c_1^{-k}\},\ k \in \mathbb{N}.$$

By the sequential compactness of C_1, this must have a convergent subsequence such that

$$\bar{c}_1 = \lim c_1^{-k}.$$

Define then $f_1 \equiv \bar{c}_1$. Consider now the associated sequence $\{f_2^k(c_1^{-k})\}$ lying in C_2. By the sequential compactness of C_2, define then

$$f_2(\bar{c}_1) \equiv \lim f_2^k(c_1^{-k})$$

using a smaller sequence if needed. Similarly, using the sequential compactness of C_3, define

$$f_3(\bar{c}_1, f_2(\bar{c}_1)) \equiv \lim f_3^k(c_1^{-k}, f_2^k(c_1^{-k}))$$

using a still smaller sequence if needed. And so on, finally defining f_T on what will be shown to be the equilibrium path for Γ. The selection of a subsequence of choices implies the selection of a subsequence of the natural numbers. Thus, for example, the ultimate subsequence found above can be represented by, say, $D(\phi) \subset \mathbb{N}$, $|D(\phi)| = \infty$. Here ϕ denotes the empty history defining the subgame which is the game itself.

Stage 1. Take any $c_1 \in C_1$. If $c_1 = \bar{c}_1$, define $c_1^k(\bar{c}_1) \equiv c_1^{-k}$, from Stage 0, and $D(\bar{c}_1) \equiv D(\phi)$.

Suppose $c_1 \neq \bar{c}_1$. Define

$$c_1^k(c_1) \equiv \tilde{c}_1^k(c_1).$$

Now, using the sequential compactness of C_2, define

$$f_2(c_1) \equiv \lim f_2^k(c_1^k(c_1))$$

using a subsequence of $D(\phi)$ if needed. Again, using the sequential compactness of C_3, define

$$f_3(c_1, f_2(c_1)) \equiv \lim f_3^k(c_1^k(c_1), f_2^k(c_1^k(c_1)))$$

using a still smaller subsequence if needed. Construct in this way strategies for players 2,...,T evaluated on what will turn out to be the equilibrium path of the subgame determined by player 1's selection of $c_1 \neq \bar{c}_1$.

This yields an ultimate sequence <u>of indices</u> $D(c_1)$, say, in which all limits taken so far can be expressed, $D(c_1) \subset D(\phi)$. ($D(e_t)$ is used, in general, to denote the sequence of indices used to construct the equilibrium path in the subgame defined by e_t. This construction is made with one eye on the crucial link between defection by some player in the infinite action game and defection in an approximating finite action game. See the last paragraph below.)

<u>Stage "t", t=2,...,T-1</u>

Available from previous stages are, for all $e_{t-1} \in E_{t-1}$

(i) A sequence of indices $D(e_{t-1}) \subset N$

(ii) A sequence of histories

$$e_{t-1}^k(e_{t-1}) \to e_{t-1},\ k \in D(e_{t-1})$$

where

$$e^k_{t-1}(e_{t-1}) \in E^k_{t-1},$$

(iii) Definitions of strategies $f_1,\ldots,f_t$.

(iv) Strategies for players t+1,...,T defined only on what will turn out to be the equilibrium path of the subgame determined by e_{t-1}.

Now consider the contribution of stage "t". Strategy f_t has been defined already. Suppose

$$c_t = f_t(e_{t-1}).$$

Then, where $e_t = (e_{t-1}, c_t)$, set

$$D(e_t) \equiv D(e_{t-1})$$

and

$$c^k_t(e_t) \equiv f^k_{t-1}(e^k_{t-1}(e_{t-1}))$$

which converges for k restricted to $D(e_{t-1})$ by some previous construction.

Finally set

$$e^k_t(e_t) = (e^k_{t-1}(e_{t-1}),\ c^k_t(e_t)).$$

Suppose instead

$$c_t \neq f_t(e_{t-1})$$

Define then

$$c^k_t(e_t) \equiv \tilde{c}^k_t(c_t)$$

and, again,

$$e^k_t(e_t) \equiv (e^k_{t-1}(e_{t-1}),\ c^k_t(e_t)).$$

By the sequential compactness of C_{t+1} define

$$f_{t+1}(e_t) \equiv \lim f^k_{t+1}(e^k_t(e_t))$$

using a subsequence of indices from $D(e_{t-1})$ if needed. Again, by sequential compactness of C_{t+2} define

$$f_{t+2}(e_t,\ f_{t+1}(e_t)) \equiv \lim f^k_{t+2}(e^k_t(e_t),\ f^k_{t+1}(e^k_t(e_t)))$$

using a smaller subsequence if needed. Define in this way strategies for t+1,...,T evaluated only on what will be the equilibrium path of the subgame determined by e_t. Define the ultimate sequence of indices as $D(e_t)$. By construction

$$D(e_t) \equiv D(e_{t-1},\ c_t) \subset D(e_{t-1}).$$

This completes the recipe for construction of all $(f_1,\ldots,f_T)$.

It must finally be shown that these strategies do indeed constitute an SPE of Γ. Suppose not. Then there exists a player t and a history e_{t-1} such that player t can do better by choosing

$$c'_t \neq f_t(e_{t-1}) \text{ where } c'_t \in C_t.$$

Hence

$$U_t(e_{t-1},\ c'_t,\ f_{t+1},\ldots,f_T) > U_t(e_{t-1},\ f_t,\ f_{t+1},\ldots,f_T)$$

using the abbreviated notation of Definition 2. Consider the sequence of indices

$$D(e_{t-1},\ c'_t) \subset D(e_{t-1}) = D(e_{t-1},\ f_t(e_{t-1})).$$

Consider also the sequences

$$e^k_{t-1}(e_{t-1}),\ \tilde{c}^k_t(c'_t)$$

as defined for stage "t" above. Recall how $f_t,\ldots,f_T$ are constructed on the equilibrium paths of the two subgames determined by e_{t-1} and by $(e_{t-1},\ c'_t)$ respectively. It follows that, if $k \in D(e_{t-1},\ c'_t)$ is large enough, and using the continuity of U_t in choices,

$$U_t(e^k_{t-1}(e_{t-1}), \tilde{c}^k_t(c_t), f^k_{t+1},\ldots,f^k_T) > U_t(e^k_{t-1}(e_{t-1}), f^k_t, f^k_{t+1},\ldots,f^k_T)$$

again with the abbreviated notation of Definition 2. But this contradicts the assumption that $(f^k_1,\ldots,f^k_T)$ is an SPE of Γ^k. Hence the proof is complete. (The above notation apparently implies $t \geq 2$, but the case for $t=1$ is precisely analogous.)

REFERENCES

Dugundji, J., Topology, Allen and Bacon, 1966.

Harris, C., "Existence and Characterization of Perfect Equilibrium in Games of Perfect Information," Econometrica, 53, 3, May 1985, 613-628.

Hellwig, M. and Leininger, W., "On the Existence of Subgame-Perfect Equilibrium in Infinite-Action Games of Perfect Information," Journal of Economic Theory, forthcoming.

Hellwig, M. and Leininger, W., "A Note on the Relation between Discrete and Continuous Games of Perfect Information," University of Bonn DP A-92, 1986.

Robson, A., "Stackelberg and Marshall," mimeo, 1986.

Selten, R., "Reexamination of the Perfectness Concept for Equilibrium Points in Extensive Games," International Journal of Game Theory 4, 1975, 25-55.