1985

A Positive Analysis of Multiproduct Firms in Market Equilibrium

Glenn M. MacDonald

Alan D. Slivinski

THE CENTRE FOR DECISION SCIENCES AND ECONOMETRICS

A Positive Analysis of Multiproduct Firms in Market Equilibrium

Glenn M. MacDonald and Alan D. Slivinski

TECHNICAL REPORT NO. 5
AUGUST 1985

Centre For Decision Sciences And Econometrics
Social Science Centre
The University of Western Ontario
London, Ontario N6A 5C2

A Positive Analysis of Multiproduct Firms in Market Equilibrium

Glenn M. MacDonald*

and

Alan D. Slivinski**

*University of Western Ontario,
Economics Research Center/NORC,
and Rochester Center for Economic Research

**University of Western Ontario
and Rochester Center for Economic Research

MacDonald, G.M. and Slivinski, A.D.

A Positive Analysis of Multiproduct Firms in Market Equilibrium

An economy in which firms may choose to produce two goods (diversification) or just one (specialization) is studied. Parameterizing costs along the fixed/variable distinction most familiar from Viner's work, a complete characterization of the model's equilibrium is provided; this characterization may be summarized in a simple diagram. The ease with which the model may be manipulated makes it a useful tool for analysis of a wide variety of issues pertaining to environments permitting multiproduct firms. This facility is illustrated through derivation of a diverse set of predictions concerning the manner in which changes in the underlying exogenous features of the economy affect the pattern of diversification and specialization, as well as through explicit treatment of several extensions.

University of Western Ontario, London, Ontario, Canada.

Will a single firm produce a diverse set of products in market equilibrium? Can some firms produce a set of goods having some but not all elements in common with the collection selected by other firms? How does the equilibrium assignment of products to firms vary with changes in the pattern of demand, technological parameters, input prices, and so on?

These general questions have not gone entirely unaddressed. Early work by R.G.D. Allen (1938), Hicks (1939), Samuelson (1947), as well as more recent efforts by Laitinen (1980), analysed in detail the isolated behavior of firms having access to an m-input/n-output production technology. The response of profit maximal input and output choices to price changes was derived but the restrictions imposed on such choices by the requirements of market equilibrium were not explored. At the other extreme, the Arrow-Debreu-McKenzie (ADM) general equilibrium model allows each producer a distinct production set, so that firms might choose to produce many goods, and the collection of goods produced by firms might overlap to some degree. However, the ADM model is a very general one, and as such restricts the data very little. More recently, the "contestability" literature (surveyed by Bailey and Friedlander (1982)) analyzes a setting in which firms produce more than one good. But due to the different goals of that work, all firms are assumed to produce the same exogenously specified set of goods.[1] Finally, there is what might be termed the "where there is sawdust there may be 'pressed logs'" approach, dating back at least to Marshall (1920, pp. 321-22), in which joint products are the result of unstructured technological complementarities.

Common to all these approaches is that while each produces partial answers to some of the questions raised at the outset, none has sufficient structure to characterize fully a market equilibrium in a manner which permits derivation of a body of falsifiable predictions regarding the behavior of firms in such an equilibrium. The contribution of this paper lies in producing a characterization which can be analysed with appropriate rigor, and which has predictive content sufficient for it to be useful empirically.

The structure of the model used to achieve this outcome is quite unremarkable. Indeed, it can be viewed correctly as a multiple output generalization of Viner's (1952) classic analysis, a restricted ADM model with technological nonconvexities and firms which are small in comparison with market demand, as in Sonnenschein (1982), or as a specialization of Baumol et al.'s (1982) "fractional firm" competitive equilibrium.

More precisely, the approach taken is as follows. There are two goods, α and β, and firms producing both are referred to as diversified; otherwise the firm is specialized.[2] In order for this distinction to have any content, it is required that diversified firms do not merely operate as a collection of contiguous specialized firms; they must differ in some more basic sense. A very fruitful way to parameterize this distinction involves distinguishing between fixed and variable costs, along the lines of Viner. Fixed costs, as usual, refer to expenditures on inputs which do not vary with the level of output; for example, fixed production inputs such as physical plant, or nonproduction inputs such as accounting, ordering of materials, product design, management and accounting, etc. In the early stage of the analysis, fixed costs need not have much structure. Variable costs are the value of variable production inputs and the cost of any other activities the level of which depends on output. Given this bifurcation of total cost, the primary issue becomes the existence, for either type of firm, of an advantage in either category of cost. It is obvious that if, for example, the output vector produced by a diversified firm could be supplied by two specialized firms at lower fixed and variable costs, then no diversified firm producing that output vector could operate in competitive equilibrium. Clearly the cases of interest are those for which diversification is cost saving in one category and not the other. The market outcome then hinges on a nontrivial tradeoff, rendering the outcome amenable to analysis.

As indicated, the payoffs to the approach taken herein are that it is very simple, and therefore easy to manipulate and extend; it offers numerous predictions; and indeed it even sheds light on several policy issues. The value of the analysis then follows in part from the current relative scarcity of such tools.

The predictions which are extracted from the basic model may be organized as follows. First, answers to the questions posed at the outset are provided; in particular, the model can indeed generate a "mixed" equilibrium in which diversified firms operate alongside one type of specialized firm. Second, some more specific predictions about mixed equilibria emerge. Among them: (i) variation in the pattern of demand for goods induces a negative correlation between the number of diversified and specialized firms; (ii) equal reductions in the fixed costs faced by all firms generate a greater tendency towards output being produced by specialized firms; (iii) increases in the price of a good-specific factor of production have no effect on the structure of equilibrium; (iv) diversified firms will typically be larger than specialized firms in the sense of total revenue, but will produce a lower output of the good also produced by specialized firms. Third, with minor elaboration still more predictions become available. Examples are: (i) a formulation which implies diversified firms are at a disadvantage in terms of variable costs yields the result that goods which are similar in terms of the factor proportions specialized firms would use are more likely to be produced in diversified firms instead; (ii) focusing on the structure of fixed costs, a simple version of Rosen's (1982) internal theory of the firm implies that diversified firms will hire more upper management and fewer supervisory personnel than will specialized firms.

The structure of the paper is as follows. Section I contains the material on cost functions relevant for specialized and diversified firms. The structure of market equilibrium is the topic of Section II. Section III details the basic theory's predictions and indicates the type of results available from some extensions. Proofs of the Propositions are contained in the Appendix. The algebra underlying many of the predictions is presented in greater detail in MacDonald and Slivinski (1983).

I. Cost Functions

In this Section the cost functions for specialized and diversified firms are presented. As usual, more implications can be obtained via explicit analysis of the production technologies, but confining attention to the cost functions suffices for the basic characterization results.

It is assumed that total costs comprise fixed and variable costs. The former consist of expenditures on bookkeeping, upper management, ordering materials, product design, and so on. The latter include the costs of materials and factor services utilized directly in the production of output. The operational distinction between fixed and variable costs is conventional. Fixed costs are incurred because there are some factors which, while their employment need not rise with the level of output, are required for positive output.

First consider specialized firms. A specialized firm producing quantity q_j of good j is called <u>type j</u> and faces finite fixed cost $F_j > 0$; $j=\alpha,\beta$.[3] As regards variable cost, the variable cost function $C^j(q_j)$ can be derived in the standard fashion. For $q_j > 0$, $C^j(q_j)$ is assumed positive, monotone increasing and strictly convex with $\partial^2 C^j/\partial q_j^2$ bounded away from zero. The last assumption guarantees minimum average cost occurs at finite q_j.

In what follows it is assumed that the level of output which minimizes average cost for specialized firms is sufficiently small that any changes in their aggregate product can be accommodated by entry and exit of firms producing the average cost minimizing output. Under this assumption, the number of specialized firms can be treated as a continuous variable. Should a type j firm choose to operate, its behavior can be summarized by

$$\bar{q}_j \equiv \underset{q_j}{\operatorname{argmin}} \frac{F_j + C^j(q_j)}{q_j}, \tag{1}$$

$$\bar{C}_j \equiv F_j + C^j(\bar{q}_j), \tag{2}$$

and $$\bar{\lambda}_j \equiv \frac{\bar{C}_j}{\bar{q}_j}; \quad j = \alpha,\ \beta. \tag{3}$$

Observe that $\bar{\lambda}_j$ equals both marginal and average cost in firms of type j.

Turning to diversified firms, a finite fixed cost $F > 0$ is assumed. The variable cost function $C(q_\alpha, q_\beta)$ can be obtained in a fashion analogous to the textbook specialized case by cost minimization given a multiproduct technology. $C(q_\alpha, q_\beta)$ is assumed monotone increasing in each argument and strictly convex; let $\partial C/\partial q_j \equiv C_j(q_\alpha, q_\beta)$ and $\partial^2 C/\partial q_j \partial q_k \equiv C_{jk}$ for $k=\alpha,\beta$. Two other restrictions on $C(q_\alpha, q_\beta)$ parallel the treatment of specialized production. First, it is required that for fixed q_α/q_β, average cost is minimized by a finite q_α. Formally, for any $k \equiv q_\alpha/q_\beta$, let $\zeta(q_\alpha, k) \equiv C(q_\alpha, q_\alpha/k)$. It is assumed that for all finite $k > 0$, and $q_\alpha > 0$, $\partial^2\zeta/\partial q_\alpha^2$ is bounded away from zero. Second, and again for any finite $k > 0$, the average cost minimizing level of q_α is sufficiently small that the number of diversified firms can be regarded as continuous.

It is clear that if the analysis is to proceed very far the relationship between the total costs faced by diversified and specialized firms must be structured to some extent. It can be argued that diversified production may have an advantage or disadvantage in either cost category. For example, simple production complementarities may generate $C(q_\alpha, q_\beta) \leq \Sigma C^j(q_j)$, but absence of advantages due to specialization in management may yield $F \geq \Sigma F_j$. The analysis can accommodate any combination of these inequalities, but the more important restriction is based on the following considerations. By definition a diversified firm has the capacity (plant and management skills etc.) needed to produce both goods. It is required that the costs of that capacity be part of F. In that case, if a diversified firm should choose to produce an output vector close to that produced by a type j firm, the variable costs differ little across firms:

$$\forall q_j > 0, \lim_{q_{j'} \to 0} C(q_\alpha, q_\beta) = C^j(q_j). \tag{4}$$

Equation (4) permits straightforward evaluation of one of the diversified firm's options, namely a production choice much like specialization.[4]

To reiterate the most important points, type j specialized firms produce $\bar{q}_j$ at total cost of $\bar{C}_j$ and marginal cost $\bar{\lambda}_j$. Total cost functions for diversified firms are $F + C(q_\alpha, q_\beta)$.

II. Market Equilibrium

In this section a complete characterization of the model's equilibrium is provided. It is shown that depending on parameter values, equilibrium can take one of three forms: (i) pure specialization--the configuration in which no diversified firms operate, familiar from Viner and every undergraduate text; (ii) pure diversification--the Hicks-Allen-Samuelson and contestability case wherein there are no specialized firms; or (iii) mixed equilibrium involving operation of diversified firms and exactly one type of specialized firm. In this section attention is confined to demonstration of the above claim, with predictions, extensions, etc. taken up subsequently (Section III).

The equilibrium configuration of firm types is obtained by making use of the familiar result that for any given aggregate production of the two goods, if a competitive equilibrium exists, the equilibrium allocation of production across firms minimizes aggregate production cost.[5]

The analytical economy involved in making use of this result is considerable. The economy can be treated as solving a very simple programming problem, and the demand for goods can be suppressed. This latter feature is particularly useful because it allows the laborious task of altering preferences in order to make predictions regarding demand shifts to be circumvented. In addition, in order to demonstrate that a proposed allocation is not an equilibrium allocation, all that need be shown is that there is some other allocation which produces the aggregate output vector more cheaply.

Finally, though the analysis requires several steps, the solution to the programming problem may be summarized in a simple diagram, facilitating applications and extensions.

To proceed with the aggregate cost minimization problem, let the aggregate quantity of good j produced be Q_j (exogenous), the number of type j specialized firms be N_j, and the number of diversified firms be N. Then the programming problem is:

$$P: \quad \min_{N_\alpha, N_\beta, N, q_\alpha, q_\beta} N[F + C(q_\alpha, q_\beta)] + \Sigma N_j \bar{C}_j$$

$$\text{S.T. } Nq_j + N_j \bar{q}_j = Q_j, \quad j = \alpha, \beta;$$

$$N \geq 0, \; N_j \geq 0, \quad j = \alpha, \beta.$$

Using asterisks to denote optimal values of the choice variables, necessary conditions characterizing the solution to the cost minimization problem are:

$$F + C(q^*_\alpha, q^*_\beta) - \Sigma \lambda_j q^*_j \begin{cases} = 0 & \text{if } N^* > 0, \\ \geq 0 & \text{if } N^* = 0; \end{cases} \tag{5}$$

$$\bar{C}_j - \lambda_j \bar{q}_j \begin{cases} = 0 & \text{if } N^*_j > 0, \\ \geq 0 & \text{if } N^*_j = 0. \end{cases}, \; j = \alpha, \beta; \tag{6}$$

$$C_j(q^*_\alpha, q^*_\beta) - \lambda_j \begin{cases} = 0 & \text{if } q^*_j > 0 \\ \geq 0 & \text{if } q^*_j = 0 \end{cases}, \; j = \alpha, \beta; \tag{7}$$

and

$$Q_j - N^* q^*_j - N^*_j \bar{q}_j = 0, \quad j = \alpha, \beta. \tag{8}$$

In (5) - (8), λ_j is the Lagrange multiplier associated with the aggregate output constraint for good j.

Conditions (5) - (8) are familiar, (5) and (6) requiring all operating firms to earn zero profits when outputs are valued at the shadow prices λ_j. Expression (7) states that the q_j are chosen to maximize profits given shadow prices λ_j. This same information, but as it applies to type j firms, is embodied in $\bar{C}_j$ and $\bar{q}_j$, and it is immediate from (6) that the solution involves $N_j^* > 0$ only if $\lambda_j = \bar{\lambda}_j$. This result is very useful in what follows, since in order to establish whether $N_j^* > 0$, all that need be done is to solve P with the added constraint $N_j = 0$, and ask whether $\lambda_j \gtrless \bar{\lambda}_j$ in this restricted problem. Finally, (8) restates the aggregate production constraints.

Before proceeding further, two preliminary results will be stated. They simplify the subsequent analysis and illustrate the extent to which the present model is in many ways simply a multiproduct analogue to the traditional long-run competitive equilibrium model which has proved so useful in the past. Neither result is unique to the environment studied here, and in general both will hold in any competitive equilibrium.[6]

Lemma 1: <u>For almost all sets of parameter values, at most two of</u> N^*, N_j^* <u>can be positive</u>.[7]

A simple way to demonstrate this result is as follows. Holding q_j fixed (at q_j^*, or indeed any other positive value), problem P reduces to a linear programming problem with N and N_j as choice variables, and two linear restrictions, (8). Then a solution involves non-zero values for at most two of the choice variables except when a "coincidence" occurs. Even under such a coincidence the minimum aggregate cost is achievable by allowing at most two of the choice variables to be positive. (See Dorfman et al. (1958, Ch. 4, Theorem 2 and Corollary to Theorem 3).

According to Lemma 1, three types of equilibria are permitted: (i) Purely Diversified Equilibrium (D^*)--$N^* > 0$ and $N_j^* = 0$; Purely Specialized Equilibrium (S^*)--$N_j^* > 0$ and $N^* = 0$; or (iii) Mixed Equilibrium of Type j (M_j^*)--$N^* > 0$, $N_j^* > 0$ and $N_{j'}^* = 0$, $j = \alpha, \beta$.[8]

For the second preliminary result, let $C^*(Q_\alpha, Q_\beta)$ be the minimized level of aggregate cost; that is

$$C^*(Q_\alpha, Q_\beta) \equiv N^*[F + C(q_\alpha^*, q_\beta^*)] + \Sigma N_j^* \overline{C}_j \ .$$

Lemma 2: $C^*(Q_\alpha, Q_\beta)$ <u>is homogeneous of degree one in</u> (Q_α, Q_β).

The argument is simply that a proportionate change in the aggregate output vector is accommodated by an equiproportionate adjustment in the number of operating firms and no alteration in q_j^*. Such occurs because Q_j enters the conditions characterizing equilibrium only through (8).

In light of Lemma 2, all changes in the aggregate output vector will be cast in terms of movements in $Q \equiv Q_\alpha / Q_\beta$. It is assumed that $Q_j > 0$ for both j, in which case $0 < Q < \infty$.

The three results to follow provide a complete characterization of the equilibrium structure of production. The basic approach is to determine which type of equilibrium arises for a given set of parameter values through examination of the shadow prices generated by the problem P for that set of parameter values. For example, $\lambda_j < \overline{\lambda}_j$ implies $N_j^* = 0$ since no type j firm could earn nonnegative profits when output is valued at λ_j.

The notation "$\triangleleft$" represents the relation "yields lower aggregate cost".

For example, $D \lhd S$ indicates that the configuration $N > 0$ and $N_j = 0$ produces the required aggregate output at lower aggregate cost than the configuration $N_j > 0$, $N = 0$. The "equal aggregate cost" relation is represented by "$\square$".

Proposition 1 provides a ranking of the D and M_α configurations for the various possible (Q,F) combinations.

Proposition 1:

(i) For any $F \leq F_\alpha$, $D \lhd M_\alpha$;

(ii) For any $F > F_\alpha$, there is an unique $Q \geq 0$, written $\underline{Q}(F,F_\alpha)$, such that $M_\alpha \lhd D$ if and only if $Q > \underline{Q}(F,F_\alpha)$;

and (iii) For any $F > F_\alpha$, $\underline{Q}(F,F_\alpha) > 0$ if and only if for all $q_\beta > 0$,

$$\lim_{q_\alpha \to 0} C_\alpha(q_\alpha, q_\beta) < \bar{\lambda}_\alpha . \tag{9a}$$

The logic of Proposition 1 is as follows. First, since only D and M_α are being compared, the additional constraint $N_\beta = 0$ is imposed on P. Then the cost minimization problem permitting only diversified firms to operate, called $\bar{P}$, is solved and the shadow prices λ_j obtained. For (Q,F) pairs yielding $\lambda_\alpha > \bar{\lambda}_\alpha$, aggregate costs could be reduced by allowing $N_\alpha > 0$; that is, $M_\alpha \lhd D$. Otherwise $D \lhd M_\alpha$ or $D \square M_\alpha$.

Proceeding in this manner, first suppose $F = F_\alpha$ and Q is large. Good β is effectively no longer part of the problem, and diversified firms are for all intents and purposes identical to type α firms. Thus problem $\bar{P}$ generates $\lambda_\alpha = \bar{\lambda}_\alpha$, and $D \square M_\alpha$.

Next, two facts concerning $\bar{P}$. Fact one is that when Q is large, an increase in F must raise λ_α. This holds because if F is augmented q_α must rise to spread the greater fixed cost, and marginal cost $C_\alpha(\cdot)$ is increasing. Fact two is that for any fixed F, a reduction in Q lowers λ_α. The demonstration of fact two requires some calculation, but the argument is simply that although an arbitrary reduction in q_α/q_β need not reduce $C_\alpha(\cdot)$, the reduction required to minimize the cost of producing a lower Q must reduce $C_\alpha(\cdot)$, and hence λ_α (from the counterpart of (7) in problem F).

The Proposition follows easily. That $M_\alpha \square D$ when $F = F_\alpha$ and Q is large has been shown. Facts one and two imply that any reduction in either Q or F generates $\lambda_\alpha < \bar{\lambda}_\alpha$, and thus $D \lhd M_\alpha$--part (i) of the Proposition. On the other hand, an increase in F yields $\lambda_\alpha > \bar{\lambda}_\alpha$. Thus for large Q and $F > F_\alpha$, $M_\alpha \lhd D$. Now retaining $F > F_\alpha$, fact two requires that if Q is reduced λ_α falls, and the level of Q at which $\lambda_\alpha = \bar{\lambda}_\alpha$ is the $\underline{Q}(F,F_\alpha)$ of part (ii).[9] If reducing Q to zero does not produce $\lambda_\alpha \leq \bar{\lambda}_\alpha$, $\underline{Q}(F, F_\alpha)$ is defined to be zero. Part (iii) simply states the condition under which reductions in Q allow $\lambda_\alpha = \bar{\lambda}_\alpha$ to be achieved at $Q > 0$ for all $F > F_\alpha$.[10]

Intuitively then, when Q is very large q_β is inconsequential, and the comparison of D and M_α turns only on the relation between F and F_α. On the other hand, when Q is very small, q_α is relatively slight, in which case, loosely, production of good β covers the fixed cost for diversified firms. In that case Q_α should be produced by diversified firms irrespective of F; that is, no type α firms will operate. For moderate Q, lower F always works in favor of diversified firms, as does lower Q.

The next Proposition ranks D and M_β, and is entirely analogous to Proposition 1.

Proposition 2:

(i) <u>For any</u> $F \leq F_\beta$, $D \lhd M_\beta$;

(ii) <u>For any</u> $F > F_\beta$, <u>there is an unique level of</u> $Q \geq 0$ <u>written</u> $\bar{Q}(F,F_\beta)$, <u>such that</u> $M_\beta \lhd D$ <u>if and only if</u> $Q < \bar{Q}(F,F_\beta)$;

<u>and</u> (iii) <u>For any</u> $F > F_\beta$, $\bar{Q}(F, F_\beta) < \infty$ <u>if and only if for all</u> $q_\alpha > 0$,

$$\lim_{q_\beta \to 0} C_\beta(q_\alpha, q_\beta) < \bar{\lambda}_\beta. \qquad (9b)$$

Figure 1 depicts $\underline{Q}(\cdot)$ and $\bar{Q}(\cdot)$. Proposition 1 states that for any (Q,F) pairs to the right of $\underline{Q}(\cdot)$, $M_\alpha \lhd D$. Similarly, $M_\beta \lhd D$ for (Q,F) pairs to the left of $\bar{Q}(\cdot)$ (Proposition 2). Though the analysis can proceed in any case, it will be assumed henceforth that (9a) and (9b) hold. This restriction confines the analysis to its simplest and most interesting case. In what follows, the point of intersection of $\bar{Q}(\cdot)$ and $\underline{Q}(\cdot)$ is important. That $\partial^2\zeta/\partial q_\alpha^2$ is bounded away from zero implies that such an intersection exists, and convexity of $C(q_\alpha,q_\beta)$ yields its uniqueness. Restrictions (9a) and (9b) guarantee that the intersection occurs at Q for which $0 < Q < \infty$.

<u>Figure 1</u>

Two further points deserve emphasis. First, by construction, $\underline{Q}(F,F_\alpha)$ represents the (Q,F) pairs for which $D \square M_\alpha$, in which case while λ_β may vary along $\underline{Q}(\cdot)$, $\lambda_\alpha = \bar{\lambda}_\alpha$ must hold along it; and similarly, $\lambda_\beta = \bar{\lambda}_\beta$ along $\bar{Q}(\cdot)$. Second, consider again the structure of production when the alternatives are just D and M_α (i.e., $N_\beta = 0$ is imposed). Given F, for $Q \leq \underline{Q}(F, F_\alpha)$, all diversified firms produce output in the proportion $q_\alpha/q_\beta = Q$. Now consider raising Q, with $Q < \underline{Q}(F,F_\alpha)$ still. Since diversified firms are the sole source of output, q_α/q_β necessarily rises along with Q. But once $Q = \underline{Q}(F, F_\alpha)$, increments of Q are met with introduction of type α firms and no change in the diversified firms' production vector. Thus for all $Q \geq \underline{Q}(F,F_\alpha)$, diversified firms produce outputs in the proportions $q_\alpha/q_\beta = \underline{Q}(F,F_\alpha)$. Analogously, when $N_\alpha = 0$ is imposed, diversified firms produce $q_\alpha/q_\beta = \bar{Q}(F,F_\beta)$ for $Q \leq \bar{Q}(F, F_\beta)$.

Now to proceed further, let $\tilde{F}(F_\alpha,F_\beta)$ be the value of F for which $\underline{Q}(F,F_\alpha) = \bar{Q}(F,F_\beta)$, noting that $\tilde{F}(F_\alpha,F_\beta) > \max\{F_\alpha,F_\beta\}$.[11] Also, for brevity's sake, let E be the notation for "equilibrium"; that is "E = D*" means, for example, that the equilibrium involves pure diversification. The major characterization result may now be stated:

Proposition 3:

(i) For all Q, and all $F \geq \tilde{F}$, $E = S^*$;

(ii) For $Q > \underline{Q}(F,F_\alpha)$ and $F_\alpha < F < \tilde{F}$, $E = M_\alpha^*$;

(iii) For $Q < \bar{Q}(F,F_\beta)$ and $F_\beta < F < \tilde{F}$, $E = M_\beta^*$;

and (iv) All other (Q, F) pairs yield $E = D^*$.

The logic here is straightforward. (Refer to Figure 2.)

Figure 2

Part (i) is fairly simple. Consider $F = \tilde{F}$, and suppose Q is such that $\underline{Q}(\tilde{F},F_\alpha) = \bar{Q}(\tilde{F},F_\beta) \equiv \tilde{Q}$. For this (Q, F) pair, problem $\bar{P}$ generates $\lambda_j = \bar{\lambda}_j$, $j = \alpha, \beta$, by construction. Thus all firm types can coexist. Now consider raising Q, say through an increase in Q_α. This change could be accommodated via pure entry of type α firms, or entry of diversified firms producing $q_\alpha/q_\beta = \tilde{Q}$ accompanied by exit of type β firms, or any of a myriad of other combinations. All leave $\lambda_j = \bar{\lambda}_j$. Thus if $F = \tilde{F}$, all firm types can coexist irrespective of Q--the "accident" excluded in Lemma 1. Now for $F > \tilde{F}$, could a diversified firm exist alongside specialized firms? Clearly not, for when shadow prices are $\bar{\lambda}_j$ and $F = \tilde{F}$, there is precisely one production vector for which diversified production yields zero profits, with all others generating negative profits. Any increase in F then implies negative profits for diversified firms.

Now, combining Propositions 1 and 2, all points in the region labelled D* generate shadow prices $\lambda_j < \bar{\lambda}_j$; when the constraint $N_\alpha = N_\beta = 0$ is imposed (Problem $\bar{P}$). Thus aggregate cost could not be reduced by introducing either type of specialized firm, and D* must be the outcome. But when $\tilde{F} > F > F_\alpha$ for example, if Q is large, so that aggregate production is skewed towards production of good α, $\bar{P}$ yields $\lambda_\alpha > \bar{\lambda}_\alpha$ and $\lambda_\beta < \bar{\lambda}_\beta$. Thus costs could be reduced by introducing type α firms, and allowing diversified firms to produce in the proportions $q_\alpha/q_\beta = \underline{Q}(F, F_\alpha)$. Such a change could not yield entry of type β firms because $\lambda_\beta < \bar{\lambda}_\beta$ still holds $(\underline{Q}(F, F_\alpha) > \bar{Q}(F, F_\beta))$. Thus M_α^* is the result. When $\tilde{F} > F > F_\beta$, low values of Q generate M_β^* by a similar argument, thus establishing (ii) - (iv).

The intuition behind the complete characterization is clear. If diversified firms face low fixed costs, they alone will produce, almost irrespective of the proportions in which output is required. But for greater fixed costs, if the required aggregate output is particularly skewed towards good j, it is more efficient to allow diversified firms to produce in less extreme proportions by permitting type j firms to operate. As fixed costs rise still further, the range of outputs diversified firms might produce in equilibrium narrows, vanishing entirely when $F = \tilde{F}$.

Figure 2 also illustrates the unifying features of this analysis. $E = S^*$ is the familiar competitive equilibrium of the undergraduate text, while $E = D^*$ represents the situation studied in the Hicks-Allen-Samuelson work and contestability literature. The mixed equilibria $E = M_j^*$, while possessing a Ricardian flavour, do not appear to have been analysed previously.

The next Section puts the model to use.

III. PREDICTIONS

As stated at the outset, the goal is to produce a simple framework--much in the spirit of the familiar competitive model--that can be manipulated with ease and rigor, and which can be adapted for analysis of specific issues. Moreover, the structure must be capable of producing testable restrictions. This Section demonstrates that these goals are indeed met.[12]

The setup required to obtain the basic characterization results was quite unadorned, and the implications which can be gleaned from it are derived first. Next, elaborations of the model are pursued. These extensions are not comprehensive analyses. Rather, they are suggestive illustrations of the ways in which the model can be extended to handle broader issues. The extensions follow the fixed/variable cost dichotomy exploited above, and obtain new predictions by placing more structure on these components of cost.

Predictions from the Basic Model

Results from the basic model are of two varieties. The first are of the "snapshot" type; that is, all parameters constant, what features is equilibrium expected to possess? The main result has already been stated. The S, D, and M_j configurations are the only outcomes consistent with equilibrium. Most interesting, from both the theoretical and empirical standpoints, are the M_j structures anticipated when aggregate production is skewed towards good j. When $E = M_j^*$, efficiency implies that firms will not all produce the same level of good j (excepting measure zero cases), though firms producing good j' all do so at the same rate. This result is unusual as a theoretical point because such intra-industry (defined by goods) heterogeneity is usually predicted only in models where firms are assumed heterogeneous from the outset. Empirically, a positive cross-industry correlation between the coexistence of diversified and specialized firms, and inter-firm output variation, is implied.

Two other results involve comparison of firm types when $E = M_j^*$. First, diversified firms will produce $q_j \gtrless \bar{q}_j$ as $C_{jj'} \lesseqgtr 0$, with $q_j < \bar{q}_j$ therefore being the leading case. Second, the restriction

$$\frac{\partial}{\partial q_j}\left(\frac{C_j}{C_{j'}}\right) > 0 \tag{10}$$

is sufficient to imply that diversified firms are larger than type j firms in the sense of total revenue: $\bar{\lambda}_j q_j + \lambda_{j'} q_{j'} > \bar{\lambda}_j \bar{q}_j$. Though $q_j < \bar{q}_j$ typically, the additional revenue earned via sale of good j' more than compensates unless $C_{jj'}$ is too large; a possibility ruled out by (10).

Further analysis is simplified by imposing the restriction

$$\sum_{k=\alpha,\beta} q_k C_{jk} > 0; \quad j = \alpha,\beta . \tag{11}$$

Inequality (11) requires that an equiproportionate increase in the q_j raise diversified firms' marginal cost for both q_j; convexity of $C(\cdot)$ alone implies that (11) must hold for at least one j. Given (11), it is readily established that $\underline{Q}(F,F_\alpha)$ and $\bar{Q}(F,F_\beta)$ are as drawn in Figure 3(a):

$$\frac{\partial \underline{Q}}{\partial F} < 0 \quad \text{and} \quad \frac{\partial \bar{Q}}{\partial F} > 0.$$

The second type of result available for the simple model is the standard comparative statics variety. Without adding more structure, the available set of parameters comprise Q, F, F_j and the prices of variable factors (previously suppressed in $C(q_\alpha,q_\beta)$ and $C^j(q_j)$).

Results on Q and F can be obtained directly from Figure 3(a).

<u>Figure 3(a,b)</u>

For given Q, a sufficiently small F always guarantees $E = D^*$ (point A). But for greater F, the range of output ratios diversified firms might produce in equilibrium narrows, at some stage excluding the given Q. At that point some good j (j = α in the Figure, point B) will be produced by type j firms; $E = M_j^*$. Still greater F (say point C) implies that even production in the proportions $\tilde{Q}$ cannot sustain diversified firms, and production of good j' will be undertaken by type j' firms; $E = S^*$.

In contrast, changes in Q for given F have no impact on the structure of production if $F \leq \min\{F_j\}$ (point A, $E = D^*$) or $F > \tilde{F}$ (point C, $E = S^*$). Under those circumstances F is too extreme for the structure of production to be influenced by the output ratio at which diversified firms would produce, though other variables (e.g. N^*) would be. Otherwise ($F_j < F \leq \tilde{F}$) changes in Q are relevant, and, as discussed above, generate production by type j firms when Q is skewed sufficiently towards good j. It should be emphasized that type j firms play a "fringe" role with respect to changes in demand for good j when $E = M_j^*$. Take $j = \alpha$. When Q_α rises or falls, type α firms bear all of the adjustment, the number and output composition of diversified firms remaining unaltered. This result, in conjunction with the generally smaller size of type j firms, squares well with the known stylized facts on intra-industry adjustment. On the other hand, when Q_β varies, the number of diversified firms responds in the same direction, with the number of type α firms moving in the opposite direction. Overall, as Q_α and Q_β fluctuate, the numbers of diversified and type α firms are predicted to be negatively correlated, with the magnitude of the correlation inversely related to the variance in demand for good α relative to the variance in demand for good β since changes in demand for good β generate alteration in the numbers of both types of firms.

Increments to F_j are also not difficult to handle. Again consider $j = \alpha$. Recall that the diagram developed above was constructed from $\underline{Q}(F,F_\alpha)$ and $\bar{Q}(F,F_\beta)$. As depicted in Fig. 3(b), the latter is not a function of F_α, and so remains fixed as F_α varies. Now recall that $\underline{Q}(F,F_\alpha)$ is the locus of (Q,F) pairs for which pure diversification yields $\lambda_\alpha = \bar{\lambda}_\alpha$, and that given F, λ_α rises with Q under pure diversification. It follows that when F_α rises, and $\bar{\lambda}_\alpha$ along with it, a greater value of Q is required to achieve $\lambda_\alpha = \bar{\lambda}_\alpha$

under pure diversification: $\underline{Q}(F,F_\alpha)$ shifts to the right. Consequently, the D^* and M_β^* regions expand, S^* contracts, and the M_α^* region may do either. Ambiguity as regards the change in M_α^* occurs because M_α^* loses some area to D^* but gains some from S^* (point A is an example). Note that at point A, type β firms are the casualties of an increase in F_α, which is somewhat counter-intuitive. The explanation is that when F is not far from $\tilde{F}$ (the level of F for which all firm types can co-exist) an increase in F_α renders optimal the expenditure by type β firms of $F - F_\beta$ required for a type β firm to begin producing good α; that is to become diversified.

Turning to factor prices, there are numerous experiments which can be performed. An interesting and suggestive route is to suppose

$$C(q_\alpha,q_\beta) = \Sigma\, r_j\delta_j q_j + \chi(q_\alpha,q_\beta), \qquad (12)$$

where r_j is the price of a good j specific factor x_j, δ_j is the exogenous factor/output ratio for x_j, and $\chi(\cdot)$ is the cost of all other variable factors.[13] Equation (12) would be an appropriate specification if, for example, x_j were a material input not used in the production of good j'.

The formulation (12) is easily shown to be sufficient for the result that a change in r_j has <u>no effect</u> on the structure of production. Such occurs because under (12) $\partial C_j/\partial r_{j'} = 0$ and $\partial C_j/\partial r_j = \delta_j$ irrespective of the output vector. Thus (for $j = \alpha$) in terms of the diagram, $\bar{Q}(F, F_\beta)$ does not shift with r_j because the value of λ_β arising from pure diversification does not change (though λ_α does, so as to achieve "zero profits" at the same q_α/q_β), in which case $\lambda_\beta = \bar{\lambda}_\beta$ for the same (Q,F) pairs. $\underline{Q}(F, F_\alpha)$ does not shift either because both $\bar{\lambda}_\alpha$ and the value of λ_α implied by pure diversification increase equally. Thus an increase in the price of a good specific input is predicted to have no effect on the structure of production.[14]

Elaborations

Fixed Costs

Thus far the fixed costs F and F_j have been taken to be independent parameters. However, given the activities in which firms engage, it is not unreasonable to suppose F and F_j to be related in some fashion. In this subsection two examples of such interrelations are examined.

First suppose fixed costs comprise expenditures π on a pure public input (eg. accounting) which all firms utilize in the same quantity, and the cost κ_j of a product j specific input (eg. product design). Then

$$F = \pi + \Sigma_j \kappa_j ,$$

$$F_j = \pi + \kappa_j ,$$

and changes in π or κ_j induce simultaneous movements in F and F_j.

Consider an increment to κ_α. Both F and F_α increase as a consequence, the latter inducing $\underline{Q}(F,F_\alpha)$ to shift to the right just as in Figure 3b. Straightforward calculation demonstrates that if (Q,F) lay on $\underline{Q}(F, F_\alpha)$ initially--D⊡M_α--then the new position of $\underline{Q}(F, F_\alpha)$ is to the right of the new (Q,F) pair if (but not only if) C_α is convex; that is D ⊲ M_α.[15] Thus while an increase in κ_α may lead to various outcomes as F changes (M_β replacing D for example), depending on the intial (Q,F) pair, such a change can never involve M_α replacing D, and frequently implies the opposite.

Along the same lines, an increase in π (equivalent to larger κ_j for both j) shifts $\underline{Q}(F,F_\alpha)$ to the right and $\bar{Q}(F,F_\beta)$ to the left. If both C_j are convex more diversification is implied in the sense that if $D \rhd M_j$ initially, then $D \lhd M_j$ afterwards. Putting the changes together in the diagram produces the following. An increase in π can only lead to one of (i) $E = S^*$ being replaced by $E = M_j^*$ or $E = D^*$; (ii) $E = M_j^*$ being replaced by $E = D^*$; or (iii) no change.

This argument has numerous applications. For example, a familiar story is that increments to the "extent of the market" foster the development of factors specialized to, or more specifically suited for, production of the good in question. If these factors are of the fixed variety, the theory here suggests that balanced (i.e., Q not varying greatly) increases in the extent of the market which reduce π (or both κ_j) do indeed foster specialization. The development of customized computer software seems a good illustration.

Recent attempts to eliminate regulation also provide a setting to which this reasoning may be applied. When removal of regulations simply lowers the costs of being in business at all, π or κ_j, greater specialization is a clear tendency. Or if deregulation permits reduction in duplication of paperwork which is not much related to the product mix, reducing F relative to F_j, greater diversification is implied. The latter characterization is often argued to typify much banking and insurance regulation. That deregulation generates banks which offer a wide variety of services is thus exactly what would be expected on the basis of the above analysis.

Another interesting specification of fixed costs can be obtained by explicit modelling of the internal structure of the firm. The material which follows is closely related to Rosen's (1982) analysis of hierarchies.

Assume that operation of the technology which produces good j requires a fixed amount $\bar{R}_j$ of nonproduction activities (again, management, accounting, etc.) R_j, and that R_j is produced using direct supervision--the number of supervisors on a production line being denoted s_j--and upper management, with m representing the number of upper level managers.

Following Rosen, the technology for producing R_j is taken to be of the form $R_j = g(m)f(m,s_j)$, where for convenience $g(\cdot)$ and $f(\cdot)$ do not vary across goods. The notion here is that part of the management activities in the firm involves m and s_j interacting in some manner, and the other part does not.[16]

Managers are available at price w per efficiency unit, and the price per efficiency unit of supervisors is normalized to unity.

The results which may be derived are typified by the following special case:

$$g(m) = m,$$

$$f(m,s_j) = ms_j.$$

Given this specification, a type j firm solves the cost minimization subproblem

$$\min_{m,s_j} \; wm + s_j$$

$$\text{S.T. } m^2 s_j = \bar{R}_j,$$

which has solution (bars denoting optimal values)

$$\bar{m}_j = (2\bar{R}_j/w)^{1/3}$$

and

$$\bar{s}_j = (w^2\bar{R}_j/4)^{1/3}.$$

It follows that the minimized level of fixed costs is

$$F_j = A(w^2 \bar{R}_j)^{1/3}$$

where $A \equiv 2^{1/3} + 2^{-2/3}$.

Turning to diversified firms, part of the problem involves the allocation of efficiency units of management across product lines. Letting t represent the fraction of management time devoted to interaction with supervisors on the good α technology, the diversified firm's subproblem is

$$\min_{t,m,s_\alpha,s_\beta} \quad wm + \Sigma s_j$$

$$\text{S.T.} \quad tm^2 s_\alpha = \bar{R}_\alpha$$

$$\text{and} \quad (1-t)m^2 s_\beta = \bar{R}_\beta .$$

Solution of this problem gives (asterisks denoting optimal values)

$$m^* = [2(\Sigma \bar{R}_j^{1/2})^2/w]^{1/3} ,$$

$$s_j^* = \bar{R}_j^{1/2}(w^2/4\Sigma \bar{R}_j^{1/2})^{1/3} ,$$

and

$$t^* = \bar{R}_\alpha^{1/2}/\Sigma \bar{R}_j^{1/2} .$$

Then

$$F = A(w \Sigma \bar{R}_j^{1/2})^{2/3} .$$

Now what does this example imply for cross firm comparisons? Focussing on the $E = M_j^*$ case for which nontrivial cross firm comparisons are possible, since $F > F_j$ is necessary for $E = M_j^*$ (Proposition 3) it follows that $m_j^* > \bar{m}_j$ and $s_j^* < \bar{s}_j$. That is, diversified firms are predicted to be more top heavy in the sense of utilizing fewer supervisors and more upper management.

Analysis of the components of F and the F_j yields other conclusions. An especially interesting one is that an increase in nonproduction requirements $\bar{R}_j$ raises F less than it does F_j, producing the same result as increases in κ_j of the previous example: greater diversification. A variety of institutional and technological changes can be given this parameterization. For example, the availability of robotic techniques, even if they offer no direct decline in cost per unit of work, may impact on nonproduction requirements by changing the level of monitoring required to obtain given work.[17] Since introduction of robotics would only be undertaken if cost saving, the decline in $\bar{R}_j$ generates a greater tendency towards $E = S^*$ and $E = M_j^*$ quite apart from any different productive attributes robots might possess.

Variable Costs

More predictions can also be obtained by imposing additional structure on variable costs. To illustrate this possibility, a simple model is analysed which incorporates the notion that goods whose production utilizes similar factors of production in ways not too dissimilar, will tend to be produced together in diversified firms. The goal is to provide a simple and refutable hypothesis which also explains why automobiles and trucks are produced together, or radios and amplifiers, or indeed even the now defunct pairing of bicycles and sewing machines.

An appealing notion is that for those goods which would be produced in specialized firms using similar factor proportions, the compromises (absence of specialized machinery, etc.) which might require diversified production to use more variable factors than specialized production are less important. As a consequence, it is those goods which are more likely to be produced in diversified firms. Put differently, a technological change which operates to exaggerate differences in factor proportions will mitigate against diversified production.

There are several ways this idea may be formalized. A straightforward, albeit restrictive, one is to suppose that if goods α and β are produced in a diversified firm, the variable factors utilize the same production technologies as are available to the specialized firms, but that the diversified firm operates the technologies using identical factor proportions; not unlike producing j in the morning and j' in the afternoon with the same tools and workforce. Obviously, this restriction to equal factor proportions implies greater use of variable factors by diversified firms whenever specialized firms would, given factor prices, choose distinct factor proportions.

To consider the issues raised above, assume there are two factors, x^1 and x^2, with prices r^1 and r^2. Letting x^i_j be the amount of factor i used in the production of good j, assume that the technologies can be written

$$q_\alpha = f(x^1_\alpha, x^2_\alpha)$$

and

$$q_\beta = h(ax^1_\beta, bx^2_\beta)$$

where a and b are positive constants, $f(\cdot)$ and $h(\cdot)$ are concave and twice continuously differentiable, and for all positive numbers y_1 and y_2

$$\frac{f_1(y_1,y_2)}{f_2(y_1,y_2)} > \frac{h_1(y_1,y_2)}{h_2(y_1,y_2)} ;$$

that is, given factor prices, firms specializing in the production of good α will be x^1-intensive. Now consider an increase in a, and adjustment of b, such that for any given q_β, the minimum cost of producing q_β is fixed. It is easy to check that for given $da > 0$,

$$db = \frac{b}{a} \frac{r^1 \hat{x}^1_\beta}{r^2 \hat{x}^2_\beta} da > 0$$

where $\hat{x}^i_\beta$ is the cost minimizing level of x^i_β. A given da thus rotates the entire isoquant map for good β clockwise, but leaves <u>unchanged</u> the cost functions for both types of specialized firms for all output levels. However, because the factor proportions used by a diversified firm are always intermediate with respect to those used by specialized firms, costs rise under the restricted technology, again for all output levels.

Given this setup, it may be shown that a reduction in a implies the S* region becomes smaller. That is, goods for which specialized firms would tend to use similar factor proportions are more likely to be produced by diversified firms.

This simple setup can be used to analyse other issues. For example, differential tax or subsidy treatment of factors depending on their use (e.g., steel in automobiles) can exaggerate or ameliorate the disadvantage diversified firms face which was used to generate the above prediction. Suppose good α (in unrestricted production) is more x^1-intensive than good β given factor prices. Then a subsidy to x^2 in the production of good α can virtually eliminate the diversified firm's disadvantage, yielding diversified production where none was viable previously. The degree of specialization in the economy is another avenue through which distortions arise.

Finally, non-convexities in production typically do not have observable consequences, as cost-minimization implies they will not be observed even if present. However, in the present context non-convexities are of some interest. The reasoning is as follows. It is easy to show that the underlying isoquants for good j can be to some degree non-convex and still, given equal factor proportions, yield a technology for the diversified firm which is convex. As a consequence, diversified firms may operate factor proportions such that one (at most one in the two good case) of the technologies is non-convex. As such, overall efficiency may imply what appear to be inefficiencies within the diversified firm. This notion provides an alternative, and efficiency-based, view of the observed inefficiencies underlying the literature on X-inefficiency.

REFERENCES

Allen, R.G.D. Mathematical Analysis for Economists. London: Macmillan, 1938.

Bailey, E.E. and Friedlander, A.F. "Market Structure and Multiproduct Industries." Journal of Economic Literature 20 (September 1982): 1024-48.

Baumol, W.J., et al. Contestable Markets and the Theory of Industry Structure. New York: Harcourt Brace Jovanovich, 1982.

Dorfman, R. et al. Linear Programming and Economic Analysis. New York, McGraw Hill, 1958.

Hicks, J.R. Value and Capital. London: Oxford, 1939.

Laitinen, K., A Theory of the Multi-Product Firm. New York: North-Holland 1980.

MacDonald, G.M. and Slivinski, A.D. "Efficiency and the Structure of Production." Economic Research Center/NORC Discussion Paper 83-21, December 1983.

Marshall, A. Principles of Economics. London: Macmillan, 1920.

Rosen, S. "Authority, Control and the Distribution of Earnings." Bell Journal of Economics 13 (1982), 311-23.

Samuelson, P.A. Foundations of Economic Analysis. Cambridge: Harvard University Press, 1947.

Sonnenschein, H. "Recent Results on the Existence of Cournot Equilibrium When Efficient Scale is Small Relative to Demand." in Hildenbrand, W. Advanced in Economic Theory. New York: Cambridge University Press, 1982.

Viner, J. "Cost Curves and Supply Curves," in A.E.A. Readings in Price Theory Chicago: Irwin, 1952.

FOOTNOTES

Comments from David Donaldson, Ig Horstmann, Boyan Jovanovic, Peter Lloyd, Michael Parkin, Charles Plott, Ed Prescott, Sherwin Rosen, Chris Robinson, Hugo Sonnenschein, and the referee are gratefully acknowledged. This paper is a revised version of Economics Research Center/NORC Discussion Paper 83-21, December 1983. The Social Science and Humanities Research Council of Canada provided no assistance of any kind.

[1] Of all the contestability material, the work of Baumol et al. (1982, Ch. 9) is the most closely related to the present analysis. Therein firms are permitted to choose a set of goods to produce, and a condition is provided which is necessary and sufficient for the (otherwise exogenous) symmetric outcome to be supported in equilibrium. That this condition is indeed a relevant restriction is shown by means of a two good numerical example in which the condition fails and the equilibrium is asymmetric.

[2] The analysis to follow does not distinguish between firms and plants. Doing so--indeed viewing firms as efficient "aggregates" of plants much in the same way as plants aggregate output--appears to be a profitable direction for extension of this type of analysis. In fact, the programming problem studied below is exactly that which would be solved by a monopolist choosing the least cost collection of plants to operate. Application to multinational firms is immediate.

[3] Throughout, the subscript j will index α and β. Also, j' will indicate which of α and β is "not j". All summations, unless indicated otherwise, add over $j = \alpha,\beta$.

[4] If $\lim_{q_{j'} \to 0} C(q_\alpha, q_\beta) - C^j(q_j) = v_j \gtrless 0 \ \forall \ q_j$, where v_j is a constant, then v_j can simply be included as part of F, or deducted from F_j provided one of $F+v_j > 0$ or $F_j - v_j > 0$ holds. That v_j, taken to equal zero in the text, does not depend on q_j is the relevant restriction.

[5] The reader will note that the cost structure analyzed contains nonconvexities at the firm level, which is why it is necessary to demonstrate (as is done in the Appendix to MacDonald and Slivinski (1983)) that an equilibrium exists. Given existence, that production is aggregate cost minimizing in the absence of externalities is trivial.

[6] Baumol et al.'s Propositions 9D8 and 9D2 are analogous to lemmas 1 and 2, for example.

[7] Specifically, "for almost all" excludes measure zero events in the parameter space. Technically the $(F, F_\alpha, F_\beta, Q_\alpha, Q_\beta)$ vectors for which a configuration permitting all of N, N_α and N_β positive can achieve the minimum level of cost are confined to a four-manifold in R^5_+.

[8] The same configurations, but without claim that they are equilibrium configurations, will be denoted by the same notation, absent asterisks.

[9] $\underline{Q}(F, F_\alpha)$ is defined by the system

$$F + C(q_\alpha, q_\beta) - \lambda_\alpha q_\alpha - \lambda_\beta q_\beta = 0,$$

$$C_\alpha(q_\alpha, q_\beta) = \bar{\lambda}_\alpha,$$

$$C_\beta(q_\alpha, q_\beta) = \lambda_\beta,$$

and
$$\underline{Q}(F, F_\alpha) = q_\alpha / q_\beta;$$

the last equation implied by $Q_\alpha = Nq_\alpha$ and $Q_\beta = Nq_\beta$ when $N_j = 0$ as in $\bar{F}$.

[10]The restriction 9(a) is stronger than required. Though tedious, the analysis can proceed unchanged if 9(a) holds only for $q_\beta \in [0, \hat{q}_\beta]$ where

$$\hat{q}_\beta \equiv \underset{q_\beta}{\operatorname{argmin}} \frac{\tilde{F} + C(0, q_\beta)}{q_\beta}$$ and $\tilde{F}$ is as defined below.

[11]Though $\underline{Q}(\cdot)$ and $\bar{Q}(\cdot)$ may be nonmonotonic in F, as depicted in Figure 1, that they may intersect exactly once implies $\tilde{F}(F_\alpha, F_\beta)$ is unique.

[12]The primary focus of attention here is the structure of production. Predictions on changes in N^*, N_j^*, q_j^*, etc., can be derived straightforwardly.

[13]$C^j(q_j) = r_j \delta_j q_j + \chi(q_\alpha, q_\beta)$ with $q_{j'} = 0$.

[14]Results can of course be obtained for cost structures more general than (12). For example, if $C^\beta(q_\beta)$ is not affected by changes in r_α, an increase in r_α raises $\tilde{F}$ (i.e., causes S* to shrink) if and only if specialized production is more x_α-intensive than is diversified production when $F = \tilde{F}$.

[15]Convexity of C_α merely ensures $\underline{Q}(F, F_\alpha)$ is not "too flat".

[16]Here the split of management time across these two activities is ignored. The specification

$$g(\xi m)\, f[(1-\xi)m, s_j],\ 0 \le \xi \le 1$$

can be accommodated.

[17]Of course monitoring, in the sense of checking for unintentional (?) shirking via breakdown, might rise with the utilization of robotics.

APPENDIX

Proof of Lemma 1

The strict convexity of $C(\cdot)$ implies that all operating diversified firms produce the same $q^* = (q_\alpha^*, q_\beta^*)$. Given q^*, P becomes:

$$\min_{N, N_\alpha, N_\beta} N_\alpha \bar{C}_\alpha + N_\beta \bar{C}_\beta + N[F + C(q^*)]$$

$$\text{subject to } N_j \bar{q}_j + N q_j^* = Q_j \qquad j=\alpha,\beta$$

As an LP problem in 3 variables with only two constraints, it follows that except for a measure zero set of $(Q_\alpha, Q_\beta, F, F_\alpha, F_\beta)$ values, at most two of $N^*, N_\alpha^*, N_\beta^*$ can be positive. (Dorfman et al., 1958, Ch. 4, Theorem 2 and Corollary to Theorem 3.) (QED)

Proof of Lemma 2

For any $Q = (Q_\alpha, Q_\beta)$, $Q_j > 0$, $j=\alpha,\beta$; if N^*, N_j^*, q^* solves P, and if $t > 0$, then

$$C^*(tQ) \leq tN_\alpha^* Q_\alpha + tN_\beta^* Q_\beta + tN^*[F + C(q^*)] = tC^*(Q)$$

Thus, $C^*[tQ/t] \leq (1/t)\, C^*(tQ)$, so $tC^*(Q) \leq C^*(tQ)$ also.
(QED)

Proof of Proposition 1

(i) Suppose, by way of contradiction, that $F \leq F_\alpha$ and $N_\alpha^* > 0$. Then from (3) and (6) it follows that $\lambda_\alpha = \bar{\lambda}_\alpha$, while (2) and (3) imply

$$F + C(\bar{q}_\alpha, 0) - \bar{\lambda}_\alpha \bar{q}_\alpha \leq 0$$

$C(\cdot)$ being strictly convex implies that for any q;

$$C(\bar{q}_\alpha, 0) = C(q) + C_\alpha(q)(\bar{q}_\alpha - q_\alpha) - C_\beta(q)q_\beta + A \qquad (A.2)$$

for some $A > 0$. Combining (A.1) and (A.2) with $q = q^*$ yields

$$0 \geq F + C(q^*) + C_\alpha(q^*)(\bar{q}_\alpha - q^*_\alpha) - C_\beta(q^*)q^*_\beta - \bar{\lambda}_\alpha\bar{q}_\alpha + A^*$$

$$= F + C(q^*) + (C_\alpha(q^*) - \bar{\lambda}_\alpha)\bar{q}_\alpha - C_\alpha(q^*)q^*_\alpha - C_\beta(q^*)q^*_\beta + A^*$$

$$> F + C(q^*) - C_\alpha(q^*)q^*_\alpha - C_\beta(q^*)q^*_\beta$$

since $A^* > 0$ and $C_\alpha(q^*) \geq \lambda_\alpha = \bar{\lambda}_\alpha$, from (7).

However, (7) also implies that $C_j(q^*) > \lambda_j$ only if $q^*_j = 0$, so that this last expression in fact equals

$$F + C(q^*) - \lambda_\alpha q^*_\alpha - \lambda_\beta q^*_\beta$$

and this being negative contradicts (5).

(ii) Consider the problem

$$\bar{F}: \quad \min_N N[F + C(Q/N)]$$

which has the FOC:

$$F + C(Q/N) - C_\alpha(Q/N)Q_\alpha/N - C_\beta(Q/N)Q_\beta/N = 0 \qquad (A.3)$$

which can be re-written as:

$$C_\alpha(Q/N) + C_\beta(Q/N)(Q_\beta/Q_\alpha) = \frac{F + C(Q/N)}{Q_\alpha/N}$$

and $$\frac{F + C(Q/N)}{Q_\alpha/N} > \frac{F_\alpha + C(Q_\alpha/N,0)}{Q_\alpha/N} \geq \bar{\lambda}_\alpha$$

Also, differentiation of (A.3) yields

$$\frac{dC_\beta(Q/N)}{dQ_\alpha} = \frac{Q_\alpha Q_\beta}{NL}[(C_{21})^2 - C_{11}C_{22}]$$

which is negative by the convexity of C(•), as $L = Q_\alpha^2 C_{11} + 2Q_\alpha Q_\beta C_{12} + Q_\beta^2 C_{22} > 0$. Thus, as $Q_\alpha \to \infty$, $C_\beta(Q/N)Q_\beta/Q_\alpha \to 0$, so that for some Q_α sufficiently large, $C_\alpha(Q/N) > \bar{\lambda}_\alpha$.

Differentiation of (A.3) also gives

$$\frac{dC_\alpha(Q/N)}{dQ_\alpha} = \frac{C_{22}C_{11} - (C_{12})^2}{NL} > 0$$

Thus, if there exists any Q_α for which $C_\alpha(Q/N) < \bar{\lambda}_\alpha$, then there is a unique $\bar{Q}_\alpha$ such that $C_\alpha(\bar{Q}_\alpha/N, Q_\beta/N) = \bar{\lambda}_\alpha$. Consider now $Q_\alpha \to 0$. If $Q_\alpha/N \to 0$ also, then $C_\alpha(Q/N) \to C_\alpha(0,Q_\beta/N)$.

Suppose, by way of contradiction, that $Q_\alpha/N \to \hat{a} > 0$. Note that the solution to $\bar{F}$ yields, for aggregate costs

$$N[F + C(Q/N)] = [\frac{F + C(Q/N)}{Q_\beta/N}]\, Q_\beta$$

which must therefore converge to:

$$[\frac{F + C(a,(\hat{Q}_\beta/\hat{Q}_\alpha)a)}{(Q_\beta/\hat{Q}_\alpha)a}]\, Q_\beta \text{ as } Q_\alpha \rightarrow 0$$

Let $\hat{q}_\beta = \underset{q_\beta}{\text{argmin}}[\frac{F + C(0,q_\beta)}{q_\beta}]$, and $\hat{N} = \hat{Q}_\beta/q_\beta$,

and note that $\hat{N}$ firms producing $(\hat{Q}_\alpha/N,\hat{Q}_\beta/N)$ will also produce the bundle (Q_α,Q_β), with total costs

$$\hat{N}[F + C(\hat{Q}_\alpha/N,\hat{Q}_\beta/N)] \rightarrow \hat{N}[F + C(0,\hat{Q}_\beta/N)] \text{ as } Q_\alpha \rightarrow 0$$

and

$$\hat{N}[F + C(0,\hat{Q}_\beta/N)] = [\frac{F + C(0,\hat{q}_\beta)}{\hat{q}_\beta}]\, Q_\beta$$

$$\leq [\frac{F + C(0,a\hat{Q}_\beta/Q_\beta)}{a(Q_\beta/\hat{Q}_\alpha)}]\, Q_\beta, \text{ by def. of } \hat{q}_\beta$$

$$< [\frac{F + C(\hat{a},a\hat{Q}_\beta/Q_\beta)}{a\hat{Q}_\beta/Q_\alpha}]\, Q_\beta, \text{ since } C_\alpha > 0.$$

Thus, $Q_\alpha/\hat{N} \to a > 0$ cannot be cost-minimizing so $C_\alpha(Q/N) \to C_\alpha(0,Q_\beta/N)$ as $Q_\alpha \to 0$. So, if $C_\alpha(0,Q_\beta/N) \leq \bar{\lambda}_\alpha$, let $\underline{Q}_\alpha$ be the unique value of Q_α which yields $C_\alpha = \hat{\lambda}_\alpha$, and if not, let $\underline{Q}_\alpha = 0$.

It is now shown that for given $F, F_\alpha (F > F_\alpha)$;

$$M_\alpha \lhd D \Leftrightarrow \underline{Q}_\alpha/Q_\beta < Q_\alpha/Q_\beta,$$

so that $\underline{Q}(F,F_\alpha) = \underline{Q}_\alpha$.

First, suppose $\underline{Q}_\alpha > 0$. Let $Q_\alpha > \underline{Q}_\alpha$ and suppose, by way of contradiction, that $D \lhd M_\alpha$. This implies that if one solves P with the added restriction $N_\beta = 0$, the solution is $\tilde{N}_\alpha = 0$, $\tilde{q} = Q/\tilde{N}$. Thus, $\tilde{q}_\alpha > 0$ is implied, so that $C_\alpha(Q/\tilde{N}) = \lambda_\alpha \leq \bar{\lambda}_\alpha$.

Solving $\bar{P}$ for aggregate α production at $\underline{Q}_\alpha < Q_\alpha$ must yield $C_\alpha = \bar{\lambda}_\alpha$ by definition. Since $\hat{N}_\alpha = 0$ in the above, however, $\hat{N}$ must be the solution to $\bar{P}$ given Q_α, and since $\frac{dC_\alpha}{dQ_\alpha} > 0$, it can't be that $C_\alpha(Q/\hat{N}) \leq \bar{\lambda}_\alpha$.

Thus, $D \lhd M_\alpha$ is not possible for $Q_\alpha > \underline{Q}_\alpha$.

Now suppose that $Q_\alpha < \underline{Q}_\alpha$ and $M_\alpha \lhd D$. Necessary conditions for the solution of P with the constraint $N_\beta = 0$ to have $\hat{N}_\alpha > 0$ is that $C_\alpha(\hat{q}) \geq \lambda_\alpha = \bar{\lambda}_\alpha$. Diversified firms then produce $\hat{Q}_\alpha = Q_\alpha - \hat{N}_\alpha\bar{q}_\alpha < Q_\alpha$, and Q_β. Solving $\bar{P}$ with $Q = (\hat{Q}_\alpha, Q_\beta)$ must yield $N = \hat{N}$, while if $Q = (Q_\alpha, Q_\beta)$, $\bar{P}$ yields, say $\tilde{N}$.

Since $\hat{Q}_\alpha < Q_\alpha < \underline{Q}_\alpha$, it follows that, since $\frac{dC_\alpha}{dQ_\alpha} > 0$ again,

$$\hat{C}_\alpha(q) = C(\hat{Q}_\alpha/\hat{N},\hat{Q}_\beta/\tilde{N}) < C_\alpha(Q_\alpha/\tilde{N},Q_\beta/\tilde{N}) < C_\alpha(\underline{Q}_\alpha/N,Q_\beta/N) = \bar{\lambda}_\alpha.$$

contradicting the requirement for $\hat{N}_\alpha > 0$. Thus, $M_\alpha \triangleleft D$ cannot hold.

Let $\underline{Q}_\alpha = 0$. Then $Q_\alpha < \underline{Q}_\alpha$ is impossible, so it need only be shown that $M_\alpha \triangleleft D$ for all Q_α. Suppose not, so that in solving P with $N_\beta = 0$, $\hat{N}_\alpha = 0$ results. Then $\hat{N}$ must solve $\bar{P}$ also, and $C_\alpha(Q/\hat{N}) = \hat{\lambda}_\alpha \leq \hat{\lambda}_\alpha$ is necessary. But $\underline{Q}_\alpha = 0$ only arises when in solving $\bar{P}$, $C_\alpha(Q/\hat{N}) > \bar{\lambda}$ is always the result. Thus, $M_\alpha \triangleleft D$ must hold.

(iii) This follows readily from the way in which $\underline{Q}_\alpha$ was defined in (ii).

(QED)

The proof of Proposition 2 is entirely analogous to that of 1, and so is not provided.

<u>Proof of Proposition 3</u>

The existence of $\tilde{F}(F_\alpha,F_\beta)$ is demonstrated first. Let $\bar{N}(Q,F)$ indicate the solution to $\bar{P}$, when $Q_\alpha/Q_\beta = Q$ and diversified firms' fixed costs are F. Then $\underline{Q}(F,F_\alpha)$ is defined by

$$F + C(\underline{Q}/\bar{N}(\underline{Q},F),\ 1/\bar{N}(\underline{Q},F)) - \bar{\lambda}_\alpha\underline{Q}/\bar{N}(\underline{Q},F) - C_\beta(\underline{Q}/\bar{N}(\underline{Q},F),\ 1/\bar{N}(\underline{Q},F))/\bar{N}(\underline{Q},F) = 0$$

Since $F_\alpha + C(\bar{q}_\alpha,0) - \bar{\lambda}_\alpha\bar{q}_\alpha = 0$, it follows that:

$$(F-F_\alpha) + C(\underline{Q}/\bar{N},1/\bar{N}) - C(\bar{q}_\alpha,0) + \bar{\lambda}_\alpha(\bar{q}_\alpha - \underline{Q}/\bar{N}) - C_\beta(\underline{Q}/\bar{N},1/\bar{N})/\bar{N} = 0$$

Then, using the Taylor approximation

$$C(\bar{q}_\alpha,0) = C(Q/N,1/N) + C_\alpha(Q/N,1/N)(\bar{q}_\alpha - Q/N) - C_\beta(Q/N,1/N)/N + A$$

and the fact that $C_\alpha(Q/N,1/N) = \bar{\lambda}_\alpha$, it follows that

$$F - F_\alpha = A.$$

Thus, $A \to 0$ as $F \to F_\alpha$, so that strict convexity of $C(\cdot)$ then implies that

$(Q(F,F_\alpha)/N(Q,F),\ 1/N(Q,F)) \to (\bar{q}_\alpha,0)$ so that $C_\beta(Q/N,1/N) \to C_2(\bar{q}_\alpha,0) < \bar{\lambda}_\beta$ by (9b), and also, $Q \to \infty$.

Rearranging the definition of Q yields:

$$C_\beta(Q/N,1/N) = \frac{F + C(Q/N,1/N)}{1/N} - \bar{\lambda}_\alpha Q$$

$$> \frac{F + C(0,\hat{q}_\beta(F))}{\hat{q}_\beta(F)} - \bar{\lambda}_\alpha Q$$

and also,

$$\frac{C_\beta(Q/N,1/N)}{Q} > \frac{F + C(\hat{q}_\alpha(F),0)}{\hat{q}_\alpha(F)} - \bar{\lambda}_\alpha$$

where $\hat{q}_j(F) = \text{argmin}\left[\frac{F + C^j(q_j)}{q_j}\right]$ for $j = \alpha,\beta$.

The first term on the RHS of both of these inequalities is unbounded as $F \to \infty$, since $C_{jj} \geq \varepsilon > 0$. If Q is bounded above as $F \to \infty$, then the first inequality implies C_2 is also unbounded above. If, however, $Q \to \infty$, then the second inequality implies $C_2 \to \infty$. Thus, it must be that for some

F', $C_\beta(Q(F',F_\alpha)/N(Q(F',F_\alpha)F'),\ 1/N(Q(F',F_\alpha),F')) = \bar{\lambda}_\beta$.

A similar argument establishes the existence of an F" at which

$C_\alpha(\bar{Q}(F'',F_\beta)/N(\bar{Q}(F'',F_\beta),F''),\ 1/N(\bar{Q}(F'',F_\beta),F'')) = \bar{\lambda}_\alpha$.

Recalling that $C_\alpha(Q/N,1/N) = \bar{\lambda}_\alpha$ and $C_\beta(\bar{Q}/N,1/N) = \bar{\lambda}_\beta$, it follows that

$(\bar{Q}(F'',F_\beta)/N(\bar{Q},F''),\ 1/N(\bar{Q},F'')) = (Q(F',F_\alpha/N(Q,F'),1/N(Q,F'))$ by strict convexity of $C(\cdot)$, and so,

$$\bar{Q}(F'',F_\beta) = Q(F',F_\alpha) = \tilde{Q} \text{ and } F'' = F' = \tilde{F}.$$

Note that this implies that this $(\tilde{F},\tilde{Q})$ pair is unique, since if there were another such pair, say $(\hat{F},\hat{Q})$, then the fact that

$$C_j(\hat{Q}/N(\hat{Q},\hat{F}),\ 1/N(\hat{Q},\hat{F})) = \bar{\lambda}_j \text{ for } j = \alpha,\beta,$$

again implies that $(\hat{Q}/\bar{N},1/\bar{N}) = (\tilde{Q}/\bar{N},1/\bar{N})$, but then

$\hat{F}+C(\hat{Q}/N,1/N) - \bar{\lambda}_\alpha\hat{Q}/N - \bar{\lambda}_\beta/N = 0$ implies $\hat{F} = \tilde{F}$, also.

The assertions in the Proposition can now be proved.

(i) Let $F \geq \tilde{F}$, and suppose $E \neq S^*$. Then $N^* > 0$ implies, from (5), that

$$F + C(q^*) - \lambda_\alpha q^*_\alpha - \lambda_\beta q^*_\beta = 0$$

and (6) requires $\lambda_j \leq \bar{\lambda}_j$ for $j = \alpha,\beta$. Since $\tilde{F}+C(\tilde{Q}/N,1/N) - \bar{\lambda}_\alpha\tilde{Q}/N - \bar{\lambda}_\beta/N = 0$ by definition, and using the fact that

$$C(q^*) = C(\tilde{Q}/N,1/N) + C_\alpha(\tilde{Q}/N,1/N)(q^*_\alpha-\tilde{Q}/N) + C_\beta(\tilde{Q}/N,1/N)(q^*_\beta-1/N) + A$$

and that $C_j(\tilde{Q}/N,1/N) = \bar{\lambda}_j$, subtraction yields

$$(\tilde{F} - F) + (\lambda_\alpha - \bar{\lambda}_\alpha)q^*_\alpha + (\lambda_\beta - \bar{\lambda}_\beta)q^*_\beta = A > 0$$

which is impossible, as every term on the LHS is non-positive.

To prove (ii), (iii) and (iv) we first show that $E = S^{*} \Rightarrow F \geq \tilde{F}$. Suppose then, that $E = S^{*}$ and $F < \tilde{F}$. (6) and (7) then require that

$C_j(q^{*}) \geq \lambda_j = \bar{\lambda}_j$ for $j = \alpha, \beta$.

But $q^{*} = (0,0) \Rightarrow C_2(q^{*}) < \bar{\lambda}_j$, while $q_{\alpha}^{*} > 0$ and $q_{\beta}^{*} = 0$ implies $q_{\alpha}^{*} = \bar{q}_{\alpha}$ and so $C_{\beta}(\bar{q}_{\alpha},0) \geq \bar{\lambda}_{\beta}$ a contradiction of (9b). A similar argument implies $q_{\alpha}^{*} > 0$, so that $C_j(q^{*}) = \bar{\lambda}_j$, for $j = \alpha, \beta$, so that $q^{*} = (\tilde{Q}/\bar{N}, 1/\bar{N})$, in fact. But then

$$\tilde{F} + C(\tilde{Q}/\bar{N}, 1/\bar{N}) - \bar{\lambda}_{\alpha}\tilde{Q}/\bar{N} - \bar{\lambda}_{\beta}/\bar{N} > F + C(q^{*}) - \bar{\lambda}_{\alpha}q_{\alpha}^{*} - \bar{\lambda}_{\beta}q_{\beta}^{*} \geq 0$$

contradicting the definition of $(\tilde{F},\tilde{Q})$.

Now, note that since it was established at the outset that $(\tilde{F},\tilde{Q})$ is the only intersection of $\underline{Q}$ and $\bar{Q}$, and that $\underline{Q} \to \infty$ as $F \to F_{\alpha}$, it must be that $\underline{Q}(F,F_{\alpha}) > \bar{Q}(F,F_{\beta})$ for all $F > \max\{F_{\alpha},F_{\beta}\}$. The rest of the proposition then follows from Propositions 1 and 2.

(QED)

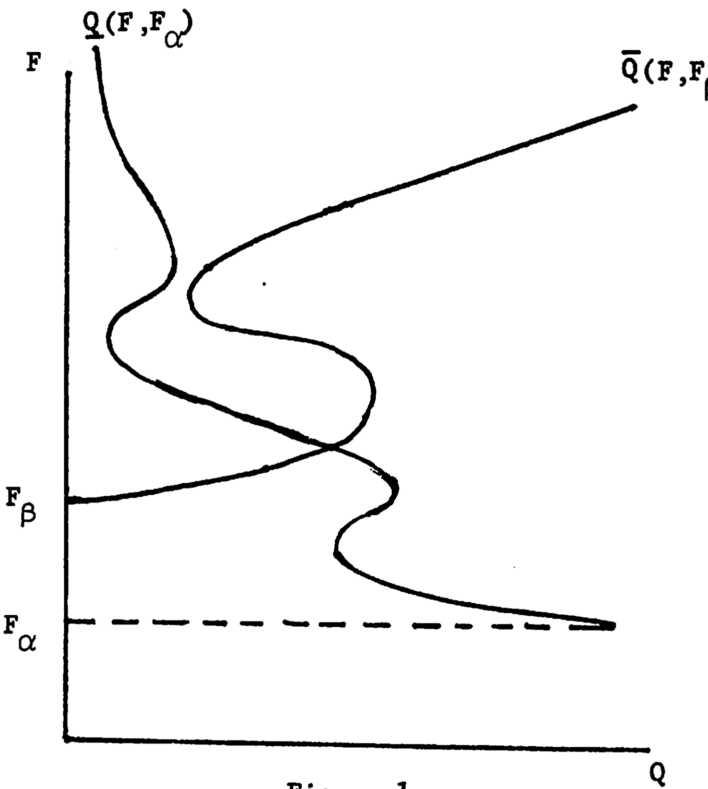


Figure 1

$\underline{Q}(F,F_\alpha)$ and $\bar{Q}(F,F_\beta)$

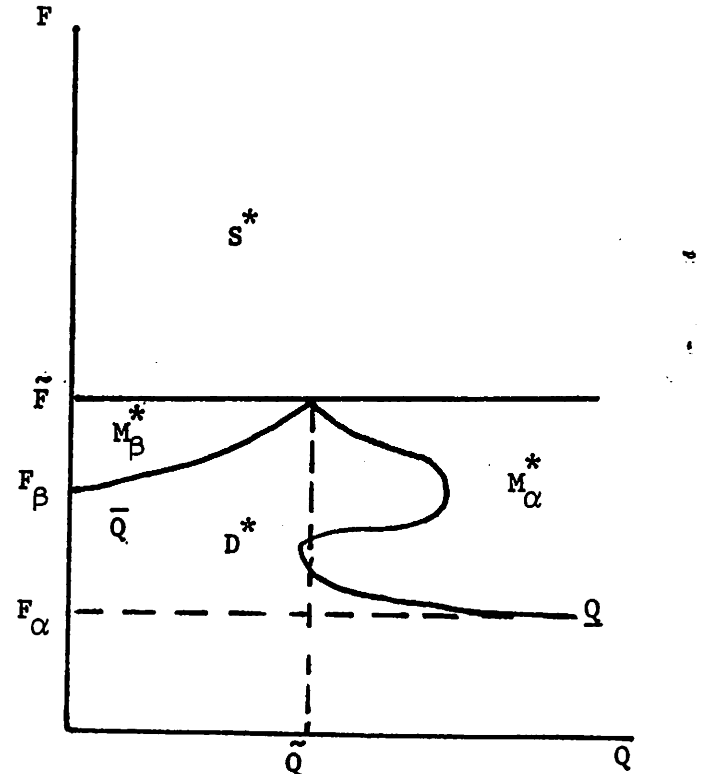
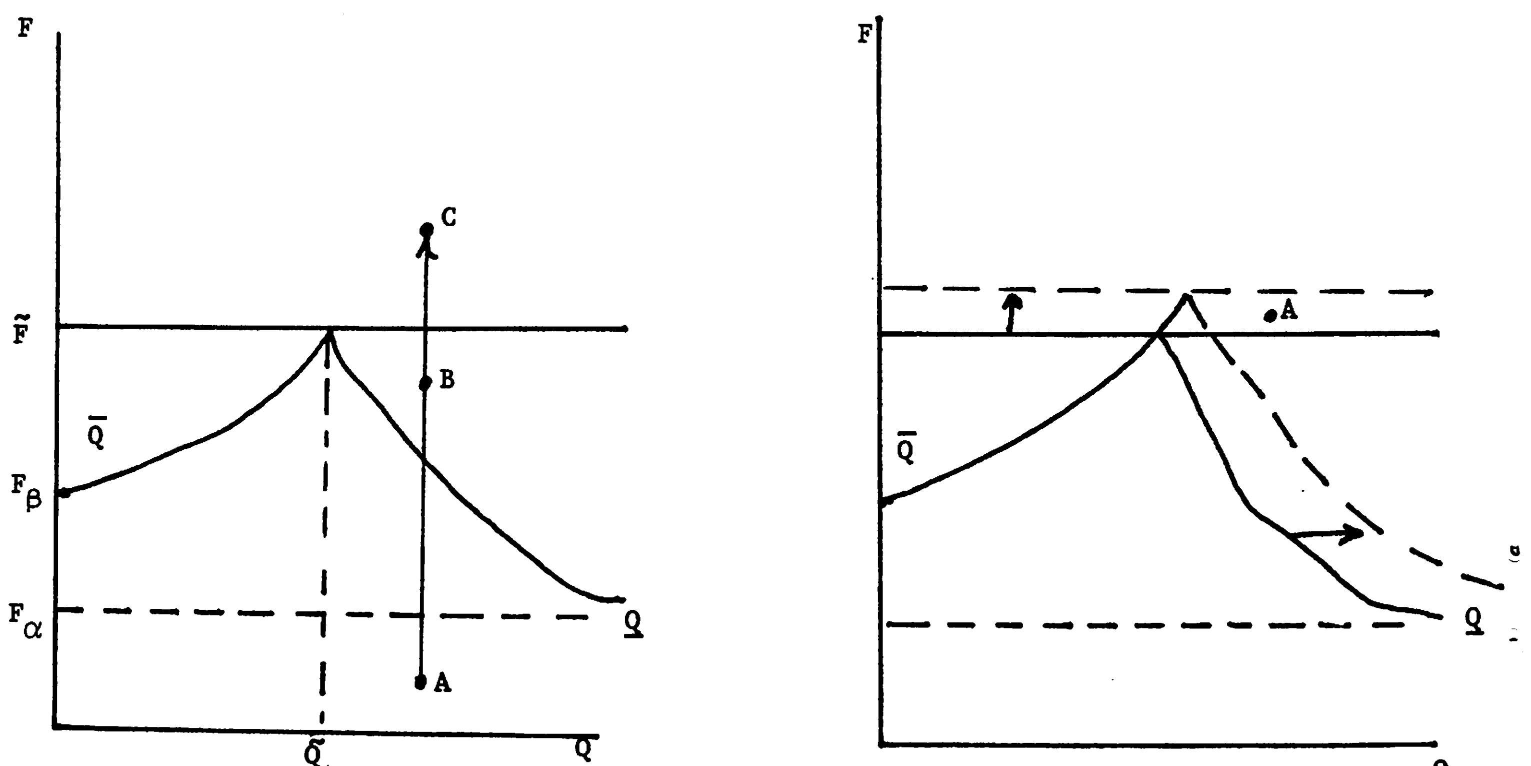


Figure 2

Characterization of Equilibria



a) Changes in Q,F Figure 3 b) An Increase in F_α

Comparative Statics of the Basic Model