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ON FORMULATING WALD TESTS OF NONLINEAR RESTRICTIONS*

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ABSTRACT

This note discusses a problem which has received little attention in the econometrics literature, namely that Wald tests of nonlinear restrictions that are algebraically equivalent under the null hypothesis give different results in small samples. The magnitude of potential discrepancies is illustrated by a simple Monte Carlo experiment.

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1. Introduction

As opposed to the Likelihood Ratio (LR) and Lagrange Multiplier (LM) or score tests, the Wald test is most convenient when the unrestricted model is easier to estimate than the restricted one, because the restricted model is linear but the restriction is nonlinear. Unfortunately, for this case, the Wald test has a drawback which does not appear to have been studied in the econometrics literature. The problem is that in small samples, changing the form of a nonlinear restriction to a form which is <u>algebraically equivalent</u> under the null hypothesis will change the numerical value of the Wald test statistic, although it will not change the LR or LM tests. This is because the Wald test is derived from a Taylor series expansion (see, for example, Silvey, 1975) and different ways of writing equivalent nonlinear expressions lead to nontrivial differences in the corresponding Taylor series. Therefore, although all the different Wald tests have the same asymptotic distribution, in actual practice there could be conflicts. While this has been noted directly (e.g., Burguete, Gallant and Souza, p. 185, 1982) and indirectly (e.g., Cox and Hinkley, p. 302, 1974), the purpose of this note is to show, on the basis of Monte Carlo evidence, that differences in the functional form of the nonlinear restrictions are likely to be important in the small sample sizes often encountered in economics.

2. An Illustrative Example

Consider the following classical linear regression equation model in N observations:

$$y_t = \beta_o + \beta_1 x_{1t} + \beta_2 x_{2t} + \epsilon_t, \qquad (1)$$

where y_t is the dependent variable, x_{1t} and x_{2t} are exogenous variables, β_o is a constant, β_1 and β_2 are slope coefficients not equal to zero and ε_t is iid $(0,\sigma^2)$. Under these assumptions, ordinary least squares (OLS) yields the best linear unbiased estimator.

Suppose one wishes to test the hypothesis that one parameter is equal to the reciprocal of the other:

$$H_o^A: \quad g^A(\beta_1,\beta_2) = \beta_1 - 1/\beta_2 = 0. \tag{2}$$

An alternative form to equation (2) which is algebraically equivalent under the null hypothesis is:

$$H_o^B: \quad g^B(\beta_1,\beta_2) = \beta_1\beta_2 - 1 = 0. \tag{3}$$

The general formula of the Wald statistic is

$$W = g(\hat{\beta})^T\{G(\hat{\beta})\widehat{V(\hat{\beta})}G(\hat{\beta})^T\}^{-1}g(\hat{\beta}) \overset{a}{\sim} \chi^2(1), \tag{4}$$

where g is the restriction expressed either as in equation (2) or equation (3), $G(\beta) = \partial g(\beta)/\partial\beta^T$ is evaluated at $\hat{\beta}$, the OLS estimator of β, $\widehat{V(\hat{\beta})}$ is the usual estimated variance-covariance matrix of $\hat{\beta}$. The test has the asymptotic distribution of a chi-square (with one degree of freedom because there is only one restriction). Using (4), the two alternative Wald test statistics can be calculated as:

$$W^A = (\hat{\beta}_1\hat{\beta}_2 - 1)^2/(\hat{\beta}_2^2 v_{11} + 2v_{12} + v_{22}/\hat{\beta}_2^2)$$

and

$$W^B = (\hat{\beta}_1\hat{\beta}_2 - 1)^2/(\hat{\beta}_2^2 v_{11} + 2\hat{\beta}_1\hat{\beta}_2 v_{12} + \hat{\beta}_1^2 v_{22}), \tag{5}$$

where the v_{ij} are the elements of $\widehat{V(\hat{\beta})}$. These statistics are clearly not identical. Moreover, infinitely many other parameterizations equivalent to (2) and (3) can be calculated, each with their own corresponding test statistic.

A Monte Carlo experiment was used to compare the relative performance of W^A and W^B in small samples.[1] The data were generated artificially using (1) above with the null hypothesis true, with x_1 and x_2 obtained using a first-order vector autoregressive process with an own-lag coefficient of 0.6 and a cross-coefficient of 0.3. The random component of each generated x and the ϵ_t were each drawn (independently) from a random standard normal deviate generator.[2] The number of rejections at the 5 percent level of confidence are reported in Table 1. Note that when the null hypothesis is true, one would expect 50 rejections out of 1000 with a 95 percent confidence interval [36,64].

The table indicates that there are sharp differences between the test results. In particular, under the null hypothesis as β_2 approaches zero, W^A tends to perform poorly in small samples, so that when $\beta_2 = .1$, W^A rejects 293 times compared to 65 times by W^B. Given that β_2 appears in the denominator of g^A, one might perhaps expect this. Nevertheless, W^A is a valid Wald test despite the fact that g^A has no derivative at $\beta_2 = 0$, because the asymptotic distribution of the test is obtained only under the null hypothesis (see pages 445-446 of Wald, 1943, or the heuristic derivation of Silvey, 1975, pages 115-116). In this case the null hypothesis specifically precludes $\beta_2 = 0$. But clearly as β_2 approaches zero, there is an approximate violation of the required continuity of derivatives (Wald, 1943, p. 463). This causes practical difficulties without changing the theoretical asymptotic justification of the test in any way.[3]

In addition, the table shows that the performance of the tests also differ when the null hypothesis is false. Under one parameter setting, W^A performs better than W^B in small samples, in the other, W^A performs very poorly.

Two other important results are observed but not presented in the table. First, under the null hypothesis, as β_2 becomes larger than one, W^A and W^B still perform similarly. Second, there are numerous conflicts between the tests, even when their distributions are similar. For example with $\beta_1 = \beta_2 = 1$ and $N = 20$, W^A and W^B are in conflict five percent of the time; this drops to one half of a percent at $N = 500$.

3. Concluding Remarks

The results above are selected from the much more detailed and extensive study in Gregory and Veall (1984) where several different assumptions about coefficients, variances and data generating processes are considered. In addition, the experiments are extended to Wald-testing of: common factor restrictions (Hendry and Mizon, 1978), nonlinear restrictions of rational expectations models (Hoffman and Schmidt, 1981), and an economic example concerning the impact of money supply shocks. All of these results confirm the conclusion that there can be substantial differences and discrepancies in Wald tests of different but algebraically equivalent nonlinear restrictions. Overall, the Monte Carlo evidence suggests that there is some statistical advantage in testing restrictions using multiplicative forms. Formalizing this last result using Edgeworth expansions is a topic for future research.

FOOTNOTES

[1]A referee suggests the following heuristic argument to explain intuitively why the tests are equivalent asymptotically but not in finite samples: as $g^B = \hat{\beta}_2 g^A$, the tests would be identical if $G^B = \hat{\beta}_2 G^A$. As $G^B = (\hat{\beta}_2, \hat{\beta}_1)$ and $G^A = (1, 1/\hat{\beta}_2^2)$, this requires $\hat{\beta}_1 = 1/\hat{\beta}_2$, which holds asymptotically under the null hypothesis, but not in finite samples due to sampling error.

[2]Random normal deviates were generated using IMSL subroutine GGNML as implemented on the CDC Cyber 170/835 at the University of Western Ontario. Processes were started 20 periods before the first observations were set.

[3]As an example of a case where the Wald test can differ analytically from W^B with no restrictions on the domain of β_1 or β_2, consider that H_o: $\beta_1\beta_2 = 1$ might be tested using H_o^C: $g^C(\beta_1, \beta_2) = e^{\beta_1\beta_2} - e = 0$.

[4]In the Hoffman and Schmidt case of testing the rational expectations hypothesis, it was found that the great similarity in Monte Carlo results for the W and LR tests (extended to the LM case in Gregory and Veall, 1985) applies only to the multiplicative form. With the ratio form of the restriction they cite (p. 267) but do not use in their computer program, results in Gregory and Veall (1984) indicate no such similarity. We thank Professor Schmidt for providing us with the computer program used in their article.

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TABLE 1

Number of Rejections at the 5 Percent Level in 1000 Trials Using Wald Tests of Simple Nonlinear Restrictions

Case	Form of Restriction	N=20	N=30	N=50	N=100	N=500
Null hypothesis true:						
$\beta_1 = 10.0,\ \beta_2 = 0.1$	A	293	253	206	142	78
	B	65	62	63	64	39
$\beta_1 = 5.0,\ \beta_2 = 0.2$	A	201	152	119	108	77
	B	64	58	57	48	56
$\beta_1 = 2.0,\ \beta_2 = 0.5$	A	86	89	78	58	43
	B	61	53	71	64	40
$\beta_1 = 1.0,\ \beta_2 = 1.0$	A	53	45	53	43	44
	B	69	47	65	46	47
Null hypothesis false:						
$\beta_1 = 1.5,\ \beta_2 = 1.0$	A	443	554	833	985	1000
	B	278	399	728	971	1000
$\beta_1 = 1.0,\ \beta_2 = 0.5$	A	65	196	601	992	1000
	B	584	775	943	998	1000