8-18-2015 12:00 AM

Fourier Inequalities in Lorentz and Lebesgue Spaces

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A thesis submitted in partial fulfillment of the requirements for the degree in Doctor of Philosophy

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(Thesis format: Monograph)

by

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Abstract

Mapping properties of the Fourier transform between weighted Lebesgue and Lorentz spaces are studied. These are generalizations to Hausdorff-Young and Pitt's inequalities. The boundedness of the Fourier transform on $\mathbb{R}^n$ as a map between Lorentz spaces leads to weighted Lebesgue inequalities for the Fourier transform on $\mathbb{R}^n$.

A major part of the work is on Fourier coefficients. Several different sufficient conditions and necessary conditions for the boundedness of Fourier transform on $\mathbb{T}$, viewed as a map between Lorentz Λ and Γ spaces are established. For a large range of Lorentz indices, necessary and sufficient conditions for boundedness are given. A number of known inequalities for generalized quasi concave functions are generalized and improved as part of the preparation for the proofs of the Fourier series results.

The Lorentz space results are used to obtain conditions that guarantee the continuity of the Fourier coefficient map between weighted L^p spaces. Applications to $L \log L$ and Lorentz-Zygmund spaces are also given.

Keywords: weighted inequalities, Fourier transform, Fourier series, Lorentz spaces, weighted Lebesgue spaces, quasi concave functions, $L \log L$, Lorentz-Zygmund space.

To the love of my life

SANAZ

Acknowledgments

I would like to express my deepest gratitude to my PhD advisor, Prof. Gord Sinnamon, for his thorough support and help during four years of graduate studies at Western University. His expertise in Mathematics and his enthusiasm for scientific discussion and problem solving, inspired and encouraged me to accomplish this research project. Besides, he supported me in other academic activities including professional writing, participation in scientific events, and teaching at university level.

I am very grateful to Dr. Rasul Shafikov who gave me valuable feedbacks on this monograph before and after its submission. I also appreciate the interesting topics that I learned from him in graduate courses.

I wish to thank my external examiner, Prof. Mieczysław Mastyło, for spending his precious time on reading this dissertation, for providing an encouraging report, and for discussing some related topics.

I also want to thank Dr. Tatyana Barron for her kindly suggestions on this thesis and her support elsewhere.

Contents

Glossary of notations

(X, μ)	5	A general measure space equipped with a σ-finite measure μ.
$L_\mu(X)$ or L_μ	5	The collection of μ-measurable functions $f : X \to \mathbb{C}$.
L^+_μ	5	The collection of non-negative μ-measurable functions on X.
$f_n \nearrow f$	5	Denotes the pointwise convergence of the increasing sequence of functions f_n.
$L^p(\mu)$	5	The function space L^p over a measure space (X, μ).
$L^p(w)$	6	The weighted L^p space on $\mathbb{R}^n$ or $\mathbb{T}$ with weight w.
$\mathbb{T}$	6	The unit circle.
m	6	The Lebesgue measure on $\mathbb{R}^n$ or $\mathbb{T}$.
L^+	6	The collection of non-negative Lebesgue measurable functions on $(0, \infty)$
$f \downarrow$	6	Indicates the function $f \in L^+$ is decreasing.
$\|f\|_{p,w}$	6	The weighted L^p norm on $(0, \infty)$ with weight w.
$A \lessapprox B$	6	Means $A \leq cB$ for some constants $c > 0$.
$A \approx B$	6	Means $c_1 A \leq B \leq c_2 A$ for some constants c_1 and c_2.
μ_f	6	The distribution function of $f \in L_\mu$.
f^*	6	The decreasing rearrangement of f.
$f^{\circledast}$	7	The increasing rearrangement of f, that is $f^{\circledast} = ((1/f)^*)^{-1}$.
f^{**}	8	The maximal function of f.
$f \prec g$	8	Denotes $f^{**} \leq g^{**}$.
B_p	11	A specific class of weight function on $(0, \infty)$.
RB_p	12	A specific class of weight function on $(0, \infty)$.
$\mathcal{F}$	12	The Fourier transform on $\mathbb{R}^n$, $\mathbb{T}$ or $\mathbb{Z}$.
$\hat{f}$	12	Fourier transform of a function f on $\mathbb{R}^n$ or $\mathbb{T}$.
τ_y	12	The translation operator.
$\mathcal{A}$	14	The collection of averaging operators on L^+.
Iu	16	The integral of u defined as $Iu(x) = \int_0^x u(t)\, dt$.
u°	16	The level function of u with respect to Lebesgue measure.
$\Lambda^p(w)$	19	The Lorentz Λ space with weight w.
$\Gamma^p(w)$	20	The Lorentz Γ space with weight w.
$\Theta^p(w)$	20	The Lorentz Θ space with weight w.
$L^{r,p}$	21	The Lorentz space $L^{r,p}$.
$L \log L$	22	The Zygmund space.
$L^{r,p}(\log L)^\alpha$	22	The Lorentz-Zygmund space.
$\Omega_{\alpha,\beta}$	24	The cone of functions f with $t^\alpha f(t)$ increasing and $t^{-\beta} f(t)$ decreasing.
P^r_ξ	25	The subclass of L^+ containing functions f such that $f(x) = x^r$ on $(0, 1)$.
P	25	The subclass of L^+ containing functions that are constant on $(0, 1)$.
$k_z^{\alpha,\beta}$	25	The function in L^+ defined as $k_z^{\alpha,\beta}(t) = \min(z^{-\alpha} t^\beta, z^\beta t^{-\alpha})$.
$\omega_z(t)$	26	The function in L^+ defined as $\omega_z(t) = \min(z^{-2}, t^{-2})$.
$K_\xi^{\alpha,\beta}$	26	A specific positive integral operator with the kernel $k_z^{\alpha,\beta}$.

Introduction

Mapping properties of the Fourier transform are well-known in simple cases. The Fourier transform maps integrable functions on $\mathbb{R}^n$ into the space of bounded functions. Plancherel's theorem asserts that $\mathcal{F} : L^2 \to L^2$ is an isometry and hence a bounded map. Thus, the Fourier transform is of strong type $(1, \infty)$ and $(2, 2)$. An application of the Riesz-Thorin interpolation theorem yields the Hausdorff-Young inequality, that is, $\mathcal{F} : L^p \to L^{p'}$ is bounded when $1 \leq p \leq 2$ and $p' = p/(p-1)$.

The Hausdorff-Young inequality, $\|\hat{f}\|_{p'} \leq \|f\|_p$, is sharp for the Fourier series, that is, the best constant is 1. Its Fourier transform version is due to Titchmarsh [T]. However, the sharp version for the Fourier transform was unknown until 1961, when Babenko in [Ba] proved $\|\hat{f}\|_{p'} \leq A_p \|f\|_p$ for even numbers p', with the best constant $A_p = (p^{n/2p})/(p')^{n/2p'}$. The extension to $2 \leq p' \leq \infty$ was obtained by Beckner in [Be].

The first weighted Fourier inequality is probably the one formulated by Hardy and Littlewood in [HL](1927). They showed that for $1 < p \leq 2$, there exists $C > 0$ such that

$$\left(\int_0^1 |f(x)|^p x^{p-2}\, dx \right)^{1/p} \leq C \left(\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^p \right)^{1/p}, \quad \forall \hat{f} \in \ell^p, \tag{1}$$

and for $p \geq 2$, there exists $C > 0$ such that

$$\left(\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^p \right)^{1/p} \leq C \left(\int_0^1 |f(x)|^p x^{p-2}\, dx \right)^{1/p}, \quad \forall f \in L^p(x^{p-2}). \tag{2}$$

Hereafter, the constant C in weighted norm inequalities such as (1) and (2) is a positive number that is independent of f but may depend on the exponents and weights involved. Moreover, this constant varies from one inequality to another.

Pitt generalized this theorem in [Pi](1937). He proved that when $1 < p \leq q < \infty$, $0 \leq a < 1/p'$, and $b = 1 - a - 1/p - 1/q \leq 0$, the following inequalities hold for $f \in L^1(\mathbb{T})$.

$$\left(\int_0^1 |f(x)|^q x^{bq}\, dx \right)^{1/q} \leq C \left(\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^p |n|^{pa} \right)^{1/p}, \text{and} \tag{3}$$

$$\left(\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^q |n|^{bq} \right)^{1/q} \leq C \left(\int_0^1 |f(x)|^p x^{pa}\, dx \right)^{1/p}. \tag{4}$$

Observe that when $q = p$ and $a = 0$, Inequality (3) reduces to (1). Similarly, when $q = p$ and $a = (p-2)/p$, Inequality (4) reduces to (2). Moreover, the Hausdorff-Young inequality, with a different constant, is recovered from (4) when $1 < p \leq 2$, $q = p'$, and $a = b = 0$.

The above inequalities deal with power weights only. The problem of obtaining Fourier inequalities in Lebesgue spaces with general weights was posed by Muckenhoupt in [Mu](1979). Specifically, the problem was to characterize those weights u, w for which $\mathcal{F} : L^p(w) \to L^q(u)$ is bounded. This turns into the weighted Fourier inequality

$$\left(\int_{\mathbb{R}^n} |\hat{f}(\gamma)|^q \, u(\gamma) \, d\gamma \right)^{1/q} \leq C \left(\int_{\mathbb{R}^n} |f(x)|^p \, w(x) \, dx \right)^{1/p}. \tag{5}$$

Benedetto and Heinig in [BH1] (1982) used a rearrangement estimate from Calderon that characterizes the quasi-linear operators, T, that are of weak type $(1, \infty)$ and $(2, 2)$. For such an operator we have

$$(Tf)^*(t) \leq c \left(\int_0^{1/t} f^*(x) \, dx + t^{-1/2} \int_{1/t}^{\infty} x^{-1/2} f^*(x) \, dx \right),$$

where f^* denotes the decreasing rearrangement of f. Using this estimate, they obtained the following sufficient condition for (5): For $1 < p \leq q < \infty$, and even weight functions u, w on $\mathbb{R}$, where u is decreasing and w is increasing on $(0, \infty)$, the inequality

$$\left(\int_{-\infty}^{\infty} |\hat{f}(\gamma)|^q \, u(\gamma) \, d\gamma \right)^{1/q} \leq C \left(\int_{-\infty}^{\infty} |f(x)|^p \, w(x) \, dx \right)^{1/p} \tag{6}$$

holds if

$$\sup_{x>0} \left(\int_0^{1/2x} u(t) \, dt \right)^{1/q} \left(\int_0^{x/2} w(t)^{1-p'} \, dt \right)^{1/p'} < \infty.$$

They also proved the converse to this result when $1 < p, q < \infty$.

Benedetto, Heinig and Johnson in [BHJ2](1987) generalized this result to arbitrary weights on $\mathbb{R}$. They obtained the following sufficient condition for (6) when $1 < p \leq q < \infty$,

$$\sup_{x>0} \left(\int_0^{1/x} u^*(t) \, dt \right)^{1/q} \left(\int_0^{x} w^{\circledast}(t)^{1-p'} \, dt \right)^{1/p'} < \infty. \tag{7}$$

Here $w^{\circledast}$ is the increasing rearrangement of w.

Benedetto and Heinig in [BH2](2003), provided new approaches to obtain Fourier inequalities. One approach is based on the following rearrangement estimate from Jodeit and Torchinsky [JT]:

$$\int_0^z (\hat{f})^*(t)^2 \, dt \leq D \int_0^z \Big(\int_0^{1/t} f^* \Big)^2 \, dt, \quad z > 0, \quad f \in L^1 + L^2. \tag{8}$$

The above inequality holds for any sublinear operator of type $(1,\infty)$ and $(2,2)$. Using this estimate together with the Hardy-Littlewood-Polya rearrangement inequality, Bendetto and Heinig proved the sufficient condition (7) for weights on $\mathbb{R}^n$.

They also introduced the Lorentz space method to obtain Fourier inequalities in weighted Lebesgue spaces. The key is to use the Hardy-Littlewood-Polya rearrangement inequality to reduce (5) to

$$\left(\int_0^\infty \hat{f}^*(t)^q\, u^*(t)\, dt\right)^{1/q} \leq C\left(\int_0^\infty f^*(t)^p\, w^{\circledast}(t)\, dt\right)^{1/p}. \tag{9}$$

Comparing this inequality to the definition of Lorentz Λ norm,

$$\|f\|_{\Lambda^p(w)} = \left(\int_0^\infty f^*(t)^p\, w(t)\, dt\right)^{1/p},$$

shows that (9) is a Fourier inequality in Lorentz spaces, that is $\|\hat{f}\|_{\Lambda^q(u^*)} \leq C\|f\|_{\Lambda^p(w^{\circledast})}$.

This motivates the problem of finding relations between weights u, w that are sufficient or necessary for $\|\hat{f}\|_{\Lambda^q(u)} \leq C\|f\|_{\Lambda^p(w)}$. In other words, we are interested in characterizing the boundedness of the Fourier transform viewed as $\mathcal{F} : \Lambda^p(w) \to \Lambda^q(u)$. Benedetto and Heing obtained sufficient conditions for the case $1 \leq p \leq q$ and $q \geq 2$, and for the case $2 \leq q < p$. They applied their results to weighted L^p spaces and provided sufficient conditions for (5).

Sinnamon took a different approach and worked on inequalities of type $\|\hat{f}\|_{\Lambda^q(u)} \leq C\|f\|_{\Gamma^p(w)}$. The Lorentz Γ space generalizes the Lorentz Λ space in that whenever $\Lambda^p(w)$ is a Banach function space, it coincides with $\Gamma^p(w)$. However, for certain weights $\Lambda^p(w)$ is no longer a Banach function space while $\Gamma^p(w)$ is.

In [Si4] (2003), Sinnamon used (8) to obtain sufficient conditions for $\|\hat{f}\|_{\Lambda^q(u)} \leq C\|f\|_{\Gamma^p(w)}$ in the case $0 < p \leq q$ and $q \geq 2$, and the case $2 \leq q < p$. He also constructed test functions that provide a necessary condition for $\|\hat{f}\|_{\Lambda^q(u)} \leq C\|f\|_{\Gamma^p(w)}$. That led to a characterization of weights u, w for which the Fourier transform $\mathcal{F} : \Gamma^p(w) \to \Lambda^2(u)$ is bounded when $p \leq 2$. The relation between u and w is stated in terms of the level function of the weight u.

In his subsequent work [Si5](2006), Sinnamon introduced the Lorentz space $\Theta^p(w)$, which he used to formulate a necessary and sufficient condition for boundedness of $\mathcal{F} : \Gamma^p(w) \to \Lambda^q(u)$ in the case $0 < p \leq 2 \leq q$.

One of our objectives in this thesis is to unify and furthermore extend the work of Benedetto, Heinig and Sinnamon. In the case $p \leq q$, we use Sinnamon's work to present new proofs for results of Benedetto and Heinig. We also provide new sufficient and necessary conditions for the boundedness of the Fourier transform between various Lorentz spaces.

Research in Fourier inequalities has mostly focused on the Fourier transform on $\mathbb{R}^n$, while very little is known about the boundedness of Fourier coefficients viewed as a map between weighted spaces. A major part of our work in this thesis is to provide Fourier series inequalities in weighted Lorentz spaces and weighted Lebesgue spaces.

Organization of the thesis

Chapter 1 contains most of the mathematical prerequisites. We briefly discuss some standard topics including Banach function spaces, the decreasing rearrangement, the Fourier transform and some useful inequalities. We discuss the definition and properties of the level function, which proves to be useful in formulating our necessary and sufficient conditions for Fourier inequalities. We also provide some details on different types of Lorentz spaces with general weights, as well as the well-known Lorentz-Zygmund spaces.

In Chapter 2, we introduce the cone of quasi concave functions as one of the main tools in our work. We reproduce and generalize some known inequalities concerning generalized quasi concave functions. Some particular cases of our results are used in Chapter 4 to prove our sufficient conditions for Fourier series inequalities.

Norm inequalities for the Fourier transform on $\mathbb{R}^n$ are studied in Chapter 3. Based on Sinnamon's work, we obtain several conditions that are sufficient or necessary for continuity of the Fourier transform as a map between Lorentz spaces. A number of examples are provided to illustrate and compare these results. We also improve, reproduce and provide new proofs for results of Benedetto and Heinig on Fourier inequalities in Lorentz spaces and weighted Lebesgue spaces.

Fourier series in Lorentz spaces are covered in Chapter 4. We adapt Sinnamon's approach and use our method from Chapter 3 to provide Lorentz norm inequalities for Fourier series. First we start with sufficient conditions for continuity of the Fourier coefficient map, viewed as a map between Lorentz spaces. Then we give the details of construction for the test functions that lead to our necessary conditions. Combining the sufficient conditions and the necessary conditions, we provide several characterizations of the boundedness of the Fourier coefficient map between Lorentz spaces.

In Chapter 5, we apply our results on Fourier series in Lorentz spaces to provide weighted L^p norm inequalities for the Fourier series. We also apply our results to $L \log L$ and Lorentz-Zygmund spaces as important instances of Lorentz Γ and Λ spaces. In particular, we provide a converse to certain results obtained by Bennett and Rudnick.

Chapter 1

Preliminaries

1.1 Basic concepts

1.1.1 The L^p spaces

Let (X,μ) be a σ-finite measure space. By $L_\mu(X)$ or simply L_μ we mean the collection of all μ-measurable complex-valued functions on X. We write L_μ^+ to denote the subcollection of L_μ that consists of non-negative functions. If $\{f_n\}$ is a sequence of functions in L_μ^+, the notation $f_n \nearrow f$ means $\{f_n\}$ is an increasing sequence and converges to f pointwise μ-a.e. Assume $0 < p < \infty$. The Lebesgue space $L^p(\mu)$ contains all functions in L_μ, for which

$$\|f\|_{L^p(\mu)} = \left(\int_X |f(x)|^p \, d\mu(x)\right)^{1/p}$$

is finite. For $p = \infty$ the above equation is replaced with

$$\|f\|_{L^\infty(\mu)} = \operatorname*{ess\,sup}_{x\in X} |f(x)|.$$

As usual we consider two functions to be equal if they are equal almost everywhere with respect to measure μ. A linear operator defined on a vector space containing $L^p(\mu)$ is said to be *of type* (p,q) if T maps $L^p(\mu)$ into L^q_ν, that is if $T : L^p(\mu) \to L^q(\nu)$ is bounded.

If $1 \le p \le \infty$, we have the well-known Hölder's inequality,

$$\int_X |f||g| \, d\mu \le \|f\|_{L^p(\mu)} \|g\|_{L^{p'}(\mu)}, \qquad \frac{1}{p} + \frac{1}{p'} = 1,$$

and Minkowski's inequality,

$$\|f+g\|_{L^p(\mu)} \le \|f\|_{L^p(\mu)} + \|g\|_{L^p(\mu)}.$$

We also require the Minkowski's integral inequality,

$$\left(\int \left(\int f(x,y)\, d\nu(y)\right)^p d\mu(x)\right)^{1/p} \le \int \left(\int f(x,y)^p \, d\mu(x)\right)^{1/p} d\nu(y),$$

for $1 \leq p < \infty$, where $f \geq 0$ is a $\mu \times \nu$-measurable function on $(X \times Y, \mu \times \nu)$. For $0 < p < 1$ the Minkowski's inequalities hold in the reverse direction, That is,

$$\|f + g\|_{L^p(\mu)} \geq \|f\|_{L^p(\mu)} + \|g\|_{L^p(\mu)},$$

and

$$\left(\int \left(\int f(x,y)\, d\nu(y)\right)^p d\mu(x)\right)^{1/p} \geq \int \left(\int f(x,y)^p\, d\mu(x)\right)^{1/p} d\nu(y).$$

There are certain measure spaces that we frequently encounter in our Fourier inequalities: $\mathbb{R}^n$ with Lebesgue measure, the unit circle $\mathbb{T}$ with normalized Lebesgue measure (i.e. $m(\mathbb{T}) = 1$), the integers $\mathbb{Z}$ with counting measure, and the half line $[0, \infty)$ with Lebesgue measure.

By a *weight function* on either of these spaces, we mean a non-negative, locally integrable function that is nonzero on a set of positive measure. If $w(x)$ is a weight on $\mathbb{R}^n$, $\mathbb{T}$ or $\mathbb{Z}$, the weighted Lebesgue space $L^p(w)$ is defined by

$$\|f\|_{L^p(w)} = \left(\int_X |f(x)|^p w(x)\, d\mu(x)\right)^{1/p},$$

where (X, μ) is either of the spaces $\mathbb{R}^n$, $\mathbb{T}$ or $\mathbb{Z}$ with the standard measure.

We introduce more specific notation for non-negative measurable functions on $[0, \infty)$. The collection of all these functions is denoted by L^+. We sometimes write $f \downarrow$ to state $f \in L^+$ is decreasing. By "decreasing" we mean "non increasing", thus a constant function is decreasing in this sense. If $w \in L^+$ is a weight on $[0, \infty)$, we write $\|f\|_{p,w}$ for the weighted L^p norm of $f \in L^+$, that is

$$\|f\|_{p,w} = \left(\int_0^\infty f(t)^p w(t)\, dt\right)^{1/p}, \quad f \in L^+.$$

For the unweighted case we drop "w" and simply write $\|f\|_p$.

We will use some standard notations for inequalities. If A and B are mathematical expressions (usually depending on parameters or classes of functions), $A \lessapprox B$ means there exists a constant $c > 0$ such that $A \leq cB$. We say A and B are equivalent and write $A \approx B$ if there exist positive constants c_1, c_2 such that $c_1 A \leq B \leq c_2 A$

1.1.2 The decreasing rearrangement

For $f \in L_\mu$, the function

$$\mu_f(\lambda) = \mu\{x \in X : |f(x)| > \lambda\}$$

is called the *distribution function* of f. Notice that μ_f is a non-negative, decreasing function on $[0, \infty)$. The *decreasing rearrangement* of f is defined as the generalized inverse of μ_f and is denoted by f^*. That is,

$$f^*(t) = \inf\{\alpha : \mu_f(\alpha) \leq t\},$$

with the convention $\inf \emptyset = \infty$. The elementary properties of μ_f and f^* are gathered in the following theorem.

Proposition 1.1. *Let $f, f_n, g \in L_\mu$, $0 \neq a \in \mathbb{C}$ and $0 < p < \infty$.*

(i) *μ_f and f^* are non-negative, decreasing, right continuous functions on the half line.*

(ii) *$f^* = m_{\mu_f}$ and $\mu_f = m_{f^*}$, where m is Lebesgue measure on the real line.*

(iii) *$\mu_{af}(\lambda) = \mu_f(\frac{\lambda}{|a|})$ and $(af)^* = |a|f^*$.*

(iv) *$|f| \leq |g|$, μ-a.e. implies $\mu_f \leq \mu_g$ and $f^* \leq g^*$.*

(v) *$|f_n| \nearrow |f|$ implies $\mu_{f_n} \nearrow \mu_f$ and $f_n^* \nearrow f^*$.*

(vi) *$(f^p)^* = (f^*)^p$ for $f \in L^+$.*

(vii) *$f^*(\mu_f(\lambda)) \leq \lambda$ and $\mu_f(f^*(t)) \leq t$, for $t, \lambda \geq 0$.*

Proof. See Propositions 2.1.3 and 2.1.6 in [BSh]. □

The L^p norm is invariant under rearrangements. More precisely:

Proposition 1.2. *Let $f \in L_\mu$ and $0 < p < \infty$. Then*

$$\int_X |f|^p \, d\mu = p \int_0^\infty \lambda^{p-1} \mu_f(\lambda) \, d\lambda = \int_0^\infty f^*(t)^p \, dt,$$

and for $p = \infty$,

$$\operatorname*{ess\,sup}_{x \in X} |f| = \inf\{\lambda : \mu_f(\lambda) = 0\} = f^*(0).$$

Proof. See Proposition 2.1.8 in [BSh]. □

This implies $\|f\|_{L^p} = \|f^*\|_p$ for $0 < p \leq \infty$.

Occasionally we will use the increasing rearrangement of a function.

Definition 1.3. Let $f \in L_\mu$. The *increasing rearrangement* of f is defined by $f^{\circledast} = ((1/f)^*)^{-1}$.

Proposition 1.4. ***The Hardy-Littlewood-Polya inequality***
For any functions $f, g \in L_\mu$ we have:

$$\int_X |fg| \, d\mu \leq \int_0^\infty f^*(t) g^*(t) \, dt, \quad \textit{and} \tag{1.1}$$

$$\int_X |fg| \, d\mu \geq \int_0^\infty f^*(t) g^{\circledast}(t) \, dt. \tag{1.2}$$

Proof. See Theorem 4.2.2 in [BSh] for the proof of the first inequality. The second inequality is proved in [H]. □

1.1.3 The maximal function

Let $f \in L_\mu$. The moving average

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*,$$

is called the *maximal function* of f. Notice that f^{**} is not the same as $(f^*)^*$. The latter is just equal to f^*. The notation $f \prec g$ in the literature often means $f^{**} \leq g^{**}$ and we will use these two notations interchangeably.

The primary properties of the maximal function are described in the following.

Proposition 1.5. *Let $f, f_n, g \in L_\mu$ and $0 \neq a \in \mathbb{C}$.*

(*i*) *f^{**} is a non-negative, decreasing, continuous function on the half line.*

(*ii*) *tf^{**} is increasing on the half line.*

(*iii*) *$f^{**}(t) > 0$ for all $t \in (0, \infty)$ unless $f = 0$ μ-a.e.*

(*iv*) *$f^* \leq f^{**}$.*

(*v*) *$(af)^{**} = |a| f^{**}$.*

(*vi*) *$|f| \leq |g|$ μ-a.e. implies $f^{**} \leq g^{**}$.*

(*vii*) *$|f_n| \nearrow |f|$ implies $f_n^{**} \nearrow f^{**}$.*

(*viii*) *$(f+g)^{**} \leq f^{**} + g^{**}$.*

Proof. See Proposition 2.3.2 and Theorem 2.3.4 in [BSh]. ☐

Observe that the last property does not hold for f^*, that is $(f+g)^* \nleq f^* + g^*$ in general. Take $f = \chi_{(0,1)}$ and $g = \chi_{(1,2)}$ as an example.

Proposition 1.6. *Hardy's lemma.* *Two functions $f, g \in L^+$ satisfy*

$$\int_0^x f(t)\,dt \leq \int_0^x g(t)\,dt, \quad x > 0,$$

if and only if

$$\int_0^\infty f(t)\varphi(t)\,dt \leq \int_0^\infty g(t)\varphi(t)\,dt$$

for all decreasing functions $\varphi \in L^+$.

Proof. See Proposition 2.3.6 in [BSh] for a proof. ☐

1.1.4 Banach function spaces

The Fourier inequalities in this thesis are stated in terms of various norms on functions. Before describing those norms and corresponding function spaces we present some general definitions.

Definition 1.7. A function $\rho : L_\mu \to [0, \infty]$ is called a *Banach function norm* if for $f, g, f_n \in L_\mu$ and $a \in \mathbb{C} \cup \{\infty\}$,

(i) $\rho(f) = 0$ if and only if $f = 0$ μ-a.e.

(ii) $\rho(af) = |a|\rho(f)$.

(iii) $\rho(f + g) \le \rho(f) + \rho(g)$.

(iv) $\rho(f) \le \rho(g)$ whenever $|f| \le |g|$ μ-a.e. (Lattice property)

(v) $\rho(f_n) \nearrow \rho(f)$ whenever $|f_n| \nearrow |f|$ μ-a.e. (Fatou property)

The *Banach function space* L_ρ is the collection of all the functions $f \in L_\mu$ with $\rho(f) < \infty$. The Fatou property guarantees that L_ρ is a complete normed space. We use $\|f\|_X$ to denote the norm corresponding to the Banach function space X. Banach function norms are often studied without assuming the Fatou property. We include it in our definition for convenience in stating certain results.

Definition 1.8. Two functions $f \in L_\mu(X)$ and $g \in L_\nu(Y)$ are called *equimeasurable* if they have the same distribution function, that is $\mu_f = \nu_g$.

A Banach function norm ρ on L_μ is said to be *rearrangement invariant* if equimeasurable functions have the same norm. That is $\rho(f) = \rho(g)$ whenever $\mu_f = \mu_g$.
In this case we say (X, μ, ρ) is a *rearrangement invariant function space.*

The first example of a rearrangement invariant space is $L^p(\mu)$ where $1 \le p \le \infty$. However, the weighted Lebesgue space $L^p(w)$ on $\mathbb{R}^n$ is not a rearrangement invariant space with respect to Lebesgue measure, provided w is not a.e. constant.

1.1.5 The associate space (Köthe dual)

Assume $1 < p < \infty$ and g is a function in $L^{p'}(\mu)$. The map

$$\varphi_g(f) = \int_X fg\, d\mu$$

defines a continuous, linear functional on $L^p(\mu)$ by Hölder's inequality. Conversely, any element of the dual space $(L^p(\mu))^*$ is of the above form. The operator norm of $\varphi_g : L^p(\mu) \to \mathbb{C}$ is equal to $\|g\|_{L^{p'}(\mu)}$, that is,

$$\|g\|_{L^{p'}(\mu)} = \sup_{h \in L^p(\mu)} \frac{\int_X |hg|\, d\mu}{\|h\|_{L^p(\mu)}}.$$

The Köthe dual generalizes this observation.

Definition 1.9. For any function norm ρ its *associate function norm* ρ' is defined as

$$\rho'(f) = \sup_{\rho(g)\le 1} \int |fg|\, d\mu = \sup_{g\in L_\rho} \frac{\int |fg|\, d\mu}{\rho(g)}, \quad f \in L_\mu.$$

The second equality is easy to verify. One can prove that ρ' itself is a Banach function norm. The space $L_{\rho'}$ is called the *associate space* or the *Köthe dual* of L_ρ. If the underlying norm is clear, we simply write X' to denote the Köthe dual of X. By "dual" of a space, we always mean the Köthe dual.

Example 1.10.

1. The Köthe dual of $L^p(\mu)$ is $L^{p'}(\mu)$ for $1 \le p \le \infty$.
2. Let w be a weight function on $\mathbb{R}^n$. The dual of weighted Lebesgue space $L^p(w)$ is $L^{p'}(w^{1-p'})$.

The definition of associate space implies a general Hölder's inequality:

$$\int_X |fg|\, d\mu \le \rho(f)\rho'(g), \quad f \in L_\rho, g \in L_{\rho'}.$$

The following theorem states that the second dual of a Banach function norm coincides with the norm itself. This provides a useful tool to prove results in Banach function spaces.

Theorem 1.11. *If ρ is a Banach function norm, then $L_{\rho''} = L_\rho$ and $\rho''(f) = \rho(f)$ for all $f \in L_\rho$.*

Proof. See Theorem 1.2.7 in [BSh] for a proof. □

The dual space is sometimes used together with the dual or adjoint of an operator.

Definition 1.12. Let (X,μ) and (Y,ν) be measure spaces. A *formal adjoint* of a linear operator $A : L^+_\nu \to L^+_\mu$ is a linear operator $B : L^+_\mu \to L^+_\nu$ such that

$$\int_Y Af(y)g(y)\, d\mu(y) = \int_X f(x)Bg(x)\, d\nu(x)$$

for all $f \in L_\nu$ and $g \in L_\mu$. In case $B = A$, we call A a *formally self adjoint* operator.

1.1.6 The Hardy inequality

The Hardy inequality concerns the boundedness of the averaging operator

$$Hf(y) = \frac{1}{y}\int_0^y f(t)\, dt, \quad f \in L^+,$$

between function spaces. There exist different versions of Hardy's inequality. The simplest case is the following:

Theorem 1.13. *Assume $1 < p \le \infty$ and $f \in L^+$. Then $\|Hf\|_p \le p'\|f\|_p$, that is,*

$$\left(\int_0^\infty \left(\frac{1}{t}\int_0^t f\right)^p dt\right)^{1/p} \le p' \left(\int_0^\infty f(t)^p\, dt\right)^{1/p}, \quad 1 < p < \infty,$$

with the usual adjustment for $p = \infty$.

Proof. See Corollary 6.21 in [F]. ☐

Corollary 1.14. *Assume $1 < p < \infty$. Then for all $f \in L^+$,*

$$\left(\int_0^x \left(\frac{1}{t}\int_0^t f\right)^p dt\right)^{1/p} \le p' \left(\int_0^x f(t)^p\, dt\right)^{1/p}, \quad x > 0.$$

Proof. This follows from Theorem 1.13 by replacing f with $f\chi_{[0,x)}$. ☐

The Hardy inequality in the weighted L^p setting is more complicated. We have the following characterization of the weights u and w for which $\|Hf\|_{q,u} \le C\|f\|_{p,w}$ holds.

Theorem 1.15. *Assume $1 < p \le q < \infty$ and let u and w be weight functions on $(0,\infty)$. There exists $C > 0$ such that*

$$\left(\int_0^\infty \left(\frac{1}{t}\int_0^t f\right)^q u(t)\, dt\right)^{1/q} \le C \left(\int_0^\infty f(t)^p w(t)\, dt\right)^{1/p}$$

holds for all $f \in L^+$ if and only if

$$\sup_{x>0} \left(\int_x^\infty \frac{u(t)}{t^q}\, dt\right)^{1/q} \left(\int_0^x w(t)^{1-p'}\, dt\right)^{1/p'} < \infty.$$

Proof. See Theorem 1 in [Br]. ☐

The weighted Hardy inequality for decreasing functions arises in the study of Lorentz spaces, which are defined in terms of f^* and f^{**}. As a first motivation for this, observe that $H(f^*) = f^{**}$. The case $w = u$ and $q = p$ is characterized by the so-called B_p weights.

Definition 1.16. Assume $p > 0$ and let w be a weight function on $(0,\infty)$. We say $w \in B_p$ if there exists $b_p > 0$ so that

$$\int_x^\infty \frac{w(t)}{t^p}\, dt \le b_p \frac{1}{x^p} \int_0^x w(t)\, dt \tag{1.3}$$

for all $x > 0$.

Remark 1.17. If $p > 1$ and w is decreasing, then $w \in B_p$. That is because

$$\int_x^\infty \frac{w(t)}{t^p}\, dt \le w(x) \int_x^\infty \frac{1}{t^p}\, dt = \frac{x^{-p}}{p-1} x w(x) \le \frac{1}{p-1}\frac{1}{x^p}\int_0^x w(t)\, dt.$$

This statement may fail when $p = 1$. As a counterexample, let $w(t) = 1$ for $t \in (0,\infty)$. Then for $p = 1$ the left-hand side of (1.3) is infinite for all $x > 0$, whereas the right-hand side is constant. Therefore, $w \notin B_1$.

Example 1.18. Let $p > 0$ and $-1 < a < p-1$. Then the power weight $w(t) = t^a$ satisfies the B_p condition. That is because

$$\int_x^\infty \frac{w(t)}{t^p}\,dt = \frac{-1}{a-p+1}x^{a-p+1} \quad \text{and} \quad \frac{1}{x^p}\int_0^x w(t)\,dt = \frac{1}{a+1}x^{a-p+1}.$$

The B_p weights are closely tied to the weighted Hardy inequality as stated in the next theorem.

Theorem 1.19. *Assume $0 < p < \infty$ and $w \in L^+$. Then $w \in B_p$ if and only if there exists $C \geq 0$ such that the weighted Hardy inequality holds for all decreasing functions $f \in L^+$. That is*

$$\int_0^\infty \Big(\frac{1}{t}\int_0^t f\Big)^p w(t)\,dt \leq C\int_0^\infty f(t)^p\,w(t)\,dt, \quad 0 \leq f\downarrow. \tag{1.4}$$

Proof. See Theorem 1.7 in [AM] for the case $p \geq 1$. The case $0 < p < 1$ is proved in Theorem 3 in [St]. $\square$

The following *reverse* B_p condition is sometimes useful in our Fourier inequalities.

Definition 1.20. Assume $p > 0$. A weight function $w \in L^+$ is said to be of class RB_p if there exists $b_p^* > 0$ so that

$$\int_x^\infty \frac{w(t)}{t^p}\,dt \geq b_p^*\frac{1}{x^p}\int_0^x w(t)\,dt$$

for all $x > 0$.

Remark 1.21.

1. Let v and w be weight functions satisfying $v(t) = t^{p-2}w(1/t)$. Then $w \in B_p$ if and only if $v \in RB_p$.

2. If w is increasing, then $w \in RB_p$ for $p > 1$. The computations are very similar to those in Remark 1.17.

1.1.7 Fourier transform on $\mathbb{R}^n$

The Fourier transform $\mathcal{F}$ on $L^1(\mathbb{R}^n)$ is defined as

$$[\mathcal{F}(f)](\omega) = \hat{f}(\omega) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i\omega\cdot x}\,dx. \tag{1.5}$$

For future reference, we give some properties of the Fourier transform in the following theorem. Here $\tau_y f$ denotes the translation of the function f by y, that is, $\tau_y f(x) = f(x-y)$.

Theorem 1.22. *Let $f, g \in L^1(\mathbb{R}^n)$ and let $\hat{f}, \hat{g}$ be their Fourier transforms. Assume $y, \gamma \in \mathbb{R}^n$. Then* *(i)* $\|\hat{f}\|_\infty \leq \|f\|_1$.

(ii) $\|\hat{f}\|_2 = \|f\|_2$, *if* $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ *(Plancherel's theorem).*

(iii) $[\mathcal{F}(\tau_y f(x))](\omega) = e^{-2\pi i y.\omega}\hat{f}(\omega)$.

(iv) $[\mathcal{F}(e^{2\pi i x.\gamma} f(x))](\omega) = \tau_\gamma \hat{f}(\omega)$.

(v) $\displaystyle\int_{\mathbb{R}^n} \hat{f}(x)g(x)\,dx = \int_{\mathbb{R}^n} f(y)\hat{g}(y)\,dy$, *for* $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$.

Notice that the integral formula (1.5) is valid for integrable functions. However since $\mathcal{F}$ is an isometry on $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ by property (ii) above, and $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ is dense in $L^2(\mathbb{R}^n)$, there is a unique continuous extension of $\mathcal{F}$ to $L^2(\mathbb{R}^n)$ which is in fact an isometry. This defines the Fourier transform on $L^2(\mathbb{R}^n)$ even though the equation (1.5) is no longer valid.

Therefore, the Fourier transform is bounded as $\mathcal{F} : L^1(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$ and $\mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ which means it is of type $(1, \infty)$ and $(2, 2)$. Notice that the operator norm is equal to 1 in both cases.

This property of the Fourier transform plays an essential role in studying its mapping properties. In fact, many of the sufficient conditions for boundedness of the Fourier transform, presented in this dissertation and other places, remain valid for any operator of type $(1, \infty)$ and $(2, 2)$.

The Fourier transform can be furthermore extended to $L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n)$. If $f \in L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n)$ then $f = f_1 + f_2$ for some $f_1 \in L^1(\mathbb{R}^n)$ and $f_2 \in L^2(\mathbb{R}^n)$. We define $\hat{f} = \hat{f}_1 + \hat{f}_2$. To show that it is well-defined, assume $f = g_1 + g_2$ for some $g_1 \in L^1(\mathbb{R}^n)$ and $g_2 \in L^2(\mathbb{R}^n)$. We have $f_1 + f_2 = g_1 + g_2$ which means $f_1 - g_1 = g_2 - f_2$. Taking the Fourier transform of both sides, we get $\hat{f}_1 - \hat{g}_1 = \hat{g}_2 - \hat{f}_2$. Hence $\hat{f}_1 + \hat{f}_2 = \hat{g}_1 + \hat{g}_2$.

1.1.8 Fourier transform on $\mathbb{T}$

Recall $\mathbb{T}$ denotes the unit circle equipped with normalized Lebesgue measure, that is, $m(\mathbb{T}) = 1$. The Fourier transform on $L^1(\mathbb{T})$ is defined as

$$[\mathcal{F}(f)](n) = \hat{f}(n) = \int_0^1 f(x)e^{-2\pi i n x}\,dx.$$

The transformed function $\hat{f}(n)$ is a function on $\mathbb{Z}$. We will stick to this viewpoint rather than considering $\hat{f}$ as a sequence. In the occasional cases where we view $\hat{f}$ as a sequence we will use the notation $\hat{f}_n$ instead of $\hat{f}(n)$. Then the Fourier series of f may be written as

$$f(x) = \sum_{n=-\infty}^{\infty} \hat{f}_n e^{2\pi i n x}.$$

We sometimes use the term *the Fourier coefficient map* to refer to the Fourier transform on $\mathbb{T}$ in order to distinguish it from the Fourier transform on $\mathbb{R}^n$. Notice that the same notation $\mathcal{F}$ is used both for Fourier transform on $\mathbb{R}^n$ and the Fourier coefficient map. The meaning is clear from the context.

The next theorem states some properties of Fourier coefficient map that will be used in our work. Recall that $\tau_y f(x) = f(x-y)$.

Theorem 1.23. *Assume $f, g \in L^1(\mathbb{T})$ with Fourier coefficients $\hat{f}(n)$ and $\hat{g}(n)$. Let $y \in \mathbb{R}/\mathbb{Z}$ and $k \in \mathbb{Z}$. Then*

(*i*) $\|\hat{f}\|_\infty \leq \|f\|_1$.

(*ii*) $\|\hat{f}\|_2 = \|f\|_2$.

(*iii*) $[\mathcal{F}(\tau_y f(x))](n) = e^{-2\pi i n y}\hat{f}(n)$.

(*iv*) $[\mathcal{F}(e^{2\pi i k x} f(x))](n) = \tau_k \hat{f}(n)$.

Notice that the finiteness of the measure on $\mathbb{T}$ implies $L^\infty(\mathbb{T}) \subset L^2(\mathbb{T}) \subset L^1(\mathbb{T})$. So the Fourier coefficient map is automatically defined on $L^2(\mathbb{T})$. The same is true for the extension to $L^1(\mathbb{T}) + L^2(\mathbb{T})$ since $L^1(\mathbb{T}) + L^2(\mathbb{T}) = L^1(\mathbb{T})$. Observe that the Fourier coefficient map is of type $(1, \infty)$ and $(2, 2)$.

1.2 The level function

The level function was introduced by Halperin in [Ha] and was studied and generalized by Sinnamon and Mastylo in [Si1], [Si2] and [MS]. It has applications to the formulation of Fourier inequalities. To provide the definition and properties of the level function, we introduce a certain type of averaging operator and define the least concave majorant of a function.

1.2.1 A class of averaging operators

Let $\{(a_j, b_j) : j \in J\}$ be a finite or countable collection of disjoint open intervals of finite length in $[0, \infty)$. The averaging operator associated to this collection is defined by

$$Af(x) = \begin{cases} \frac{1}{b_j - a_j} \int_{a_j}^{b_j} f(t)\, dt, & x \in (a_j, b_j), \\ \\ f(x), & x \notin \cup_{j \in J}(a_j, b_j), \end{cases}$$

where $f \in L^+$. On each interval (a_j, b_j) the averaging operator A replaces the function with its average on that interval. For points that are outside any interval the value of function remains intact.

The collection of all averaging operators on L^+ is denoted by $\mathcal{A}$. For future reference, we state the properties of averaging operators in the next proposition.

Proposition 1.24. *Let $A \in \mathcal{A}$ be associated to $\{(a_j, b_j) : j \in J\}$ and assume $f, g, f_n \in L^+$.*

(i) If f is decreasing, so is Af.

(ii) A is formally self adjoint, that is

$$\int_0^\infty (Af)(t)\, g(t)\, dt = \int_0^\infty f(t)\, (Ag)(t)\, dt.$$

(iii) If $f_n \nearrow f$, then $Af_n \nearrow Af$.

(iv) If $1 \leq p < \infty$ then $(Af)(t)^p \leq A(f^p)(t)$ for $t \geq 0$.

(v) $(Af)^{**} \leq f^{**}$.

Proof. Parts (i) and (ii) follow easily from the definition. Part (iii) is implied by the monotone convergence theorem. To prove Part (iv), we invoke Hölder's inequality to obtain:

$$\int_{a_j}^{b_j} f \leq (b_j - a_j)^{1/p'} \left(\int_{a_j}^{b_j} f^p \right)^{1/p}, \quad j \in J,$$

which implies $(Af)(t)^p \leq A(f^p)(t)$ for $t \in (a_j, b_j)$. Since $(Af)(t)^p = f(t)^p = (Af)(t)^p$ for $t \notin \bigcup_{j \in J}(a_j, b_j)$, the proof of (iv) is complete. For a proof of Part (v), see Theorem 2.3.7 in [BSh]. □

1.2.2 The least concave majorant

Let $\varphi \in L^+$ and set

$$\mathcal{C}_\varphi = \{G \in L^+ : \varphi \leq G, G \text{ is concave}\} \quad \text{and} \quad \tilde{\varphi}(x) = \inf_{G \in \mathcal{C}_\varphi} G(x).$$

Then $\tilde{\varphi} \in L^+$, $\varphi \leq \tilde{\varphi}$ and $\tilde{\varphi}$ is concave. The concavity of $\tilde{\varphi}$ follows from:

$$\begin{aligned} \tilde{\varphi}(tx + (1-t)y) &= \inf_{G \in \mathcal{C}_\varphi} G(tx + (1-t)y) \\ &\geq \inf_{G \in \mathcal{C}_\varphi} tG(x) + (1-t)G(y) \\ &\geq t \inf_{G \in \mathcal{C}_\varphi} G(x) + (1-t) \inf_{G \in \mathcal{C}_\varphi} G(y) \\ &= t\tilde{\varphi}(x) + (1-t)\tilde{\varphi}(y). \end{aligned}$$

Observe that if F is another non-negative concave function on $[0, \infty)$ with $\varphi \leq F$, then $\tilde{\varphi} \leq F$. The function $\tilde{\varphi}$ is called the least concave majorant of φ.

Definition 1.25. The *least concave majorant* of a function $\varphi \in L^+$, is the smallest non-negative concave function which dominates φ.

The discussion above shows that the least concave majorant is uniquely defined. However it may be infinite everywhere. As an example, consider $\varphi(t) = t^2$ for $t \geq 0$ and observe that $\tilde{\varphi} \equiv \infty$.

1.2.3 The level function

For a function $f \in L^+$ we adopt the notation $If(x) = \int_0^x f(t)\,dt$. Notice that for decreasing functions f and g, the relations $If \le Ig$ and $f^{**} \le g^{**}$ are equivalent.

Definition 1.26. Assume $f \in L^+$ such that $If(x) < \infty$ for $0 < x < \infty$. The *level function* of f, denoted by f°, is a decreasing function in L^+ such that $If \le If^\circ$ and for any decreasing function $g \in L^+$ satisfying $If \le Ig$ we have $If^\circ \le Ig$.

The two concepts, level function and least concave majorant are closely tied together. In fact If° is the least concave majorant of If. This suggest the way to construct the level function. Given $f \in L^+$ that is integrable near zero, first we compute the function If, then we compute the least concave majorant of If which we call F and finally we take the derivative of F to obtain the level function, that is $f^\circ = F'$. Concavity of F implies that F is absolutely continuous and therefore it is differentiable almost everywhere. So f° is defined almost everywhere on the half line. For a definition of f° that remains valid for all $f \in L^+$, see Definition 2.3 and Proposition 5.1 of [Si2].

Example 1.27.

1. If u is decreasing and integrable near zero, then $u^\circ = u$ a.e. That is because Iu is a concave function and hence its least concave majorant is Iu again.

2. Fix $z > 0$ and consider the following functions:

   - $u_1(t) = ae^t\chi_{(0,z)}\,,\ a = z(e^z - 1)^{-1}$,
   - $u_2(t) = t(z^2 - t^2)^{-1/2}\chi_{(0,z)}$. and
   - $u_3(t) = ct^r\chi_{(0,z)}\,,\ r \ge 0\,,\ c = z^{-r}(r+1)$.

   Their integrals $Iu_j(x) = \int_0^x u_j$ are computed as

   $$Iu_1(x) = \begin{cases} a(e^x - 1), & 0 < x \le z, \\ z, & x > z, \end{cases}$$

   $$Iu_2(x) = \begin{cases} z - (z^2 - x^2)^{1/2}, & 0 < x \le z, \\ z, & x > z, \quad \text{and} \end{cases}$$

   $$Iu_3(x) = \begin{cases} z^{-r}x^{r+1}, & 0 < x \le z, \\ z, & x > z. \end{cases}$$

   Examining the graphs, we see the functions Iu_j have the same least concave majorant given by

   $$F(x) = Iu_j^\circ = \begin{cases} x, & 0 < x \le z, \\ z, & x > z. \end{cases}$$

   Finally u_j° is the derivative of F which exists at all points $x \in (0,\infty)$ except for $x = z$. Therefore, the level function is $u_j^\circ(t) = \chi_{(0,z)}$ for $j = 1, 2, 3$.

The connection of the level function to averaging operators is stated in the next proposition.

Proposition 1.28. *Let $f \in L^+$ be bounded and assume it has a compact support. There exists an averaging operator A such that $Af = f^\circ$.*

Proof. See Proposition 2.1 in [Si2]. □

The next proposition gives an important property of the level function. It asserts that the level function is well-behaved for an increasing sequence of functions.

Proposition 1.29. *Let $f, f_n \in L^+$ and assume $f_n \nearrow f$. Then $f_n^\circ \nearrow f^\circ$.*

Proof. See Proposition 5.1 in [Si2]. □

The two propositions above, together with properties of averaging operators, provide the following functional description of the level function.

Theorem 1.30. *Let $h, u \in L^+$ with h decreasing. Then*

$$\sup_{A \in \mathcal{A}} \int_0^\infty (Ah)u = \sup_{0 \le \varphi\downarrow, \varphi \prec h} \int_0^\infty \varphi u = \int_0^\infty hu^\circ. \tag{1.6}$$

Proof. We repeat the proof from Lemma 2.2 in [Si4]. First assume u is bounded and compactly supported. For each $A \in \mathcal{A}$, observe that Ah is decreasing and $Ah \prec h$ by Proposition 1.24. Hence,

$$\sup_{A \in \mathcal{A}} \int_0^\infty (Ah)u \le \sup_{0 \le \varphi\downarrow, \varphi \prec h} \int_0^\infty \varphi u. \tag{1.7}$$

On the other hand assume φ is a decreasing function in L^+ with $\varphi \prec h$. This means $I\varphi \le Ih$ because both φ and h are decreasing. Recall $Iu \le Iu^\circ$ by definition. Now two applications of Proposition 1.6 yields

$$\int_0^\infty \varphi u \le \int_0^\infty hu \le \int_0^\infty hu^\circ. \tag{1.8}$$

It follows from (1.7) and (1.8) that

$$\sup_{A \in \mathcal{A}} \int_0^\infty (Ah)u \le \sup_{0 \le \varphi\downarrow, \varphi \prec h} \int_0^\infty \varphi u \le \int_0^\infty hu^\circ. \tag{1.9}$$

Now by Proposition 1.28 there exists $A_1 \in \mathcal{A}$ such that $A_1 u = u^\circ$. We have

$$\int_0^\infty (A_1 h)u = \int_0^\infty h(A_1 u) = \int_0^\infty hu^\circ.$$

Therefore, the supremum in (1.9) is attained at A_1 and equality holds.

Now if $u \in L^+$ is arbitrary, there exists an increasing sequence of bounded and compactly supported functions $\{u_n\}$ in L^+ which converges to u pointwise. By the monotone convergence theorem and Proposition 1.29 we have:

$$\begin{aligned}
\sup_{A\in\mathcal{A}} \int_0^\infty (Ah)u &= \sup_{A\in\mathcal{A}} \sup_n \int_0^\infty (Ah)u_n \\
&= \sup_n \sup_{A\in\mathcal{A}} \int_0^\infty (Ah)u_n \\
&= \sup_n \int_0^\infty hu_n^\circ \\
&= \int_0^\infty hu^\circ .
\end{aligned}$$

□

We can use duality to generalize the first part of Theorem 1.30 to arbitrary Banach function norms.

Corollary 1.31. *Let $h \in L^+$ be decreasing and ρ be a Banach function norm on L^+. Then*

$$\sup_{0\le\varphi\downarrow,\,\varphi\prec h} \rho(\varphi) = \sup_{A\in\mathcal{A}} \rho(Ah).$$

Proof. The proof is taken from Corollary 2.3 in [Si4]. By Theorem 1.11 for each $\varphi \in L^+$ we have

$$\rho(\varphi) = \rho^{''}(\varphi) = \sup_{u\in L^+,\,\rho^{'}(u)\le 1} \int \varphi u.$$

Now

$$\begin{aligned}
\sup\{\rho(\varphi) : 0 \le \varphi \downarrow, \varphi \prec h\} &= \sup\left\{\int_0^\infty \varphi u : 0 \le \varphi \downarrow, \varphi \prec h, \rho^{'}(u) \le 1\right\} \\
&= \sup\left\{\int_0^\infty (Ah)u : A \in \mathcal{A}, \rho^{'}(u) \le 1\right\} \\
&= \sup\{\rho(Ah) : A \in \mathcal{A}\}.
\end{aligned}$$

□

Although the second equality in (1.6) does not generalize to arbitrary function norms, we have a coarse estimate for the weighted Lebesgue norm, which will be used in our Fourier inequalities.

Corollary 1.32. *Let $1 \le s < \infty$ and $h, u \in L^+$ with h decreasing. Then*

$$\sup_{0\le\varphi\downarrow,\varphi\prec h} \|\varphi\|_{s,u} = \sup_{A\in\mathcal{A}} \|Ah\|_{s,u} \le \|h\|_{s,u^\circ}.$$

Proof. This is proved in Corollary 2.4 in [Si4], but we give a different proof here. The equality is a special case of Corollary 1.31, since $\|.\|_{s,u}$ is a Banach function norm. To prove the inequality, observe that Proposition 1.24 implies

$$\sup_{A\in\mathcal{A}}\int_0^\infty (Ah)^s u \le \sup_{A\in\mathcal{A}}\int_0^\infty A(h^s)u.$$

Note that h is decreasing, and so is h^s. Thus, applying Theorem 1.30 to h^s and u results in:

$$\sup_{A\in\mathcal{A}}\int_0^\infty A(h^s)u = \int_0^\infty h^s u^\circ.$$

Finally

$$\sup_{A\in\mathcal{A}}\left(\int_0^\infty (Ah)^s u\right)^{1/s} \le \left(\int_0^\infty h^s u^\circ\right)^{1/s}.$$

□

We finish this section by providing an equivalent expression for Iu°, as a double supremum involving Iu.

Proposition 1.33. *Let $f \in L^+$. Then for all $x > 0$*

$$\frac{1}{x}\int_0^x u^\circ \le \sup_{A\in\mathcal{A}}\int_0^x Au \le 2\sup_{y\ge x}\frac{1}{y}\int_0^y u \le 2\frac{1}{x}\int_0^x u^\circ.$$

Proof. See Lemma 2.5 in [Si4] for a proof. □

This proposition in particular implies

$$\frac{1}{x}\int_0^x u^\circ \approx \sup_{y\ge x}\frac{1}{y}\int_0^y u.$$

1.3 Lorentz spaces

1.3.1 Lorentz Λ space

Let (X,μ) be a σ-finite measure space and $w \in L^+$ be a weight function. The Lorentz Λ space is defined by

$$\|f\|_{\Lambda^p(w)} = \|f^*\|_{p,w} = \left(\int_0^\infty f^*(t)^p w(t)\,dt\right)^{1/p}, \text{ and}$$
$$\Lambda^p(w) = \{f \in L_\mu : \|f\|_{\Lambda^p(w)} < \infty\}.$$

The space $\Lambda^p(w)$ was first introduced by G. Lorentz in [L]. For $p \ge 1$, it is a Banach function space whenever w is decreasing. In particular the triangle inequality holds even though $(f+g)^* \not\le f^* + g^*$ in general. For $p \ge 1$, if w is not decreasing the functional $\|f\|_{\Lambda^p(w)}$ is not a norm. However it is equivalent to a Banach function norm if w satisfies the B_p condition. See Theorem 1.34.

According to Proposition 1.2, the unweighted Lorentz Λ space reduces to the usual L^p space. In other words, if $w \equiv 1$ then $\Lambda^p(w) = L^p(\mu)$.

1.3.2 Lorentz Γ space

The space $\Gamma^p(w)$ was studied in [Sa]. For $p > 0$ and weight $w \in L^+$, it is defined by

$$\|f\|_{\Gamma^p(w)} = \|f^{**}\|_{p,w} = \left(\int_0^\infty f^{**}(t)^p w(t)\,dt\right)^{1/p}, \text{ and}$$
$$\Gamma^p(w) = \{f \in L_\mu : \|f\|_{\Gamma^p(w)} < \infty\}.$$

Since $f^* \leq f^{**}$ we have $\|f\|_{\Lambda^p(w)} \leq \|f\|_{\Gamma^p(w)}$ which means $\Gamma^p(w) \hookrightarrow \Lambda^p(w)$. Unlike $\Lambda^p(w)$, for $p \geq 1$, the Lorentz Γ norm is a Banach function norm and we do not need w to be decreasing. The next theorem shows that the Γ space is a generalization of Λ space. It is an immediate corollary of the Hardy inequality for decreasing functions (1.4).

Theorem 1.34. *Assume $0 < p < \infty$ and $w \in B_p$. Then $\|f\|_{\Lambda^p(w)} \approx \|f\|_{\Gamma^p(w)}$.*

Proof. Since w satisfies the B_p condition and f^* is decreasing, Theorem 1.19 implies

$$\int_0^\infty \left(\frac{1}{t}\int_0^t f^*\right)^p w(t)\,dt \leq C\int_0^\infty f^*(t)^p\, w(t)\,dt$$

for all $f \in L_\mu$. This means $\|f\|_{\Gamma^p(w)} \leq C^{1/p}\|f\|_{\Lambda^p(w)}$. The proof is complete since $\|f\|_{\Lambda^p(w)} \leq \|f\|_{\Gamma^p(w)}$. $\square$

1.3.3 Lorentz Θ space

The Θ space is an intermediate space between $\Lambda^p(w)$ and $\Gamma^p(w)$. It was introduced and used by Sinnamon in [Si5] to formulate Fourier inequalities. The norm is defined by

$$\|f\|_{\Theta^p(w)} = \sup_{h\in L^+,\, h^{**}\leq f^{**}} \|h^*\|_{p,w} = \sup_{h^{**}\leq f^{**}} \left(\int_0^\infty h^*(t)^p w(t)\,dt\right)^{1/p}, \text{ and}$$
$$\Theta^p(w) = \{f \in L_\mu : \|f\|_{\Theta^p(w)} < \infty\}.$$

When $p \geq 1$ the expression $\|.\|_{\Theta^p(w)}$ is a function norm. See Theorem 3 in [Si5] for the proof.

The Θ norm lies between the Λ norm and Γ norm. It is readily seen that

$$\|f\|_{\Lambda^p(w)} \leq \|f\|_{\Theta^p(w)} \leq \|f\|_{\Gamma^p(w)}. \tag{1.10}$$

This implies the embeddings,

$$\Gamma^p(w) \hookrightarrow \Theta^p(w) \hookrightarrow \Lambda^p(w).$$

Sometimes an alternative form for $\|f\|_{\Theta^p(w)}$ is more useful. According to Corollary 1.31 we have

$$\|f\|_{\Theta^p(w)} = \sup_{0<h\downarrow,\, h\prec f} \|h\|_{p,w} = \sup_{A\in\mathcal{A}} \|A(f^*)\|_{p,w}, \tag{1.11}$$

where $h \in L^+$. In particular, if $f \in L^+$ is decreasing,

$$\|f\|_{\Theta^p(w)} = \sup_{0<h\downarrow,\, h\prec f} \|h\|_{p,w} = \sup_{A\in\mathcal{A}} \|Af\|_{p,w}. \tag{1.12}$$

Notice that when $w \in B_p$, Inequalities (1.10) and Theorem 1.34 imply that $\Lambda^p(w) = \Gamma^p(w) = \Theta^p(w)$.

1.3.4 Lorentz space $L^{r,p}$

Assume $1 \le p < \infty$ and $0 < r \le \infty$. The Lorentz space $L^{r,p}(\mathbb{T})$ consists of functions $f \in L_m(\mathbb{T})$ such that

$$\|f\|_{L^{r,p}(\mathbb{T})} = \left(\int_0^1 [t^{1/r} f^*(t)]^p \, \frac{dt}{t} \right)^{1/p} \tag{1.13}$$

is finite. Hereafter, we will drop the "$\mathbb{T}$" and agree that $L^{r,p} = L^{r,p}(\mathbb{T})$. The Lorentz space $L^{r,p}$ is an extension of L^p spaces since $L^{p,p} = L^p$.

Observe that $L^{r,p}$ is a Lorentz-Λ space with a power weight.

$$\|f\|_{L^{r,p}} = \|f\|_{\Lambda^p(w)}, \quad w(t) = t^{p/r-1}\chi_{[0,1)}. \tag{1.14}$$

The Lorentz norm $\|.\|_{\Lambda^p(w)}$ is a function norm whenever w is decreasing. Thus (1.13) defines a function norm when $r \ge p \ge 1$. By Example 1.18, $w(t) = t^{p/r-1}$ is a B_p weight if $r > 1$. Thus Theorem 1.34 asserts, for $p \ge 1$ and $r > 1$, that the functional (1.13) is equivalent to a Banach function norm.

For functions on $\mathbb{Z}$, with counting measure, we adopt the notation $\ell^{r,p}$, where

$$\|f\|_{\ell^{r,p}} = \left(\sum_{n=1}^{\infty} [n^{1/r} f_n^*]^p \, \frac{1}{n} \right)^{1/p}, \text{ and} \tag{1.15}$$
$$\ell^{r,p} = \{f : \mathbb{Z} \to \mathbb{C} : \|f\|_{\ell^{r,p}} < \infty\}.$$

Here, f_n^* is the decreasing rearrangement of f_n viewed as a two-sided sequence. In our treatment of the decreasing rearrangement, we consider $f(n) = f_n$ as a function on the atomic measure space $\mathbb{Z}$. Hence $f^*(t)$ is a decreasing, right continuous step function on $[0, \infty)$ which is constant on each interval $[n, n+1)$, where n is a non-negative integer. Notice that the value of $f^*(t)$ is determined by its value on non-negative integers. Now the only difference between $f^*(n)$ and f_n^* is a shift, that is $f_{n+1}^* = f^*(n)$.

Now (1.15) turns into:

$$\|f\|_{\ell^{r,p}} = \left(\sum_{n=0}^{\infty} [(n+1)^{1/r} f^*(n)]^p \, \frac{1}{n+1} \right)^{1/p} = \left(\int_0^{\infty} f^*(t)^p w(t) \, dt \right)^{1/p} = \|f\|_{\Lambda^p(w)},$$

where the weight $w(t)$ is

$$w(t) = (n+1)^{p/r-1}, \quad n \le t < n+1, \ 0 \le n \in \mathbb{Z}. \tag{1.16}$$

There exist well-known embeddings between Lorentz spaces of type $L^{r,p}$. The following theorem states that $L^{r,p}$ is increasing in p when r is fixed.

Theorem 1.35. *Assume $0 < r \le \infty$ and $0 < p \le q < \infty$. Then*

(*i*) $L^{r,p} \hookrightarrow L^{r,q}$,

(*ii*) $\ell^{r,p} \hookrightarrow \ell^{r,q}$.

Proof. See Proposition 4.4.2 in [BSh] for the proof. □

1.3.5 Zygmund space $L\log L$

The Zygmund space $L\log L$ consists of functions $f \in L_m(\mathbb{T})$ such that

$$\int_{\mathbb{T}} |f(x)| \log^+ |f(x)|\, dx < \infty,$$

where $\log^+ = \max(\log, 0)$. Lemma 4.6.2 in [BSh] shows that $f \in L\log L$ if and only if the functional

$$\|f\|_{L\log L} = \int_0^1 f^*(t)\log(1/t)\, dt = \int_0^1 f^{**}(t)\, dt$$

is finite. The equality follows from Tonelli's theorem. This shows that $L\log L$ is a Lorentz Γ space. We have

$$\|f\|_{L\log L} = \|f\|_{\Gamma^1(w)}, \quad w = \chi_{(0,1)}.$$

In particular $L\log L$ is a rearrangement invariant Banach function space. Theorem 4.6.5 in [BSh] shows that $L\log L$ is closer to $L^1(\mathbb{T})$ than any other $L^p(\mathbb{T})$ space in the sense that

$$L^\infty \hookrightarrow L^p \hookrightarrow L\log L \hookrightarrow L^1, \quad p > 1$$

1.3.6 Lorentz-Zygmund space

The spaces $L^{r,p}$ and $L\log L$ can be generalized to Lorentz-Zygmund space. Assume $0 < p < \infty$, $0 < r \le \infty$ and $-\infty < \alpha < \infty$. For $f \in L_m(\mathbb{T})$ set:

$$\|f\|_{L^{r,p}(\log L)^\alpha} = \left(\int_0^1 [t^{1/r}(1-\log t)^\alpha f^*(t)]^p \frac{dt}{t}\right)^{1/p}.$$

The Lorentz-Zygmund space $L^{r,p}(\log)^\alpha$ is a Lorentz Λ space since

$$\|f\|_{L^{r,p}(\log L)^\alpha} = \|f\|_{\Lambda^p(w)} \quad \text{where} \quad w(t) = t^{p/r-1}(1-\log t)^{p\alpha}\chi_{[0,1)}. \tag{1.17}$$

Note that for $\alpha = 0$ we have $L^{r,p}(\log L)^0 = L^{r,p}$ and for $p = r = \alpha = 1$ one can show that $L^{1,1}(\log L)^1 = L\log L$. In the case of counting measure on $\mathbb{Z}$, the Lorentz-Zygmund norm is defined by

$$\|f\|_{\ell^{r,p}(\log \ell)^\alpha} = \left(\sum_{n=1}^\infty [n^{1/r}(1+\log n)^\alpha f_n^*]^p \frac{1}{n}\right)^{1/p}.$$

Similar to Lorentz space $\ell^{r,p}$ we have

$$\begin{aligned}\|f\|_{\ell^{r,p}(\log \ell)^\alpha} &= \left(\sum_{n=0}^\infty [(n+1)^{1/r}(1+\log(n+1))^\alpha f^*(n)]^p \frac{1}{n+1}\right)^{1/p} \\ &= \left(\int_0^\infty f^*(t)^p w(t)\, dt\right)^{1/p},\end{aligned}$$

where the weight $w(t)$ is defined as

$$w(t) = (n+1)^{p/r-1}(1+\log(n+1))^{p\alpha} \quad \text{when} \quad n \le t < n+1,\ 0 \le n \in \mathbb{Z}. \tag{1.18}$$

This means $\ell^{s,q}(\log \ell)^\beta = \Lambda^q(w)$.

The space $L^{r,p}(\log L)^\alpha$ is decreasing with respect to r. This is stated in the following theorem.

Theorem 1.36. *Assume* $0 < r < s \le \infty$, $0 < p, q < \infty$ *and* $-\infty < \alpha, \beta < \infty$. *Then* $L^{s,q}(\log L)^\beta \hookrightarrow L^{r,p}(\log L)^\alpha$.

Proof. See Theorem 9.1 in [BR]. □

The behavior of $L^{r,p}(\log L)^\alpha$ when p changes is more subtle and depends on the value of α as well.

Theorem 1.37. *Assume* $0 < r \le \infty$, $0 < p, q \le \infty$ *and* $-\infty < \alpha, \beta < \infty$. *Then* $L^{r,p}(\log L)^\alpha \hookrightarrow L^{r,q}(\log L)^\beta$ *in the following cases.*

(*i*) $p \le q$ *and* $\alpha \ge \beta$.

(*ii*) $p > q$ *and* $\alpha + 1/p > \beta + 1/q$.

Proof. See Theorem 9.3 in [BR]. □

Chapter 2

Generalized quasi concave functions

In this chapter we provide inequalities concerning cones of functions with certain monotonicity properties. We will use these results in formulating our sufficient conditions for Fourier series inequalities. However the statements of the results are given in a very general sense. Those are Theorems 2.9 and 2.10, in which we will extend, reproduce and improve results of Maligranda in [Ma1] and [Ma2], and Sinnamon in [Si3] and [Si5].

2.1 Functions with two monotonicity conditions

A function $f \in L^+$ is called *quasi concave* if it is increasing and $\frac{1}{t}f(t)$ is decreasing. This means the slope of the line passing through the origin and the point $(t, f(t))$ is decreasing.

It is easy to verify that a non-negative concave function on $[0, \infty)$, is also quasi concave. On the other hand, any quasi concave function is equivalent to a concave function in the following sense.

Proposition 2.1. *Let φ be a quasi concave function and let $\tilde{\varphi}$ be its least concave majorant. Then $\frac{1}{2}\tilde{\varphi} \le \varphi \le \tilde{\varphi}$.*

Proof. We take the proof from Theorem 2.5.10 in [BSh]. The definition of the least concave majorant implies $\varphi \le \tilde{\varphi}$. To prove the other inequality let $x > 0$. Since φ is a quasi concave function, we have $\varphi(t) \le \varphi(x)$ when $0 < t \le x$, and $\varphi(t)/t \le \varphi(x)/x$ when $t \ge x$. Therefore,

$$\varphi(t) \le \varphi(x) + \frac{t}{x}\varphi(x) = \left(1 + \frac{t}{x}\right)\varphi(x), \quad t > 0.$$

Thus, $\varphi(t)$ is dominated by the concave function $\psi(t) = (1 + t/x)\varphi(x)$. This implies that $\tilde{\varphi} \le \psi$ since $\tilde{\varphi}$ is the least concave majorant of φ. It follows that $\tilde{\varphi}(t) \le (1 + t/x)\varphi(x)$ for $x, t > 0$. Let $x = t$ to obtain $\tilde{\varphi}(t) \le 2\varphi(t)$ and the proof is complete. $\square$

A nonmepty subset of a vector space is called a *cone* if it is closed under vector addition and multiplication by non-negative scalars. The class of quasi concave functions is therefore a cone in L^+. This is a special case of the cone $\Omega_{\alpha,\beta}$. For $\alpha + \beta > 0$, the cone $\Omega_{\alpha,\beta}$ contains all functions $f \in L^+$ such that $t^\alpha f(t)$ is increasing and $t^{-\beta} f(t)$ is decreasing. In the case $\alpha = 0$ and $\beta = 1$, the definition implies that $\Omega_{0,1}$ is exactly the class of quasi concave functions. Another interesting case is $\alpha = 2$ and $\beta = 0$ which arises naturally in our Fourier inequalities.

The quasi concave functions are not just an instance of $\Omega_{\alpha,\beta}$. Most of the time, a statement concerning functions in $\Omega_{\alpha,\beta}$, is proved by reducing the problem to quasi concave functions. This is possible since there are simple transformations to move between different cones of type $\Omega_{\alpha,\beta}$. For instance, if $f(t) \in \Omega_{\alpha,\beta}$, then $t^c f(t) \in \Omega_{\alpha-c,\beta+c}$, and for $\lambda > 0$ both $f(t^\lambda)$ and $f(t)^\lambda$ belong to $\Omega_{\lambda\alpha,\lambda\beta}$. Notice that all these transformations are invertible. In particular we can transform functions in $\Omega_{\alpha,\beta}$ to quasi concave functions and back.

Sinnamon used the cone $\Omega_{2,0}$ in [Si4] to formulate his sufficient condition for boundedness of Fourier transform between Lorentz spaces. See Theorem 3.2 in Chapter 3. However, for the case of Fourier series we will frequently encounter functions that have an additional property, namely being constant on the interval $(0, 1)$. This happens due to the finiteness of the measure on $\mathbb{T}$. To deal with this, we introduce a subclass P_ξ^r of L^+.

Let $\xi, r \geq 0$. We say $f \in P_\xi^r$ if there exists $c \geq 0$ so that $f(x) = cx^r$ for $0 < x < \xi$. Notice that in the trivial case $\xi = 0$, we have $P_0^r = L^+$. The case that we will use in our Fourier inequalities is $r = 0$ and $\xi = 1$. In this case we write P instead of P_1^0. Thus $f \in P$ means f is constant on the interval $(0, 1)$.

Now we introduce a 1-parameter family of functions in $P_\xi^\beta \cap \Omega_{\alpha,\beta}$, which plays an essential role in the study of quasi concave functions. Fix $\xi \geq 0$ and recall that $\alpha + \beta > 0$. Define

$$k_z^{\alpha,\beta}(t) = \min(z^{-\alpha} t^\beta, z^\beta t^{-\alpha}), \quad z, t > 0.$$

Observe that, as a function of t,

$$t^\alpha k_z^{\alpha,\beta}(t) = z^{-\alpha} \min(t^{\alpha+\beta}, z^{\alpha+\beta})$$

is increasing and

$$t^{-\beta} k_z^{\alpha,\beta}(t) = z^\beta \min(z^{-\alpha-\beta}, t^{-\alpha-\beta})$$

is decreasing. This means $k_z^{\alpha,\beta} \in \Omega_{\alpha,\beta}$. Furthermore, if $z > \xi$ then for $t \in (0, \xi)$ we have

$$k_z^{\alpha,\beta}(t) = z^{-\alpha} t^\beta \min(1, z^{\alpha+\beta} t^{-\alpha-\beta}) = z^{-\alpha} t^\beta.$$

Hence $k_z^{\alpha,\beta} \in P_\xi^\beta$ whenever $z > \xi$. We conclude that $k_z^{\alpha,\beta} \in P_\xi^\beta \cap \Omega_{\alpha,\beta}$ for all $z > \xi$.

The importance of functions $k_z^{\alpha,\beta}$ becomes clear in Theorems 2.9 and 2.10. Those theorems show that the collection $\{k_z^{\alpha,\beta} : z > \xi\}$ is a sufficiently large subcone of $P_\xi^\beta \cap \Omega_{\alpha,\beta}$ in a certain sense.

For our Fourier inequalities we use the special case $k_z^{2,0}$ which we denote by ω_z. That is,

$$\omega_z(t) = \min(z^{-2}, t^{-2}).$$

In particular, $\omega_z \in \Omega_{2,0}$ for all $z > 0$. For Fourier inequalities on $\mathbb{T}$ we work with $z > \xi = 1$. This means $\omega_z(t)$ is constant on the interval $(0, 1)$ whenever $z > 1$. So $\omega_z \in P \cap \Omega_{2,0}$ for $z > 1$.

2.2 Inequalities for quasi concave functions

In [Ma2], Maligranda proved the following.

Proposition 2.2. *Suppose $1 \le p \le q < \infty$ and $u, v \in L^+$. Then*

$$\sup_{f \in \Omega_{\alpha,\beta}} \frac{\|f\|_{q,u}}{\|f\|_{p,v}} \approx \sup_{z>0} \frac{\|k_z^{\alpha,\beta}\|_{q,u}}{\|k_z^{\alpha,\beta}\|_{p,v}}. \tag{2.1}$$

More precisely, if the left and right sides of the above estimate are equal to C and D respectively, then $D \le C \le 2D$.

Also, as part of the proof of Theorem 6 in [Si5], Sinnamon showed this.

Proposition 2.3. *Let $0 < p \le 1 \le q < \infty$ and $u, v \in L^+$ and assume A is an averaging operator. Then*

$$\sup_{f \in \Omega_{2,0}} \frac{\|Af\|_{q,u}}{\|f\|_{p,v}} \approx \sup_{z>0} \frac{\|A\omega_z\|_{q,u}}{\|\omega_z\|_{p,v}}.$$

More precisely, if the left and right sides of the above estimate are equal to C and D respectively, then $D \le C \le 2D$.

In Theorem 2.9 which is our main result in this chapter we will unify and generalize the above theorems. The consequences are Corollaries 2.11 and 2.12 that are useful in our inequalities for Fourier series. In addition, as a by-product, we improve the constant in Proposition 2.2 and extend the range of exponents p, q. This is stated in Theorem 2.10.

A powerful tool to prove our results is the following positive integral operator on L^+. For $\alpha + \beta > 0$ and $\xi \ge 0$ set

$$K_\xi^{\alpha,\beta} h(z) = \int_\xi^\infty \min(z^{-\alpha} t^\beta, z^\beta t^{-\alpha})\, h(t)\, dt,$$

where $h \in L^+$. The kernel of this operator is the function $k_z^{\alpha,\beta}$ introduced in the previous section. Observe that $z^\alpha K_\xi^{\alpha,\beta} h(z)$ is an increasing function of z and $z^{-\beta} K_\xi^{\alpha,\beta} h(z)$ is decreasing. Thus $K_\xi^{\alpha,\beta} h \in \Omega_{\alpha,\beta}$. Moreover, if $\xi > 0$ then for $z \in (0, \xi)$ we have

$$K_\xi^{\alpha,\beta} h(z) = z^\beta \int_\xi^\infty \min(z^{-\alpha-\beta}, t^{-\alpha-\beta})\, t^\beta h(t)\, dt = z^\beta \int_\xi^\infty t^{-\alpha} h(t)\, dt.$$

It follows that $K_\xi^{\alpha,\beta}h \in P_\xi^\beta \cap \Omega_{\alpha,\beta}$. We will show in Proposition 2.6 that any function of class $P_\xi^\beta \cap \Omega_{\alpha,\beta}$ can be estimated from below and above by functions of form $K_\xi^{\alpha,\beta}h$.

We start with the geometrically obvious fact that if a function is linear on some interval, so is its least concave majorant.

Lemma 2.4. *Assume $g \in L^+$ satisfies $g(x) = cx$ on $(0, a)$ for some positive a and c. Let $\tilde{g}$ be the least concave majorant of g. Then $\tilde{g}(x) = \lambda x$ on $(0, a)$ where $\lambda = \tilde{g}(a)/a$.*

Proof. Since $\tilde{g}$ is non-negative and concave, we have $\lambda x \le \tilde{g}(x)$ on $(0, a]$ and $\lambda x \ge \tilde{g}(x)$ on $[a, \infty)$. The function $h(x) = \min(\lambda x, \tilde{g}(x))$ is a concave function in L^+ since it is the minimum of two concave functions in L^+. We have $h(x) = \tilde{g}(x) \ge g(x)$ on $[a, \infty)$. Moreover,

$$\lambda = \frac{\tilde{g}(a)}{a} = \lim_{x\to a-} \frac{\tilde{g}(x)}{x} \ge \lim_{x\to a-} \frac{g(x)}{x} = c,$$

which means $h(x) \ge g(x)$ on $(0, a)$. So h is a concave majorant of g. Since $h \le \tilde{g}$ we have $h = \tilde{g}$. In particular $\tilde{g}(x) = \lambda x$ on $(0, a)$. □

The next lemma extends Lemma 2.3 in [Si3] which concerns the case $\xi = 0$. The proof here is adapted from [Si3] with a major adjustment in order to deal with the extra condition $\tilde{g}(x) = \lambda x$ on $(0, \xi)$.

Lemma 2.5. *Assume $\xi \ge 0$ and $\tilde{g} \in L^+$ is an increasing concave function satisfying $\tilde{g}(x) = \lambda x$ for $x \in (0, \xi)$. Then there exists a sequence of functions $f_n \in L^+$ such that $K_\xi^{0,1} f_n$ increases to $\tilde{g}$ pointwise.*

Proof. The concavity of $\tilde{g}$ implies that its right derivative $D_+\tilde{g}$ is a right continuous decreasing function defined on $(0, \infty)$. The derivative of $\tilde{g}$ exists almost everywhere on $(0, \infty)$ and by absolute continuity we have $\tilde{g}(x) = \int_0^x \tilde{g}'(t)\,dt + \tilde{g}(0+)$ which means $\tilde{g}(x) = \int_0^x D_+\tilde{g}(t)\,dt + \tilde{g}(0+)$.

Note that $D_+\tilde{g} \in L^+$ since $\tilde{g}$ is increasing. Moreover $D_+\tilde{g}(x) = \lambda$ on $(0, \xi)$. Let $b = \lim_{x\to\infty} D_+\tilde{g}(x) \ge 0$ and set $\psi(x) = \tilde{g}(x) - bx$. Observe that $D_+\psi(x) = D_+\tilde{g}(x) - b$ is a non-negative, decreasing, right continuous function on $(0, \infty)$ and $\lim_{x\to\infty} D_+\psi(x) = 0$. Moreover, $D_+\psi(x) = \lambda - b$ on $(0, \xi)$.

If $\xi > 0$ let $a = \xi(\lambda - b - D_+\psi(\xi)) = \psi(\xi) - \xi D_+\psi(\xi)$ and define $\eta(x) = D_+\psi(x) - (a/\xi)\chi_{(0,\xi)}(x)$. We have

$$\eta(x) = \begin{cases} D_+\psi(\xi), & 0 < x < \xi, \\ \\ D_+\psi(x), & x \ge \xi, \end{cases}$$

where we used $\eta(x) = D_+\psi(x) - a/\xi = \lambda - b - a/\xi = D_+\psi(\xi)$ for $0 < x < \xi$.

If $\xi = 0$ set $a = \psi(0+) = \tilde{g}(0+)$ and $\eta(x) = D_+\psi(x)$. In either case, the properties of $D_+\psi(x)$ imply that η is a non-negative, decreasing, right continuous function on $(0, \infty)$ with $\lim\limits_{x\to\infty} \eta(x) = 0$.

Now set $\varphi(x) = \int_0^x \eta(t)\,dt$ and $k(x) = a\,\min(1, x/\xi)$. If $\xi > 0$ we have

$$\varphi(x) = \int_0^x D_+\tilde{g}(t)\,dt - \int_0^x b\,dt - \frac{a}{\xi}\int_0^x \chi_{(0,\xi)}\,dt = \tilde{g}(x) - bx - k(x),$$

and if $\xi = 0$ we get

$$\varphi(x) = \int_0^x D_+\tilde{g}(t)\,dt - \int_0^x b\,dt = \tilde{g}(x) - \tilde{g}(0+) - bx = \tilde{g}(x) - bx - k(x).$$

Thus $\tilde{g}(x) = bx + k(x) + \varphi(x)$. Set $h_n(t) = b\chi_{(n,n+1)}(t)$ and observe that

$$K_\xi^{0,1}h_n(x) = \int_\xi^\infty b\min(x,t)\chi_{(n,n+1)}(t)\,dt$$

is the moving average of the increasing function $b\min(x,t)$ for each x. Therefore, $K_\xi^{0,1}h_n(x)$ is an increasing sequence and it is easily seen that it converges to bx.

Then consider the sequence of functions $k_n(t) = (a/t)n\chi_{(\xi,\xi+1/n)}(t)$. We have

$$K_\xi^{0,1}k_n(x) = \int_\xi^\infty \frac{an}{t}\min(x,t)\chi_{(\xi,\xi+1/n)}(t)\,dt = a\int_\xi^{\xi+\frac{1}{n}} n\min(1, x/t)\,dt.$$

For each $x > 0$, this is a shrinking average of the decreasing function $a\min(1, x/t)$ over the interval $(\xi, \xi + 1/n)$. Thus $K_\xi^{0,1}k_n(x)$ increases to $a\min(1, x/\xi) = k(x)$.

It remains to find the corresponding sequence for $\varphi(t)$. First we write $\min(x,t) = \int_0^{\min(x,t)} dy$ and use Tonelli's theorem to get

$$K_\xi^{0,1}f(x) = \int_\xi^\infty \min(x,t)f(t)\,dt = \int_0^x \int_{\max(y,\xi)}^\infty f(t)\,dtdy.$$

Now set

$$\varphi_n(t) = \frac{\eta(t) - \eta((n+1)t/n)}{t\log((n+1)/n)} \quad \text{and} \quad \Phi_n(y) = \int_{\max(y,\xi)}^\infty \varphi_n(t)\,dt.$$

For $y \geq \xi$ we have

$$\begin{aligned}
\Phi_n(y) &= \lim_{z\to\infty}\int_y^z \varphi_n(t)\,dt \\
&= \frac{1}{\log((n+1)/n)}\lim_{z\to\infty}\left(\int_y^z \frac{\eta(t)}{t}dt - \int_y^z \frac{\eta(t(n+1)/n)}{t}dt\right) \\
&= \frac{1}{\log((n+1)/n)}\lim_{z\to\infty}\left(\int_y^z \frac{\eta(t)}{t}dt - \int_{\frac{n+1}{n}y}^{\frac{n+1}{n}z} \frac{\eta(t)}{t}dt\right) \\
&= \frac{1}{\log((n+1)/n)}\lim_{z\to\infty}\left(\int_y^{\frac{n+1}{n}y} \frac{\eta(t)}{t}dt - \int_z^{\frac{n+1}{n}z} \frac{\eta(t)}{t}dt\right).
\end{aligned}$$

Notice that $\lim_{z\to\infty}\int_z^{\frac{n+1}{n}z}\dfrac{\eta(t)}{t}dt = 0$, since $\lim_{t\to\infty}\eta(t)=0$. Hence,

$$\begin{aligned}\Phi_n(y) &= \lim_{z\to\infty}\frac{1}{\log((n+1)/n)}\int_y^{\frac{n+1}{n}y}\frac{\eta(t)}{t}dt\\ &= \frac{1}{\int_y^{\frac{n+1}{n}y}\frac{dt}{t}}\left(\int_y^{\frac{n+1}{n}y}\eta(t)\frac{dt}{t}\right).\end{aligned}$$

Notice that for a fixed y, the above expression is a shrinking average of the decreasing function $\eta(t)$ with respect to the measure $\dfrac{dt}{t}$ over the interval $\big(y, y(n+1)/n\big)$. So $\Phi_n(y)$ forms an increasing sequence converging to $\eta(y+)$ which is equal to $\eta(y)$ by right continuity of η. In particular, $\Phi_n(\xi)$ increases to $\eta(\xi)$.

If $\xi = 0$ the above argument is complete. If $\xi > 0$ we finish the argument by observing for $0 < y < \xi$ we have

$$\Phi_n(y) = \Phi_n(\xi) \nearrow \eta(\xi) = D_+\psi(\xi) = \eta(y).$$

So we proved $\Phi_n(y)$ increases to $\eta(y)$ for every $y > 0$. Hence for $x \geq 0$, the monotone convergence theorem gives

$$K_\xi^{0,1}\varphi_n(x) = \int_0^x \Phi_n(y)dy \nearrow \int_0^x \eta(y) = \varphi(x).$$

The proof is complete if we set: $f_n = h_n + k_n + \varphi_n$. $\square$

Now we use this lemma to show that an arbitrary function of class $P_\xi^\beta \cap \Omega_{\alpha,\beta}$ is equivalent to a pointwise limit of functions of form $K_\xi^{\alpha,\beta}h$. This is achieved by reducing the problem to the quasi concave case. It is a generalization of Lemma 5 in [Si5].

Proposition 2.6. *Assume $f \in P_\xi^\beta \cap \Omega_{\alpha,\beta}$. Then there exists $\tilde f \in L^+$ and a sequence of functions $\{h_n\}$ in L^+ such that $\frac{1}{2}\tilde f \leq f \leq \tilde f$ and $K_\xi^{\alpha,\beta}h_n \nearrow \tilde f$.*

Proof. If $\xi > 0$ assume $f(t) = ct^\beta$ for $0 < t < \xi$. Since $f \in \Omega_{\alpha,\beta}$ the function $g(t) = t^{\alpha/(\alpha+\beta)}f(t^{1/(\alpha+\beta)})$ is increasing and $t^{-1}g(t)$ is decreasing. So g is a quasi concave function. Moreover, in case $\xi > 0$ we have $g(t) = ct$ for $t \in (0, \xi^{\alpha+\beta})$. Let $\tilde g$ be the least concave majorant of g. Lemma 2.4 asserts that $\tilde g(t) = \lambda t$ on $(0, \xi^{\alpha+\beta})$ and by Lemma 2.5 there exists a sequence of functions $\{g_n\}$ such that

$$K_{\xi^{\alpha+\beta}}^{0,1}\, g_n(z) = \int_{\xi^{\alpha+\beta}}^\infty \min(z,t)\; g_n(t)\; dt = \int_\xi^\infty \min(z, t^{\alpha+\beta})\;(\alpha+\beta)t^{\alpha+\beta-1}\; g_n(t^{\alpha+\beta})\; dt$$

increases to $\tilde g(z)$ for each $z \geq 0$.

Set $\tilde f(z) = z^{-\alpha}\tilde g(z^{\alpha+\beta})$ and $h_n(t) = (\alpha+\beta)\; t^{2\alpha+\beta-1}\; g_n(t^{\alpha+\beta})$. We have

$$\begin{aligned}K_\xi^{\alpha,\beta}h_n(z) &= \int_\xi^\infty \min(z^{-\alpha}t^\beta, z^\beta t^{-\alpha})\; h_n(t)dt\\ &= z^{-\alpha}\int_\xi^\infty \min(t^{\alpha+\beta}, z^{\alpha+\beta})\; t^{-\alpha}\; h_n(t)dt\\ &= z^{-\alpha}K_{\xi^{\alpha+\beta}}^{0,1}\, g_n(z^{\alpha+\beta}).\end{aligned}$$

Hence the sequence $K_\xi^{\alpha,\beta} h_n(z)$ increases to $z^{-\alpha}\tilde{g}(z^{\alpha+\beta}) = \tilde{f}(z)$.

Finally, Proposition 2.1 implies that $\frac{1}{2}\tilde{g}(t) \le g(t) \le \tilde{g}(t)$ for $t > 0$. Thus, for $s > 0$ we have $\frac{1}{2}s^{-\alpha}\tilde{g}(s^{\alpha+\beta}) \le s^{-\alpha}g(s^{\alpha+\beta}) \le s^{-\alpha}\tilde{g}(s^{\alpha+\beta})$ which implies $\frac{1}{2}\tilde{f}(s) \le f(s) \le \tilde{f}(s)$. □

The following lemma is a modified version of Lemma 4 in [Si5] with a new proof. In the new version the operator A is not necessarily an integral operator, but it has a formal adjoint, thus making it possible to apply the lemma to averaging operators later on.

Lemma 2.7. *Let $0 < p \le 1 \le q < \infty$. Suppose (Y,μ), (X,ν) and (T,λ) are σ-finite measure spaces, $k(x,t)$ is a non-negative $\nu\times\lambda$-measurable function, and $A : L_\nu^+ \to L_\mu^+$ has a formal adjoint. Define K and k_t by $Kh(x) = \int_T k(x,t)h(t)\,d\lambda(t)$ and $k_t(x) = k(x,t)$. Then*

$$\sup_{h\ge 0} \frac{\|AKh\|_{L^q(\mu)}}{\|Kh\|_{L^p(\nu)}} \le \operatorname*{ess\,sup}_{t\in T} \frac{\|Ak_t\|_{L^q(\mu)}}{\|k_t\|_{L^p(\nu)}}.$$

Proof. Set

$$C = \operatorname*{ess\,sup}_{t\in T} \frac{\|Ak_t\|_{L^q(\mu)}}{\|k_t\|_{L^p(\nu)}},$$

and let B be a formal adjoint of A. For $g \in L_\nu^+$ and $h \in L_\lambda^+$ we have

$$\begin{aligned}\int_Y AKh(y)g(y)\,d\mu(y) &= \int_X Kh(x)Bg(x)\,d\nu(x)\\ &= \int_X \int_T k(x,t)h(t)\,d\lambda(t)Bg(x)\,d\nu(x).\end{aligned}$$

A change of the order of integration according to Tonelli's theorem yields

$$\begin{aligned}\int_X \int_T k(x,t)h(t)\,d\lambda(t)Bg(x)\,d\nu(x) &= \int_T \int_X k_t(x)Bg(x)\,d\nu(x)h(t)\,d\lambda(t)\\ &= \int_T \int_Y Ak_t(y)g(y)\,d\mu(y)h(t)\,d\lambda(t).\end{aligned}$$

By Hölder's inequality we get

$$\int_Y Ak_t(y)g(y)\,d\mu(y) \le \|Ak_t\|_{L^q(\mu)}\|g\|_{L^{q'}(\mu)} \le C\|k_t\|_{L^p(\nu)}\|g\|_{L^{q'}(\mu)}.$$

for almost every $t \in T$. It follows that

$$\begin{aligned}\int_Y AKh(y)g(y)\,d\mu(y) &\le \int_T \int_Y Ak_t(y)g(y)\,d\mu(y)h(t)\,d\lambda(t)\\ &\le C\int_T \|k_t\|_{L^p(\nu)}h(t)\,d\lambda(t)\|g\|_{L^{q'}(\mu)}\\ &= C\|g\|_{L^{q'}(\mu)} \int_T \left(\int_X k(x,t)^p\,d\nu(x)\right)^{1/p} h(t)\,d\lambda(t).\end{aligned}$$

Now Minkowski's inequality for $0 < p \le 1$ asserts

$$\int_T \left(\int_X k(x,t)^p \, d\nu(x) \right)^{1/p} h(t)\, d\lambda(t) \le \left(\int_X \left(\int_T k(x,t) h(t)\, d\lambda(t) \right)^p d\nu(x) \right)^{1/p},$$

and therefore,

$$\begin{aligned} \int_Y AKh(y)g(y)\, d\mu(y) &\le C\|g\|_{L^{q'}(\mu)} \left(\int_X \left(\int_T k(x,t)h(t)\, d\lambda(t) \right)^p d\nu(x) \right)^{1/p} \\ &= C\|g\|_{L^{q'}(\mu)} \|Kh\|_{L^p(\nu)}. \end{aligned}$$

The above inequality holds for all $g \in L^+_\nu$ and $h \in L^+_\lambda$. Hence the duality of $L^q(\mu)$ and $L^{q'}(\mu)$ implies

$$\|AKh\|_{L^q(\mu)} = \sup_{g \in L^+_\nu} \frac{\int_Y AKh(y)g(y)\, d\mu(y)}{\|g\|_{L^{q'}(\mu)}} \le C\|Kh\|_{L^p(\nu)},$$

which completes the proof. $\square$

Recall that

$$k_z^{\alpha,\beta}(t) = \min(z^{-\alpha}t^\beta, z^\beta t^{-\alpha}) \quad \text{and} \quad K_\xi^{\alpha,\beta}h(z) = \int_\xi^\infty \min(z^{-\alpha}t^\beta, z^\beta t^{-\alpha})\, h(t)\, dt.$$

Corollary 2.8. *Let $0 < p \le 1 \le q < \infty$ and $u, v \in L^+$and assume A is an averaging operator. Then*

$$\sup_{h \ge 0} \frac{\|AK_\xi^{\alpha,\beta}h\|_{q,u}}{\|K_\xi^{\alpha,\beta}h\|_{p,v}} \le \sup_{z > \xi} \frac{\|Ak_z^{\alpha,\beta}\|_{q,u}}{\|k_z^{\alpha,\beta}\|_{p,v}}.$$

Proof. In Lemma 2.7 set $X = Y = T = (0,\infty)$ and define the measures μ, ν and λ as

$$d\mu = u(t)dt, \quad d\nu = v(t)dt, \quad d\lambda = \chi_{[\xi,\infty)}dt.$$

Note that the interval $(0,\xi)$ has λ-measure zero. We have

$$K_\xi^{\alpha,\beta}h(z) = \int_\xi^\infty k_z^{\alpha,\beta}(t)h(t)dt = \int_T k_z^{\alpha,\beta}(t)h(t)\chi_{[\xi,\infty)}dt = \int_T k_z^{\alpha,\beta}(t)h(t)d\lambda(t).$$

Now Lemma 2.7 implies

$$\sup_{h \ge 0} \frac{\|AK_\xi^{\alpha,\beta}h\|_{q,u}}{\|K_\xi^{\alpha,\beta}h\|_{p,v}} \le \operatorname{ess\,sup}_{t \in T} \frac{\|Ak_t^{\alpha,\beta}\|_{q,u}}{\|k_t^{\alpha,\beta}\|_{p,v}} = \sup_{t > \xi} \frac{\|Ak_t^{\alpha,\beta}\|_{q,u}}{\|k_t^{\alpha,\beta}\|_{p,v}}.$$

In the last equality we used $\lambda(0,\xi) = 0$. $\square$

Now we have all the machinery to prove the main result of this chapter. It shows that to compute the operator norm of $A \in \mathcal{A}$, the set $\{k_z^{\alpha,\beta} : z > \xi\}$ is a sufficiently large subset of the cone $P_\xi^\beta \cap \Omega_{\alpha,\beta}$. Sinnamon proved and used a special case of this theorem in [Si5] to provide some Fourier inequalities. See Theorem 3.3 in next chapter.

Theorem 2.9. *Let $0 < p \leq 1 \leq q < \infty$ and $u, v \in L^+$. Assume A is an averaging operator. Then*

$$\sup_{f \in P_\xi^\beta \cap \Omega_{\alpha,\beta}} \frac{\|Af\|_{q,u}}{\|f\|_{p,v}} \approx \sup_{z>\xi} \frac{\|Ak_z^{\alpha,\beta}\|_{q,u}}{\|k_z^{\alpha,\beta}\|_{p,v}}.$$

More precisely, if the left and right sides of the above estimate are equal to C and D respectively, then $D \leq C \leq 2D$.

Proof. Suppose $f \in P_\xi^\beta \cap \Omega_{\alpha,\beta}$ and denote $K = K_\xi^{\alpha,\beta}$, $k_z = k_z^{\alpha,\beta}$ for simplicity. Choose $\tilde{f}$ and h_n such that $\frac{1}{2}\tilde{f} \leq f \leq \tilde{f}$ and $Kh_n \nearrow \tilde{f}$ according to Proposition 2.6. Note that $Kh_n \uparrow \tilde{f}$ implies $AKh_n \nearrow A\tilde{f}$. So $\|Kh_n\| \nearrow \|\tilde{f}\|$ and $\|AKh_n\| \nearrow \|A\tilde{f}\|$. Thus,

$$\frac{\|Af\|_{q,u}}{\|f\|_{p,v}} \leq 2\frac{\|A\tilde{f}\|_{q,u}}{\|\tilde{f}\|_{p,v}} = 2\sup_{n\in\mathbb{Z}^+} \frac{\|AKh_n\|_{q,u}}{\|\tilde{f}\|_{p,v}} \leq 2\sup_{n\in\mathbb{Z}^+} \frac{\|AKh_n\|_{q,u}}{\|Kh_n\|_{p,v}} \leq 2\sup_{h\geq 0} \frac{\|AKh\|_{q,u}}{\|Kh\|_{p,v}}.$$

Taking the supremum over all $f \in P_\xi^\beta \cap \Omega_{\alpha,\beta}$ and incorporating Corollary 2.8 yields,

$$\sup_{f \in P_\xi^\beta \cap \Omega_{\alpha,\beta}} \frac{\|Af\|_{q,u}}{\|f\|_{p,v}} \leq 2\sup_{h\geq 0} \frac{\|AKh\|_{q,u}}{\|Kh\|_{p,v}} \leq 2\sup_{z>\xi} \frac{\|Ak_z\|_{q,u}}{\|k_z\|_{p,v}}.$$

Finally,

$$\sup_{z>\xi} \frac{\|Ak_z\|_{q,u}}{\|k_z\|_{p,v}} \leq \sup_{f \in P_\xi^\beta \cap \Omega_{\alpha,\beta}} \frac{\|Af\|_{q,u}}{\|f\|_{p,v}} \leq 2\sup_{z>\xi} \frac{\|Ak_z\|_{q,u}}{\|k_z\|_{p,v}}$$

since k_z lies in the class $P_\xi^\beta \cap \Omega_{\alpha,\beta}$ when $z > \xi$. □

A consequence of Theorem 2.9 is the following generalization of Proposition 2.2. The proposition is stated and proved by Maligranda in [Ma2] for $1 \leq p < q < \infty$. A restatement of the theorem in his subsequent paper, Theorem 3 in [Ma1], assumes $0 < p < q < \infty$ which is most likely a typographical error. In fact a careful examination of Maligranda's proof shows that it fails when $p < 1$. Our proof not only extends the range of p but also improves the constant. Namely, $D \leq C \leq 2^{1/q}D$ instead of $D \leq C \leq 2D$.

Theorem 2.10. *Suppose $0 < p \leq q < \infty$ and $u, v \in L^+$. Then*

$$\sup_{f \in P_\xi^\beta \cap \Omega_{\alpha,\beta}} \frac{\|f\|_{q,u}}{\|f\|_{p,v}} \approx \sup_{z>\xi} \frac{\|k_z^{\alpha,\beta}\|_{q,u}}{\|k_z^{\alpha,\beta}\|_{p,v}}. \tag{2.2}$$

More precisely, if the left and right sides of the above estimate are equal to C and D respectively, then $D \leq C \leq 2^{1/q}D$.

Proof. The estimate $D \le C$ is trivial since $k_z^{\alpha,\beta} \in P_\xi \cap \Omega_{\alpha,\beta}$ for all $z > \xi$. To prove the other inequality observe that

$$C^q = \left(\sup_{f \in P_\xi^\beta \cap \Omega_{\alpha,\beta}} \frac{\|f\|_{q,u}}{\|f\|_{p,v}} \right)^q = \sup_{f \in P_\xi^\beta \cap \Omega_{\alpha,\beta}} \frac{\int_0^\infty f(t)^q\, u(t)\, dt}{\left(\int_0^\infty f(t)^p\, v(t)\, dt\right)^{q/p}} = \sup_{f \in P_\xi^\beta \cap \Omega_{\alpha,\beta}} \frac{\|f^q\|_{1,u}}{\|f^q\|_{p/q,v}}.$$

It is easy to see that $g(t) = f(t)^q$ lies in the cone $P_\xi^{q\beta} \cap \Omega_{q\alpha,q\beta}$ if and only if $f \in P_\xi^\beta \cap \Omega_{\alpha,\beta}$. Therefore,

$$C^q = \sup_{g \in P_\xi^{q\beta} \cap \Omega_{q\alpha,q\beta}} \frac{\|g\|_{1,u}}{\|g\|_{p/q,v}}$$

Now we invoke Theorem 2.9 with A as the identity operator, p, q, α and β replaced with p/q, 1, $q\alpha$ and $q\beta$ respectively. The theorem yields

$$C^q = \sup_{g \in P_\xi^{q\beta} \cap \Omega_{q\alpha,q\beta}} \frac{\|g\|_{1,u}}{\|g\|_{p/q,v}} \le 2 \sup_{z>\xi} \frac{\|k_z^{q\alpha,q\beta}\|_{1,u}}{\|k_z^{q\alpha,q\beta}\|_{p/q,v}}.$$

From

$$k_z^{q\alpha,q\beta}(t) = \min(z^{-q\alpha} t^{q\beta}, z^{q\beta} t^{-q\alpha}) = \min(z^{-\alpha} t^\beta, z^\beta t^{-\alpha})^q = \left(k_z^{\alpha,\beta}(t)\right)^q$$

we obtain

$$\sup_{z>\xi} \frac{\|(k_z^{\alpha,\beta})^q\|_{1,u}}{\|(k_z^{\alpha,\beta})^q\|_{p/q,v}} = \sup_{z>\xi} \frac{\int_0^\infty (k_z^{\alpha,\beta})^q\, u(t)\, dt}{\left(\int_0^\infty (k_z^{\alpha,\beta})^p\, v(t)\, dt\right)^{q/p}} = \sup_{z>\xi} \left(\frac{\|k_z^{\alpha,\beta}\|_{q,u}}{\|k_z^{\alpha,\beta}\|_{p,v}} \right)^q.$$

This implies

$$C^q \le 2 \sup_{z>\xi} \left(\frac{\|k_z^{\alpha,\beta}\|_{q,u}}{\|k_z^{\alpha,\beta}\|_{p,v}} \right)^q = 2D^q,$$

and the proof is complete. $\square$

We end this section with Corollaries 2.11 and 2.12 which will be used for our results in inequalities for Fourier series. These correspond to Propositions 2.2 and 2.3 that were used by Sinnamon for Fourier transform inequalities. These corollaries are just special cases of Theorems 2.9 and 2.10. Recall the notations $\omega_z(t) = \min(z^{-2}, t^{-2})$ and $P = P_1^0$

Corollary 2.11. *Let $0 < p \le 1 \le q < \infty$, $u, v \in L^+$, and assume A is an averaging operator. Then*

$$\sup_{f \in P \cap \Omega_{2,0}} \frac{\|Af\|_{q,u}}{\|f\|_{p,v}} \approx \sup_{z>1} \frac{\|A\omega_z\|_{q,u}}{\|\omega_z\|_{p,v}}.$$

More precisely, if the left and right sides of the above estimate are equal to C and D respectively, then $D \le C \le 2D$.

Proof. In Theorem 2.9 set $\xi = 1$, $\alpha = 2$, $\beta = 0$ and notice that $\omega_z = k_z^{2,0}$. $\square$

Corollary 2.12. *Let $0 < p \le q < \infty$ and $u, v \in L^+$. Then*

$$\sup_{f \in P \cap \Omega_{2,0}} \frac{\|f\|_{q,u}}{\|f\|_{p,v}} \approx \sup_{z>1} \frac{\|\omega_z\|_{q,u}}{\|\omega_z\|_{p,v}}.$$

More precisely, if the left and right sides of the above estimate are equal to C and D respectively, then $D \le C \le 2^{1/q} D$.

Proof. In Theorem 2.10 set $\xi = 1$, $\alpha = 2$, $\beta = 0$ and notice that $\omega_z = k_z^{2,0}$. $\square$

Chapter 3

Fourier transform inequalities

As mentioned in the introduction, one challenging problem in Fourier inequalities is to characterize those weights u, w, for which the weighted Lebesgue norm inequality $\|\hat{f}\|_{L^q(u)} \leq C\|f\|_{L^p(w)}$ holds. One approach taken by Benedetto and Heinig in [BH2] is to find Fourier inequalities of type $\|\hat{f}\|_{\Lambda^q(u)} \leq C\|f\|_{\Lambda^p(w)}$ and then use the Hardy-Littlewood-Polya inequality to get the weighted Lebesgue inequality $\|\hat{f}\|_{L^q(u)} \leq C\|f\|_{L^p(w)}$.

In [BH2], Benedetto and Heinig gave a necessary and sufficient condition for $\|\hat{f}\|_{\Lambda^q(u)} \leq C\|f\|_{\Lambda^p(w)}$ when $p \leq q$, u is decreasing and $w \in B_p$. Using this result, they provided a sufficient condition for weighted L^p inequalities. They also have another sufficient condition for $\|\hat{f}\|_{L^q(u)} \leq C\|f\|_{L^p(w)}$, for which they gave a direct and rather lengthy proof. One of our main results in this chapter is the missing Lorentz space inequality which easily gives this weighted L^p sufficient condition. See Theorems 3.18 and 3.32.

Sinnamon worked on the Lorentz Γ space and obtained Fourier inequalities of type $\|\hat{f}\|_{\Lambda^q(u)} \leq C\|f\|_{\Gamma^p(w)}$ in [Si4]. In the case $q = 2$ the necessary and sufficient conditions coincide and provide a characterization of weights u, w which satisfy $\|\hat{f}\|_{\Lambda^2(u)} \leq C\|f\|_{\Gamma^p(w)}$. Subsequently, in [Si5], he obtained a characterization in terms of averaging operators and the Lorentz Θ space in the case $0 < p \leq 2$. Our focus in this dissertation is on the case $p \leq q$. We will show in this chapter that the results in [BH2] can be deduced from Sinnamon's work.

We start this chapter with a review of Sinnamon's results in [Si4] and [Si5]. In Section 3.2 we give various sufficient and necessary conditions for Fourier inequalities in Lorentz spaces. We also provide examples on usage of these theorems. In Section 3.3 we present the weighted L^p inequalities given in [BH2]. In particular we give a very short proof of Theorem 3.32 based on a corresponding Lorentz space inequality.

Throughout this chapter $\mathcal{F}$ denotes the Fourier transform on $\mathbb{R}^n$ and $\hat{f} = \mathcal{F}(f)$.

3.1 Previous work on Fourier inequalities

The sufficient conditions for Fourier inequalities in Lorentz or Lebesgue spaces are based on the following rearrangement estimate due to Jodeit and Torchinsky [JT]. In fact it characterizes all linear operators of type $(1,\infty)$ and $(2,2)$.

Proposition 3.1. *Let (X,μ) and (Y,ν) be σ-finite measure spaces and assume $T : L^1(\mu)+L^2(\mu)\to L_\nu(Y)$ is a linear operator. Then T is of type $(1,\infty)$ and $(2,2)$ if and only if there exists a constant D such that*

$$\int_0^z (Tf)^*(t)^2\,dt \le D\int_0^z \Big(\int_0^{1/t} f^*\Big)^2 dt, \quad z>0,$$

for all $f\in L^1(\mu)+L^2(\mu)$.

Proof. See Theorem 4.6 in [JT] for the proof. □

If the operator norms of maps $T : L^1(\mu)\longrightarrow L^\infty(\nu)$ and $T : L^2(\mu)\longrightarrow L^2(\nu)$ are at most 1, then $D\le 4$. In particular $D\le 4$ for the Fourier transform.

In [Si4], Sinnamon used this inequality to obtain sufficient conditions for

$$\Big(\int_0^\infty (\hat f)^*(t)^q u(t)\,dt\Big)^{1/q} \le C\Big(\int_0^\infty \Big(\int_0^{1/t} f^*\Big)^p v(t)\,dt\Big)^{1/p}.$$

If $w(t)=t^{p-2}v(1/t)$, we get the equivalent inequality,

$$\Big(\int_0^\infty (\hat f)^*(t)^q u(t)\,dt\Big)^{1/q} \le C\Big(\int_0^\infty f^{**}(t)^p w(t)\,dt\Big)^{1/p},$$

which is a Fourier inequality in Lorentz spaces, that is: $\|\hat f\|_{\Lambda^q(u)}\le C\|f\|_{\Gamma^p(w)}$.

Recall that $\Omega_{2,0}$ is a subcone of L^+ containing decreasing functions $f(t)$ such that $t^2f(t)$ is increasing. Also $\mathcal{A}$ is the collection of averaging operators on L^+. The following is a sufficient condition for $\|\hat f\|_{\Lambda^q(u)}\le C\|f\|_{\Gamma^p(w)}$.

Theorem 3.2. *Suppose $0<p<\infty$, $2\le q<\infty$ and $u,v,w\in L^+$ with $v(t)=t^{p-2}w(1/t)$. If*

$$\sup_{h\in\Omega_{2,0},\,A\in\mathcal{A}} \frac{\|Ah\|_{q/2,u}}{\|h\|_{p/2,v}}<\infty,$$

then there exists $C>0$ such that

$$\Big(\int_0^\infty (\hat f)^*(t)^q u(t)dt\Big)^{1/q} \le C\Big(\int_0^\infty \Big(\int_0^{1/t} f^*\Big)^p v(t)dt\Big)^{1/p},$$

or equivalently

$$\|\hat f\|_{\Lambda^q(u)}\le C\|f\|_{\Gamma^p(w)}$$

for all $f\in L^1(\mathbb{R}^n)+L^2(\mathbb{R}^n)$.

Proof. See Theorem 3.1 and Corollary 3.2 in [Si4]. □

When $0 < p \le 2$, there is a simpler expression in terms of the Θ space. Recall that $\omega_z(t) = \min(z^{-2}, t^{-2})$.

Theorem 3.3. *Let $0 < p \le 2 \le q < \infty$ and $u, v, w \in L^+$ with $v(t) = t^{p-2}w(1/t)$. If*

$$\sup_{z>0} \frac{\|\omega_z\|_{\Theta^{q/2}(u)}}{\|\omega_z\|_{p/2,v}} < \infty,$$

then there exists $C > 0$ such that

$$\|\hat{f}\|_{\Lambda^q(u)} \le C\|f\|_{\Gamma_p(w)}$$

for all $f \in L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n)$.

Proof. See Theorem 6 in [Si5]. □

Using the level function, one obtains a stronger but simpler sufficient condition.

Theorem 3.4. *Suppose that $0 < p \le q < \infty$ and $2 \le q$, $u, v, w \in L^+$ with $v(t) = t^{p-2}w(1/t)$. If*

$$\sup_{z>0} \frac{\|\omega_z\|_{q/2,u^\circ}}{\|\omega_z\|_{p/2,v}} < \infty,$$

then there exists $C > 0$ such that

$$\|\hat{f}\|_{\Lambda^q(u)} \le C\|f\|_{\Gamma_p(w)}$$

for all $f \in L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n)$.

Proof. See Theorem 3.4 in [Si4] for a proof. □

To obtain a necessary condition, Sinnamon constructed the appropriate test functions in [Si4] and proved the following result.

Theorem 3.5. *Let n be a positive integer, $z > 0$ and $A \in \mathcal{A}$. For each $\varepsilon > 0$ there exists a function $f : \mathbb{R}^n \longrightarrow \mathbb{C}$ such that*

$$f^* \le \chi_{[0,1/z)} \quad \text{and} \quad (A\omega_z)^{1/2} \le c_n(\hat{f}^* + \varepsilon),$$

where c_n is a constant number depending only on n.

Proof. See Theorem 4.6 and Corollary 4.7 in [Si4] for a proof. □

The following necessary condition is a consequence of Theorem 3.5.

Theorem 3.6. *Suppose $0 < p < \infty$, $0 < q < \infty$, $0 < C$, $u, v, w \in L^+$ with $v(t) = t^{p-2}w(1/t)$, satisfy*

$$\|\hat{f}\|_{\Lambda^q(u)} \le C\|f\|_{\Gamma^p(w)}$$

for all $f \in L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n)$. Then

$$\sup_{A\in\mathcal{A},\, z>0} \frac{\|A\omega_z\|_{q/2,u}}{\|\omega_z\|_{p/2,v}} < \infty.$$

Proof. See Corollary 4.8 in [Si4]. □

This leads to following necessary and sufficient condition when $0 < p \leq 2 \leq q$. This is Theorem 8 in [Si5].

Theorem 3.7. *Let $0 < p \leq 2 \leq q < \infty$ and $u, v, w \in L^+$ with $v(t) = t^{p-2}w(1/t)$. Then there exists $C > 0$ such that the inequality*

$$\|\hat{f}\|_{\Lambda^q(u)} \leq C\|f\|_{\Gamma_p(w)}$$

holds for all $f \in L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n)$ if and only if

$$\sup_{z>0} \frac{\|\omega_z\|_{\Theta^{q/2}(u)}}{\|\omega_z\|_{p/2,v}} < \infty.$$

Proof. This follows from Theorems 3.3 and 3.6. □

When $q = 2$ this gives the following characterization in terms of the level function of $u(t)$.

Theorem 3.8. *Suppose $0 < p \leq 2$ and $u, v, w \in L^+$ with $v(t) = t^{p-2}w(1/t)$. Then there exists $C > 0$ such that the inequality*

$$\|\hat{f}\|_{\Lambda^2(u)} \leq C\|f\|_{\Gamma^p(w)}$$

holds for all $f \in L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n)$ if and only if

$$\sup_{z>0} \frac{\|\omega_z\|_{1.u^\circ}}{\|\omega_z\|_{p/2,v}} < \infty.$$

Proof. See Theorem 5.1 in [Si4]. □

3.2 More sufficient and necessary conditions

The goal of this section is to provide somewhat simpler sufficient and necessary conditions for inequalities of type $\|\hat{f}\|_{\Lambda^q(u)} \leq C\|f\|_{\Gamma^p(w)}$ and $\|\hat{f}\|_{\Lambda^q(u)} \leq C\|f\|_{\Lambda^p(w)}$. We will also give several examples to illustrate those results. As a byproduct we will deduce Theorem 2 in [BH2] from Sinnamon's work. (See Theorem 3.13)

We start with an immediate corollary of the necessary condition in Theorem 3.6.

Corollary 3.9. *Suppose $0 < p < \infty$, $0 < q < \infty$, $C > 0$, $u, v, w \in L^+$ with $v(t) = t^{p-2}w(1/t)$, satisfy*

$$\|\hat{f}\|_{\Lambda^q(u)} \leq C\|f\|_{\Gamma^p(w)}$$

for all $f \in L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n)$. Then

$$\sup_{z>0} \frac{\|\omega_z\|_{q/2,u}}{\|\omega_z\|_{p/2,v}} < \infty.$$

Proof. In Theorem 3.6 observe that the identity operator lies in $\mathcal{A}$. $\square$

For a decreasing weight u we obtain the following characterization.

Proposition 3.10. *Let $0 < p \leq q < \infty$, $q \geq 2$ and assume u and w are weight functions on $(0, \infty)$ with $v(t) = t^{p-2}w(1/t)$. Suppose u is decreasing. Then there exists $C > 0$ so that the inequality*

$$\|\hat{f}\|_{\Lambda^q(u)} \leq C\|f\|_{\Gamma^p(w)}$$

holds for all $f \in L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n)$ if and only if

$$\sup_{z>0} \frac{\|\omega_z\|_{q/2,u}}{\|\omega_z\|_{p/2,v}} < \infty.$$

Proof. The necessary side is proved in Corollary 3.9. For the sufficient part, observe that $u^\circ = u$ since u is decreasing. Hence Theorem 3.4 completes the proof. $\square$

The fraction $\dfrac{\|\omega_z\|_{q/2,u}}{\|\omega_z\|_{p/2,v}}$ appears frequently both in sufficient and necessary conditions. Before proceeding we do some calculations. Set

$$F_{q,u}(z) = \left(\|\omega_z\|_{q/2,u}\right)^{1/2}, \quad G_{p,w}(z) = \left(\|\omega_z\|_{p/2,v}\right)^{1/2} \quad \text{and} \quad v(t) = t^{p-2}w(1/t).$$

Notice that the conditions in Corollary 3.9 and Theorems 3.4 and 3.8 may be reformulated as $\sup_{z>0} \dfrac{F_{q,u}(z)}{G_{p,w}(z)} < \infty$ with the appropriate weights and indices.

Observe that

$$\begin{aligned} F_{q,u}(z)^2 &= \left(\int_0^\infty \min(z^{-q}, t^{-q})\, u(t)\, dt\right)^{2/q} \\ &= \left(z^{-q}\int_0^z u(t)\, dt + \int_z^\infty \frac{u(t)}{t^q} dt\right)^{2/q}, \end{aligned} \tag{3.1}$$

which trivially leads to the following estimates:

$$F_{q,u}(z) \geq z^{-1}\left(\int_0^z u(t)\, dt\right)^{1/q}, \quad \text{and} \tag{3.2}$$

$$F_{q,u}(z) \geq \left(\int_z^\infty \frac{u(t)}{t^q} dt\right)^{1/q}. \tag{3.3}$$

If u is decreasing and $q > 1$, then $u \in B_q$ by Remark 1.17. So

$$F_{q,u}(z) \leq \left(z^{-q}\int_0^z u(t)\, dt + \frac{1}{q-1} z^{-q}\int_0^z u(t)\, dt\right)^{1/q} = \left(\frac{q}{q-1}\right)^{1/q} z^{-1}\left(\int_0^z u(t)\, dt\right)^{1/q}.$$

Therefore, for decreasing u and $q > 1$ we have

$$F_{q,u}(z) \approx z^{-1} \left(\int_0^z u(t)\, dt \right)^{1/q}. \tag{3.4}$$

For $G_{p.w}$ we have

$$G_{p,w}(z)^2 = \left(z^{-p} \int_0^z v(t)\, dt + \int_z^\infty \frac{v(t)}{t^p} dt \right)^{2/p},$$

which, using $v(t) = t^{p-2} w(1/t)$, turns into

$$G_{p,w}(z)^2 = \left(z^{-p} \int_{1/z}^\infty \frac{w(t)}{t^p}\, dt + \int_0^{1/z} w(t)\, dt \right)^{2/p}, \tag{3.5}$$

and we obtain the following immediate estimates:

$$G_{p,w}(z) \geq z^{-1} \left(\int_{1/z}^\infty \frac{w(t)}{t^p}\, dt \right)^{1/p}, \quad \text{and} \tag{3.6}$$

$$G_{p,w}(z) \geq \left(\int_0^{1/z} w(t)\, dt \right)^{1/p}. \tag{3.7}$$

For weights in B_p or RB_p we can get better estimates. If $w \in B_p$ then

$$G_{p,w}(z) \leq \left(b_p \int_0^{1/z} w(t)\, dt + \int_0^{1/z} w(t)\, dt \right)^{1/p} = (1 + b_p)^{1/p} \left(\int_0^{1/z} w(t)\, dt \right)^{1/p},$$

which implies

$$G_{p,w}(z) \approx \left(\int_0^{1/z} w(t)\, dt \right)^{1/p}. \tag{3.8}$$

If $w \in RB_p$ then

$$G_{p,w}(z) \leq \left(z^{-p} \int_{1/z}^\infty \frac{w(t)}{t^p}\, dt + b_p^* z^{-p} \int_{1/z}^\infty \frac{w(t)}{t^p}\, dt \right)^{1/p} = (1+b_p^*)^{1/p} z^{-1} \left(\int_{1/z}^\infty \frac{w(t)}{t^p}\, dt \right)^{1/p},$$

which implies

$$G_{p,w}(z) \approx z^{-1} \left(\int_{1/z}^\infty \frac{w(t)}{t^p}\, dt \right)^{1/p}. \tag{3.9}$$

Theorem 3.11. *Let $0 < p \leq q < \infty$ and $2 \leq q$. Assume u and w are weight functions on $(0, \infty)$. If*

$$\sup_{x>0} x \left(\int_0^{1/x} u^\circ(t)\, dt \right)^{1/q} \left(\int_0^x w(t)\, dt \right)^{-1/p} < \infty,$$

then there exists $C > 0$ such that

$$\|\hat{f}\|_{\Lambda^q(u)} \leq C \|f\|_{\Gamma^p(w)}$$

for all $f \in L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n)$.

Proof. Using the equivalence (3.4), with u° replacing u, and the inequality (3.7) we get

$$\sup_{z>0} \frac{F_{q,u^\circ}(z)}{G_{p,w}(z)} \lessapprox \sup_{z>0} z^{-1} \left(\int_0^z u^\circ(t)\,dt \right)^{1/q} \left(\int_0^{1/z} w(t)\,dt \right)^{-1/p}$$

$$= \sup_{x>0} x \left(\int_0^{1/x} u^\circ(t)\,dt \right)^{1/q} \left(\int_0^x w(t)\,dt \right)^{-1/p} < \infty.$$

So the condition in Theorem 3.4 is satisfied. □

Example 3.12. We provide an example to illustrate Theorem 3.11. Let $w(t) = t^{a-1}$ where $p/2 \le a < p$. Assume $0 < p \le q \le p/(p-a)$ and $q \ge 2$. Fix $z > 0$ and observe that $u = \chi_{(0,z)}$ is decreasing and therefore $u^\circ = u$. Let u_1, u_2, u_3 be the functions defined in Example 1.27, that is,

- $u_1(t) = ae^t\chi_{(0,z)}$, $a = z(e^z-1)^{-1}$,
- $u_2(t) = t(z^2-t^2)^{-1/2}\chi_{(0,z)}$, and
- $u_3(t) = ct^r\chi_{(0,z)}$, $r \ge 0$, $c = z^{-r}(r+1)$.

We showed $u_1^\circ = u_2^\circ = u_3^\circ = u^\circ = \chi_{(0,z)}$. Observe that

$$\int_0^{1/x} u^\circ(t)\,dt = \begin{cases} z, & 0 < x \le 1/z \\ x^{-1}, & x > 1/z \end{cases} \quad \text{and} \quad \int_0^x w(t)\,dt = (1/a)x^a.$$

Now we have

$$\sup_{0<x\le 1/z} x \left(\int_0^{1/x} u^\circ(t)\,dt \right)^{1/q} \left(\int_0^x w(t)\,dt \right)^{-1/p} = \sup_{0<x\le 1/z} x(z)^{1/q}((1/a)x^a)^{-1/p},$$

which is finite since $a < p$. On the other hand,

$$\sup_{x>1/z} x \left(\int_0^{1/x} u^\circ(t)\,dt \right)^{1/q} \left(\int_0^x w(t)\,dt \right)^{-1/p} = \sup_{x>1/z} x(x^{-1})^{1/q}((1/a)x^a)^{-1/p}$$

is finite since $q \le p/(p-a)$.

So the condition in Theorem 3.11 is satisfied. Hence $\mathcal{F} : \Gamma^p(w) \to \Lambda^q(u)$ is bounded. Since $w \in B_p$ as we showed in Example 1.18, we have $\|.\|_{\Gamma^p(w)} \approx \|.\|_{\Lambda^p(w)}$. Therefore, $\mathcal{F} : \Lambda^p(w) \to \Lambda^q(u)$ is bounded. This statement is true for any weight function whose level function is equal to $u(t)$. In particular $\mathcal{F} : \Lambda^p(w) \to \Lambda^q(u_j)$ is bounded for $j = 1, 2, 3$.

As a corollary of Theorems 3.11 and 3.5 we prove the following result on Fourier inequalities in Lorentz Λ space from Benedetto and Heinig. That is, Theorem 2 in [BH2].

Theorem 3.13. *Let u and w be weight functions on $(0,\infty)$.*

(*i*) *Let* $1 < p \le q < \infty$, $q \ge 2$, *and assume* u *is decreasing and* $w \in B_p$. *If*

$$\sup_{x>0} x \left(\int_0^{1/x} u(t)\,dt \right)^{1/q} \left(\int_0^x w(t)\,dt \right)^{-1/p} < \infty, \tag{3.10}$$

then there exists $C > 0$ *such that for all* $f \in L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n)$ *the inequality*

$$\|\hat{f}\|_{\Lambda^q(u)} \le C\|f\|_{\Lambda^p(w)}$$

holds.

(*ii*) *Conversely, assume* $p, q > 1$ *and* u *and* w *are arbitrary weight functions. If* $\|\hat{f}\|_{\Lambda^q(u)} \le C\|f\|_{\Lambda^p(w)}$ *for all* $f \in L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n)$, *then (3.10) holds.*

Proof. To prove (i), observe that $u^\circ = u$ since u is decreasing. Moreover $\|.\|_{\Gamma^p(w)} \approx \|.\|_{\Lambda^p(w)}$ since $w \in B_p$. So the assertion is implied by Theorem 3.11.

To prove (ii), fix $z > 0$ and let c_n be the constant in Theorem 3.5. Set $\varepsilon = (2c_n z)^{-1}$ and let A be the identity operator. There exists $f : \mathbb{R}^n \to \mathbb{C}$ such that

$$f^* \le \chi_{[0,1/z)} \quad \text{and} \quad (\omega_z)^{1/2} \le c_n(\hat{f}^* + \varepsilon).$$

For this f,

$$\|f\|_{\Lambda^p(w)} = \left(\int_0^\infty f^*(t)^p w(t)\,dt \right)^{1/p} \le \left(\int_0^{1/z} w(t)\,dt \right)^{1/p}.$$

On the other hand for $0 < t < z$ we have

$$\hat{f}^*(t) \ge c_n^{-1} \min(z^{-1}, t^{-1}) - \varepsilon \ge \tfrac{1}{2} c_n^{-1} z^{-1},$$

which implies

$$\|\hat{f}\|_{\Lambda^q(u)} = \left(\int_0^\infty \hat{f}^*(t)^q u(t)\,dt \right)^{1/q} \ge \tfrac{1}{2} c_n^{-1} z^{-1} \left(\int_0^z u(t)\,dt \right)^{1/q}.$$

Finally

$$z^{-1} \left(\int_0^z u(t)\,dt \right)^{1/q} \left(\int_0^{1/z} w(t)\,dt \right)^{-1/p} \le 2c_n \left(\|\hat{f}\|_{\Lambda^q(u)}\right) \left(\|f\|_{\Lambda^p(w)}\right)^{-1} \le 2c_n C.$$

Since $z > 0$ is arbitrary, the proof is complete by taking $x = z^{-1}$. □

Example 3.14. A natural example to consider is the case of power weights. We take this example from [BH2]. Assume $1 < p \le q < \infty$ and $q \ge 2$. Set $u(t) = t^{b-1}$ and $w(t) = t^{a-1}$ where $0 < b \le 1$ and $0 < a < p$.

Obviously u is decreasing and by Example (1.18) we have $w \in B_p$. The supremum

$$\sup_{x>0} x \left(\int_0^{1/x} u(t)\,dt \right)^{1/q} \left(\int_0^x w(t)\,dt \right)^{-1/p} = \sup_{x>0} x \left(\frac{x^{-b}}{b} \right)^{1/q} \left(\frac{x^a}{a} \right)^{-1/p} \approx \sup_{x>0} x^{1-b/q-a/p}$$

is finite exactly when $1 - (b/q) + (a/p) = 0$. So Theorem 3.13 implies that $\mathcal{F} : \Lambda^p(w) \to \Lambda^q(u)$ is bounded if and only if $b/q + a/p = 1$.

Example 3.15. This is a modification of Example 3.12. Assume $1 < p \le q < \infty$ and $q \ge 2$. Let $u(t) = \chi_{(0,z)}$ and $w(t) = t^{a-1}$ where $0 < a < p$. Note that u is decreasing and $w \in B_p$. Similar computations as in Example 3.12 assert that

$$\sup_{0<x\le 1/z} x \left(\int_0^{1/x} u(t)\,dt\right)^{1/q} \left(\int_0^{x} w(t)\,dt\right)^{-1/p} = \sup_{0<x\le 1/z} x(z)^{1/q}((1/a)x^a)^{-1/p}$$

is finite since $a < p$, and

$$\sup_{x>1/z} x \left(\int_0^{1/x} u(t)\,dt\right)^{1/q} \left(\int_0^{x} w(t)\,dt\right)^{-1/p} = \sup_{x>1/z} x(x^{-1})^{1/q}((1/a)x^a)^{-1/p}$$

is finite exactly when $q \le p/(p-a)$. Now Theorem 3.13 implies that $\mathcal{F} : \Lambda^p(w) \to \Lambda^q(u)$ is bounded when $q \le p/(p-a)$ and is unbounded when $q > p/(p-a)$.

Taking a similar approach as we did in Theorem 3.11, we obtain our second sufficient condition.

Theorem 3.16. *Let $0 < p \le q < \infty$ and $2 \le q$. Assume u and w are weight functions on $(0,\infty)$. If*

$$\sup_{x>0} \left(\int_0^{1/x} u^\circ(t)\,dt\right)^{1/q} \left(\int_x^{\infty} \frac{w(t)}{t^p}\,dt\right)^{-1/p} < \infty,$$

then there exists $C > 0$ such that

$$\|\hat{f}\|_{\Lambda^q(u)} \le C\|f\|_{\Gamma^p(w)}$$

for all $f \in L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n)$.

Proof. Using the equivalence (3.4), with u° replacing u, and the inequality (3.6) we get

$$\begin{aligned}\sup_{z>0}\frac{F_{q,u^\circ}(z)}{G_{p,w}(z)} &\lessapprox \sup_{z>0} z^{-1}\left(\int_0^{z} u^\circ(t)\,dt\right)^{1/q} z\left(\int_{1/z}^{\infty} \frac{w(t)}{t^p}\,dt\right)^{-1/p}\\ &= \sup_{x>0} \left(\int_0^{1/x} u^\circ(t)\,dt\right)^{1/q} \left(\int_x^{\infty} \frac{w(t)}{t^p}\,dt\right)^{-1/p} < \infty,\end{aligned}$$

and the assertion is proved by Theorem 3.4 $\square$

Example 3.17. Here is an example where $\mathcal{F} : \Lambda^p(w) \to \Lambda^q(u)$ is unbounded, but the restriction of $\mathcal{F}$ to the smaller space $\Gamma^p(w)$ is bounded.

Assume $p/2 \le a < p$, $1 < p \le q \le p/(p-a)$ and $q \ge 2$. Fix $z > 0$ and set $u(t) = \chi_{(0,z)}$ and $w(t) = t^{a-1}\chi_{(1/z,\infty)}$. It is obvious that $u^\circ = u(t)$ and $w \notin B_p$. We have

$$\sup_{0<x<1/z} \left(\int_0^{1/x} u^\circ(t)\,dt\right)^{1/q} \left(\int_x^{\infty} \frac{w(t)}{t^p}\,dt\right)^{-1/p} = z^{1/q}\left(\frac{z^{p-a}}{p-a}\right)^{-1/p},$$

which is constant, and

$$\sup_{x>1/z}\left(\int_0^{1/x} u^\circ(t)\,dt\right)^{1/q}\left(\int_x^\infty \frac{w(t)}{t^p}\,dt\right)^{-1/p} = \sup_{x>1/z}(p-a)^{1/p}x^{-1/q}x^{-(a-p)/p},$$

which is finite since $q \le p/(p-a)$. Therefore, Theorem 3.16 implies that $\mathcal{F} : \Gamma^p(w) \to \Lambda^q(u)$ is continuous.

Now we show $\mathcal{F} : \Lambda^p(w) \to \Lambda^q(u)$ is unbounded. Notice that $w \notin B_p$ means the Lorentz spaces $\Lambda^p(w)$ and $\Gamma^p(w)$ do not coincide. Moreover, $\int_0^x w = 0$ whenever $x < 1/z$. Hence,

$$\sup_{x>0} x\left(\int_0^{1/x} u(t)\,dt\right)^{1/q}\left(\int_0^x w(t)\,dt\right)^{-1/p} = \infty.$$

Therefore, the necessary condition in Theorem 3.13 does not hold. This means $\mathcal{F} : \Lambda^p(w) \to \Lambda^q(u)$ is unbounded. We can also check this directly. Choose $f(t) = \chi_{(0,1/z)}$. Then $\|f\|_{\Lambda^p(w)} = 0$ whereas $\|\hat{f}\|_{\Lambda^q(u)} \neq 0$.

The next theorem is a sufficient condition for boundedness of $\mathcal{F}$ between Lorentz Λ spaces. It is not only of interest in its own right but also leads to a new proof of Theorem 1(i) in [BH2], which provides a sufficient condition for Fourier inequalities in weighted L^p spaces. See Theorem 3.32 for our proof of the theorem.

Theorem 3.18. *Let $1 < p \le q < \infty$ and $2 \le q$. Assume u and w are weight functions on $(0,\infty)$. If*

$$\sup_{x>0}\left(\int_0^{1/x} u^\circ(t)\,dt\right)^{1/q}\left(\int_0^x w(t)^{1-p'}\,dt\right)^{1/p'} < \infty, \tag{3.11}$$

then there exists $C > 0$ such that

$$\|\hat{f}\|_{\Lambda^q(u)} \le C\|f\|_{\Lambda^p(w)}$$

for all $f \in L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n)$.

Proof. Since $w(t)$ is a weight function and therefore locally integrable, it is finite a.e. Thus, $w(t)^{1-p'} > 0$ a.e., which means $\int_0^x w(t)^{1-p'}\,dt > 0$ for all $x > 0$. Hence, (3.11) implies that $\int_0^{1/x} u^\circ(t)\,dt < \infty$ for $x > 0$. Since the concave function $s \mapsto \int_0^s u^\circ(t)dt$ is absolutely continuous, it is differentiable almost everywhere. So we may set

$$\sigma(t) = t^{q-2}u^\circ(1/t) = -t^q\frac{d}{dt}\left(\int_0^{1/t} u^\circ(t)\,dt\right).$$

Then,

$$-\int_a^x \frac{\sigma(t)}{t^q}\,dt = \int_0^{1/x} u^\circ(t)\,dt - \int_0^{1/a} u^\circ(t)\,dt.$$

The second term on the right hand side vanishes as $a \to \infty$, so

$$\int_x^\infty \frac{\sigma(t)}{t^q}\,dt = \int_0^{1/x} u^\circ(t)\,dt.$$

which means

$$\sup_{x>0} \left(\int_0^{1/x} u^\circ(s)\, ds \right)^{1/q} \left(\int_x^\infty \frac{\sigma(t)}{t^q}\, dt \right)^{-1/q} = 1 < \infty.$$

So the condition in Theorem 3.16 with $w(t)$ and p replaced with $\sigma(t)$ and q is satisfied. It follows there exists $C_1 >$ so that

$$\|\hat{f}\|_{\Lambda^q(u)} \leq C_1 \|f\|_{\Gamma^q(\sigma)}$$

for all $f \in L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n)$.

On the other hand, the hypothesis of the theorem and the relation between σ and u° implies:

$$\begin{aligned} &\sup_{x>0} \left(\int_x^\infty \frac{\sigma(t)}{t^q}\, dt \right)^{1/q} \left(\int_0^x w(t)^{1-p'}\, dt \right)^{1/p'} \\ &= \sup_{x>0} \left(\int_0^{1/x} u^\circ(t)\, dt \right)^{1/q} \left(\int_0^x w(t)^{1-p'}\, dt \right)^{1/p'} < \infty. \end{aligned}$$

Now by the weighted Hardy inequality (Theorem 1.15) there exists $C_2 > 0$ such that

$$\left(\int_0^\infty \left(\frac{1}{t} \int_0^t g \right)^q \sigma(t)\, dt \right)^{1/q} \leq C_2 \left(\int_0^\infty g(t)^p w(t)\, dt \right)^{1/p}$$

for all $g \in L^+$. Replacing g with f^* in the above inequality we get

$$\|f\|_{\Gamma^q(\sigma)} \leq C_2 \|f\|_{\Lambda^p(w)}$$

for all $f \in L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n)$.

Finally we have $\|\hat{f}\|_{\Lambda^q(u)} \leq C_1 \|f\|_{\Gamma^q(\sigma)} \leq C_1 C_2 \|f\|_{\Lambda^p(w)}$ for all $f \in L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n)$. The proof is complete by taking $C = C_1 C_2$. □

Remark 3.19. We compare Theorem 3.13 with 3.18. In the above theorem, we do not need u to be decreasing or w to be B_p. However the supremum condition (3.11) is stronger than (3.10). In fact if (3.11) holds then $\mathcal{F} : \Lambda^p(w) \to \Lambda^q(u)$ is bounded by Theorem 3.18. Then the necessary part of Theorem 3.13 implies (3.10).
We may investigate this directly. We have

$$\int_0^{1/x} u(t)\, dt \leq \int_0^{1/x} u^\circ(t)\, dt.$$

On the other hand, Hölder's inequality shows

$$x = \int_0^x dt \leq \left(\int_0^x w(t)\, dt \right)^{1/p} \left(\int_0^x w(t)^{1-p'}\, dt \right)^{1/p'}.$$

It follows that

$$x\left(\int_0^{1/x} u(t)\,dt\right)^{1/q}\left(\int_0^x w(t)\,dt\right)^{-1/p} \le \left(\int_0^{1/x} u^\circ(t)\,dt\right)^{1/q}\left(\int_0^x w(t)^{1-p'}\,dt\right)^{1/p'}.$$

So the left-hand side is finite whenever the right-hand side is.

Example 3.20. Here is an application of Theorem 3.18 where Theorems 3.11, 3.13 and 3.16 are inconclusive. Assume $1 < p \le q < \infty$ and $2 \le q$. Let u be an arbitrary weight function in $L^1(0,\infty)$ and $w(t) = e^t\chi_{(0,\infty)}$.
Note that by Proposition 1.33 we have $u^\circ \in L^1$ since $u \in L^1$. Therefore,

$$\left(\int_0^{1/x} u^\circ(t)\,dt\right)^{1/q} \le \left(\int_0^\infty u^\circ(t)\,dt\right)^{1/q} = M_1 < \infty.$$

Moreover,

$$\left(\int_0^x w(t)^{1-p'}\,dt\right)^{1/p'} = \left(\int_0^x e^{-t(p'-1)}\,dt\right)^{1/p'} \le \left(\int_0^\infty e^{-t(p'-1)}\,dt\right)^{1/p'} = M_2 < \infty.$$

It follows that

$$\sup_{x>0}\left(\int_0^{1/x} u^\circ(t)\,dt\right)^{1/q}\left(\int_0^x w(t)^{1-p'}\,dt\right)^{1/p'} \le M_1M_2 < \infty.$$

Thus Theorem 3.18 asserts that $\mathcal{F} : \Lambda^p(w) \to \Lambda^q(u)$ is bounded.

Now observe that $w \notin B_p$ because

$$\int_x^\infty \frac{e^t}{t^p}\,dt = \infty \quad \text{but} \quad \frac{1}{x^p}\int_0^x e^t\,dt = \frac{e^x - 1}{x^p}, \quad x > 0.$$

So Theorem 3.13 is inapplicable even for a decreasing u. We may use Theorem 3.11 to prove $\mathcal{F} : \Gamma^p(w) \to \Lambda^q(u)$ is bounded. But this does not imply the boundedness of $\mathcal{F} : \Lambda^p(w) \to \Lambda^q(u)$.

The next theorem provides a necessary condition comparable to that in Theorem 3.13. We will use it in the following section, to provide a necessary condition for continuity of Fourier transform between weighted Lebesgue spaces. (See Theorem 3.35.)

Proposition 3.21. *Let $0 < p, q < \infty$ and assume u and w are weight functions on $(0,\infty)$. Suppose $w \in B_p$. If there exists $C > 0$ so that*

$$\|\hat{f}\|_{\Lambda^q(u)} \le C\|f\|_{\Gamma^p(w)}$$

for all $f \in L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n)$, then

$$\sup_{x>0}\left(x^q\int_0^{1/x} u(t)\,dt + \int_{1/x}^\infty \frac{u(t)}{t^q}\,dt\right)^{1/q}\left(\int_0^x w(t)\,dt\right)^{-1/p} < \infty. \tag{3.12}$$

Proof. Corollary 3.9 implies $\sup_{z>0} \dfrac{F_{q,u}(z)}{G_{p,w}(z)} < \infty$. Now Equations (3.1) and (3.8) assert that

$$F_{q,u}(z) = \left(z^{-q} \int_0^z u(t)\, dt + \int_z^\infty \frac{u(t)}{t^q}\, dt \right)^{1/q} \quad \text{and} \quad G_{p,w} \approx \left(\int_0^{1/z} w(t)\, dt \right)^{1/p}.$$

Set $x = 1/z$, and the proof is complete. $\square$

Remark 3.22.

1. Within the range $1 < p < \infty$, $0 < q < \infty$, the condition (3.12) is a necessary condition for $\|\hat{f}\|_{\Lambda^q(u)} \leq C\|f\|_{\Lambda^p(w)}$, since $w \in B_p$ implies $\|f\|_{\Gamma^p(w)} \approx \|f\|_{\Lambda^p(w)}$.
2. In comparison with the necessary condition in Theorem 3.13, the above theorem assumes the a priori condition $w \in B_p$. However the supremum in (3.12) is greater than the supremum in (3.10).

Proposition 3.23. *Let $0 < p, q < \infty$ and assume u and w are weight functions on $(0, \infty)$. Suppose $w \in RB_p$. If there exists $C > 0$ so that*

$$\|\hat{f}\|_{\Lambda^q(u)} \leq C\|f\|_{\Gamma^p(w)}$$

for all $f \in L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n)$, then

$$\sup_{x>0} x^{-1} \left(x^q \int_0^{1/x} u(t)\, dt + \int_{1/x}^\infty \frac{u(t)}{t^q}\, dt \right)^{1/q} \left(\int_x^\infty \frac{w(t)}{t^p}\, dt \right)^{-1/p} < \infty.$$

Proof. Corollary 3.9 implies $\sup_{z>0} \dfrac{F_{q,u}(z)}{G_{p,w}(z)} < \infty$. Now equations (3.1) and (3.9) assert that

$$F_{q,u}(z) = \left(z^{-q} \int_0^z u(t)\, dt + \int_z^\infty \frac{u(t)}{t^q}\, dt \right)^{1/q} \quad \text{and} \quad G_{p,w} \approx z^{-1} \left(\int_{1/z}^\infty \frac{w(t)}{t^p}\, dt \right)^{1/p}.$$

Set $x = 1/z$, and the proof is complete. $\square$

The following characterization may be considered as an analogy to Theorem 3.13 where $w \in RB_p$.

Theorem 3.24. *Let $0 < p \leq q < \infty$ and $q \geq 2$. Assume u and w are weight functions on $(0, \infty)$ with u decreasing and $w \in RB_p$. Then there exists $C > 0$ so that the inequality*

$$\|\hat{f}\|_{\Lambda^q(u)} \leq C\|f\|_{\Gamma^p(w)}$$

holds for all $f \in L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n)$ if and only if

$$\sup_{x>0} \left(\int_0^{1/x} u(t)\, dt \right)^{1/q} \left(\int_x^\infty \frac{w(t)}{t^p}\, dt \right)^{-1/p} < \infty.$$

Proof. By Proposition 3.10 the inequality $\|\hat{f}\|_{\Lambda^q(u)} \leq C\|f\|_{\Gamma^p(w)}$ holds if and only if $\sup_{z>0} \dfrac{F_{q,u}(z)}{G_{p,w}(z)} < \infty$. Now inequalities (3.4) and (3.9) assert that

$$F_{q,u}(z) \approx z^{-1} \left(\int_0^z u(t)\, dt \right)^{1/q} \quad \text{and} \quad G_{p,w} \approx z^{-1} \left(\int_{1/z}^\infty \frac{w(t)}{t^p}\, dt \right)^{1/p}.$$

The proof is complete by taking $x = 1/z$. □

For the case $q = 2$ we are able to characterize the boundedness of $\mathcal{F} : \Gamma^p(w) \to \Lambda^q(u)$ in terms of the level function. To do this, we invoke Theorem 3.8 in two separate cases in which w is a B_p or RB_p weight.

Theorem 3.25. *Let $0 < p \leq 2$ and assume u and w are weight functions on $(0, \infty)$. Suppose $w \in B_p$. Then there exists $C > 0$ so that the inequality*

$$\|\hat{f}\|_{\Lambda^2(u)} \leq C\|f\|_{\Gamma^p(w)}$$

holds for all $f \in L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n)$ if and only if

$$\sup_{x>0} x \left(\int_0^{1/x} u^\circ(t)\, dt \right)^{1/2} \left(\int_0^x w(t)\, dt \right)^{-1/p} < \infty.$$

Proof. By Theorem 3.8 the inequality $\|\hat{f}\|_{\Lambda^2(u)} \leq C\|f\|_{\Gamma^p(w)}$ holds if and only if $\sup_{z>0} \dfrac{F_{2,u^\circ}(z)}{G_{p,w}(z)} < \infty$. Since the level function is decreasing, the estimate (3.4) is applicable. We can also use the estimate (3.8) since $w \in B_p$. It follows that

$$F_{2,u^\circ}(z) \approx z^{-1} \left(\int_0^z u^\circ(t)\, dt \right)^{1/q} \quad \text{and} \quad G_{p,w} \approx \left(\int_0^{1/z} w(t)\, dt \right)^{1/p}.$$

Taking $x = 1/z$ completes the proof. □

Theorem 3.26. *Let $0 < p \leq 2$ and assume u and w are weight functions on $(0, \infty)$. Suppose $w \in RB_p$. Then there exists $C > 0$ so that the inequality*

$$\|\hat{f}\|_{\Lambda^2(u)} \leq C\|f\|_{\Gamma^p(w)}$$

holds for all $f \in L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n)$ if and only if

$$\sup_{x>0} \left(\int_0^{1/x} u^\circ(t)\, dt \right)^{1/2} \left(\int_x^\infty \frac{w(t)}{t^p}\, dt \right)^{-1/p} < \infty.$$

Proof. By Theorem 3.8 the inequality $\|\hat{f}\|_{\Lambda^2(u)} \leq C\|f\|_{\Gamma^p(w)}$ holds if and only if $\sup_{z>0} \dfrac{F_{2,u^\circ}(z)}{G_{p,w}(z)} < \infty$. Since the level function is decreasing, the estimate (3.4) is applicable. We can also use the estimate (3.9) since $w \in RB_p$. It follows that

$$F_{2,u^\circ}(z) \approx z^{-1}\left(\int_0^z u^\circ(t)\,dt\right)^{1/q} \quad \text{and} \quad G_{p,w} \approx z^{-1}\left(\int_{1/z}^\infty \frac{w(t)}{t^p}\,dt\right)^{1/p}.$$

Set $x = 1/z$, and the proof is complete. □

Example 3.27. Here is an application of Theorem 3.25 where Theorem 3.13 is inconclusive. Let $1 \leq p \leq 2$. Set $u(t) = 4z^{-3}t^3\chi_{(0,z)}$ and $w(t) = \chi_{(0,1/z)}$ where $z > 0$. In Example 1.27 we showed that $u^\circ(t) = \chi_{(0,z)}$. Moreover, w is decreasing so $w \in B_p$. Observe that

$$\sup_{x>1/z} x\left(\int_0^{1/x} u^\circ(t)\,dt\right)^{1/2}\left(\int_0^x w(t)\,dt\right)^{-1/p} = \sup_{x>1/z} x(x^{-1/2})z^{1/p} \approx \sup_{x>1/z} x^{1/2} = \infty.$$

So the condition in Theorem 3.25 is violated which means $\mathcal{F} : \Lambda^p(w) \to \Lambda^2(u)$ is unbounded.

However, this can not be deduced from the necessary condition in Theorem 3.13. That is because

$$\begin{aligned}&\sup_{0<x<1/z} x\left(\int_0^{1/x} u(t)\,dt\right)^{1/2}\left(\int_0^x w(t)\,dt\right)^{-1/p}\\ &= \sup_{0<x<1/z} x(z^{1/2})(x^{-1/p}) = \sup_{0<x<1/z} x^{1-1/p} < \infty,\end{aligned}$$

and

$$\sup_{x>1/z} x\left(\int_0^{1/x} u(t)\,dt\right)^{1/2}\left(\int_0^x w(t)\,dt\right)^{-1/p} = \sup_{x>1/z} x(z^{-3}x^{-4})^{1/2}z^{1/p} \approx \sup_{x>1/z} x^{-1} < \infty.$$

So the condition in Theorem 3.13 is satisfied.

3.3 Fourier inequalities on Lebesgue spaces

With the aid of Lorentz spaces, one may obtain Fourier inequalities in L^p spaces. The technique is to replace the weight functions on $\mathbb{R}^n$ with their decreasing or increasing rearrangements as illustrated in the following lemma.

Lemma 3.28. *Let $0 < p, q < \infty$ and suppose u and w are weight functions on $\mathbb{R}^n$. Assume $\|\hat{f}\|_{\Lambda^q(u^*)} \leq C\|f\|_{\Lambda^p(w^\circledast)}$ where $C > 0$ and f is a measurable function on $\mathbb{R}^n$. Then $\|\hat{f}\|_{L^q(u)} \leq C\|f\|_{L^p(w)}$.*

Proof. Using the Hardy-Littlewood-Polya inequality (Proposition 1.4) and noting that $(|f|^p)^* = (f^*)^p$ we have

$$\begin{aligned}\|\hat{f}\|_{L^q(u)} &= \left(\int_{\mathbb{R}^n} |\hat{f}(\gamma)|^q \, u(\gamma)\, d\gamma\right)^{1/q} \le \left(\int_0^\infty \hat{f}^*(t)^q \, u^*(t)\, dt\right)^{1/q} \\ &\le C\left(\int_0^\infty f^*(x)^p \, w^{\circledast}(t)\, dt\right)^{1/p} \le C\left(\int_{\mathbb{R}^n} |f(x)|^p \, w(x)\, dx\right)^{1/p} \\ &= C\|f\|_{L^p(w)}.\end{aligned}$$

□

Now we combine Lemma 3.28 with Theorem 3.13 to obtain a sufficient condition for $\|\hat{f}\|_{L^q(u)} \le C\|f\|_{L^p(w)}$. This is what Benedetto and Heinig did in Theorem 4(i) of [BH2].

Theorem 3.29. *Let $1 < p \le q < \infty$, $q \ge 2$ and assume u and w are weight functions on $\mathbb{R}^n$ with $w^{\circledast} \in B_p$. If*

$$\sup_{x>0} x \left(\int_0^{1/x} u^*(t)\, dt\right)^{1/q} \left(\int_0^x w^{\circledast}(t)\, dt\right)^{-1/p} < \infty,$$

then there exists $C > 0$ such that

$$\left(\int_{\mathbb{R}^n} \hat{f}(\gamma)^q \, u(\gamma)\, d\gamma\right)^{1/q} \le C\left(\int_{\mathbb{R}^n} f(x)^p \, w(x)\, dx\right)^{1/p}$$

or, equivalently,

$$\|\hat{f}\|_{L^q(u)} \le C\|f\|_{L^p(w)},$$

for all $f \in L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n)$.

Proof. Since $w^{\circledast} \in B_p$ and u^* is decreasing the conditions of Theorem 3.13, with u and w replaced with u^* and $w^{\circledast}$, are satisfied. Thus, there exists $C > 0$ such that $\|\hat{f}\|_{\Lambda^q(u^*)} \le C\|f\|_{\Lambda^p(w^{\circledast})}$ for all $f \in L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n)$. The proof is complete by Lemma 3.28. □

It's possible to obtain results for a different range of indices p and q using duality properties of the Fourier transform. The idea is stated in the following lemma.

Lemma 3.30. *Let ρ and σ be Banach function norms on $\mathbb{R}^n$ such that $L^1(\mathbb{R}^n) \cap L_\sigma$ is dense in L_σ. Assume there exists $C > 0$ such that $\rho(\hat{g}) \le C\sigma(g)$ for all $g \in L^1(\mathbb{R}^n) \cap L_\sigma$. Then $\sigma'(\hat{f}) \le C\rho'(f)$ for all $f \in L^1(\mathbb{R}^n) \cap L_{\rho'}$.*

Proof. For $f, g \in L^1(\mathbb{R}^n)$ we have $\int_{\mathbb{R}^n} |\hat{f}g| = \int_{\mathbb{R}^n} |f\hat{g}|$. Since $L^1(\mathbb{R}^n) \cap L_\sigma$ is dense in L_σ, for each $f \in L^1(\mathbb{R}^n) \cap L_{\rho'}$ we have

$$\begin{aligned}\sigma'(\hat{f}) &= \sup_{g\in L^1(\mathbb{R}^n)\cap L_\sigma} \frac{\int_{\mathbb{R}^n} |\hat{f}g|}{\sigma(g)} \le C \sup_{g\in L^1(\mathbb{R}^n)\cap L_\sigma} \frac{\int_{\mathbb{R}^n} |\hat{f}g|}{\rho(\hat{g})} = C \sup_{g\in L^1(\mathbb{R}^n)\cap L_\sigma} \frac{\int_{\mathbb{R}^n} |f\hat{g}|}{\rho(\hat{g})} \\ &\le C \sup_{h\in L_\rho} \frac{\int_{\mathbb{R}^n} |fh|}{\rho(h)} = C\rho'(f).\end{aligned}$$

□

Now we have the next theorem which deals with the case $q < 2$. The theorem was stated in part (iii) of Theorem 4 of [BH2].

Theorem 3.31. *Let $1 < p \le q < 2$ and assume u and w are weight functions on $\mathbb{R}^n$ with $(u^*)^{1-q'} \in B_{q'}$. If*

$$\sup_{x>0} \frac{1}{x} \left(\int_0^{1/x} u^*(t)^{1-q'} \right)^{-1/q'} \left(\int_0^x w^{\circledast}(t)^{1-p'} \right)^{1/p'} < \infty,$$

then there exists $C > 0$ such that

$$\|\hat{f}\|_{L^q(u)} \le C \|f\|_{L^p(w)}$$

for all $f \in L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n)$.

Proof. We repeat the proof of Theorem 4(iii) in [BH2] which uses a duality argument and deduces the result from the case $1 < p \le q < \infty$, $q \ge 2$. Let $P = q'$, $Q = p'$, $U(t) = w(t)^{1-p'}$ and $W(t) = u(t)^{1-q'}$. Since $1 < p \le q < 2$ we have $2 \le q' \le p'$ which means $2 \le P \le Q$. Observe that

$$\begin{aligned} W^{\circledast} &= ((1/W)^*)^{-1} = ((u^{q'-1})^*)^{-1} = (u^*)^{1-q'} \in B_{q'} = B_P, \quad \text{and} \\ U^* &= (w^{1-p'})^* = ((1/w)^{p'-1})^* = ((1/w)^*)^{p'-1} = (w^{\circledast})^{1-p'}. \end{aligned}$$

Now

$$\begin{aligned} &\sup_{x>0} x \left(\int_0^{1/x} U^*(t)\, dt \right)^{1/Q} \left(\int_0^x W^{\circledast}(t)\, dt \right)^{-1/P} \\ &= \sup_{x>0} \frac{1}{x} \left(\int_0^x w^{\circledast}(t)^{1-p'}\, dt \right)^{1/p'} \left(\int_0^{1/x} u^*(t)^{1-q'}\, dt \right)^{-1/q'} < \infty. \end{aligned}$$

Therefore, Theorem 3.29 guarantees the existence of $C > 0$ so that $\|\hat{f}\|_{L^Q(U)} \le C\|f\|_{L^P(W)}$ which is the same as $\|\hat{f}\|_{L^{p'}(w^{1-p'})} \le C\|f\|_{L^{q'}(u^{1-q'})}$. It follows from Lemma 3.30 that $\|\hat{f}\|_{L^q(u)} \le C\|f\|_{L^p(w)}$. Here we used the facts that $L^p(w)$ and $L^{p'}(w^{1-p'})$ are duals of each other, and $L^1 \cap L^{q'}(u^{1-q'})$ is dense in $L^{q'}(u^{1-q'})$. $\square$

Theorems 3.29 and 3.31 are based on the sufficient condition for $\|\hat{f}\|_{\Lambda^q(u)} \le C\|f\|_{\Lambda^p(w)}$ stated in Theorem 3.13. In the next theorem we use our other sufficient condition from Theorem 3.18 to obtain results in Lebesgue spaces. This result is proved in [BH2], Theorem 1(i), but our proof is considerably shorter.

Theorem 3.32. *Let $1 < p \le q < \infty$ and assume u and w are weight functions on $\mathbb{R}^n$. If*

$$\sup_{x>0} \left(\int_0^{1/x} u^*(t)\, dt \right)^{1/q} \left(\int_0^x w^{\circledast}(t)^{1-p'}\, dt \right)^{1/p'} < \infty,$$

then there exists $C > 0$ such that

$$\|\hat{f}\|_{L^q(u)} \le C \|f\|_{L^p(w)}$$

for all $f \in L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n)$.

Proof. First consider the case $q \geq 2$. We invoke Theorem 3.18, replacing u and w with u^* and $w^{\circledast}$ respectively. Thus there exists $C > 0$ such that $\|\hat{f}\|_{\Lambda^q(u^*)} \leq C\|f\|_{\Lambda^p(w^{\circledast})}$ for all $f \in L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n)$. It follows from Lemma 3.28 that $\|\hat{f}\|_{L^q(u)} \leq C\|f\|_{L^p(w)}$.

Now assume $q < 2$. As in the proof of Theorem 3.31, we use the the case $q \geq 2$ and the duality argument. Let $P = q'$, $Q = p'$, $U(t) = w(t)^{1-p'}$ and $W(t) = u(t)^{1-q'}$. Since $1 < p \leq q < 2$ we have $2 < q' \leq p'$ which means $2 < P \leq Q$. Observe that

$$U^* = (w(t)^{1-p'})^* = ((1/w)^{p'-1})^* = ((1/w)^*)^{p'-1} = (w^{\circledast})^{1-p'},$$

and

$$W^{\circledast} = ((1/w)^*)^{-1} = ((u^{q'-1})^*)^{-1} = (u^*)^{1-q'},$$

which implies

$$(W^{\circledast})^{1-P'} = \left((u^*)^{1-q'}\right)^{1-q} = u^*.$$

Now

$$\begin{aligned}
&\sup_{y>0} \left(\int_0^{1/y} U^*(t)\,dt\right)^{1/Q} \left(\int_0^y W^{\circledast}(t)^{1-P'}\,dt\right)^{1/P'} \\
&= \sup_{y>0} \left(\int_0^{1/y} w^{\circledast}(t)^{1-p'}\,dt\right)^{1/p'} \left(\int_0^y u^*(t)\,dt\right)^{1/q} \\
&= \sup_{x>0} \left(\int_0^{1/x} u^*(t)\,dt\right)^{1/q} \left(\int_0^x w^{\circledast}(t)^{1-p'}\,dt\right)^{1/p'},
\end{aligned}$$

which is finite by the hypothesis. Therefore, by the first part of proof, we have $\|\hat{f}\|_{L^Q(U)} \leq C\|f\|_{L^P(W)}$ which is the same as $\|\hat{f}\|_{L^{p'}(w^{1-p'})} \leq C\|f\|_{L^{q'}(u^{1-q'})}$. It follows from Lemma 3.30 that $\|\hat{f}\|_{L^q(u)} \leq C\|f\|_{L^p(w)}$. $\square$

Example 3.33. We use Example 3.20 to illustrate Theorem 3.32 on $\mathbb{R}$. Assume $1 < p \leq q < \infty$, $q \geq 2$. Let $w(t) = e^t\chi_{(0,\infty)}$ and suppose $u \in L^1(\mathbb{R})$. Then $w^{\circledast} = e^t\chi_{(0,\infty)}$. The same calculations as in Example 3.11 shows that

$$\sup_{x>0} \left(\int_0^{1/x} u^*(t)\,dt\right)^{1/q} \left(\int_0^x w^{\circledast}(t)^{1-p'}\,dt\right)^{1/p'} \leq M_1M_2 < \infty.$$

Hence, Theorem 3.32 implies that $\mathcal{F} : L^p(w) \to L^q(u)$ is continuous. Note that $w^{\circledast} \notin B_p$, so Theorem 3.29 is inconclusive.

Similar to the way we proved the sufficient conditions, we can prove necessary conditions for $\|\hat{f}\|_{L^q(u)} \leq C\|f\|_{L^p(w)}$ using necessary conditions in Lorentz spaces and Hardy-Littlewood-Polya inequality. However, the necessary conditions are much different from the sufficient conditions.

Lemma 3.34. *Assume $0 < p, q < \infty$ and u, w are weight functions on $\mathbb{R}^n$. Let f be a measurable function on $\mathbb{R}^n$. If $\|\hat{f}\|_{L^q(u)} \leq C\|f\|_{L^p(w)}$ for some $C > 0$, then $\|\hat{f}\|_{\Lambda^q(u^{\circledast})} \leq C\|f\|_{\Lambda^p(w^*)}$.*

Proof. Using the Hardy-Littlewood-Polya inequality (Proposition 1.4) and noting that $(|f|^p)^* = (f^*)^p$ we have

$$\begin{aligned}\|\hat{f}\|_{\Lambda^q(u^{\circledast})} &= \left(\int_0^\infty \hat{f}^*(t)^q\, u^{\circledast}(t)\, dt\right)^{1/q} \leq \left(\int_{\mathbb{R}^n} |\hat{f}(\gamma)|^q\, u(\gamma)\, d\gamma\right)^{1/q} \\ &\leq C\left(\int_{\mathbb{R}^n} |f(x)|^p\, w(x)\, dx\right)^{1/p} \leq C\left(\int_0^\infty f^*(x)^p\, w^*(t)\, dt\right)^{1/p} \\ &= C\|f\|_{\Lambda^p(w^*)}.\end{aligned}$$

□

Theorem 3.35. *Suppose $1 < p < \infty$, $1 < q < \infty$ and let u and w be weight functions on $\mathbb{R}^n$. Assume there exists $C > 0$ such that*

$$\|\hat{f}\|_{L^q(u)} \leq C\|f\|_{L^p(w)}$$

for all $f \in L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n)$. Then

$$\sup_{x>0} \left(\int_{1/x}^\infty \frac{u^{\circledast}(t)}{t^q}\, dt\right)^{1/q} \left(\int_0^x w^*(t)\, dt\right)^{-1/p} < \infty. \tag{3.13}$$

Proof. The hypothesis of the theorem together with Lemma 3.34 implies $\|\hat{f}\|_{\Lambda^q(u^{\circledast})} \leq C\|f\|_{\Lambda^p(w^*)}$ for all $f \in L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n)$. Since w^* is decreasing it is a B_p weight and we have $\|f\|_{\Lambda^p(w^*)} \approx \|f\|_{\Gamma^p(w^*)}$. We invoke Proposition 3.21 with u and w replaced with $u^{\circledast}$ and w^* respectively, to obtain

$$\sup_{x>0} \left(x^q \int_0^{1/x} u^{\circledast}(t)\, dt + \int_{1/x}^\infty \frac{u^{\circledast}(t)}{t^q}\, dt\right)^{1/q} \left(\int_0^x w^*(t)\, dt\right)^{-1/p} < \infty.$$

This completes the proof since

$$\int_{1/x}^\infty \frac{u^{\circledast}(t)}{t^q}\, dt \leq x^q \int_0^{1/x} u^{\circledast}(t)\, dt + \int_{1/x}^\infty \frac{u^{\circledast}(t)}{t^q}\, dt. \tag{3.14}$$

□

Remark 3.36.

1. Note that the two sides of Inequality (3.14) are in fact equivalent. That is because $u^{\circledast}$ is increasing and therefore satisfies the RB_p condition by Remark 1.21. Hence,

$$\begin{aligned}x^q \int_0^{1/x} u^{\circledast}(t)\, dt + \int_{1/x}^\infty \frac{u^{\circledast}(t)}{t^q}\, dt &\leq (q-1)\int_{1/x}^\infty \frac{u^{\circledast}(t)}{t^q}\, dt + \int_{1/x}^\infty \frac{u^{\circledast}(t)}{t^q}\, dt \\ &= q\int_{1/x}^\infty \frac{u^{\circledast}(t)}{t^q}\, dt,\end{aligned}$$

which implies

$$x^q \int_0^{1/x} u^{\circledast}(t)\, dt + \int_{1/x}^{\infty} \frac{u^{\circledast}(t)}{t^q}\, dt \approx \int_{1/x}^{\infty} \frac{u^{\circledast}(t)}{t^q}\, dt.$$

2. We will get a weaker result if we use the necessary condition from Theorem 3.13. That is,

$$\sup_{x>0} x \left(\int_0^{1/x} u^{\circledast}(t)\, dt \right)^{1/q} \left(\int_0^{x} w^{*}(t)\, dt \right)^{-1/p} < \infty \tag{3.15}$$

is also a necessary condition for $\|\hat{f}\|_{L^q(u)} \leq C\|f\|_{L^p(w)}$, but (3.13) always implies (3.15).

Chapter 4

Fourier series in Lorentz spaces

Many of the results in Fourier inequalities deal with the Fourier transform on $\mathbb{R}^n$ rather than with Fourier series. In particular, there is not much known about boundedness of the Fourier coefficient map between weighted Lebesgue spaces and Lorentz spaces. The aim of this chapter is to provide results, analogous to those in Chapter 3, for the Fourier coefficient map. This analogy is far from being trivial, despite the fact that the Fourier transform on $\mathbb{R}^n$ and the Fourier coefficient map on $\mathbb{T}$ share many common properties. That is because of the finite measure on $\mathbb{T}$ and the atomic measure on $\mathbb{Z}$.

The finiteness of the measure on $\mathbb{T}$ imposes an extra condition on the class $\Omega_{2,0}$ that was used in Theorem 3.2. The condition requires the functions in $\Omega_{2,0}$ to be constant on interval $(0,1)$ as stated in Theorem 4.2. To pass from this theorem to the next results we need the results on quasi concave functions. In fact this was the main motivation for the material of Chapter 2.

The atomic measure on $\mathbb{Z}$ makes the estimates for the rearrangement of $\hat{f}$ coarser. This makes Lemma 4.13 fail for values of z close to 1. To overcome this issue we first prove Theorem 4.17 for $z \geq 3$ and then extend it to $z > 1$. As a consequence, the constant c in Theorem 4.18 is quite large.

We start this chapter with sufficient conditions and provide results analogous to the sufficient conditions in Chapter 3. However, we work with the more general inequality $\|\hat{f}\|_{\Gamma^q(u)} \leq C\|f\|_{\Gamma^p(w)}$. In Section 4.2 we introduce the collection of test functions that lead to our necessary conditions in Section 4.3. Finally, we combine our results to obtain necessary and sufficient conditions for boundedness of the Fourier coefficient map between Lorentz spaces.

4.1 Sufficient conditions

We start with a slightly different version of Proposition 3.1, which provides a rearrangement estimate for operators of type $(1,\infty)$ and $(2,2)$. Using this estimate, we will give sufficient conditions for the inequality $\|\hat{f}\|_{\Gamma^q(u)} \leq C\|f\|_{\Gamma^p(w)}$. Although the statement of the theorems in this section deal with the Fourier coefficient map, they remain true for any operator of type $(1,\infty)$ and $(2,2)$.

Proposition 4.1. *Assume (X,μ) and (Y,ν) are σ-finite measure spaces and Let $T : L^1(\mu)+L^2(\mu)\to L_\nu(Y)$ be a sublinear operator. Then T is of type $(1,\infty)$ and $(2,2)$ if and only if there exists a constant D such that*

$$\int_0^z (Tf)^{**}(t)^2\,dt \le D\int_0^z \Big(\int_0^{1/t} f^*(s)\,ds\Big)^2\,dt \tag{4.1}$$

for all $z>0$ and $f\in L^1(\mu)+L^2(\mu)$.

Proof. By Proposition 3.1, T is of type $(1,\infty)$ and $(2,2)$ if and only if there exists $D_1>0$ such that

$$\int_0^z (Tf)^{*}(t)^2\,dt \le D_1\int_0^z \Big(\int_0^{1/t} f^*(s)\,ds\Big)^2\,dt \tag{4.2}$$

for all $z>0$ and $f\in L^1(\mu)+L^2(\mu)$.

First, observe that (4.1) implies (4.2) with $D_1=D$, since $(Tf)^*\le (Tf)^{**}$. Conversely, assume (4.2) holds. Observe that for $f\in L_\mu$ the Hardy inequality (Corollary 1.14) implies

$$\int_0^z (Tf)^{**}(t)^2\,dt = \int_0^z \Big(\frac{1}{t}\int_0^t (Tf)^*(s)\,ds\Big)^2\,dt \le 2^2\int_0^z (Tf)^*(t)^2\,dt.$$

Hence (4.1) holds with $D=4D_1$. $\square$

The constant D_1 in (4.2) is not greater than than 4 when both operator norms $(1,\infty)$ and $(2,2)$ are at most 1. This in particular means $D\le 16$ for the Fourier transform. Using the above proposition we get the following sufficient condition for $\|\hat f\|_{\Gamma^q(u)}\le C\|f\|_{\Gamma^p(w)}$ with arbitrary exponents p,q. It serves as a platform to obtain our main results.

Theorem 4.2. *Suppose $0<p,q<\infty$ and $u,v,w\in L^+$ with $v(t)=t^{p-2}w(1/t)$. If*

$$\sup_{h\in P\cap\Omega_{2,0}}\ \sup_{0\le\varphi\downarrow,\varphi\prec h}\frac{\|\varphi\|_{q/2,u}}{\|h\|_{p/2,v}}<\infty \tag{4.3}$$

then there exists $C>0$ such that

$$\|\hat f\|_{\Gamma^q(u)}\le C\|f\|_{\Gamma^p(w)}$$

for all $f\in L^1(\mathbb{T})$.

Proof. Let C_1 be the value of the supremum in the hypothesis. Fix $f\in L^1(\mathbb{T})$ and let h_f and φ_f be defined by

$$h_f=\left(\int_0^{1/t} f^*(s)\,ds\right)^2 \quad\text{and}\quad \varphi_f=\frac{1}{16}(\hat f)^{**}(t)^2.$$

It's readily seen that h_f is decreasing and $t^2h_f(t)=f^{**}(1/t)$ is an increasing function. So $h_f\in\Omega_{2,0}$. In addition, f^* vanishes outside the interval $(0,1)$ since $m(\mathbb{T})=1$. So

$h_f(t)$ is constant on $(0,1)$ which means $h_f \in P \cap \Omega_{2,0}$. Notice that φ_f is also decreasing and Proposition 4.1 implies that $\varphi_f \prec h_f$. Thus

$$\frac{\|\varphi_f\|_{q/2,u}}{\|h_f\|_{p/2,v}} \leq \sup_{h \in P \cap \Omega_{2,0}} \sup_{0 \leq \varphi \downarrow, \varphi \prec h} \frac{\|\varphi\|_{q/2,u}}{\|h\|_{p/2,v}} = C_1.$$

This implies

$$\left(\int_0^\infty \varphi_f(t)^{q/2} u(t)\, dt\right)^{2/q} \leq C_1 \left(\int_0^\infty h_f(t)^{p/2} v(t)\, dt\right)^{2/p},$$

or equivalently

$$\left(\int_0^\infty \hat{f}^{**}(t)^q u(t)\, dt\right)^{2/q} \leq 16 C_1 \left(\int_0^\infty \left(\int_0^{1/t} f^*(s)\, ds\right)^p v(t)\, dt\right)^{2/p}.$$

Taking the square root of both sides, using $v(t) = t^{p-2} w(1/t)$, and making the change of variable $t \to 1/t$ we obtain

$$\left(\int_0^\infty \hat{f}^{**}(t)^q u(t)\, dt\right)^{1/q} \leq 4\sqrt{C_1} \left(\int_0^\infty f^{**}(t)^p w(t)\, dt\right)^{1/p},$$

which is the desired assertion with $C = 4\sqrt{C_1}$. $\square$

It is possible to state the condition (4.3) in term of the Θ space norm.

Corollary 4.3. *Suppose $0 < p, q < \infty$ and $u, v, w \in L^+$ with $v(t) = t^{p-2} w(1/t)$. If*

$$\sup_{h \in P \cap \Omega_{2,0}} \frac{\|h\|_{\Theta^{q/2}(u)}}{\|h\|_{p/2,v}} < \infty,$$

then there exists $C > 0$ such that

$$\|\hat{f}\|_{\Gamma^q(u)} \leq C \|f\|_{\Gamma^p(w)}$$

for all $f \in L^1(\mathbb{T})$.

Proof. This follows from (1.11) where we defined the Θ space. $\square$

If we restrict ourselves to $q \geq 2$ we can write the condition (4.3) in terms of averaging operators.

Theorem 4.4. *Suppose $0 < p < \infty$, $q \geq 2$ and $u, v, w \in L^+$ with $v(t) = t^{p-2} w(1/t)$. If*

$$\sup_{h \in P \cap \Omega_{2,0}} \sup_{A \in \mathcal{A}} \frac{\|Ah\|_{q/2,u}}{\|h\|_{p/2,v}} < \infty, \tag{4.4}$$

then there exists $C > 0$ such that

$$\|\hat{f}\|_{\Gamma^q(u)} \leq C \|f\|_{\Gamma^p(w)}$$

for all $f \in L^1(\mathbb{T})$.

Proof. Since each $h \in P \cap \Omega_{2,0}$ is decreasing and $q \geq 2$, Corollary 1.32 asserts that

$$\sup_{0 \leq \varphi\downarrow, \varphi \prec h} \|\varphi\|_{q/2,u} = \sup_{A \in \mathcal{A}} \|Ah\|_{q/2,u},$$

and the result follows from Theorem 4.2. □

Now the results of Chapter 2 enable us to simplify the condition (4.4) as stated in next theorem. Recall that $\omega_z(t) = (z^{-2}, t^{-2})$.

Theorem 4.5. *Let $0 < p \leq 2 \leq q < \infty$ and $u, v, w \in L^+$ with $v(t) = t^{p-2}w(1/t)$. If*

$$\sup_{z>1, A \in \mathcal{A}} \frac{\|A\omega_z\|_{q/2,u}}{\|\omega_z\|_{p/2,v}} < \infty \tag{4.5}$$

then there exists $C > 0$ such that

$$\|\hat{f}\|_{\Gamma^q(u)} \leq C\|f\|_{\Gamma_p(w)} \tag{4.6}$$

for all $f \in L^1(\mathbb{T})$.

Proof. Using Corollary 2.11 and taking the supremum over all $A \in \mathcal{A}$ we get

$$\sup_{A \in \mathcal{A}} \sup_{f \in P \cap \Omega_{2,0}} \frac{\|Af\|_{q/2,u}}{\|f\|_{p/2,v}} \approx \sup_{z>1, A \in \mathcal{A}} \frac{\|A\omega_z\|_{q/2,u}}{\|\omega_z\|_{p/2,v}} < \infty.$$

So the existence of such C is guaranteed by Theorem 4.4. □

Corollary 4.6. *Let $0 < p \leq 2 \leq q < \infty$ and $u, v, w \in L^+$ with $v(t) = t^{p-2}w(1/t)$. If*

$$\sup_{z>1} \frac{\|\omega_z\|_{\Theta^{q/2}(u)}}{\|\omega_z\|_{p/2,v}} < \infty$$

then there exists $C > 0$ such that

$$\|\hat{f}\|_{\Gamma^q(u)} \leq C\|f\|_{\Gamma^p(w)}$$

for all $f \in L^1(\mathbb{T})$.

Proof. Since ω_z is decreasing and $q \geq 2$, we can use (1.12) to re-write the Θ space norm. Then the assertion follows from Theorem 4.5 □

In the following theorem, we incorporate the level function of u to obtain a weaker result. However the condition is much easier to verify because the supremum is taken over a one parameter family of functions, namely $z > 1$. Thus standard calculus arguments may be used to verify it. Moreover we will obtain important sufficient conditions (Theorems 4.9 and 4.10) in the same way we did for the Fourier transform.

Theorem 4.7. *Suppose $0 < p \le q < \infty$, $2 \le q$ and $u, v, w \in L^+$ with $v(t) = t^{p-2}w(1/t)$. If*

$$\sup_{z>1} \frac{\|\omega_z\|_{q/2,u^\circ}}{\|\omega_z\|_{p/2,v}} < \infty$$

then there exists $C > 0$ such that

$$\|\hat{f}\|_{\Gamma^q(u)} \le C\|f\|_{\Gamma^p(w)}$$

for all $f \in L^1(\mathbb{T})$.

Proof. Corollaries 1.32 and 2.12 imply

$$\sup_{h\in P\cap\Omega_{2,0}} \sup_{A\in\mathcal{A}} \frac{\|Ah\|_{q/2,u}}{\|h\|_{p/2,v}} \le \sup_{h\in P\cap\Omega_{2,0}} \frac{\|h\|_{q/2,u^\circ}}{\|h\|_{p/2,v}} \approx \sup_{z>1} \frac{\|\omega_z\|_{q/2,u^\circ}}{\|\omega_z\|_{p/2,v}} < \infty.$$

So the condition in Theorem 4.4 is satisfied and the proof is complete. □

For our next results in this section we require some inequalities from Chapter 3. Recall that

$$F_{q,u}(z) = \left(\|\omega_z\|_{q/2,u}\right)^{1/2}, \quad G_{p,w}(z) = \left(\|\omega_z\|_{p/2,v}\right)^{1/2} \quad \text{and} \quad v(t) = t^{p-2}w(1/t).$$

In the theorem below, we replace the integral of the level function with an equivalent expression that is sometimes more convenient.

Theorem 4.8. *Suppose $0 < p \le q < \infty$, $2 \le q$ and $u, w \in L^+$. If,*

$$\sup_{z>1} \left(\sup_{y\ge z} \frac{z}{y}\int_0^y u(t)\,dt\right)^{1/q} \left(\int_{1/z}^\infty \frac{w(t)}{t^p}\,dt + z^p\int_0^{1/z} w(t)\,dt\right)^{-1/p} < \infty$$

then there exists $C > 0$ such that

$$\|\hat{f}\|_{\Gamma^q(u)} \le C\|f\|_{\Gamma^p(w)}$$

for all $f \in L^1(\mathbb{T})$.

Proof. Observe, by estimate (3.4), that we have

$$F_{q,u^\circ}(z) \approx z^{-1}\left(\int_0^z u^\circ(t)\,dt\right)^{1/q},$$

which by Proposition 1.33, turns into

$$F_{q,u^\circ}(z) \approx z^{-1}\left(\sup_{y\ge z}\frac{z}{y}\int_0^y u(t)\,dt\right)^{1/q}.$$

On the other hand, Equation (3.5) asserts

$$G_{p,w}(z) = \left(z^{-p}\int_{1/z}^\infty \frac{w(t)}{t^p}\,dt + \int_0^{1/z} w(t)\,dt\right)^{1/p}.$$

Therefore,

$$\sup_{z>1} \frac{F_{q,u^\circ}(z)}{G_{p,w}(z)} \approx \sup_{z>1} z^{-1} \left(\sup_{y\geq z} \frac{z}{y} \int_0^y u(t)\,dt \right)^{1/q} \left(z^{-p} \int_{1/z}^\infty \frac{w(t)}{t^p}\,dt + \int_0^{1/z} w(t)\,dt \right)^{-1/p}$$
$$= \sup_{z>1} \left(\sup_{y\geq z} \frac{z}{y} \int_0^y u(t)\,dt \right)^{1/q} \left(\int_{1/z}^\infty \frac{w(t)}{t^p}\,dt + z^p \int_0^{1/z} w(t)\,dt \right)^{-1/p},$$

which is finite by the hypothesis. This means $\sup_{z>1} \dfrac{\|\omega_z\|_{q/2,u^\circ}}{\|\omega_z\|_{p/2,v}}$ is finite and the proof is complete by Theorem 4.7. $\square$

Now we use Theorem 4.7 to obtain two more sufficient conditions. These conditions are stronger but easier to use compared to Theorem 4.7.

Theorem 4.9. *Let $0 < p \leq q < \infty$ and $2 \leq q$. Assume u and w are weight functions on $(0,\infty)$. If*

$$\sup_{0<x<1} x \left(\int_0^{1/x} u^\circ(t)\,dt \right)^{1/q} \left(\int_0^x w(t)\,dt \right)^{-1/p} < \infty,$$

then there exists $C > 0$ such that

$$\|\hat{f}\|_{\Gamma^q(u)} \leq C\|f\|_{\Gamma^p(w)}$$

for all $f \in L^1(\mathbb{T})$.

Proof. The proof is the same as in Theorem 3.11. Since u° is decreasing, (3.4) yields

$$\left(\|\omega_z\|_{q/2,u^\circ}\right)^{1/2} = F_{q,u^\circ}(z) \approx z^{-1} \left(\int_0^z u^\circ(t)\,dt \right)^{1/q}.$$

Moreover, Inequality (3.7) implies

$$\left(\|\omega_z\|_{p/2,v}\right)^{1/2} = G_{p,w}(z) \geq \left(\int_0^{1/z} w(t)\,dt \right)^{-1/p}.$$

The above estimates together with $z = 1/x$ shows that

$$\sup_{z>1} \frac{F_{q,u^\circ}(z)}{G_{p,w}(z)} \leq \sup_{z>1} z^{-1} \left(\int_0^z u^\circ(t)\,dt \right)^{1/q} \left(\int_0^{1/z} w(t)\,dt \right)^{-1/p}$$
$$= \sup_{0<x<1} x \left(\int_0^{1/x} u^\circ(t)\,dt \right)^{1/q} \left(\int_0^x w(t)\,dt \right)^{-1/p},$$

which is finite by hypothesis. Therefore, $\sup_{z>1} \dfrac{\|\omega_z\|_{q/2,u^\circ}}{\|\omega_z\|_{p/2,v}}$ is finite and the proof is complete by Theorem 4.7. $\square$

Theorem 4.10. *Let $0 < p \le q < \infty$ and $2 \le q$. Assume u and w are weight functions on $(0, \infty)$. If*

$$\sup_{0<x<1} \left(\int_0^{1/x} u^\circ(t)\, dt \right)^{1/q} \left(\int_x^\infty \frac{w(t)}{t^p}\, dt \right)^{-1/p} < \infty,$$

then there exists $C > 0$ such that

$$\|\hat{f}\|_{\Gamma^q(u)} \le C \|f\|_{\Gamma^p(w)}$$

for all $f \in L^1(\mathbb{T})$.

Proof. The proof is the same as in Theorem 3.16. Since u° is decreasing, (3.4) yields

$$\left(\|\omega_z\|_{q/2,u}\right)^{1/2} = F_{q,u}(z) \approx z^{-1} \left(\int_0^z u^\circ(t)\, dt \right)^{1/q}.$$

Moreover, (3.6) implies

$$\left(\|\omega_z\|_{p/2,v}\right)^{1/2} = G_{p,w}(z) \ge z^{-1} \left(\int_{1/z}^\infty \frac{w(t)}{t^p}\, dt \right)^{1/p}.$$

The above estimates, together with $z = 1/x$, show that

$$\begin{aligned} \sup_{z>1} \frac{F_{q,u^\circ}(z)}{G_{p,w}(z)} &\le \sup_{z>1}\ z^{-1} \left(\int_0^z u^\circ(t)\, dt \right)^{1/q} .z \left(\int_{1/z}^\infty \frac{w(t)}{t^p}\, dt \right)^{-1/p} \\ &= \sup_{0<x<1} \left(\int_0^{1/x} u^\circ(t)\, dt \right)^{1/q} \left(\int_x^\infty \frac{w(t)}{t^p}\, dt \right)^{-1/p} < \infty, \end{aligned}$$

which is finite by hypothesis. Therefore, $\sup_{z>1} \dfrac{\|\omega_z\|_{q/2,u^\circ}}{\|\omega_z\|_{p/2,v}}$ is finite and the proof is complete by Theorem 4.7. □

Remark 4.11. Since $\|\hat{f}\|_{\Lambda^q(u)} \le \|\hat{f}\|_{\Gamma^q(u)}$, Theorems 4.2 to 4.10 provide sufficient conditions for $\|\hat{f}\|_{\Lambda^q(u)} \le C\|f\|_{\Gamma^p(w)}$.

The next theorem gives a sufficient condition for boundedness of the Fourier coefficient map $\mathcal{F} : \Lambda^p(w) \to \Lambda^q(u)$. We will use this theorem to generate Fourier series inequalities with weighted Lebesgue norms in Chapter 5.

Theorem 4.12. *Let $1 < p \le q < \infty$ and $2 \le q$. Assume u and w are weight functions on $(0, \infty)$. If*

$$\sup_{0<x<1} \left(\int_0^{1/x} u^\circ(t)\, dt \right)^{1/q} \left(\int_0^x w(t)^{1-p'}\, dt \right)^{1/p'} < \infty, \tag{4.7}$$

then there exists $C > 0$ such that

$$\|\hat{f}\|_{\Lambda^q(u)} \le C \|f\|_{\Lambda^p(w)}$$

for all $f \in L^1(\mathbb{T})$.

Proof. First, notice that $w(t)$ is finite a.e. since it is a weight function and therefore locally integrable. Thus, $w(t)^{1-p'} > 0$ a.e., which means $\int_0^x w(t)^{1-p'}\,dt > 0$ for all $x > 0$. Hence, (4.7) implies that $\int_0^{1/x} u^\circ(t)\,dt < \infty$ for $x \in (0,1)$. Set $\sigma(t) = t^{q-2}u^\circ(1/t)$ and observe that

$$-\int_a^x \frac{\sigma(t)}{t^q}\,dt = \int_0^{1/x} u^\circ(t)\,dt - \int_0^{1/a} u^\circ(t)\,dt.$$

Let $a \to \infty$ to obtain

$$\int_x^\infty \frac{\sigma(t)}{t^q}\,dt = \int_0^{1/x} u^\circ(t)\,dt. \tag{4.8}$$

This implies

$$\sup_{x>0} \left(\int_0^{1/x} u^\circ(t)\,dt\right)^{1/q} \left(\int_x^\infty \frac{\sigma(t)}{t^q}\,dt\right)^{-1/q} = 1 < \infty.$$

So the condition in Theorem 4.10, with $w(t)$ and p replaced with $\sigma(t)$ and q, is satisfied. It follows that there exists $C_1 >$ so that $\|\hat{f}\|_{\Gamma^q(u)} \le C_1\|f\|_{\Gamma^q(\sigma)}$ for all $f \in L^1(\mathbb{T})$. From $\|\hat{f}\|_{\Lambda^q(u)} \le \|\hat{f}\|_{\Gamma^q(u)}$ we have the inequality

$$\|\hat{f}\|_{\Lambda^q(u)} \le C_1\|f\|_{\Gamma^q(\sigma)} \tag{4.9}$$

for all $f \in L^1(\mathbb{T})$.

On the other hand, (4.7) and (4.8) show that

$$\sup_{0<x<1} \left(\int_x^\infty \frac{\sigma(t)}{t^q}\,dt\right)^{1/q} \left(\int_0^x w(t)^{1-p'}\,dt\right)^{1/p'} = M_1 < \infty. \tag{4.10}$$

By continuity,

$$\left(\int_1^\infty \frac{\sigma(t)}{t^q}\,dt\right)^{1/q} \left(\int_0^1 w(t)^{1-p'}\,dt\right)^{1/p'} \le M_1.$$

Now define

$$\tilde{w}(t) = \begin{cases} w(t), & 0 < t < 1, \\ t^{2/(p'-1)}, & t \ge 1. \end{cases}$$

For $x \ge 1$ we have

$$\left(\int_0^x \tilde{w}(t)^{1-p'}\,dt\right)^{1/p'} = \left(\int_0^1 w(t)^{1-p'}\,dt + \int_1^x t^{-2}\,dt\right)^{1/p'} \le \left(\int_0^1 w(t)^{1-p'}\,dt\right)^{1/p'} + 1,$$

which implies

$$\begin{aligned}\left(\int_x^\infty \frac{\sigma(t)}{t^q}\,dt\right)^{1/q} \left(\int_0^x \tilde{w}(t)^{1-p'}\,dt\right)^{1/p'} &\le \left(\int_1^\infty \frac{\sigma(t)}{t^q}\,dt\right)^{1/q} \left(\left(\int_0^1 w(t)^{1-p'}\,dt\right)^{1/p'} + 1\right) \\ &\le M_1 + \left(\int_1^\infty \frac{\sigma(t)}{t^q}\,dt\right)^{1/q}.\end{aligned}$$

So we showed

$$\sup_{x\geq 1}\left(\int_x^\infty \frac{\sigma(t)}{t^q}\,dt\right)^{1/q}\left(\int_0^x \tilde{w}(t)^{1-p'}\,dt\right)^{1/p'}<\infty. \tag{4.11}$$

The definition of $\tilde{w}$ together with (4.10) and (4.11) imply

$$\sup_{x>0}\left(\int_x^\infty \frac{\sigma(t)}{t^q}\,dt\right)^{1/q}\left(\int_0^x \tilde{w}(t)^{1-p'}\,dt\right)^{1/p'}<\infty.$$

So the condition for the weighted Hardy inequality (Theorem 1.15) holds. It follows that there exists $C_2>0$ such that

$$\left(\int_0^\infty \left(\frac{1}{t}\int_0^t g(s)\,ds\right)^q \sigma(t)\,dt\right)^{1/q}\leq C_2\left(\int_0^\infty g(t)^p\tilde{w}(t)\,dt\right)^{1/p}$$

for all $g\in L^+$. Replacing g with f^* in the above inequality we get

$$\left(\int_0^\infty \left(\frac{1}{t}\int_0^t f^*(s)\,ds\right)^q \sigma(t)\,dt\right)^{1/q}\leq C_2\left(\int_0^\infty f^*(t)^p\tilde{w}(t)\,dt\right)^{1/p}.$$

Notice that the function f^* is supported in $[0,1)$ since it is defined on $\mathbb{T}$. Therefore, the right hand side of the above inequality does not change if we replace $\tilde{w}$ with w. This means

$$\|f\|_{\Gamma^q(\sigma)}\leq C_2\|f\|_{\Lambda^p(w)} \tag{4.12}$$

for all $f\in L^+$.

Finally (4.9) and (4.12) show that $\|\hat{f}\|_{\Lambda^q(u)}\leq C_1\|f\|_{\Gamma^q(\sigma)}\leq C_1C_2\|f\|_{\Lambda^p(w)}$ for all $f\in L^1(\mathbb{T})$. The proof is complete by taking $C=C_1C_2$ □

4.2 Construction of test functions

In this section we construct the collection of test functions that provides our necessary condition for Fourier inequalities of type $\|\hat{f}\|_{\Lambda^q(u)}\leq C\|f\|_{\Gamma^p(w)}$. As we will see for a certain range of indices the necessary conditions coincides with our sufficient condition. The approach to generate the test functions is inspired from Sinnamon's work in [Si4]. However the details are substantially different because of the finite measure on $\mathbb{T}$ and atomic measure on $\mathbb{Z}$.

Throughout this section μ denotes counting measure on $\mathbb{Z}$ and for computations in $\mathbb{T}$ we use $\mathbb{T}\cong\mathbb{R}/\mathbb{Z}$. We start with the following lemma, which corresponds to Lemma 4.1 and Corollary 4.2 in [Si4]. It provides a rearrangement estimate for Fourier coefficients of the pulse function. We prove this lemma and its consequences for $z\geq 3$ and in Theorem 4.18 we extend our result to $z>1$. The particular choice of "3", is to make the constant $c>0$ in Theorem 4.17 smaller.

Lemma 4.13. *Assume $z \geq 3$ and let $f(x) = \chi_{(0,1/z)}(x)$ for $x \in \mathbb{T}$. Then $\hat{f}^*(y) \geq \dfrac{1}{3\pi y + 9\pi z}$.*

Proof. The Fourier coefficients of f are computed as:

$$\hat{f}(n) = \begin{cases} \dfrac{e^{-in\pi/z}}{n\pi} \sin \dfrac{n\pi}{z}, & n \neq 0, \\ \\ \dfrac{1}{z}, & n = 0. \end{cases}$$

To find an estimate for the rearrangement of $\hat{f}$ we first need an estimate for the distribution function $\mu_{\hat{f}}$ where μ is counting measure on $\mathbb{Z}$. Assume $\alpha > 0$. Then

$$\begin{aligned}
\mu_{\hat{f}}(\alpha) &= \mu\{k \in \mathbb{Z},\ |\hat{f}(k)| > \alpha\} \\
&\geq \mu\{k \in \mathbb{Z} \setminus \{0\},\ |(1/k\pi)\sin(k\pi/z)| > \alpha\} \\
&\geq 2\mu\{0 < k \in \mathbb{Z} : |\sin(k\pi/z)| > k\pi\alpha\} \\
&= 2\sum_{n=1}^{\infty} \mu\{k \in \mathbb{Z}, n-1 < k/z \leq n : |\sin(k\pi/z)| > k\pi\alpha\} \\
&\geq 2\sum_{n=1}^{\infty} \mu\{k \in \mathbb{Z}, n-1 < k/z \leq n : |\sin(k\pi/z)| > zn\pi\alpha\} \\
&= 2\sum_{n=1}^{\infty} \mu(E_n)
\end{aligned}$$

where

$$E_n = \{k \in \mathbb{Z}, n-1 < k/z \leq n : |\sin(k\pi/z)| > zn\pi\alpha\}.$$

Let N be the (unique) integer satisfying $1/(z\pi\alpha) - 1 < N \leq 1/(z\pi\alpha)$. Then $n \geq N+1$ implies $zn\pi\alpha > 1$ which means $E_n = \emptyset$. Therefore,

$$\mu_{\hat{f}}(\alpha) \geq 2\sum_{n=1}^{N} \mu(E_n)$$

On each interval $((n-1)\pi\,,\, n\pi)$, the function $x \mapsto 1 - |(2/\pi)x - (2n-1)|$ consists of two line segments. By the concavity of $|\sin(x)|$ on such intervals we have $|\sin(x)| \geq 1 - |(2/\pi)x - (2n-1)|$. Let $x = k\pi/z$ to obtain

$$\begin{aligned}
\mu(E_n) &\geq \mu\{k \in \mathbb{Z}, n-1 < k/z \leq n : 1 - |2k/z - (2n-1)| > zn\pi\alpha\} \\
&= \mu\{k \in \mathbb{Z}, n-1 < k/z \leq n : n-1 + zn\pi\alpha/2 < k/z < n - zn\pi\alpha/2\}.
\end{aligned}$$

Since the interval $(n-1, n)$ contains the interval $(n-1+zn\pi\alpha/2, n-zn\pi\alpha/2)$ we get

$$\begin{aligned}
\mu(E_n) &\geq \mu\{k \in \mathbb{Z}, n-1+zn\pi\alpha/2 < \frac{k}{z} < n - zn\pi\alpha/2\} \\
&= \mu\{k \in \mathbb{Z}, nz - z + z^2 n\pi\alpha/2 < k < nz - z^2 n\pi\alpha/2\}.
\end{aligned}$$

Notice that in the above inequality, the quantity on the right-hand side is the number of integers in the interval $(nz - z + z^2 n\pi\alpha/2, nz - z^2 n\pi\alpha/2)$. In general the number of integers in an interval of length L is equal to either L or $L-1$. Hence,

$$\mu(E_n) \geq z - z^2 n\pi\alpha - 1.$$

Now, taking the sum over n, we get

$$\begin{aligned}
\mu_{\hat{f}}(\alpha) &\geq 2\sum_{n=1}^{N}\big(z - z^2 n\pi\alpha - 1\big) \\
&= 2N(z-1) - z^2\pi\alpha N(N+1) \\
&\geq 2(1/(z\pi\alpha) - 1)(z-1) - z^2\pi\alpha(1/(z\pi\alpha))(1/(z\pi\alpha)+1) \\
&= 1/(\pi\alpha) - 2/(z\pi\alpha) - 3z + 2 \\
&\geq (1/(\pi\alpha))(1 - 2/z) - 3z.
\end{aligned}$$

The hypothesis $z \geq 3$ implies $\mu_{\hat{f}}(\alpha) \geq 1/(3\pi\alpha) - 3z$.

Finally for $y > 0$ let $\alpha = \hat{f}^*(y)$ in the above inequality. By properties of rearrangement we have $\mu_{\hat{f}}(\hat{f}^*(y)) \leq y$. Thus

$$y \geq \frac{1}{3\pi\hat{f}^*(y)} - 3z,$$

which yields

$$\hat{f}^*(y) \geq \frac{1}{3\pi y + 9\pi z}.$$

□

The next lemma is analogous to Lemma 4.3 in [Si4] which was stated in a general sense. That generality does not work here because of finiteness of measure on $\mathbb{T}$. Therefore, we restrict ourselves to those functions that we need later.

Lemma 4.14. *Assume k is a positive integer and $z > 1$. Let $f(x) = \chi_{[0,1/(kz))}(x)$. Then for any $\varepsilon > 0$ there exists a function $g \in L^1(\mathbb{T})$ such that*

$$g^*(s) = f^*(s/k) \quad \textit{and} \quad \hat{g}^*(y) \geq \hat{f}^*(y/k) - \varepsilon$$

for $0 \leq s < 1$ and $y > 0$.

Proof. We show that for a sufficiently large integer M,

$$g(x) = \sum_{j=0}^{k-1} e^{2\pi i j M x} f\big(x - j/(kz)\big)$$

would be the desired function.

First, notice that the supports of translations of f in the sum above don't overlap. So

$$|g(x)| = \sum_{j=0}^{k-1}\left|e^{2\pi i j M x} f(x - j/(kz))\right| = \sum_{j=0}^{k-1} f(x - j/(kz)) = \chi_{[0,1/z)}(x).$$

Therefore, g is well defined on $\mathbb{T}$ because $1/z < 1$. Furthermore, $|g(s)|$ and $f(s)$ are both decreasing functions for $0 \le s < 1$. Hence,

$$g^*(s) = |g(s)| = \chi_{[0,1/z)}(s) = f(s/k) = f^*(s/k).$$

Recall the translation properties of Fourier coefficients from Chapter 1. If $h(x)$ is a function on the unit circle, $x, x_0 \in [0,1)$ and $n, n_0 \in \mathbb{Z}$, then

$$h_1(x) = e^{2\pi i n_0 x} f(x) \quad \text{and} \quad h_2(x) = f(x - x_0)$$

imply

$$\hat{h}_1(n) = \hat{f}(n - n_0) \quad \text{and} \quad \hat{h}_2(n) = e^{-2\pi i n x_0} \hat{f}(n).$$

Making use of these two properties we get

$$\hat{g}(n) = \sum_{j=0}^{k-1} e^{-2\pi i (n - jM) j/(kz)} \hat{f}(n - jM).$$

Choose M such that $M > 2k/(\pi\varepsilon)$. For all n satisfying $|n| \ge M/2$ we have

$$|\hat{f}(n)| = \left| \frac{e^{in\pi/kz}}{n\pi} \sin(n\pi/z) \right| \le 1/n\pi < \varepsilon/k.$$

Assume $n \in (jM - M/2, jM + M/2)$ for some $j \in \{0, 1, \ldots, k-1\}$. Then $|n - lM| > M/2$ for $l \ne j$, and we have

$$\begin{aligned}
|\hat{g}(n) &\ge |\hat{f}(n - jM)| - \left| \sum_{l \ne j} e^{-2\pi i (n - lM) l/(kz)} \hat{f}(n - lM) \right| \\
&\ge |\hat{f}(n - jM)| - (k-1)\varepsilon/k \\
&\ge |\hat{f}(n - jM)| - \varepsilon.
\end{aligned}$$

Now we can estimate the distribution function of $\hat{g}$, with respect to the counting measure μ. For $\alpha > 0$,

$$\begin{aligned}
\mu_{\hat{g}}(\alpha) &= \mu\{n \in \mathbb{Z} : |\hat{g}(n)| > \alpha\} \\
&\ge \sum_{j=0}^{k-1} \mu\{n \in (jM - M/2, jM + M/2) : |\hat{g}(n)| > \alpha\} \\
&\ge \sum_{j=0}^{k-1} \mu\{n \in (jM - M/2, jM + M/2) : |\hat{f}(n - jM)| - \varepsilon > \alpha\} \\
&= k\mu\{n \in (-M/2, M/2) : |\hat{f}(n)| > \alpha + \varepsilon\}.
\end{aligned}$$

Note that $n \notin (-M/2, M/2)$ implies $|\hat{f}(n)| < \varepsilon$. Hence,

$$\mu_{\hat{g}}(\alpha) \ge k\mu\{n \in \mathbb{Z} : |\hat{f}(n)| > \alpha + \varepsilon\} = k\mu_{\hat{f}}(\alpha + \varepsilon).$$

Finally, the properties of rearrangement and distribution functions imply $\mu_f(f^*(y)) \leq y$ and $f^*(\mu_f(\alpha)) \leq \alpha$. So for $y > 0$,

$$y \geq \mu_{\hat{g}}(\hat{g}^*(y)) \geq k\mu_{\hat{f}}(\hat{g}^*(y) + \varepsilon).$$

Since $\hat{f}^*$ is a decreasing function, we have

$$\hat{f}^*(y/k) \leq \hat{f}^*\left(\mu_f\left(\hat{g}(y) + \varepsilon\right)\right) \leq \hat{g}^*(y) + \varepsilon.$$

Thus $\hat{g}^*(y) \geq \hat{f}^*(y/k) - \varepsilon$. □

We combine the last two lemmas to obtain an estimate for the Fourier coefficients of a family of functions equimeasurable with $\chi_{[0,1/z)}$.

Corollary 4.15. *For $z \geq 3$, $r > 0$ and $\varepsilon > 0$ there exists $g \in L^1(\mathbb{T})$ such that*

$$g^* = \chi_{[0,1/z)} \quad \textit{and} \quad \hat{g}^*(y) \geq \frac{1}{3\pi y/r + 9\pi(r+1)z} - \varepsilon.$$

Proof. Let k be the integer satisfying $r \leq k < r + 1$ and let $f = \chi_{[0,1/kz)}$. Then by Lemma 4.14 there exists g such that

$$g^*(s) = f^*(s/k) = \chi_{[0,1/z)} \quad \text{and} \quad \hat{g}^*(y) \geq \hat{f}^*(y/k) - \varepsilon.$$

Lemma 4.13 yields

$$\hat{g}^*(y) \geq \frac{1}{3\pi y/k + 9\pi kz} - \varepsilon \geq \frac{1}{3\pi y/r + 9\pi(r+1)z} - \varepsilon.$$

□

The next lemma generalizes Lemma 4.14 to a possibly infinite number of functions. However the estimate for $\hat{g}^*$ in Lemma 4.14 is sharper than the estimate in this more general setting.

Lemma 4.16. *Let $\{p_j\}$ be a sequence of non-negative real numbers satisfying $\sum_{j=1}^{\infty} p_j = p_0 < 1$. For each p_j let $f_j = \chi_{[0,p_j)}$. Then for any $\varepsilon > 0$ there exists $g \in L^1(\mathbb{T})$ such that*

$$g^* = \chi_{[0,p_0)} \quad \textit{and} \quad \hat{g}^*(y) \geq \hat{f}_j^*(y) - \varepsilon, \quad j = 1, 2, \ldots.$$

Proof. Let $X_1 = 0$ and $X_j = \sum_{l=1}^{j-1} p_l$ for $j \geq 2$. Define g as

$$g(x) = \sum_{j=1}^{\infty} e^{2\pi i M_j x} f_j(x - X_j),$$

where the M_j's are to be chosen later. The numbers X_j were defined so that the supports of the translated functions $f_j(x - X_j)$ don't overlap. It is readily seen that

$$|g(x)| = \sum_{j=1}^{\infty} |e^{2\pi i M_j x} f_j(x - X_j)| = \chi_{[0,p_0)}.$$

So g is well defined on S. In addition $g^* = \chi_{[0,p_0)}$. The Fourier coefficients of g are given by:

$$\hat{g}(n) = \sum_{j=1}^{\infty} e^{-2\pi i(n-M_j)X_j} \hat{f}_j(n - M_j).$$

Note that the series defining $\hat{g}(n)$ converges for all n, since $g \in L^1(\mathbb{T})$.

For each j choose R_j such that $R_j \geq 2^j/(\pi\varepsilon)$. It follows that for all $|n| \geq R_j$,

$$|\hat{f}_j(n)| = 1/(n\pi)|\sin(n\pi p_j)| \leq 1/(n\pi) < \varepsilon 2^{-j}.$$

Now set $M_1 = 0$ and assume $M_1, M_2, ..., M_{j-1}$ are chosen positive integers. Then choose a positive integer M_j such that the interval $(M_j - R_j, M_j + R_j)$ does not intersect intervals $(M_i - R_i, M_i + R_i)$ for $i = 1, 2, ..., j-1$. For each $n \in (M_j - R_j, M_j + R_j)$ we have

$$|\hat{g}(n)| \geq |\hat{f}_j(n - M_j)| - \sum_{l \neq j} |\hat{f}_l(n - M_l)| \geq |\hat{f}_j(n - M_j)| - \sum_{l \neq j} \varepsilon 2^{-j} \geq \hat{f}_j(n - M_j) - \varepsilon.$$

For $\alpha > 0$,

$$\begin{aligned}
\mu_{\hat{g}}(\alpha) &= \mu\{n \in \mathbb{Z} : |\hat{g}(n)| > \alpha\} \\
&\geq \sum_{j=1}^{\infty} \mu\{n \in (M_j - R_j, M_j + R_j) : |\hat{g}(n)| > \alpha\} \\
&\geq \sum_{j=1}^{\infty} \mu\{n \in (M_j - R_j, M_j + R_j) : |\hat{f}_j(n - M_j)| - \varepsilon > \alpha\} \\
&= \sum_{j=1}^{\infty} \mu\{n \in (-R_j, R_j) : |\hat{f}_j(n)| - \varepsilon > \alpha\} \\
&\geq \sup_j \mu\{n \in (-R_j, R_j) : |\hat{f}_j(n)| - \varepsilon > \alpha\} \\
&= \sup_j \mu\{n : |\hat{f}_j(n)| > \alpha + \varepsilon\} \\
&= \sup_j \mu_{f_j}(\alpha + \varepsilon).
\end{aligned}$$

Here we used the fact that $|\hat{f}_j(n)| < \varepsilon$ for $n \notin (-R_j, R_j)$. Finally, a similar argument as in Lemma 4.14 implies that $\hat{g}^*(y) \geq \hat{f}_j^{\,*}(y) - \varepsilon$. □

Now we are ready to prove the main result of this section which will be used in proving our necessary conditions. First we get the result for $z \geq 3$ and then we extend it to $z > 1$. This theorem comparable to Theorem 4.6 in [Si4] which deals with the Fourier transform on $\mathbb{R}^n$.

Theorem 4.17. *Let $z \geq 3$ and $A \in \mathcal{A}$. For each $\varepsilon > 0$ there exists $f \in L^1(\mathbb{T})$ such that*

$$f^* \leq \chi_{[0,1/z)} \quad \textit{and} \quad (A\omega_z)^{1/2} \leq c_1(\hat{f}^* + \varepsilon),$$

with $c_1 = 320$.

Proof. Let $\{(a_i, b_i)\}$ be the collection of intervals associated to A. Let (a_0, b_0) be the interval containing z if there is one. If there is no such interval we set $a_0 = b_0 = 1$. Choose $y \geq 0$. There are 3 possible cases.

Case 1: $y \notin (a_0, b_0)$ and $A\omega_z(y) \leq 2\omega_z(y)$. Let $f_0 = \chi_{[0,1/4z)}$. Then $f_0^* = \chi_{[0,1/4z)}$ and by Lemma 4.13 we have

$$f_0^*(y) \geq (3\pi y + 36\pi z)^{-1} \geq (39\pi \max(z, y))^{-1} = (39\pi)^{-1} \min(z^{-1}, y^{-1}).$$

It follows that

$$A\omega_z(y)^{1/2} \leq \sqrt{2} \min(z^{-1}, y^{-1}) \leq 39\sqrt{2}\pi f_0^*(y) \leq c_1 f_0^*(y).$$

So in this case

$$(A\omega_z)(y)^{1/2} \leq c_1(\hat{f_0}^*(y) + \varepsilon/2). \tag{4.13}$$

Case 2: $y \in (a_0, b_0)$. Note that this case does not occur if none of the intervals associated to A contains z. Invoke Corollary 4.15 with $r_0 = \sqrt{b_0/8z}$ and $z_0 = 8z/3$. There exists a function g_0 such that $g_0^* = \chi_{[0,3/8z)}$ and

$$\begin{aligned}
\hat{g_0}^*(y) + \varepsilon/2 &\geq (3\pi y/r_0 + 9\pi(r_0 + 1)z_0)^{-1} \\
&= \left(3\pi y\sqrt{8z/b_0} + 9\pi(\sqrt{b_0/8z} + 1)(8z/3)\right)^{-1} \\
&= (b_0 z)^{-1/2}\left(6\pi\sqrt{2}(y/b_0) + 6\pi\sqrt{2} + 24\pi\sqrt{z/b_0}\right)^{-1}.
\end{aligned}$$

Since z and y lie in the interval (a_0, b_0) we conclude that both y/b_0 and z/b_0 are less than one. Therefore,

$$\hat{g_0}^*(y) + \varepsilon/2 \geq \frac{1}{\sqrt{b_0 z}}\left(12\sqrt{2}\pi + 24\pi\right)^{-1}.$$

On the other hand, since ω_z is decreasing we get

$$A\omega_z(y)^{1/2} = \left(\frac{1}{b_0 - a_0}\int_{a_0}^{b_0} \omega_z\right)^{1/2} \leq \left(\frac{1}{b_0}\int_0^{b_0} \omega_z\right)^{1/2} \leq \left(\frac{1}{b_0}\int_0^{\infty} \omega_z\right)^{1/2} = \frac{\sqrt{2}}{\sqrt{b_0 z}}.$$

It follows that

$$A\omega_z(y)^{1/2} \leq \sqrt{2}\left(12\sqrt{2}\pi + 24\pi\right)\frac{1}{\sqrt{b_0 z}}\left(12\sqrt{2}\pi + 24\pi\right)^{-1} \leq c_1 \hat{g_0}^*(y) + \varepsilon/2.$$

Hence, in this case we have

$$(A\omega_z)^{1/2} \leq c_1(\hat{g_0}^* + \varepsilon/2). \tag{4.14}$$

Case 3: $y \notin (a_0, b_0)$ and $A\omega_z(y) > 2\omega_z(y)$. Observe that if y does not belong to any interval (a_j, b_j), then $A\omega_z(y) = \omega_z(y)$. The same is true if y lies in some interval (a_j, b_j) contained in $(0, z)$, because ω_z is constant on such an interval. Therefore, in the third case, y lies in some interval (a_j, b_j) of A with $b_j \geq z$. Moreover $y \notin (a_0, b_0)$ implies $a \geq z$. Therefore, with $z \leq a < y < b_j$, we have:

$$A\omega_z(y) = \frac{1}{b_j - a_j} \int_{a_j}^{b_j} \min(z^{-2}, t^{-2})\, dt = \frac{1}{b_j - a_j} \int_{a_j}^{b_j} t^{-2}\, dt = \frac{1}{a_j b_j}.$$

On the other hand,

$$\omega_z(y) = \min(z^{-2}, y^{-2}) = y^{-2} > b_j^{-2}.$$

Now the condition $A\omega_z(y) > 2\omega_z(y)$ implies $1/(a_j b_j) > 2/(b_j^2)$ which means $a_j < b_j/2$. We distinguish all intervals (a_j, b_j) with this property by defining

$$J = \{j : z \leq a_j < b_j/2\}.$$

This means in this case, $y \in (a_j, b_j)$ for some $j \in J$.

Now for each $j \in J$, we again invoke Corollary 4.15, this time with $r_j = \sqrt{b_j/(16a_j)}$ and $z_j = 16a_j/3$, to produce a function g_j so that $g_j^* = \chi_{[0,3/16a_j)}$ and

$$\begin{aligned}
\hat{g_j}^*(y) + \varepsilon/2 &\geq (3\pi y/r_j + 9\pi(r_j + 1)z_j)^{-1} \\
&= \left(3\pi y\sqrt{16a_j/b_j} + 9\pi(\sqrt{b_j/(16a_j)} + 1)(16a_j/3)\right)^{-1} \\
&= (a_j b_j)^{-1/2} \left(12\pi(y/b_j) + 12\pi + 24\sqrt{2}\pi\sqrt{2a_j/b_j}\right)^{-1}.
\end{aligned}$$

Since both y/b_j and $2a_j/b_j$ are less than 1, we get

$$\hat{g_j}^*(y) + \varepsilon/2 \geq \frac{1}{\sqrt{a_j b_j}} \left(24\pi + 24\sqrt{2}\pi\right)^{-1}.$$

It follows that

$$A\omega_z(y)^{1/2} = \frac{1}{\sqrt{a_j b_j}} \leq \left(24\pi + 24\sqrt{2}\pi\right)(\hat{g_j}^*(y) + \varepsilon/2) \leq c_1 \left(\hat{g_j}^*(y) + \varepsilon/2\right).$$

Hence, in this case we get

$$(A\omega_z)^{1/2} \leq c_1(\hat{g_j}^* + \varepsilon/2). \tag{4.15}$$

We are going to apply Lemma 4.16 to the collection of functions $\mathcal{F} = \{f_0, g_0, g_j : j \in J\}$. First we need to prove that the sum

$$p_0 = \frac{1}{4z} + \frac{3}{8z} + \sum_{j \in J} \frac{3}{16a_j}$$

is less than 1. For each $j \in J$, let m_j be the largest integer such that $2^{m_j} z \leq a_j$. Note that $m_j \geq 0$ since $a_j \geq z$. Now let $j, k \in J$ be distinct and assume $a_j \leq a_k$. Since the intervals of A are disjoint, we have $b_j \leq a_k$. This implies $2^{m_j} z \leq a_j < b_j/2 \leq a_k/2$ which means $2^{m_j+1}z < a_k$. It follows that $m_j + 1 \le m_k$ and therefore $m_j < m_k$. We conclude that all m_j's are different which implies

$$\sum_{j\in J} \frac{1}{a_j} \le \frac{1}{z} \sum_{j\in J} 2^{-m_j} \le \frac{1}{z} \sum_{m=0}^{\infty} 2^{-m} = \frac{2}{z}.$$

Thus $p_0 \le \frac{1}{z} \le \frac{1}{3} < 1$.

Now Lemma 4.16 guarantees the existence of a function f so that $f^* = \chi_{[0,p_0)} \le \chi_{[0,1/z)}$, and

$$\begin{aligned} \hat{f}^*(y) &\ge \hat{f_0}^*(y) - \varepsilon/2, \\ \hat{f}^*(y) &\ge \hat{g_0}^*(y) - \varepsilon/2, \quad \text{and} \\ \hat{f}^*(y) &\ge \hat{g_j}^*(y) - \varepsilon/2, \quad j \in J. \end{aligned}$$

These inequalities together with Inequalities (4.13) , (4.14) and (4.15) yield

$$(A\omega_z)^{1/2} \le c_1(\hat{f}^* + \varepsilon).$$

□

Theorem 4.18. *Let $z > 1$ and $A \in \mathcal{A}$. For each $\varepsilon > 0$ there exists a function $f \in L^1(\mathbb{T})$ such that*

$$f^* \le \chi_{[0,1/z)} \quad \text{and} \quad (A\omega_z)^{1/2} \le c(\hat{f}^* + \varepsilon),$$

with $c = 3c_1 = 549$

Proof. If $z \ge 3$ then Theorem 4.17 implies existence of the desired function f. If $1 < z < 3$ then we invoke Theorem 4.17 with $z = 3$ to produce a function f so that $f^* \le \chi_{[0,\frac{1}{3})}$ and $(A\omega_3)^{1/2} \le c_1(\hat{f}^* + \varepsilon)$. Obviously $f^* \le \chi_{[0,1/z)}$. We also have $\omega_z \le (9/z^2)\omega_3$ which implies $A\omega_z \le 9A\omega_3$ and completes the proof. □

4.3 Necessary conditions

Now we use the test functions from the previous section to give necessary conditions for Fourier series inequalities in the Lorentz space setting. Our first theorem is a necessary condition for the boundedness of $\mathcal{F} : \Gamma^p(w) \to \Lambda^q(u)$.

Theorem 4.19. *Assume $0 < p < \infty$, $0 < q < \infty$ and let $u, w \in L^+$. Suppose $C > 0$ and*

$$\left(\int_0^\infty \hat{f}^*(t)^q \, u(t) \, dt \right)^{1/q} \le C \left(\int_0^\infty f^{**}(t)^p \, w(t) \, dt \right)^{1/p},$$

for all functions $f \in L^1(\mathbb{T})$. Then

$$\sup_{z>1} \sup_{A\in\mathcal{A}} \frac{\|A\omega_z\|_{q/2,u}}{\|\omega_z\|_{p/2,v}} \le c^2 C^2,$$

where c is the constant of Theorem 4.18 and $v(t) = t^{p-2} w(1/t)$.

Proof. Fix $A \in \mathcal{A}$, $z > 1$. Assume $\varepsilon > 0$ and let f be the corresponding function in Theorem 4.18. We have

$$\int_0^{\frac{1}{t}} f^*(s)\,ds \le \int_0^{\frac{1}{t}} \chi_{[0,1/z)}\,ds = \min(z^{-1}, t^{-1}) = \omega_z(t)^{1/2},$$

which implies

$$\int_0^\infty f^{**}(t)^p\,w(t)\,dt \;= \int_0^\infty \Big(\int_0^{1/t} f^*(s)\,ds\Big)^p v(t)\,dt \le \int_0^\infty \omega_z(t)^{p/2}\,v(t)\,dt.$$

It follows that

$$\left(\int_0^\infty f^{**}(t)^p\,w(t)\,dt\right)^{2/p} \le \|\omega_z\|_{p/2,v}. \tag{4.16}$$

Let $g_\varepsilon(t) = \max(c^{-1}A\omega_z(t)^{1/2} - \varepsilon, 0)$ and observe that $\hat{f}^* \ge g_\varepsilon$ since $(A\omega_z)^{1/2} \le c(\hat{f}^* + \varepsilon)$. Also notice that $g_\varepsilon(t)$ increases to $c^{-1}A\omega_z(t)^{1/2}$ as ε decreases to 0. Now,

$$\left(\int_0^\infty \hat{f}^*(t)^q\,u(t)\,dt\right)^{2/q} \ge \left(\int_0^\infty g_\varepsilon(t)^q\,u(t)\,dt\right)^{2/q}.$$

The last two inequalities together with the hypothesis yield

$$\left(\int_0^\infty g_\varepsilon(t)^q\,u(t)\,dt\right)^{2/q} \le C^2\|\omega_z\|_{p/2,v}.$$

Finally let $\varepsilon \to 0$ and use the monotone convergence theorem to obtain

$$\left(\int_0^\infty [c^{-1}A\omega_z(t)^{1/2}]^q\,u(t)\,dt\right)^{2/q} \le C^2\|\omega_z\|_{p/2,v},$$

which asserts

$$\|A\omega_z\|_{q/2,u} \le c^2C^2\|\omega_z\|_{p/2,v},$$

and the proof is complete. □

The following useful corollary is immediate.

Corollary 4.20. *Assume* $0 < p < \infty$, $0 < q < \infty$ *and let* $u, v, w \in L^+$ *with* $v(t) = t^{p-2}w(1/t)$. *Suppose* $C > 0$ *and*

$$\left(\int_0^\infty \hat{f}^*(t)^q\,u(t)\,dt\right)^{1/q} \le C\left(\int_0^\infty f^{**}(t)^p\,w(t)\,dt\right)^{1/p}$$

for all functions $f \in L^1(\mathbb{T})$. *Then*

$$\sup_{z>1} \frac{\|\omega_z\|_{q/2,u}}{\|\omega_z\|_{p/2,v}} < \infty.$$

Proof. In Theorem 4.19, observe that the identity operator is a particular averaging operator. □

When w satisfies the reverse B_p condition, we obtain the following necessary condition.

Proposition 4.21. *Let $0 < p, q < \infty$ and assume u and w are weight functions on $(0,\infty)$. Suppose $w \in RB_p$. If there exists $C > 0$ so that*

$$\|\hat{f}\|_{\Lambda^q(u)} \leq C\|f\|_{\Gamma^p(w)}$$

for all $f \in L^1(\mathbb{T})$, then

$$\sup_{0<x<1} \left(x^q \int_0^{1/x} u(t)\, dt + \int_{1/x}^{\infty} \frac{u(t)}{t^q}\, dt \right)^{1/q} \left(\int_x^{\infty} \frac{w(t)}{t^p}\, dt \right)^{-1/p} < \infty.$$

Proof. We repeat the proof of Proposition 3.23. Corollary 4.20 implies $\sup_{z>0} \dfrac{F_{q,u}(z)}{G_{p,w}(z)} < \infty$. Now Equations (3.1) and (3.9) assert that

$$F_{q,u}(z) = \left(z^{-q} \int_0^z u(t)\, dt + \int_z^{\infty} \frac{u(t)}{t^q}\, dt \right)^{1/q} \quad \text{and} \quad G_{p,w} \approx \left(\int_{1/z}^{\infty} \frac{w(t)}{t^p}\, dt \right)^{1/p}.$$

The proof is complete by taking $x = 1/z$. □

It is possible obtain a similar necessary condition for $w \in B_p$, like Proposition 3.21. But here, we first use the test functions introduced in Theorem 4.18 to get a general necessary condition for boundedness of $\mathcal{F} : \Lambda^p(w) \to \Lambda^q(u)$. Then we deduce the result corresponding to Proposition 3.21 but with no restriction on $w(t)$.

Theorem 4.22. *Assume $0 < p < \infty$, $0 < q < \infty$ and let $u, w \in L^+$. Suppose $C > 0$ and*

$$\left(\int_0^{\infty} \hat{f}^*(t)^q\, u(t)\, dt \right)^{1/q} \leq C \left(\int_0^1 f^*(t)^p\, w(t)\, dt \right)^{1/p}$$

for all functions $f \in L^1(\mathbb{T})$. Then

$$\sup_{z>1} \sup_{A \in \mathcal{A}} \|A\omega_z\|_{q/2,u} \left(\int_0^{1/z} w(t)\, dt \right)^{-2/p} \leq c^2 C^2$$

where c is the constant of Theorem 4.18.

Proof. The proof is the same as the proof of Theorem 4.19 except for the estimate (4.16) which is replaced with

$$\left(\int_0^1 f^*(t)^p\, w(t)\, dt \right)^{2/p} \leq \left(\int_0^{1/z} w(t)\, dt \right)^{2/p}.$$

We omit the details. □

Corollary 4.23. *Assume $0 < p < \infty$, $0 < q < \infty$ and let $u, w \in L^+$. Suppose $C > 0$ and*

$$\|\hat{f}\|_{\Lambda^q(u)} \leq C\|f\|_{\Lambda^p(w)}$$

for all $f \in L^1(\mathbb{T})$. Then

$$\sup_{0<x<1} \left(x^q \int_0^{1/x} u(t)\,dt + \int_{1/x}^{\infty} \frac{u(t)}{t^q} dt \right)^{1/q} \left(\int_0^x w(t)\,dt \right)^{-1/p} < \infty.$$

Proof. Let A be the identity operator in Theorem 4.22. We have

$$\sup_{z>0} F_{q,u}(z) \left(\int_0^{1/z} w(t)\,dt \right)^{-1/p} < \infty.$$

Now equation (3.1) asserts that

$$F_{q,u}(z) = \left(z^{-q} \int_0^z u(t)\,dt + \int_z^{\infty} \frac{u(t)}{t^q}\,dt \right)^{1/q}.$$

Set $x = 1/z$, and the proof is complete. □

4.4 Necessary and sufficient conditions

We are now ready combine our sufficient conditions and necessary conditions to provide various characterizations for boundedness of $\mathcal{F} : \Gamma^p(w) \to \Lambda^q(u)$. We start with the following theorem for the case $p \leq 2$.

Theorem 4.24. *Let $0 < p \leq 2 \leq q < \infty$ and $u, v, w \in L^+$ with $v(t) = t^{p-2}w(1/t)$. Then there exists $C > 0$ such that the inequality*

$$\|\hat{f}\|_{\Lambda^q(u)} \leq C\|f\|_{\Gamma_p(w)} \tag{4.17}$$

holds for all $f \in L^1(\mathbb{T})$ if and only if

$$\sup_{z>1, A\in\mathcal{A}} \frac{\|A\omega_z\|_{q/2,u}}{\|\omega_z\|_{p/2,v}} < \infty. \tag{4.18}$$

Proof. The sufficient part follows from Theorems 4.5 with the observation that $\|\hat{f}\|_{\Lambda^q(u)} \leq \|\hat{f}\|_{\Gamma_q(u)}$. The necessary part is proved in Theorem 4.19. □

Corollary 4.25. *Let $0 < p \leq 2 \leq q < \infty$ and $u, v, w \in L^+$ with $w(t) = t^{p-2}v(1/t)$. Then there exists $C > 0$ such that the inequality*

$$\|\hat{f}\|_{\Lambda^q(u)} \leq C\|f\|_{\Gamma_p(w)}$$

holds for all $f \in L^1(\mathbb{T})$ if and only if

$$\sup_{z>1} \frac{\|\omega_z\|_{\Theta^{q/2}(u)}}{\|\omega_z\|_{p/2,v}} < \infty. \tag{4.19}$$

Proof. It follows from Theorem 4.24 and definition of the Lorentz Θ norm. $\square$

When $u(t)$ is decreasing, we get the following readily verifiable characterization of the boundedness of $\mathcal{F} : \Gamma^p(w) \to \Lambda^q(u)$.

Theorem 4.26. *Let $0 < p \leq q < \infty$, $q \geq 2$ and assume u and w are weight functions on $(0, \infty)$ with $v(t) = t^{p-2}w(1/t)$. Suppose u is decreasing. Then there exists $C > 0$ so that the inequality*

$$\|\hat{f}\|_{\Lambda^q(u)} \leq C\|f\|_{\Gamma^p(w)}$$

holds for all $f \in L^1(\mathbb{T})$ if and only if

$$\sup_{z>1} \frac{\|\omega_z\|_{q/2,u}}{\|\omega_z\|_{p/2,v}} < \infty.$$

Proof. Since u is decreasing we have $u^\circ = u$. So we may invoke the sufficient condition of Theorem 4.7. The necessity is stated in Corollary 4.20. $\square$

Another case where we get a simple necessary and sufficient condition, is $q = 2$. The following theorem and its corollary characterize all weights u, w for which the Fourier coefficient map $\mathcal{F} : \Gamma^p(w) \to \Lambda^2(u)$ is bounded.

Theorem 4.27. *Suppose $0 < p \leq 2$ and $u, v, w \in L^+$ with $v(t) = t^{p-2}w(1/t)$. Then there exists $C > 0$ such that the inequality*

$$\|\hat{f}\|_{\Lambda^2(u)} \leq C\|f\|_{\Gamma^p(w)} \tag{4.20}$$

holds for all functions $f \in L^1(\mathbb{T})$ if and only if

$$\sup_{z>1} \frac{\|\omega_z\|_{1,u^\circ}}{\|\omega_z\|_{p/2,v}} < \infty. \tag{4.21}$$

Proof. Theorem 4.7 with $q = 2$ asserts that (4.21) implies (4.20). Conversely, assume (4.20) holds for some constant C. Since ω_z is decreasing, Theorem 1.30 with $h = \omega_z$ implies

$$\|\omega_z\|_{1,u^\circ} = \int_0^\infty \omega_z u^\circ = \sup_{A\in\mathcal{A}} \int_0^\infty (A\omega_z)u = \sup_{A\in\mathcal{A}} \|A\omega_z\|_{1,u}. \tag{4.22}$$

Then we use Theorem 4.19 with $q = 2$ to get

$$\sup_{z>1} \frac{\|\omega_z\|_{1,u^\circ}}{\|\omega_z\|_{p/2,v}} = \sup_{\substack{A\in\mathcal{A}\\ z>1}} \frac{\|A\omega_z\|_{1,u}}{\|\omega_z\|_{p/2,v}} \leq c^2C^2 < \infty,$$

which proves (4.21). $\square$

Remark 4.28. This theorem is a special case of Corollary 4.25 with $q = 2$. In fact (4.22) shows that $\|\omega_z\|_{1,u^\circ} = \|\omega_z\|_{\Theta^1(u)}$, and therefore the two conditions (4.21) and (4.19) are identical.

Corollary 4.29. *Suppose $0 < p \le 2$ and $u, w \in L^+$. Then there exists $C > 0$ such that the inequality*

$$\|\hat{f}\|_{\Lambda^2(u)} \le C\|f\|_{\Gamma^p(w)} \tag{4.23}$$

holds for all functions $f \in L^1(\mathbb{T})$ if and only if

$$\sup_{z>1}\left(\sup_{y\ge z}\frac{z}{y}\int_0^y u(t)\,dt\right)^{1/2}\left(\int_{1/z}^\infty \frac{w(t)}{t^p}\,dt + z^p\int_0^{1/z} w(t)\,dt\right)^{-1/p} < \infty.$$

Proof. The assertion follows from Theorem 4.27 and the following estimate we showed in proof of Theorem 4.8.

$$\sup_{z>1}\frac{\|\omega_z\|_{1,u^\circ}}{\|\omega_z\|_{p/2,v}} \approx \sup_{z>1}\left(\sup_{y\ge z}\frac{z}{y}\int_0^y u(t)\,dt\right)^{1/2}\left(\int_{1/z}^\infty \frac{w(t)}{t^p}\,dt + z^p\int_0^{1/z} w(t)\,dt\right)^{-1/p}.$$

□

Now we give the Fourier series version of Theorem 2 in [BH2]. We stated that theorem in Theorem 3.13.

Theorem 4.30. *Let u and w be weight functions on $(0,\infty)$.*

(*i*) *Let $1 < p \le q < \infty$, $q \ge 2$ and assume u is decreasing and $w \in B_p$. If*

$$\sup_{0<x<1} x\left(\int_0^{1/x} u(t)\,dt\right)^{1/q}\left(\int_0^x w(t)\,dt\right)^{-1/p} < \infty \tag{4.24}$$

then there exists $C > 0$ such that for all $f \in L^1(\mathbb{T})$,

$$\|\hat{f}\|_{\Lambda^q(u)} \le C\|f\|_{\Lambda^p(w)}.$$

(*ii*) *Conversely, assume $p, q > 1$ and u and w are arbitrary weight functions. If $\|\hat{f}\|_{\Lambda^q(u)} \le C\|f\|_{\Lambda^p(w)}$ for all $f \in L^1(\mathbb{T})$, then (4.24) holds.*

Proof. To prove (i), observe that $u^\circ = u$ since u is decreasing. Moreover $\|.\|_{\Gamma^p(w)} \approx \|.\|_{\Lambda^p(w)}$ since $w \in B_p$. So the assertion is implied by Theorem 4.9.

To prove (ii), take $A \in \mathcal{A}$ to be the identity in Theorem 4.22. The theorem implies

$$\sup_{z>1}\|\omega_z\|_{q/2,u}\left(\int_0^{1/z} w(t)\,dt\right)^{-2/p} < \infty.$$

Recall Inequality (3.2) which asserts

$$\left(\|\omega_z\|_{q/2,u}\right)^{1/2} = F_{q,u}(z) \ge z^{-1}\left(\int_0^z u(t)\,dt\right)^{1/q}.$$

It follows that

$$\sup_{z>1} z^{-2}\left(\int_0^z u(t)\,dt\right)^{2/q}\left(\int_0^{1/z} w(t)\,dt\right)^{-2/p} < \infty.$$

Taking the square root and setting $x = 1/z$ completes the proof. □

A somewhat similar result holds for RB_p weights. But the necessary condition is for a restricted class of weights, compared to the above theorem.

Theorem 4.31. *Let $0 < p \le q < \infty$ and $q \ge 2$. Assume u and w are weight functions on $(0,\infty)$ with u decreasing and $w \in RB_p$. Then there exists $C > 0$ so that the inequality*

$$\|\hat{f}\|_{\Lambda^q(u)} \le C\|f\|_{\Gamma^p(w)},$$

holds for all $f \in L^1(\mathbb{T})$ if and only if

$$\sup_{0<x<1} \left(\int_0^{1/x} u(t)\,dt\right)^{1/q} \left(\int_x^\infty \frac{w(t)}{t^p}\,dt\right)^{-1/p} < \infty. \tag{4.25}$$

Proof. By estimate (3.2) for a decreasing u, we have

$$\left(\|\omega_z\|_{q/2,u}\right)^{1/2} = F_{q,u}(z) \approx z^{-1}\left(\int_0^z u\right)^{1/q}.$$

Let $v(t) = t^{p-2}w(1/t)$. Since w satisfies the RB_p condition, the inequality (3.9) holds, so

$$\left(\|\omega_z\|_{p/2,v}\right)^{1/2} = G_{p,w}(z) \approx z^{-1}\left(\int_{1/z}^\infty \frac{w(t)}{t^p}\,dt\right)^{1/p}.$$

Now

$$\begin{aligned}\left(\sup_{z>1}\frac{\|\omega_z\|_{q/2,u}}{\|\omega_z\|_{p/2,v}}\right)^{1/2} &\approx \sup_{z>1}\left(\int_0^z u(t)\,dt\right)^{1/q}\left(\int_{1/z}^\infty \frac{w(t)}{t^p}\,dt\right)^{-1/p}\\ &= \sup_{0<x<1}\left(\int_0^{1/x} u(t)\,dt\right)^{1/q}\left(\int_x^\infty \frac{w(t)}{t^p}\,dt\right)^{-1/p}.\end{aligned}$$

where we used $x = 1/z$. The above supremum is finite by hypothesis. Hence the proof is complete by Theorem 4.26. □

The last results in this chapter are necessary and sufficient conditions for the boundedness of $\mathcal{F} : \Gamma^p(w) \to \Lambda^q(u)$ in the case $q = 2$. We have two theorems corresponding to the B_p and RB_p conditions on w. But the weight $u(t)$ need not be decreasing anymore.

Theorem 4.32. *Let $0 < p \le 2$ and assume u and w are weight functions on $(0,\infty)$. Suppose $w \in RBp$. Then there exists $C > 0$ so that the inequality*

$$\|\hat{f}\|_{\Lambda^2(u)} \le C\|f\|_{\Gamma^p(w)}$$

holds for all $f \in L^1(\mathbb{T})$ if and only if

$$\sup_{0<x<1} \left(\int_0^{1/x} u^\circ(t)\,dt\right)^{1/2} \left(\int_x^\infty \frac{w(t)}{t^p}\,dt\right)^{-1/p} < \infty.$$

Proof. The proof is similar to proof of Theorem 4.31 with $q = 2$ and u replaced with u°. Let $v(t) = t^{p-2}w(1/t)$. Since u° is decreasing and $w \in RB_p$ we get

$$\left(\sup_{z>1} \frac{\|\omega_z\|_{1,u^\circ}}{\|\omega_z\|_{p/2,v}}\right)^{1/2} \approx \sup_{z>1} \left(\int_0^z u^\circ(t)\, dt\right)^{1/2} \left(\int_{1/z}^\infty \frac{w(t)}{t^p}\, dt\right)^{-1/p}$$
$$= \sup_{0<x<1} \left(\int_0^{1/x} u^\circ(t)\, dt\right)^{1/2} \left(\int_x^\infty \frac{w(t)}{t^p}\, dt\right)^{-1/p},$$

where we used $x = 1/z$. The proof is complete by Theorem 4.27. $\square$

Theorem 4.33. *Let $0 < p \leq 2$ and assume u and w are weight functions on $(0, \infty)$. Suppose $w \in B_p$. Then there exists $C > 0$ so that the inequality*

$$\|\hat{f}\|_{\Lambda^2(u)} \leq C\|f\|_{\Lambda^p(w)}$$

holds for all $f \in L^1(\mathbb{T})$ if and only if

$$\sup_{0<x<1} x \left(\int_0^{1/x} u^\circ(t)\, dt\right)^{1/2} \left(\int_0^x w(t)\, dt\right)^{-1/p} < \infty.$$

Proof. Let $v(t) = t^{p-2}w(1/t)$. Since u° is decreasing and $w \in B_p$ we get

$$\left(\sup_{z>1} \frac{\|\omega_z\|_{1,u^\circ}}{\|\omega_z\|_{p/2,v}}\right)^{1/2} \approx \sup_{z>1} z^{-1} \left(\int_0^z u^\circ(t)\, dt\right)^{1/2} \left(\int_0^{1/z} w(t)\, dt\right)^{-2/p}$$
$$= \sup_{0<x<1} x \left(\int_0^{1/x} u^\circ(t)\, dt\right)^{1/2} \left(\int_0^x w(t)\, dt\right)^{-1/p},$$

where we used $x = 1/z$. By Theorem 4.27 the above supremum is finite if and only if $\|\hat{f}\|_{\Lambda^2(u)} \leq C\|f\|_{\Gamma^p(w)}$ holds. The proof is complete since $w \in B_p$ and therefore $\|f\|_{\Gamma^p(w)} \approx \|f\|_{\Lambda^p(w)}$. $\square$

Remark 4.34. When using conditions like those in Theorem 4.33 we verify the function

$$\Phi_{u^\circ,w}(x) = x \left(\int_0^{1/x} u^\circ(t)\, dt\right)^{1/2} \left(\int_0^x w(t)\, dt\right)^{-1/p}$$

is bounded on the interval $(0, 1)$. We show that the condition $w \in B_p$ implies $\int_0^x w(t)dt > 0$ for all $x \in (0, 1)$. Recall that the B_p condition means for some $b_p > 0$ we have

$$\int_x^\infty \frac{w(t)}{t^p}\, dt \leq b_p \frac{1}{x^p} \int_0^x w(t)\, dt, \quad x > 0.$$

If $\int_0^y w(t)dt = 0$ for some $y > 0$, then $\int_0^x w(t)dt = 0$ for $x \in (0, y)$ and therefore,

$$\int_0^\infty \frac{w(t)}{t^p}\, dt = 0,$$

which is not possible since w is not identically zero. We conclude that the B_p condition for w makes $\Phi_{u^\circ,w}(x)$ a continuous function. Therefore, $\Phi_{u^\circ,w}(x)$ is bounded on the interval $(0,1)$ if and only if its limits as $x \to 0$ and $x \to 1$ are bounded. This observation is very useful when we apply the general results to specific weight functions. See Theorem 5.11 as an example.

Chapter 5

Applications to other function spaces

In this chapter we apply our results from Chapter 4 to some important special cases. In Section 1, we follow the method of Benedetto and Heinig from [BH2], to obtain some sufficient and necessary conditions for the boundedness of the Fourier coefficient map $\mathcal{F} : L^p(w) \to \ell^q(u)$. These are analogous to the results on the Fourier transform discussed in Chapter 3. However, we don't get those results that were obtained from duality properties of the Fourier transform, because the Lorentz norm inequalities for the Fourier transform on $\mathbb{Z}$ are unknown.

In Section 2, we consider well-known examples of weighted Lorentz spaces, and provide Fourier series inequalities in Lorentz space $L^{r,p}$, Zygmund space $L \log L$, and Lorentz-Zygmund space $L^{r,p}(\log L)^{\alpha}$. We reproduce some of the sufficient conditions from Bennett and Rudnick in [BR] and provide their converse as well. The consequence is a characterization, for a large range of exponents, exactly when the Fourier coefficient map $\mathcal{F} : L^{r,q}(\log L)^{\beta} \longrightarrow \ell^{s,q}(\log \ell)^{\beta}$ is bounded.

5.1 Weighted Lebesgue inequalities

Similar to the approach used for the Fourier transform, we use our weighted Lorentz inequalities to obtain weighted L^p inequalities for Fourier coefficients. The key is the Hardy-Littlewood-Polya inequality, which implies the following lemma.

Lemma 5.1. *Let $0 < p, q < \infty$ and suppose u and w are weight functions on $\mathbb{Z}$ and $\mathbb{T}$ respectively. Assume $\|\hat{f}\|_{\Lambda^q(u^*)} \leq C\|f\|_{\Lambda^p(w^{\circledast})}$ where $C > 0$ and f is a measurable function on $\mathbb{T}$. Then $\|\hat{f}\|_{\ell^q(u)} \leq C\|f\|_{L^p(w)}$.*

Proof. Let $m(x)$ and $\mu(k)$ denote normalized Lebesgue measure on $\mathbb{T}$ and counting measure on $\mathbb{Z}$, respectively. Since $(|f|^p)^* = (f^*)^p$, the Hardy-Littlewood-Polya inequality (1.1) yields

$$\|\hat{f}\|_{\ell^q(u)} = \left(\int_{\mathbb{Z}} |\hat{f}(k)|^q \, u(k) \, d\mu(k)\right)^{1/q} \leq \left(\int_0^{\infty} \hat{f}^*(t)^q \, u^*(t) \, dt\right)^{1/q} = \|\hat{f}\|_{\Lambda^q(u^*)}.$$

The hypothesis, together with (1.2), shows that

$$\|\hat{f}\|_{\Lambda^q(u^*)} \le C\|f\|_{\Lambda^p(w^\circledast)} = C\left(\int_0^1 f^*(x)^p\, w^\circledast(t)\, dt\right)^{1/p} \le \left(\int_{\mathbb{T}} |f(x)|^p\, w(x)\, dm(x)\right)^{1/p},$$

which completes the proof. $\square$

Assume $w(x)$ is a weight function defined on $\mathbb{T}$. Then $(1/w)^*$ vanishes outside $(0,1)$. So $w^\circledast(t) = \infty$ on $[1,\infty)$. However the expression $\|f\|_{\Lambda^p(w^\circledast)}$ remains valid, because for a function $f : \mathbb{T} \to \mathbb{C}$ the decreasing rearrangement is supported in $(0,1)$. In particular, if $\bar{w} = w^\circledast \chi_{(0,1)}$ then $\|f\|_{\Lambda^p(w^\circledast)} = \|f\|_{\Lambda^p(\bar{w})}$. We will use this in the following theorem which gives a sufficient condition for $\|\hat{f}\|_{\ell^q(u)} \le C\|f\|_{L^p(w)}$.

Theorem 5.2. *Let $1 < p \le q < \infty$, $q \ge 2$ and suppose u and w are weight functions on $\mathbb{R}^n$. Set $\bar{w} = w^\circledast \chi_{(0,1)}$ and assume $\bar{w} \in B_p$. If*

$$\sup_{0<x<1} x\left(\int_0^{1/x} u^*(t)\, dt\right)^{1/q} \left(\int_0^x w^\circledast(t)\, dt\right)^{-1/p} < \infty,$$

then there exists $C > 0$ such that

$$\left(\int_{\mathbb{Z}} |\hat{f}(k)|^q\, u(k)\, d\mu(k)\right)^{1/q} \le C\left(\int_{\mathbb{T}} |f(x)|^p\, w(x)\, dm(x)\right)^{1/p}$$

or, equivalently,

$$\|\hat{f}\|_{\ell^q(u)} \le C\|f\|_{L^p(w)}$$

for all $f \in L^1(\mathbb{T})$.

Proof. Since $x \in (0,1)$ in the above supremum, we may replace $w^\circledast$ with $\bar{w}$. Now $\bar{w} \in B_p$ and u^* is decreasing. So the conditions of Theorem 4.30, with u and w replaced with u^* and $\bar{w}$, are satisfied. Thus there exists $C > 0$ such that $\|\hat{f}\|_{\Lambda^q(u^*)} \le C\|f\|_{\Lambda^p(\bar{w})} = C\|f\|_{\Lambda^p(w^\circledast)}$ for all $f \in L^1(\mathbb{T})$. The proof is complete by Lemma 5.1. $\square$

Note that we can not get the Fourier series analogue of Theorem 3.31 which provides a sufficient condition in the case $q < 2$. The proof of Theorem 3.31 is based on Theorem 3.29 and self duality of the Fourier transform on $\mathbb{R}^n$.

Here, a substantial difference between the Fourier transform on $\mathbb{R}^n$ and the Fourier transform on $\mathbb{T}$ arises. In order to use the duality argument we need the corresponding result of Theorem 5.2 for the dual operator; this is the trigonometric series map defined on ℓ^1 by

$$\left[\mathcal{F}\big((a_n)_{n\in\mathbb{Z}}\big)\right](t) = \sum_{n\in\mathbb{Z}} a_n e^{-2\pi i n t}.$$

This provides a strong motivation to study the mapping properties of the trigonometric series, in Lorentz spaces and weighted Lebesgue spaces.

Another sufficient condition is stated in the next theorem, which is analogous to Theorem 3.32. Notice that the theorem holds for a smaller range of indices ($q \ge 2$). The reason is that we used a duality argument to prove the case $1 \le q < 2$ in Theorem 3.32.

Theorem 5.3. *Let $1 < p \leq q < \infty$, $q \geq 2$ and assume u and w are weight functions on $\mathbb{R}^n$. If*

$$\sup_{0<x<1} \left(\int_0^{1/x} u^*(t)\, dt \right)^{1/q} \left(\int_0^x w^{\circledast}(t)^{1-p'}\, dt \right)^{1/p'} < \infty,$$

then there exists $C > 0$ such that

$$\|\hat{f}\|_{\ell^q(u)} \leq C\|f\|_{L^p(w)}$$

for all $f \in L^1(\mathbb{T})$.

Proof. We invoke Theorem 4.12, replacing u and w with u^* and $w^{\circledast}$, respectively. Thus there exists $C > 0$ such that $\|\hat{f}\|_{\Lambda^q(u^*)} \leq C\|f\|_{\Lambda^p(w^{\circledast})}$ for all $f \in L^1(\mathbb{T})$. It follows from Lemma 3.28 that $\|\hat{f}\|_{L^q(u)} \leq C\|f\|_{L^p(w)}$.

□

The next lemma prepares us to give necessary conditions for the boundedness of the Fourier coefficient map between weighted L^p spaces.

Lemma 5.4. *Let $0 < p, q < \infty$ and suppose u and w are weight functions on $\mathbb{Z}$ and $\mathbb{T}$, respectively. Assume $\|\hat{f}\|_{\ell^q(u)} \leq C\|f\|_{L^p(w)}$ where $C > 0$ and f is a measurable function on $\mathbb{T}$. Then $\|\hat{f}\|_{\Lambda^q(u^{\circledast})} \leq C\|f\|_{\Lambda^p(w^*)}$.*

Proof. Let $m(x)$ and $\mu(k)$ denote normalized Lebesgue measure on $\mathbb{T}$ and counting measure on $\mathbb{Z}$, respectively. By hypothesis we have

$$\left(\int_{\mathbb{Z}} |\hat{f}(k)|^q\, u(k)\, d\mu(k) \right)^{1/q} \leq C \left(\int_{\mathbb{T}} |f(x)|^p\, w(x)\, dm(x) \right)^{1/p}.$$

For the right hand side, the Hardy-Littlewood-Polya inequality (1.2) and $(|\hat{f}|^q)^* = (\hat{f}^*)^q$ imply

$$\|\hat{f}\|_{\Lambda^q(u^{\circledast})} = \left(\int_0^\infty \hat{f}^*(t)^q\, u^{\circledast}(t)\, dt \right)^{1/q} \leq \left(\int_{\mathbb{Z}} |\hat{f}(k)|^q\, u(k)\, d\mu(k) \right)^{1/q}.$$

For the left hand side, we invoke (1.1) together with $(|f|^p)^* = (f^*)^p$ to get

$$\left(\int_{\mathbb{T}} |f(x)|^p\, w(x)\, dm(x) \right)^{1/p} \leq \left(\int_0^\infty f^*(x)^p\, w^*(t)\, dt \right)^{1/p} = \|f\|_{\Lambda^p(w^*)}.$$

It follows that $\|\hat{f}\|_{\Lambda^q(u^{\circledast})} \leq C\|f\|_{\Lambda^p(w^*)}$ which completes the proof. □

Now we have the following necessary condition.

Theorem 5.5. *Suppose $0 < p < \infty$, $0 < q < \infty$ and let u and w be weight functions on $\mathbb{Z}$ and $\mathbb{T}$, respectively. Assume there exists $C > 0$ such that*

$$\|\hat{f}\|_{\ell^q(u)} \leq C\|f\|_{L^p(w)}$$

for all $f \in L^1(\mathbb{T})$. Then

$$\sup_{0<x<1} \left(\int_{1/x}^\infty \frac{u^{\circledast}(t)}{t^q}\, dt \right)^{1/q} \left(\int_0^x w^*(t)\, dt \right)^{-1/p} < \infty. \tag{5.1}$$

Proof. The hypothesis of the theorem together with Lemma 5.4, implies $\|\hat{f}\|_{\Lambda^q(u^{\circledast})} \leq C\|f\|_{\Lambda^p(w^*)}$ for all $f \in L^1(\mathbb{T})$. Corollary 4.23 with u and w replaced with $u^{\circledast}$ and w^*, respectively, yields

$$\sup_{0<x<1} \left(x^q \int_0^{1/x} u^{\circledast}(t)\, dt + \int_{1/x}^{\infty} \frac{u^{\circledast}(t)}{t^q}\, dt \right)^{1/q} \left(\int_0^x w^*(t)\, dt \right)^{-1/p} < \infty.$$

This completes the proof since

$$\int_{1/x}^{\infty} \frac{u^{\circledast}(t)}{t^q}\, dt \leq x^q \int_0^{1/x} u^{\circledast}(t)\, dt + \int_{1/x}^{\infty} \frac{u^{\circledast}(t)}{t^q}\, dt.$$

□

5.2 Lorentz-Zygmund spaces

Now we apply our sufficient and necessary conditions for $\|\hat{f}\|_{\Lambda^q(u)} \leq C\|f\|_{\Lambda^p(w)}$ and $\|\hat{f}\|_{\Lambda^q(u)} \leq C\|f\|_{\Gamma^p(w)}$ to Lorentz spaces with power and logarithmic weights, namely Lorentz space $L^{r,p}$, Zygmund space $L \log L$ and Lorentz-Zygmund space $L^{r,p}(\log L)^{\alpha}$.

We start with $L \log L$. Recall from Section 1.3 that $L \log L = \Gamma^1(w)$ where $w = \chi_{(0,1)}$.

Theorem 5.6. *Assume $q \geq 2$ and let $u(t)$ be a weight function. If*

$$\sup_{z>1} \frac{z}{(1+\log z)^q} \left(\sup_{y>z} \frac{1}{y} \int_0^y u(t)\, dt \right) < \infty,$$

then the Fourier coefficient map $\mathcal{F} : L \log L \longrightarrow \Lambda^q(u)$ is bounded.

Proof. The proof is based on Theorem 4.8 with $p = 1$ and $w = \chi_{(0,1)}$. For $z > 1$ we have

$$\int_{1/z}^{\infty} \frac{w(t)}{t}\, dt + z \int_0^{1/z} w(t)\, dt = \int_{1/z}^{1} \frac{1}{t}\, dt + z \int_0^{1/z} dt = \log z + 1. \tag{5.2}$$

Hence, by Theorem 4.8 the inequality $\|\hat{f}\|_{\Gamma^q(u)} \leq C\|f\|_{\Gamma^1(w)}$ holds if

$$\sup_{z>1} \left(\sup_{y \geq z} \frac{z}{y} \int_0^y u(t)\, dt \right)^{1/q} (\log z + 1)^{-1} < \infty.$$

Taking the qth power we get

$$\sup_{z>1} \frac{z}{(1+\log z)^q} \left(\sup_{y>z} \frac{1}{y} \int_0^y u(t)\, dt \right) < \infty,$$

which is true by the hypothesis. Therefore, $\mathcal{F} : \Gamma^1(w) \to \Gamma^q(u)$ is bounded. The proof is complete observing $\Gamma^q(u) \hookrightarrow \Lambda^q(u)$ and $L \log L = \Gamma^1(w)$. □

For decreasing weights we get a much simpler condition.

Corollary 5.7. *Assume $q \geq 2$ and u is a decreasing weight function on $(0,\infty)$. If*

$$\sup_{z>1} \frac{1}{(1+\log z)^q} \int_0^z u(t)\,dt < \infty,$$

then $\mathcal{F} : L\log L \longrightarrow \Lambda^q(u)$ is bounded.

Proof. Since u is decreasing the moving average $\frac{1}{y}\int_0^y u(t)\,dt$ is also decreasing and therefore,

$$\sup_{y>z} \frac{1}{y} \int_0^y u(t)\,dt = \frac{1}{z} \int_0^z u(t)\,dt. \tag{5.3}$$

Now Theorem 5.6 implies the assertion. □

When $q = 2$ the sufficient condition in Theorem 5.6 is also necessary and we have the following characterization of weights $u(t)$ for which $\|\hat{f}\|_{\Lambda^2(u)} \leq C\|f\|_{L\log L}$ holds.

Theorem 5.8. *Let $u(t)$ be a weight function. Then $\mathcal{F} : L\log L \longrightarrow \Lambda^2(u)$ is bounded if and only if*

$$\sup_{z>1} \frac{z}{(1+\log z)^2} \left(\sup_{y>z} \frac{1}{y} \int_0^y u(t)\,dt \right) < \infty.$$

Proof. The proof is similar to proof of Theorem 5.6. Corollary 4.29 with $p = 1$ and $w = \chi_{(0,1)}$ together with equation (5.2), implies that $\|\hat{f}\|_{\Lambda^2(u)} \leq C\|f\|_{\Gamma^1(w)}$ holds for all $f \in L^1(\mathbb{T})$ if and only if

$$\sup_{z>1} \left(\sup_{y\geq z} \frac{z}{y} \int_0^y u(t)\,dt \right)^{1/2} (\log z + 1)^{-1} < \infty.$$

The proof is complete by taking the square of the above supremum. □

Corollary 5.9. *Assume u is a decreasing weight on $(0,\infty)$. Then $\mathcal{F} : L\log L \longrightarrow \Lambda^2(u)$ is bounded if and only if*

$$\sup_{z>1} \frac{1}{(1+\log z)^2} \int_0^z u(t)\,dt < \infty.$$

Proof. This follows from Theorem 5.8 and Equation (5.3). □

Now we turn to Lorentz-Zygmund spaces. Recall from Section 1.3 that $L^{r,p}(\log L)^\alpha = \Lambda^p(w)$ with weight

$$w(t) = t^{p/r-1}(1-\log t)^{p\alpha}\chi_{[0,1)}, \tag{5.4}$$

and $\ell^{s,q}(\log \ell)^\beta = \Lambda^q(u)$ with weight

$$u(t) = (n+1)^{q/s-1}(1+\log(n+1))^{q\beta} \quad , \quad n \leq t < n+1,\ 0 \leq n \in \mathbb{Z}. \tag{5.5}$$

The goal is to apply Theorem 4.30 to weight functions u and w to find relation among p,q,r,s,α,β that are sufficient or necessary for inequality $\|\hat{f}\|_{\ell^{s,q}(\log \ell)^\beta} \leq C\|f\|_{L^{r,p}(\log L)^\alpha}$. Before proceeding, we verify the B_p condition for w.

Lemma 5.10. *Assume $1 < p < \infty$ and $-\infty < \alpha < \infty$. The weight $w(t)$ described in (5.4) satisfies the B_p condition exactly when $r > 1$.*

Proof. We need to show the existence of some constant b_p such that

$$\int_x^\infty t^{-p} w(t)\, dt \le b_p \frac{1}{x^p} \int_0^x w(t)\, dt$$

holds for all $x > 0$. Since $w(t)$ vanishes on $[1, \infty)$, the left hand side of this inequality is zero for $x \ge 1$. Therefore, it is enough to show the inequality holds for some constant b_p and $x \in (0, 1)$. That is,

$$\int_x^1 t^{p/r-1-p}(1 - \log t)^{p\alpha}\, dt \le b_p \frac{1}{x^p} \int_0^x t^{p/r-1}(1 - \log t)^{p\alpha}\, dt, \quad 0 < x < 1,$$

for some $b_p > 0$. Let

$$F(x) = \frac{x^p \int_x^1 t^{p/r-1-p}(1 - \log t)^{p\alpha}\, dt}{\int_0^x t^{p/r-1}(1 - \log t)^{p\alpha}\, dt}, \quad 0 < x < 1,$$

and observe that the B_p condition is equivalent to $\sup\limits_{0<x<1} F(x) < \infty$.

Notice that $F(x)$ is a continuous function. Therefore, F is bounded on $(0, 1)$ exactly when its limits are bounded at the end points $x = 0$ and $x = 1$. To compute the limit as $x \to 1$ observe that $t^{p/r-1-p}(1 - \log t)^{p\alpha}$ is bounded on the interval $[x, 1]$ when $x > 0$. Therefore,

$$\lim_{x \to 1} \int_x^1 t^{p/r-1-p}(1 - \log t)^{p\alpha}\, dt = 0,$$

which implies that $\lim\limits_{x\to 1} F(x) = 0$.

We conclude that $w \in B_p$ if and only if $\lim\limits_{x\to 0} F(x)$ is finite. To compute $\lim\limits_{x\to 0} F(x)$, first observe that $\int_0^1 t^{p/r-1}(1 - \log t)^{p\alpha}\, dt$ is finite since p and r are positive and hence $p/r - 1 > -1$. This implies

$$\lim_{x \to 0} \int_0^x t^{p/r-1}(1 - \log t)^{p\alpha}\, dt = 0.$$

Now we apply L'Hôspital rule to get

$$\begin{aligned}
\lim_{x\to 0} \frac{\int_0^x t^{p/r-1}(1 - \log t)^{p\alpha}\, dt}{x^{p/r}(1 - \log x)^{p\alpha}} &= \lim_{x\to 0} \frac{x^{p/r-1}(1 - \log x)^{p\alpha}}{(p/r)x^{p/r-1}(1 - \log x)^{p\alpha} - p\alpha x^{p/r-1}(1 - \log x)^{p\alpha-1}} \\
&= \lim_{x\to 0} \frac{1}{p/r - p\alpha(1 - \log x)^{-1}} \\
&= \frac{r}{p}. \qquad (5.6)
\end{aligned}$$

It follows that

$$\lim_{x\to 0} F(x) = \lim_{x\to 0} \frac{x^p \int_x^1 t^{p/r-1-p}(1-\log t)^{p\alpha}\,dt}{(r/p)x^{p/r}(1-\log x)^{p\alpha}} = \lim_{x\to 0} \frac{\int_x^1 t^{p/r-1-p}(1-\log t)^{p\alpha}\,dt}{(r/p)x^{p/r-p}(1-\log x)^{p\alpha}}.$$

This limit is infinite when $r < 1$, since in this case we have $p/r - p > 0$ and the denominator approaches zero.

If $r = 1$ we have

$$\lim_{x\to 0} F(x) = \lim_{x\to 0} \frac{\int_x^1 t^{-1}(1-\log t)^{p\alpha}\,dt}{(1/p)(1-\log x)^{p\alpha}}. \tag{5.7}$$

Observe that

$$\int_x^1 t^{-1}(1-\log t)^{p\alpha}\,dt = \begin{cases} \dfrac{1}{p\alpha+1}\big((1-\log x)^{p\alpha+1}-1\big), & p\alpha \neq -1, \\ \log(1-\log x), & p\alpha = -1. \end{cases}$$

When $p\alpha \neq 1$ the limit (5.7) turns into

$$\lim_{x\to 0} F(x) = \lim_{x\to 0} \frac{\big(1/(p\alpha+1)\big)\big((1-\log x)^{p\alpha+1}-1\big)}{(1/p)(1-\log x)^{p\alpha}} = \infty,$$

and when $p\alpha = -1$ we get

$$\lim_{x\to 0} F(x) = \lim_{x\to 0} \frac{\log(1-\log x)}{(1/p)(1-\log x)^{-1}} = \infty.$$

It follows that $\lim\limits_{x\to 0} F(x) = \infty$ when $r \geq 1$.

Finally assume $r > 1$. Since $p/r - 1 - p < -1$ we have

$$\lim_{x\to 0} \int_x^1 t^{p/r-1-p}(1-\log t)^{p\alpha}\,dt = \infty.$$

The L'Hôspital rule yields

$$\begin{aligned} \lim_{x\to 0} \frac{\int_x^1 t^{p/r-1-p}(1-\log t)^{p\alpha}\,dt}{x^{p/r-p}(1-\log x)^{p\alpha}} &= \lim_{x\to 0} \frac{-1}{(p/r-p) - p\alpha(1-\log x)^{-1}} \\ &= \frac{r}{p(r-1)}. \end{aligned} \tag{5.8}$$

Therefore,

$$\lim_{x\to 0} F(x) = \lim_{x\to 0} \frac{\int_x^1 t^{p/r-1-p}(1-\log t)^{p\alpha}\,dt}{(r/p)x^{p/r-p}(1-\log x)^{p\alpha}} = \frac{(\frac{r}{p(r-1)})x^{p/r-p}(1-\log x)^{p\alpha}}{(r/p)x^{p/r-p}(1-\log x)^{p\alpha}} = \frac{1}{r-1}.$$

We conclude that $\lim_{x\to 0} F(x) < \infty$ exactly when $r > 1$. This completes the proof. $\square$

Theorem 5.11. *Assume $1 < p \le q < \infty$, $2 \le q < s$, $r > 1$ and $-\infty < \alpha, \beta < \infty$. If either of the conditions,*

(*i*) $\dfrac{1}{s} + \dfrac{1}{r} < 1$, *or*

(*ii*) $\dfrac{1}{s} + \dfrac{1}{r} = 1$ *and* $\beta \le \alpha$,

holds then $\mathcal{F} : L^{r,p}(\log L)^\alpha \longrightarrow \ell^{s,q}(\log \ell)^\beta$ is bounded.

Conversely for $0 < p, q < \infty$, $0 < r, s < \infty$ and $-\infty < \alpha, \beta < \infty$, if $\mathcal{F} : L^{r,p}(\log L)^\alpha \longrightarrow \ell^{s,q}(\log \ell)^\beta$ is bounded then one of the conditions (i) or (ii) must hold.

Proof. Recall $\ell^{s,q}(\log \ell)^\beta = \Lambda^q(u)$ and $L^{r,p}(\log L)^\alpha = \Lambda^p(w)$ where u and w are the weights defined in (5.5) and (5.4). We are going to apply Theorem 4.30 to weights u and w. Since $r > 1$, Lemma 5.10 asserts that $w \in B_p$. For computations involving $u(t)$ we introduce the following estimate. Define

$$g(t) = t^{q/s-1}(1 + \log t)^{q\beta}, \quad t \ge 1,$$

and observe that $g(n + 1) = u(n)$ for integers $n \ge 0$. We have

$$g'(t) = t^{q/s-2}(1 + \log t)^{q\beta-1}\left((q/s - 1)(1 + \log t) - q\beta\right).$$

Note that $s > q$ implies that $g'(t)$ is negative for large values of t. Hence there exists an integer $N \ge 2$ such that $g(t)$ is decreasing on $[N, \infty)$. This allows us to replace u with the decreasing function u_1 defined by

$$u_1(t) = \begin{cases} M, & 0 \le t < N, \\ \\ g(t), & t \ge N \end{cases}, \text{ where } \quad M = \max_{1 \le t \le N} g(t).$$

Obviously u_1 is decreasing on $[0, \infty)$ and $u(t) \le u_1(t)$. Moreover $g(t + 1) \le u(t) \le g(t)$ for $t > N$.

To prove the sufficiency, assume either of the conditions (i) or (ii) holds. Observe that $u \le u_1$ implies $\|\hat{f}\|_{\Lambda^q(u)} \le \|\hat{f}\|_{\Lambda^q(u_1)}$, so it is enough to prove that $\mathcal{F} : \Lambda^p(w) \to \Lambda^q(u_1)$ is bounded. Since u_1 is deceasing and $w \in B_p$, we can use the sufficient condition in Theorem 4.30. Thus it suffices to show the function

$$\Phi_{u_1,w}(x) = x \left(\int_0^{1/x} u_1(t)\, dt \right)^{1/q} \left(\int_0^x w(t)\, dt \right)^{-1/p}$$

is bounded on the interval $(0, 1)$.

As explained in Remark 4.34, $\Phi_{u_1,w}(x)$ is continuous because $w \in B_p$. Hence it is bounded on $(0, 1)$ if and only if it has finite limits at the endpoints $x = 0$ and $x = 1$. The limit at $x = 1$ is finite since

$$\lim_{x\to 1} \Phi_{u_1,w}(x) = \left(\int_0^1 u_1(t)\, dt \right)^{1/q} \left(\int_0^1 w(t)\, dt \right)^{-1/p} = N^{1/q} \left(\int_0^1 w(t)\, dt \right)^{-1/p}.$$

To show the finiteness of $\Phi_{u_1,w}(0+)$, observe that for $0 < x < \dfrac{1}{N}$ we have

$$\int_0^{1/x} u_1(t)\,dt = NM + \int_N^{1/x} g(t)\,dt.$$

Hence,

$$\lim_{x\to 0} \Phi_{u_1,w}(x) = \lim_{x\to 0} \frac{x(NM + \int_N^{1/x} g(t)\,dt)^{1/q}}{(\int_0^x t^{p/r-1}(1-\log t)^{p\alpha}\,dt)^{1/p}}.$$

The L'Hôspital's rule asserts that

$$\begin{aligned}
\lim_{x\to 0} \frac{NM + \int_N^{1/x} g(t)\,dt}{x^{-q/s}(1-\log x)^{q\beta}} &= \lim_{y\to\infty} \frac{NM + \int_N^{y} t^{q/s-1}(1+\log t)^{q\beta}\,dt}{y^{q/s}(1+\log y)^{q\beta}} \\
&= \lim_{y\to\infty} \frac{y^{q/s-1}(1+\log y)^{q\beta}}{(q/s)y^{q/s-1}(1+\log y)^{q\beta} + q\beta y^{q/s-1}(1+\log y)^{q\beta-1}} \\
&= \lim_{y\to\infty} \frac{1}{(q/s) + q\beta(1+\log y)^{-1}} = \frac{s}{q}. \qquad (5.9)
\end{aligned}$$

Now Equations (5.6) and (5.9) imply that

$$\begin{aligned}
\lim_{x\to 0} \Phi_{u_1,w}(x) &= \lim_{x\to 0} \frac{x((s/q)x^{-q/s}(1-\log x)^{q\beta})^{1/q}}{((r/p)x^{p/r}(1-\log x)^{p\alpha})^{1/p}} \\
&= \left(\frac{s}{q}\right)^{1/q}\left(\frac{p}{r}\right)^{1/p} \lim_{x\to 0} x^{1-1/r-1/s}(1-\log x)^{\beta-\alpha},
\end{aligned}$$

which is finite when either $\dfrac{1}{s} + \dfrac{1}{r} < 1$ or $\dfrac{1}{s} + \dfrac{1}{r} = 1$ with $\beta \le \alpha$. These are exactly conditions (i) and (ii) of the hypothesis.

To prove the necessity, assume $\mathcal{F} : \Lambda^p(w) \to \Lambda^q(u)$ is bounded. By Part (ii) of Theorem 4.30, we have

$$\sup_{0<x<1} \Phi_{u,w}(x) = \sup_{0<x<1} x \left(\int_0^{1/x} u(t)\,dt\right)^{1/q} \left(\int_0^x w(t)\,dt\right)^{-1/p} < \infty.$$

This in particular implies $\lim_{x\to 0} \Phi_{u,w}(x) < \infty$. Now for $0 < x < 1/(N-1)$ we have

$$\int_0^{1/x} u(t)\,dt \ge \int_{N-1}^{1/x} g(t+1)\,dt \ge \int_N^{1/x} g(t)\,dt.$$

Using the same computations as in (5.9), we obtain

$$\lim_{x\to 0} \frac{\int_N^{1/x} g(t)\,dt}{x^{-q/s}(1-\log x)^{q\beta}} = \frac{s}{q}. \qquad (5.10)$$

Now (5.6) and (5.10) yield

$$\lim_{x\to 0}\Phi_{u,w}(x) \geq \lim_{x\to 0}\frac{x(\int_N^{1/x} g(t)\,dt)^{1/q}}{(\int_0^x t^{p/r-1}(1-\log t)^{p\alpha}\,dt)^{1/p}}$$
$$= \left(\frac{s}{q}\right)^{1/q}\left(\frac{p}{r}\right)^{1/p} x^{1-1/r-1/s}(1-\log x)^{\beta-\alpha}.$$

Since the above limit is finite we need to have either $\frac{1}{s}+\frac{1}{r}-1<0$ or $\frac{1}{s}+\frac{1}{r}-1=0$ and $\beta\leq\alpha$. This completes the proof. $\square$

When $\alpha=\beta=0$, we obtain the following necessary and sufficient conditions for Fourier inequalities with Lorentz $L^{r,p}$ norm.

Corollary 5.12. *Assume* $1<p\leq q<\infty$, $2\leq q<s$ *and* $r>1$. *If* $\frac{1}{s}+\frac{1}{r}\leq 1$ *then* $\mathcal{F}: L^{r,p}\longrightarrow \ell^{s,q}$ *is bounded.*
Conversely for $0<p,q<\infty$ *and* $0<r,s<\infty$, *if* $\mathcal{F}: L^{r,p}\longrightarrow \ell^{s,q}$ *is bounded then* $\frac{1}{s}+\frac{1}{r}\leq 1$.

Proof. It follows from Theorem 5.11 with $\alpha=\beta=0$. $\square$

The well-known Hausdorff-Young inequality asserts that $\mathcal{F}: L^p\longrightarrow \ell^{p'}$ is bounded for $1\leq p\leq 2$. As a consequence of the necessary condition in Corollary 5.12, we show the range space in the Hausdorff-Young inequality is optimal among L^p spaces.

Corollary 5.13. *Let* $0<p,q<\infty$. *If* $\mathcal{F}: L^p\longrightarrow \ell^q$ *is bounded then* $q\geq p'$.

Proof. Since $L^{p,p}=L^p$ the hypothesis implies $\mathcal{F}: L^{p,p}\longrightarrow \ell^{q,q}$. Now Corollary 5.12 with $r=p$ and $s=q$ asserts that $\frac{1}{q}+\frac{1}{p}\leq 1$. This means $\frac{1}{q}\leq\frac{1}{p'}$ which is the same as $q\geq p'$. $\square$

Now we compare Theorem 5.11 to some sufficient conditions obtained by Bennett and Rudnick in [BR]. One of their results is the following theorem in interpolation theory. We state the theorem in a slightly less general setting that is suitable for our work.

Theorem 5.14. *Suppose* $0<q\leq\infty$ *and* $-\infty<\beta<\infty$. *Assume* $0<p_1<p_2\leq\infty$ *and* $0<q_1,q_2\leq\infty$ *with* $q_1\neq q_2$. *Let* T *be a linear operator of type* (p_1,q_1) *and* (p_2,q_2). *Suppose* $0<\theta<1$ *and let*

$$\frac{1}{r}=\frac{1-\theta}{p_1}+\frac{\theta}{p_2}\quad\text{and}\quad\frac{1}{s}=\frac{1-\theta}{q_1}+\frac{\theta}{q_2}. \tag{5.11}$$

Then

$$T: L^{r,q}(\log L)^{\beta}\longrightarrow \ell^{s,q}(\log \ell)^{\beta}$$

is bounded.

Proof. See the proof of Theorem B in [BR]. $\square$

Remark 5.15. The above theorem is true for a larger class of operators, namely *quasi linear* operators of *weak type* $(p_1, q_1; p_2, q_2)$. See [BSh] for the definitions.

The Fourier transform is of type $(1, \infty)$ and $(2, 2)$. So for $0 < \theta < 1$ the equations (5.11) read as

$$\frac{1}{r} = 1 - \frac{\theta}{2} \quad \text{and} \quad \frac{1}{s} = \frac{\theta}{2},$$

which after eliminating θ, imply $\frac{1}{r} + \frac{1}{s} = 1$. Notice that the range of θ implies $s > 2$. So Theorem 5.14 includes the following as special case.

Theorem 5.16. *Suppose* $0 < q \leq \infty$ *and* $-\infty < \beta < \infty$. *Assume* $s > 2$ *and let* $\frac{1}{r} + \frac{1}{s} = 1$. *Then* $\mathcal{F} : L^{r,q}(\log L)^{\beta} \longrightarrow \ell^{s,q}(\log \ell)^{\beta}$ *is bounded.*

The scope of this theorem can be extended using the inclusion relations between Lorentz-Zygmund spaces. The results are stated in the next two corollaries.

Corollary 5.17. *Suppose* $0 < p, q \leq \infty$ *and* $-\infty < \alpha, \beta < \infty$. *Assume* $s > 2$, $r > 1$ *and* $\frac{1}{r} + \frac{1}{s} < 1$. *Then* $\mathcal{F} : L^{r,p}(\log L)^{\alpha} \longrightarrow \ell^{s,q}(\log \ell)^{\beta}$ *is bounded.*

Proof. Define s' by $\frac{1}{s'} + \frac{1}{s} = 1$ and observe that the relation between r and s implies $r > s'$. By Theorem 1.36, the space $L^{r,p}(\log L)^{\alpha}$ is embedded into $L^{s',q}(\log L)^{\beta}$. The proof is complete since $\mathcal{F} : L^{s',q}(\log L)^{\beta} \longrightarrow \ell^{s,q}(\log \ell)^{\beta}$ is bounded by Theorem 5.16. $\square$

Corollary 5.18. *Suppose* $0 < p, q \leq \infty$ *and* $-\infty < \alpha, \beta < \infty$. *Assume* $s > 2$ *and let* $\frac{1}{r} + \frac{1}{s} = 1$. *Then* $\mathcal{F} : L^{r,p}(\log L)^{\alpha} \longrightarrow \ell^{s,q}(\log \ell)^{\beta}$ *is bounded in the following cases:*

(*i*) $p \leq q$ *and* $\beta \leq \alpha$.

(*ii*) $p > q$ *and* $\alpha + \frac{1}{p} > \beta + \frac{1}{q}$.

Proof. By Theorem 1.37, either of the above conditions implies that $L^{r,p}(\log L)^{\alpha}$ is embedded into $L^{r,q}(\log L)^{\beta}$. The proof is complete since $\mathcal{F} : L^{s',q}(\log L)^{\beta} \longrightarrow \ell^{s,q}(\log \ell)^{\beta}$ is bounded by Theorem 5.16. $\square$

Notice that our sufficient condition in Theorem 5.11 holds for a smaller range of p, q, s compared to Corollaries 5.17 and 5.18. However, we can combine our necessary condition in Theorem 5.11 with Corollaries 5.17 and 5.18 to characterize the boundedness of $\mathcal{F}$ between Lorentz-Zygmund spaces for a broad range of indices.

Theorem 5.19. *Suppose* $0 < p \leq q < \infty$ *and* $-\infty < \alpha, \beta < \infty$. *Assume* $s > 2$ *and* $r > 1$. *Then* $\mathcal{F} : L^{r,p}(\log L)^{\alpha} \longrightarrow \ell^{s,q}(\log \ell)^{\beta}$ *is bounded if and only if one of the following conditions holds:*

(*i*) $\dfrac{1}{s} + \dfrac{1}{r} < 1.$

(*ii*) $\dfrac{1}{s} + \dfrac{1}{r} = 1$ *and* $\beta \leq \alpha$.

Note that in comparison to Theorem 5.11 the conditions $q \geq 2$, $s > q$ and $p > 1$ are relaxed.

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Curriculum Vitae

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