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## Applications of Stochastic Control in Energy Real Options and Market Illiquidity

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Supervisor Matt Davison *The University of Western Ontario* 

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## Applications of Stochastic Control in Energy Real Options and Market Illiquidity

(Thesis Format: Integrated-Article)

by

Christian  $\underline{Maxwell}$ 

Graduate Program in Applied Mathematics

A thesis submitted in partial fulfillment of the requirements for Doctor of Philosophy

The School of Graduate and Postdoctoral Studies Western University London, Ontario, Canada

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### Abstract

We present three interesting applications of stochastic control in finance. The first is a real option model that considers the optimal entry into and subsequent operation of a biofuel production facility. We derive the associated Hamilton Jacobi Bellman (HJB) equation for the entry and operating decisions along with the econometric analysis of the stochastic price inputs. We follow with a Monte Carlo analysis of the risk profile for the facility. The second application expands on the analysis of the biofuel facility to account for the associated regulatory and taxation uncertainty experienced by players in the renewables and energy industries. A federal biofuel production subsidy per gallon has been available to producers for many years but the subsidy price level has changed repeatedly. We model this uncertain price as a jump process. We present and solve the HJB equations for the associated multidimensional jump diffusion problem which also addresses the model uncertainty pervasive in real option problems such as these. The novel real option framework we present has many applications for industry practitioners and policy makers dealing with country risk or regulatory uncertainty—which is a very real problem in our current global environment. Our final application (which, although apparently different from the first two applications, uses the same tools) addresses the problem of producing reliable bid-ask spreads for derivatives in illiquid markets. We focus on the hedging of over the counter (OTC) equity derivatives where the underlying assets realistically have transaction costs and possible illiquidity which standard finance models such as Black-Scholes neglect. We present a model for hedging under market impact (such as bid-ask spreads, order book depth, liquidity) using temporary and permanent equity price impact functions and derive the associated HJB equations for the problem. This model transitions from continuous to impulse trading (control) with the introduction of fixed trading costs. We then price and hedge via the economically sound framework of utility indifference pricing. The problem of hedging under liquidity impact is an on-going concern of market makers following the Global Financial Crisis.

**Keywords:** Stochastic control, Financial mathematics, Hamilton Jacobi Bellman equations, Ethanol, Real options, Policy, Crush spread, Optimal switching, Optimal stopping, Renewable energy, Regulatory uncertainty, Country risk, Jump diffusions, Partial integro differential equations, Quasivariational inequalities, Numerical methods, Finite differences, Market impact, Illiquidity, Derivatives, Transaction costs, Utility indifference, Market Incompleteness

## The Co-Authorship Statement

All three of my papers included here are co-authored with my Supervisor Matt Davison. I, Christian Maxwell, am the primary author.

## Acknowledgements

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## Preface

This thesis is comprised of some of my collected work during my graduate studies in the Applied Mathematics Department at Western University studying under Dr. Matt Davison. I began my studies at Western in September 2010 in the M.Sc. program and after two years transferred directly into the Ph.D. program. The first article represents some of my work during my Master's research while the latter two articles represent my doctoral research.

# Chapter 1 Introduction

This thesis contains three distinct but related applications of modeling real world problems of decision making under uncertainty, and seeking solutions and the related optimal decision strategies for those problems using optimal control. The models considered here use stochastic processes to model future uncertainty. This chapter begins with some motivating problems from engineering and finance for which stochastic control may provide some insights and solutions. We follow with a very intuitive introduction to the topic of stochastic optimal control and review of stochastic processes. The chapter concludes by introducing the three papers which together comprise the novel contribution of this thesis, and explaining how they relate to stochastic control and each other.

## 1.1 Optimal Control: A Motivation

Consider the example of a plant manager operating a facility that produces electricity from natural gas. The manager is tasked with maximizing her expected profit. The price of electricity and natural gas may be known today, but their future prices are uncertain and subject to price volatility. She knows the fixed running costs associated both with operating the plant at capacity or with idling. The profits over the day while operating can be characterized as

profit = volume produced  $\times$  (electricity price – natural gas and fixed running costs).

The problem becomes more complicated by the fact there is a large fixed cost associated with changing operating status from on/off to off/on. Given uncertain future price outcomes, she must decide whether to incur the cost of turning production off if the plant is running at a loss today. Will the prices bounce back favourably tomorrow or are the expected future losses enough to overcome the cost of changing the plant's production state?

Phrased mathematically, we are given some known operating costs while running in regime  $\alpha$ , and the prices of electricity and gas  $X_t, Y_t$ . Since prices are uncertain we will model them with stochastic processes. Her instantaneous profit rate is given by  $f(X_t, Y_t, \alpha, t)dt$  and is a function of price, cost and whichever operating regime she chooses for the plant (e.g. on, off, half capacity). The income over the life of the plant is J

$$J(X_t, Y_t, t) = \int_t^T f(X_s, Y_s, \alpha, s) ds - \sum \text{switching costs.}$$

The manager may seek a control  $\alpha$  that maximizes her expected earnings over the life of the plant V while minimizing the accrued switching costs.

Another example is a fund manager who wishes to liquidate his position in a certain stock. If his position were sufficiently large that selling his entire inventory at once would cause a significant drop in price due to market impact, the manager might seek an optimal selling rate  $\alpha$  to minimize the expected losses from the liquidation. Since stock prices are challenging to predict and appear random, we model the price as a stochastic process  $S_t$ . The optimal rate  $\alpha$  may be a function of the stock price, his inventory level and target position, and the time remaining until the trade must be closed out T. When buying or selling, there is a temporary market impact on the price because of the bid-ask spread<sup>1</sup>

#### realized execution price = bid-ask impact $\times$ stock price.

This is because market makers offer a lower price to traders seeking to sell stocks and vice versa. In other words, the

bid impact 
$$\leq 1$$
 and the ask impact  $\geq 1$ .

There may additionally be a fixed brokerage cost associated with executing any trade. The problem becomes more complicated when the liquidation rate also has a permanent impact on the stock price. This may occur in especially illiquid markets (i.e. markets with little trading activity).

<sup>&</sup>lt;sup>1</sup>The ask is the price one might expect to pay for a stock, whereas the bid is the price one might expect to receive for selling a stock.

The total income J from liquidating is given by

realized income = bid factor  $\times$  stock price  $\times$  volume sold

or mathematically, if  $f(S_t, \alpha, t)dt$  is the instantaneous revenue

$$J(S_t, t) = \int_t^T f(S_s, \alpha, s) ds$$

Similar to the plant manager, the trader may seek a liquidation strategy that maximizes his expected income.

In both cases, we can write the total optimal value V realized over the lifetime of the plant (or trade) as a functional

$$V(X_t, t) = \max_{\alpha} E[J(X_t, t)].$$

In some cases, simply seeking a strategy that maximizes the expected payoffs may result in too much potential risk. For example, the trader seeking to liquidate his portfolio by strictly minimizing his market impact may trade too slowly and expose himself to excess market risk. He should then seek a balance between minimizing his market impact costs and market exposure over time. The trade-off between market impact and market risk can be modeled using a penalty term for example that penalizes excess risk.

In the following section, we aim to find a method of maximizing or minimizing the functional above, which we will accomplish by using dynamic programming to find the associated Hamilton-Jacobi-Bellman (HJB) equation for the maximum.

#### **1.1.1** Real Options versus Financial Options

The application of Ito calculus and arbitrage free pricing of financial options was pioneered by [4, 15], both published in 1973, and later codified in terms of martingale pricing in [12]. In the years following, many papers and books have been published in the field of financial mathematics. Financial derivatives are written on market underlyings such as stocks, bonds, rates, indexes and the like. Examples include equity call options, interest rate swaps, commodity futures and many other exotic derivative contracts traded on financial exchanges or over the counter.

Later the same mathematical tools developed in the study of financial options were applied to "real" or more tangible assets such as mines, lumber fields, energy plants, and other areas. Since the assets and problems considered were more "real," the framework is called "real options" in contrast with financial options written on intangibles such as rates or stocks.

This real options method of modeling resource project management decisions was introduced in the 1980s by [6] in a seminal paper which considered the problem of optimally starting and stopping production to maximize the profits of a natural resource project. The optimal entry and exit from investment projects was also considered by [8] in another influential real options paper. A classical text on real options was introduced in [9]. Typically the early applications were limited to natural resource and energy projects as the projects were still contingent upon underlyings that may trade in financial markets (e.g. oil futures). Later applications, however, can be found on a wide variety of topics ranging from capital budgeting, corporate strategic planning and competition [20], patents, pharmaceuticals, and R&D project and portfolio management [14, 19, 22], and in a much darker application, suicide [11].

To see how real options relate to financial options, we consider an analogy between an oil field and a call option. A call option allows the holder the right but not the obligation to purchase a stock in the future for some predetermined amount called the strike. If the stock price is worth more than the strike price at the expiry date of the financial contract, the holder may exercise it for a profit. If the stock price is less than the strike, the holder would allow it to expire unexercised as the stock could be had for less on the exchange. Thus the payoff is

call option payoff = 
$$\max(\text{stock price} - \text{strike price}, 0)$$
.

Now consider a firm that holds an undeveloped oil field with some known oil reserves. The firm has the option but not the obligation to develop the field by drilling a well and building a rig. The cost to develop the field is known, and is analogous to the strike price. The value of the oil reserves in terms of reserve volume and price today is known. Its future price, however, is uncertain like the stock price in the call option example. The value of the option to expand and develop the oil field has a payoff very similar to the call option. The real option payoff is

real option payoff = max(reserve volume  $\times$  oil price – development costs, 0).

Using the same financial mathematics tools from option pricing, it is possible to value an oil field.

Real options along with financial options are complicated by market incomplete-

ness. In fact, examining real options often requires the issue of market incompleteness to be confronted head on. Incompleteness is the case when not all the underlying sources of risk can be traded or adequately hedged (e.g. electricity, trading restrictions or blackouts, etc.). Incompleteness may also follow from model uncertainty, transaction costs, or basis risk (say, we have an ethanol biofuel plant and can trade gasoline futures but not ethanol biofuel; gasoline is a similar asset but not a perfect substitute for hedging ethanol price risk). In this case, there is no unique no arbitrage price or hedging strategy for the option. We treat market incompleteness in later chapters (primarily 3 and 4).

## **1.2** Deterministic Optimal Control

To introduce the ideas of stochastic control in a simpler setting, we begin by considering a deterministic optimal control problem. We seek to maximize a functional J of a dynamical system  $x_t$  via a control function  $\alpha(x_t, t)$ . We may drop the arguments from  $\alpha$  for notational simplicity and occasionally refer to its output as simply  $\alpha_t = \alpha(x_t, t)$ . If the profit function is  $f(x_t, \alpha, t)$  and the value of the system in its end state is  $g(x_T, T)$ , the total profit from the system is

$$J(x_0, \alpha, 0) = \int_0^T f(x_t, \alpha_t, t)dt + g(x_T, T)$$

where  $\alpha$  is any admissible control and the dynamics of  $x_t$  are governed by

$$\dot{x}_t = F(x_t, \alpha_t)$$

given some initial condition  $x_0$ .

If the optimum is defined as  $V(x_t, t) = \sup_{\alpha} J(x_t, \alpha, t)$ , then

$$V(x_0, 0) = \sup_{\alpha} \left\{ \int_0^T f(x_t, \alpha_t, t) dt + g(x_T, T) \right\}.$$
 (1.1)

A powerful method of deriving a PDE for the evolution of  $V(x_t, t)$  is to use the dynamic programming principle (DPP) and Bellman's principle of optimality. The dynamic programming method takes this problem on [0, T] and breaks it down into smaller subproblems, each defined on a subinterval of [0, T].

Bellman's Principle of Optimality:

An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision [2].

By Bellman's principle of optimality splitting V from time t to t + dt yields

$$V(x_t, t) = \sup_{\alpha_t} \left\{ f(x_t, \alpha_t, t) dt + V(x_{t+dt}, t+dt) \right\}.$$
 (1.2)

This is a dynamic algorithm because we have broken the problem down into two smaller subproblems on the subintervals [t, t+dt) and [t+dt, T]. Intuitively, assuming we know the optimum function V at some time t + dt and for all time into the future to T, we only need to solve the optimal  $\alpha_t$  at an instant  $(x_t, t)$  to complete the solution for all [t, T]. Since the end state  $V(x_T, T) = g(x_T, T)$  is known, the equation is then solved backward in time to t = 0 (i.e.  $V(x_T, T) \to V(x_{T-dt}, T-dt) \to ... \to V(x_0, 0)$ ).

The Taylor expansion of  $V(x_{t+dt}, t+dt)$  is

$$V(x_{t+dt}, t+dt) = V(x_t, t) + \frac{\partial V}{\partial t}dt + \nabla V \cdot \dot{x}_t dt + O(dt^2)$$

where  $\nabla$  denotes the gradient operator with respect to  $x_t$ . By combining the two equations above and taking the limit as  $dt \to 0$ , we obtain the HJB equation for V,

$$\frac{\partial V}{\partial t} + \sup_{\alpha_t} \left\{ \nabla V \cdot F(x_t, \alpha_t, t) + f(x_t, \alpha_t, t) \right\} = 0, \tag{1.3}$$

subject to the final condition  $V(x_T, T) = g(x_T, T)$ . If we solve over the entire state space,  $x_t$ , and all admissible controls  $\alpha(x_t, t)$  we obtain a necessary and sufficient condition for the maximum [3].

For example, if J is some energy functional, then this becomes similar to the Hamiltonian and action of the dynamical system, which could then potentially be solved using variational calculus [1]. The resulting first variation and associated non-linear PDE is the Hamilton-Jacobi equation or a statement of the principle of minimum energy. The Hamilton-Jacobi-Bellman equation (1.3) is in essence an extension of the Hamilton-Jacobi equation.

In the next section we investigate the topic of stochastic optimal control, but see a similar structure to Equation 1.3, albeit with the addition of a Laplacian term which accounts for the randomness.

### **1.3** Stochastic Optimal Control

We now consider the problem of deriving an optimal control strategy in a stochastic setting. An excellent reference on stochastic control can be found in [18]. Say the state process  $x_t$  evolves stochastically. We rename  $x_t$  as  $X_t$  to conform to the typical convention for denoting a stochastic process. The dynamics of  $X_t$  are governed by the stochastic differential equations (SDE)

$$dX_t = \mu(X_t, \alpha_t, t)dt + \sigma(X_t, \alpha_t, t)dW_t$$
(1.4)

$$X_0 = x, \tag{1.5}$$

for which there exists a solution where  $W_t$  is a Brownian motion defined on a filtered probability space  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$  satisfying the usual conditions [16]. In other words, every singular outcome or possible collection of events has an associated probability and the process history is known. (See [16] for a review of stochastic differential equations.) Our control  $\alpha_t$  is progressively measurable with respect to our filtration and is in the set of all admissible controls A. That is, we know  $\alpha_t$  at time t and it only depends on the information up until t. In particular, we assume  $\alpha_t$  is non-anticipating and Markovian (i.e. a function of only the information at time  $t, \alpha_t = \alpha(X_t, t)$ ).

#### Ito's Lemma

Ito's lemma is one of the main tools used in stochastic analysis and control [16]. It gives the evolution of a function  $u(X_t, t)$  of an SDE  $X_t$  through time. It follows from a Taylor expansion and the finite quadratic variation of Brownian motion. Then for the SDE of Equation 1.5<sup>2</sup>

$$u(X_T, T) - u(x, 0) = \int_0^T \left(\frac{\partial u}{\partial t} + \mu(X_t, \alpha_t, t)\frac{\partial u}{\partial X} + \frac{1}{2}\sigma^2(X_t, \alpha_t, t)\frac{\partial^2 u}{\partial X^2}\right)dt + \int_0^T \sigma(X_t, \alpha_t, t)\frac{\partial u}{\partial X}dW_t.$$
 (1.6)

<sup>&</sup>lt;sup>2</sup>For an *n*-dimensional SDE with *d*-dimensional Brownian motion, the generator is of the form  $du = \left(\frac{\partial u}{\partial t} + \mu^T \nabla u + \frac{1}{2} \operatorname{Tr} \sigma \sigma^T \nabla^2 u\right) dt + \nabla u^T \sigma dW_t$  where  $\nabla$  is the gradient with respect to  $X_t$ ,  $\nabla^2$  is the Hessian, Tr denotes the matrix trace and superscript T is the matrix transpose.

Taking expectations of both sides with respect to  $t = 0, X_0 = x$  yields

$$E[u(X_T,T)] - u(x,0) = E\left[\int_0^T \left(\frac{\partial u}{\partial t} + \mu(X_t,\alpha_t,t)\frac{\partial u}{\partial X} + \frac{1}{2}\sigma^2(X_t,\alpha_t,t)\frac{\partial^2 u}{\partial X^2}\right)dt\right]$$
(1.7)

since E[u(x,0)] = u(x,0) is known and the expectation of the Ito integral of a smooth function  $h(\cdot)$  is zero  $E[\int_0^T h dW_t] = 0$  (specifically h need only be square integrable). Moreover, if this PDE is satisfied

$$\frac{\partial u}{\partial t} + \mu(X_t, \alpha_t, t) \frac{\partial u}{\partial X} + \frac{1}{2} \sigma^2(X_t, \alpha_t, t) \frac{\partial^2 u}{\partial X^2} = 0$$
(1.8)

with final condition

$$u(X_T, T) = g(X_T) \tag{1.9}$$

then its solution is

$$u(x,0) = E[u(X_T,T)] = E[g(X_T)].$$
(1.10)

This result is known as the Feynman-Kac theorem and can be considered as an application of Green's function methods to solving the PDE equations 1.8–1.9.

#### The Stochastic Optimization Problem

Assume our running payoff function  $f(X_t, \alpha_t, t)$  and our final condition  $g(X_T)$  satisfy some mild growth conditions and g is also bounded below. Then our total expected controlled payoff or profit is

$$J(x,\alpha,0) = E\left[\int_0^T f(X_t,\alpha_t,t)dt + g(X_T)\right]$$
(1.11)

where E is the expectation given the information today  $X_0$ . Now we seek to find an optimal control and associated value function V

$$V(x,0) = \sup_{\alpha} J(x,\alpha,0).$$
(1.12)

Hence if  $J(x, \alpha^*, 0) = V(x, 0)$ , we say that the function  $\alpha^*$  is an optimal control.

#### Example controlled diffusion

To build some intuition, we briefly investigate a simple example of a controlled diffusion. Consider a stochastic dynamical process  $X_t$  that one wishes to steer to some final end target B at time T. If  $X_T \neq B$ , some penalty function is applied  $g(X_T, B)$ .



Figure 1.1: The controlled ( $\alpha = 1$ ) contrasted with the uncontrolled diffusion ( $\alpha = 0$ ). The target is B = 1 and the other parameters are  $X_0 = 0, T = 1, \sigma = 0.5$ . The solid blue lines are different possible walks while the thick solid black line is the expected value  $E[X_t]$ . The black dashed lines are the 90% confidence limits.

We consider a possible drift control  $\alpha$  that steers the walk  $X_t$  to some target B

$$dX_t = -\alpha_t (X_t - B)dt + \sigma dW_t \tag{1.13}$$

$$X_0 = 0.$$
 (1.14)

When  $\alpha_t = 0$ , this becomes an uncontrolled diffusion which freely evolves as a continuous random walk. When  $\alpha_t > 0$ , depending on the intensity of  $\alpha_t$ , the walk targets B and any deviation from B is always corrected by the drift; the larger the deviation, the stronger the correction. See Figure 1.1 for an illustration.

Say a cost is incurred by correcting the drift  $f(X_t, \alpha_t)$ , then we may seek a control strategy function  $\alpha_t = \alpha(X_t, B, t)$  that minimizes our total cost V given our current state, target and time remaining

$$V(X_0, 0) = \inf_{\alpha} E\left[\int_0^T f(X_t, \alpha_t) dt + g(X_T, B)\right].$$
 (1.15)

Not every stochastic optimization problem necessarily has diffusion parameters that can themselves be controlled. In Chapter 2, we consider a stochastic control problem where an agent seeks to maximize his profit by switching production on and off given a stochastic price processes in his operating revenues. The decisions have no effect on the price diffusion parameters. Chapter 3 includes an example where the diffusion parameters are "controlled" in the sense that the operator of the facility of Chapter 2 assumes the econometric parameter estimates come from some worst case distribution. The parameters for the stochastic profit process are still not affected by the decisions. Finally we consider a situation in Chapter 4 where the price process parameters are adversely affected by the agent's decision making. The agent of Chapter 4 is a trader trying to hedge his portfolio position but every buy/sell pushes the price up/down (generally adversely) for the trader.

#### **1.3.1** The Dynamic Programming Principle

As with its deterministic counterpart, the fundamental tool of stochastic optimal control is the DPP and Bellman optimality. The DPP can be stated in a stochastic setting as

$$V(x,0) = \sup_{\alpha \in A} E\left[\int_0^\theta f(X_t, \alpha_t, t)dt + V(X_\theta, \theta)\right]$$
(1.16)

where  $\theta \in (0,T)$  is any stopping time. As we are now considering stochastic uncertain quantities, we use expectations to arrive at a result otherwise similar to the deterministic Bellman statement Equation 1.2.

As in the previous section, we can intuitively interpret the stochastic DPP as follows: Divide the problem into two subintervals (hence it is "dynamic"),  $[0, \theta)$  and  $[\theta, T]$ . Provided we have already solved for the optimal control from time T working back to  $\theta$ , we need only to optimize the controls over the region  $[0, \theta)$ . We begin at the known end condition t = T and then incrementally work back in time to t = 0.

### 1.3.2 The HJB Equation in Stochastic Control

Consider if we start from an initial condition  $X_0 = x$  and take  $\theta = \Delta t$  and  $\alpha_t = a$ where a is any constant (possibly suboptimal) control in A. Then by the DPP (1.16),

$$V(x,0) \ge E\left[\int_0^{\Delta t} f(X_t, a, t)dt + V(X_{\Delta t}, \Delta t)\right]$$
(1.17)

since, again, a may be a suboptimal control. If V is  $C^{1,2}$  continuous, we may apply Ito's lemma (1.7) through  $\Delta t$  and cancel both V(x,0) and  $V(X_{\Delta t}, \Delta t)$  to obtain

$$0 \ge E\left[\int_0^{\Delta t} f(X_t, a, t)dt + \int_0^{\Delta t} \left(\frac{\partial V}{\partial t} + \mathcal{L}_a[V(X_t, t)]\right)dt\right].$$
 (1.18)

where  $\mathcal{L}_a$  is the generator associated with the controlled diffusion  $X_t$  with  $\alpha_t = a$ 

$$\mathcal{L}_a[V(x,t)] = \mu(x,a)\frac{\partial V}{\partial x} + \frac{1}{2}\sigma^2(x,a)\frac{\partial^2 V}{\partial x^2}.$$
(1.19)

Then if we divide by  $\Delta t$  and let  $\Delta t \to 0$ , we arrive at the inequality

$$\frac{\partial V}{\partial t} + \mathcal{L}_a[V(X_t, t)] + f(X_t, a, t) \le 0$$
(1.20)

using the mean value theorem. Further, assume now that  $a = \alpha_t^*$ , the local optimal control, then by comparing the above inequality with the DPP (1.17), it is intuitively reasonable that there must be equality

$$\frac{\partial V}{\partial t} + \mathcal{L}_{\alpha_t^*}[V(X_t^*, t)] + f(X_t^*, \alpha_t^*, t) = 0.$$
(1.21)

Comparing the PDE (1.21) with the Bellman statement (1.16), we claim that

$$\frac{\partial V}{\partial t} + \sup_{\alpha_t} \left\{ \mathcal{L}_{\alpha_t}[V(X_t, t)] + f(X_t, \alpha_t, t) \right\} = 0.$$
(1.22)

This is the associated HJB equation for the stochastic control problem which must be solved for all  $X_t, t$  to recover the globally optimal control function  $\alpha$  where  $\alpha_t = \alpha(X_t, t)$ . Thus if we can find a supremum function  $\alpha(x, t) = \alpha^*(x, t)$  as defined in Equation 1.22, then by the Feynman-Kac theorem

$$V(x,0) = E\left[\int_0^T f(X_t^*, \alpha_t^*, t)dt + g(X_T^*) \middle| X_0 = x\right]$$
(1.23)

where  $X_t^*$  is the solution to the controlled diffusion using the optimal control  $\alpha^*$ .

This simplifies the problem of finding the global optimal control and associated value function V. One must find the local maximizing argument  $\alpha_t$  of Equation 1.22 and solve the associated PDE at this argument. The PDE then yields the value function and the optimal control.

#### Verification Theorem

The verification theorem states that a smooth solution to the HJB equation coincides with the value function. The version we state below is taken from [18]. Suppose that u is a sufficiently smooth function that satisfies certain conditions and suppose further that

$$\frac{\partial u}{\partial t} + \sup_{\alpha_t \in A} \left\{ \mathcal{L}_{\alpha_t}[u(x,t)] + f(x,\alpha_t,t) \right\} \le 0$$
(1.24)

and that  $u(x,T) \ge g(x)$ . Then  $u \ge V$  of Equation 1.22  $\forall (x,t)$ . To illustrate why this is true, consider if we were to rearrange the above and apply a time reversal transformation,  $\tau = T - t$ ,

$$\frac{\partial u}{\partial \tau} \ge \sup_{\alpha_t \in A} \left\{ L_{\alpha_t}[u(x,t)] + f(x,\alpha_t,t) \right\}.$$
(1.25)

Hence, as we work back in time  $\frac{\partial u}{\partial \tau} \geq \frac{\partial V}{\partial \tau}$  due to the inequality above and, if  $u(x,T) \geq g(x)$ , the result  $u \geq V$  is guaranteed.

Assume now that u(x,T) = g(x) and we can find a measurable function  $\alpha^*(x,t)$ such that

$$\frac{\partial u}{\partial t} + \sup_{\alpha_t \in A} \left\{ \mathcal{L}_{\alpha_t}[u(x,t)] + f(x,\alpha_t,t) \right\} = \frac{\partial u}{\partial t} + \mathcal{L}_{\alpha_t^*}[u(x,t)] + f(x,\alpha_t^*,t) = 0.$$
(1.26)

Then by uniqueness u = V and  $\alpha^*$  is an optimal control.

## **1.4** Stochastic Control and Optimal Switching

Consider a situation where the diffusion and running profit function can be controlled by switching states or regimes via an impulse. Suppose there are m possible states denoted as  $\mathbf{I}_m = \{1, 2, ..., m\}$  where  $1 < m < \infty$ . The control then consists of a sequence of stopping times  $\alpha = (\tau_n, i_n)$  with  $n \ge 0$ . Here,  $0 \le \tau_n \le T$  denotes an increasing sequence of stopping times (when to switch) with corresponding states  $i_n$ (where to switch).

Each regime is characterized by its own running payoff function  $f_i(X_t, t)$  (which is assumed to satisfy some mild growth conditions [18]) and the cost to switch from regime *i* to *j*,  $D_{ij}$ . A "triangular" cost inequality is required that  $D_{ij} \ge 0$  and  $D_{iq} + D_{qj} > D_{ij}$  to ensure that no strategies instantaneously switch via intermediate states. The triangular inequality ensures that trying to reach state *j* from *i* via state *q* is more costly than simply going directly from *i* to *j*. This also ensures that the continuation regions are non-empty and the switching boundaries are "regular" which are conditions required for existence of impulse control solutions [16]. The diffusion satisfies

$$dX_t = \mu(X_t, I_t, t)dt + \sigma(X_t, I_t, t)dW_t$$
(1.27)

$$X_0 = x \tag{1.28}$$

where  $I_t = \sum_{0 \le k \le n} i_k \mathbb{1}_{\{\tau_k \le t < \tau_{k+1}\}}$  given  $\tau_0 = 0$  and  $i_0 = i$ .

The gain functional is then

$$J_i(x,0,\alpha) = E\left[\int_0^T f_{I_t}(X_t,t)dt + \sum_{k=1}^n D_{i_{k-1},i_k}\right]$$
(1.29)

where the expectation is taken with respect to the initial state x, i at t = 0. Hence, the value function follows from the optimal control

$$V_i(x,0) = \sup_{\alpha} J_i(x,0,\alpha).$$
 (1.30)

For simplicity, we assume here that g(x) = 0 and hence  $V_i(x, T) = 0$  (although we later relax this assumption to include more general forms of  $g(\cdot)$  in our articles).

## 1.4.1 Dynamic Programming Principle and Variational Inequalities

In the optimal switching case, the dynamic programming principle leads to a system of free boundary problem PDEs with interconnected obstacles or quasi-variational inequalities [18]. Then following the DPP

$$V_i(x,0) = \sup_{\tau} E\left[\int_0^{\tau} f_i(X_t,t)dt + \max_{j \in \mathbf{I}_m} \{V_j(X_{\tau},\tau) - D_{ij}\} 1_{\{\tau < T\}}\right]$$
(1.31)

with  $V_i(x,T) = 0$  and  $j \neq i$ .

As an intuitive derivation, consider the evolution of V from 0 to  $\Delta t$  with reference to Equation 1.31.

$$V_{i}(x,0) = E\left[\int_{0}^{\Delta t} f_{I_{t}}(X_{t},t)dt + V_{I_{\Delta t}}(X_{\Delta t},\Delta t)\right].$$
 (1.32)

If it is optimal not to switch over  $0 \le t < \Delta t$  (i.e.  $\tau \notin [0, \Delta t)$ ), then by Ito's lemma

(1.7) and canceling  $V_i$  on both sides

$$0 = E\left[\int_0^{\Delta t} f_i(X_t, t)dt + \int_0^{\Delta t} \frac{\partial V_i}{\partial t} + \mathcal{L}_i[V_i(X_t, t)]dt\right].$$
(1.33)

By dividing by  $\Delta t$ , letting  $\Delta t$  go to zero, and having noted that the expectation of an Ito integral is zero, we see that  $V_i(X_t, t)$  satisfies the PDE

$$\frac{\partial V_i}{\partial t} + \mathcal{L}_i[V_i(X_t, t)] + f_i(X_t, t) = 0.$$
(1.34)

The implication, given that it is not optimal to switch, is that

$$V_i(X_t, t) > V_j(X_t, t) - D_{ij} \ \forall \ j \neq i.$$

$$(1.35)$$

On the other hand, in the limit that  $\Delta t \to 0$  and if  $\tau \in [0, \Delta t)$ , then the integral vanishes and

$$V_i(x,t) = E\left[\int_0^\tau f_i(X_t,t)dt + \max_{j \in \mathbf{I}_m} \{V_j(X_\tau,\tau) - D_{ij}\}\right] = V_j(x,t) - D_{ij}, \ j \neq i. \ (1.36)$$

Given that it is optimal to switch and  $\tau \in [0, \Delta t)$ , then the strategy of continuing in the current state  $(I_t = i)$  is suboptimal and using (1.32) we obtain

$$V_i(x,0) > E\left[\int_0^{\Delta t} f_i(X_t,t)dt + V_{I_{\Delta t}}(X_{\Delta t},\Delta t)\right].$$

Hence it must be that

$$\frac{\partial V_i}{\partial t} + \mathcal{L}_i[V_i(X_t, t)] + f_i(X_t, t) < 0$$
(1.37)

when it is optimal to switch.

Thus the problem can be restated as a system of variational inequalities

$$\max\left\{\frac{\partial V_i}{\partial t} + \mathcal{L}_i[V_i(X_t, t)] + f_i(X_t, t), \max_{j \in \mathbf{I}_m} \left[ (V_j(X_t, t) - D_{ij}) \right] - V_i(X_t, t) \right\} = 0$$
(1.38)

for  $i = \{1, \ldots, m\}$  and  $j \neq i$  where  $\mathcal{L}_i$  is the infinitesimal generator associated with regime *i*.

The variational inequality yields a set of regions in space. We define the continuation region  $C_i$  where it is both optimal to continue (i.e. not to switch) and the PDE is satisfied. Hence,

$$C_i = \left\{ x, t : \frac{\partial V_i}{\partial t} + \mathcal{L}_i[V_i(x, t)] + f_i(x, t) = 0 \right\}$$
(1.39)

$$= \{x, t: V_i(x, t) > V_j(x, t) - D_{ij} \forall j\}.$$
(1.40)

Similarly, the switching region from i to j,  $S_{ij}$  is

$$S_{ij} = \{x, t : V_i(x, t) = V_j(x, t) - D_{ij}\}.$$
(1.41)

Further, we can say that:

- $S_i$ , the general switching region from *i* to any alternative state is  $S_i = \bigcup_{i \neq j} S_{ij}$ .
- $S_{ij} \subseteq C_j$  (following from "triangle" inequality).
- $C_i = S_i^c$ , that is  $S_i$  is the complement of  $C_i$ ;  $S_i \cup C_i = [0, T] \times R_n$ ; and  $S_i \cap C_i = \emptyset$ .

#### **Continuity of Classical Solutions**

Classical solutions  $V_i$  are  $C^{1,2}$  continuous on  $C_i$  and  $C^{1,1}$  continuous along the switching boundary  $\partial S_i$  [16, 18]. For a solution of the above obstacle problem to be optimal, the so-called "smooth pasting" or "high contact" [15] condition must hold along the exercise (switching) boundary  $\partial S_{ij}$ 

$$\nabla V_i(x,t) = \nabla V_i(x,t). \tag{1.42}$$

For example, see [5, 21].

Intuitively, the smooth-pasting condition is just the first order optimality condition. As a simpler example, consider an American put option written on a stock  $S_t$ struck at K expiring at time T. We follow an exercise strategy  $\alpha_t = \alpha(t)$  where we exercise the put when  $S_t = \alpha_t$ . Thus we seek an optimal exercise price boundary  $\alpha^*$ that maximizes the put option's value. The payoff g is

$$g(S_t) = K - S_t$$

The expected value function v following any exercise strategy  $\alpha(t)$  is

$$v(S_0, \alpha_0) = E[g(S_\tau)].$$

where  $\tau = \inf\{t \in [0,T] : S_t = \alpha_t\}$  upon exercise  $S_t = \alpha_t, v(S_t, \alpha_t) = g(S_t)$ . The smooth pasting principle can be verified by differentiating with respect to  $\alpha_t$  at  $S_t = \alpha_t$ 

$$\frac{d}{d\alpha_t}v(S_t = \alpha_t, \alpha_t) = \frac{d}{d\alpha_t}g(S_t = \alpha_t)$$
$$\frac{\partial v}{\partial S_t} + \frac{\partial v}{\partial \alpha_t} = \frac{\partial g}{\partial S_t}$$

since  $\frac{dS_t}{d\alpha_t} = 1$  at  $S_t = \alpha_t$ . If  $\alpha_t = \alpha_t^*$  is the optimum, then by the first order optimality condition

$$\left. \frac{\partial v}{\partial \alpha_t} \right|_{\alpha_t^*} = 0$$

and thus the smooth pasting condition holds

$$\left. \frac{\partial v}{\partial S_t} \right|_{\alpha_t^*} = \left. \frac{\partial g}{\partial S_t} \right|_{\alpha_t^*}$$

This is a special case of the switching case above.

## 1.5 The Mixed Case

In two of the articles, we present a stochastic control model with a mixed case where the controlled diffusion is subject to both impulse and stopping controls. Consider a value function with optimal control

$$V(x,0) = \sup_{\alpha,\tau} E\left[\int_0^{\tau} f(X_t, \alpha_t, t)dt + h(X_{\tau}, \tau) \mathbf{1}_{\{\tau < T\}}\right].$$
 (1.43)

Following the DPP and the pattern before, the associated HJB variational inequality is

$$\max\left\{\frac{\partial V}{\partial t} + \sup_{\alpha_t} (\mathcal{L}_{\alpha_t}[V(x,t)] + f(x,\alpha_t,t)), \quad h(x,t) - V(x,t)\right\} = 0.$$
(1.44)

We briefly note that in many cases, the HJB equation does not have smooth classical solutions [10]. In these cases, we must interpret the solutions in a viscosity sense [7].

#### Viscosity Solutions

For the highly nonlinear partial integro differential variational inequalities (HJB equations) considered in the chapters that follow, classical smooth solutions may not exist. This begs the question of how to interpret nonsmooth solutions to differential equations that may not be differentiable everywhere. Crandall and Lions [7] provide an insightful way to interpret weak solutions to these HJB equations called "viscosity" solutions. The definition of viscosity solutions presented here below follows [17].

We say that V is a viscosity supersolution of

$$\max\left\{\frac{\partial V}{\partial t} + \sup_{\alpha_t} (\mathcal{L}_{\alpha_t}[V(x,t)] + f(x,\alpha_t,t)), \quad h(x,t) - V(x,t)\right\} = 0$$
(1.45)

$$V(x,T) = g(x) \tag{1.46}$$

if for every test function  $\phi \in C^{2,1}$  and any  $(x^*, t^*)$  such that  $V(x, t) \leq \phi(x, t) \forall (x, t)$ and  $V(x^*, t^*) = \phi(x^*, t^*)$  (so  $V - \phi$  achieves its maximum, zero, at  $(x^*, t^*)$ ), the condition holds

$$\max\left\{\frac{\partial\phi}{\partial t} + \sup_{\alpha_t} (\mathcal{L}_{\alpha_t}[\phi(x,t)] + f(x,\alpha_t,t)), \quad h(x,t) - V(x,t)\right\} \le 0.$$
(1.47)

Note that the differential operator  $\mathcal{L}$  should satisfy an ellipticity condition (i.e. non-negative volatility).

Similarly we say that V is a viscosity subsolution of 1.45 if V satisfies the terminal condition g and for every test function  $\phi \in C^{2,1}$  and any  $(x_*, t_*)$  such that  $\phi \geq V$  and  $V(x_*, t_*) = \phi(x_*, t_*)$ , the condition is satisfied

$$\max\left\{\frac{\partial\phi}{\partial t} + \sup_{\alpha_t} (\mathcal{L}_{\alpha_t}[\phi(x,t)] + f(x,\alpha_t,t)), \quad h(x,t) - V(x,t)\right\} \ge 0.$$
(1.48)

If V is both a viscosity subsolution and supersolution, then we say V is a viscosity solution of 1.45-1.46.

Loosely speaking, if we cannot differentiate the optimal solution V itself, we can see it as a limiting sequence of a sufficiently differentiable family of (possibly suboptimal) test functions at  $(x^*, t^*)$ . Its limit is the optimal control solution to the HJB equation which was verified by derivation

$$V(x,t) = \sup_{\alpha,\tau} E\left[\int_t^\tau f(X_s)ds + h(X_\tau)\mathbf{1}_{\{\tau < T\}} + g(X_T)\mathbf{1}_{\{\tau \not < T\}} \middle| X_t = x\right].$$

(Of course, this definition does not require V to belong to this or any family of test functions or to be differentiable at all. It need only satisfy the conditions above.) For additional reading and resources on stochastic control, see [10, 13, 16, 17, 18, 23].

#### Discounting

When the model includes discounted cash flows, a reaction term is added to the parabolic PDEs. For example, with continuously compounded discounting at rate r in the switching problem, the DPP statement is

$$V_i(x,0) = \sup_{\tau} E\left[\int_0^{\tau} e^{-rt} f_i(X_t, t) dt + \max_{j \in \mathbf{I}_m} e^{-r\tau} \left\{ V_j(X_\tau, \tau) - D_{ij} \right\} \mathbf{1}_{\{\tau < T\}} \right], \quad (1.49)$$

and the variational inequalities become

$$\max\left\{\frac{\partial V_i}{\partial t} + \mathcal{L}_i[V_i(x,t)] + f_i(x,t) - rV_i(x,t), \max_{j\in\mathbf{I}_m}\left[\left(V_j(x,t) - D_{ij}\right)\right] - V_i(x,t)\right\} = 0$$
(1.50)

where  $i \neq j$ .

## 1.6 The Relationship Among the Three Integrated Articles

All three articles presented herein are applications of stochastic control. In Chapter 2, we first consider the example of a biofuel production plant which produces ethanol from corn. Both commodities are traded on financial markets and are subject to spot price uncertainty, hence we model both as stochastic processes. The plant manager has several layered exercise decisions: She may build the biofuel plant given its expected future income (optimal starting—a variant of optimal stopping) and second, given the price and profitability conditions, the manager may switch production on or off (optimal switching).

Our second (Chapter 3) article considers the same biofuel plant but with some added complexity in addition to the optimal operating (switching) problem. First, an additional stochastic factor is introduced by means of a jump process representing the biofuel subsidy offered to the producer. This increases the dimensionality of the problem. Further, the parameters of that jump process are uncertain. The manager assumes a worst case pricing scenario and adjusts the parameters accordingly (controlled jump diffusion). The final application in Chapter 4 is related to the first two in that the numerical tools and associated HJB equations are very similar. We consider a market maker attempting to hedge his equity derivative position in the presence of market impact and transaction costs. His rate of trading affects the stock price (controlled diffusion) and if fixed costs are associated with every trade, he must make finite block trades (impulse control).

All three applications utilize the DPP, stochastic optimal control and similar numerical methods via finite differences to obtain solutions.

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# Chapter 2

# Using Real Option Analysis to Quantify Ethanol Policy Impact on the Firm's Entry into and Optimal Operation of Corn Ethanol Facilities

#### Chapter Summary:

Ethanol crush spreads are used to model the value of a facility which produces ethanol from corn. A real option analysis is used to investigate the effects of model parameters on the related managerial decisions of (i) how to operate the facility through optimal switching from idled to operational status and (ii) the decision to enter into the project given its expected real option net present value. We present evidence of increased correlation between corn and ethanol prices, perhaps as a result of government policy which has induced more players to enter into the market. This paper investigates the subsequent negative effects on firms. Further, this paper illustrates the impact of an abrupt change in government policy, as happened in January 2012, on a firm's decision to enter the business.

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# 2.1 Introduction

In recent decades, efforts to promote US energy independence from foreign oil [24, 50] and initiatives to obtain more fuel from environmentally friendly sources have led to increased subsidies to the production of ethanol from corn. These projects have been subsidized via policies such as the volumetric ethanol excise tax credit provided to domestic ethanol biofuel blenders and the small ethanol producer tax credit. The amount of subsidy for blenders has changed from \$0.40/gallon in 1978 (Energy Tax Act) to \$0.60/gallon at its peak in 1984 (Tax Reform Act) [35]. With the introduction of the 2008 Farm Bill, the subsidy was reduced to \$0.45/gallon and many subsidies expired at the beginning of 2012. Before its December 2011 expiry, the small ethanol producer tax credit was \$0.10/gallon. This credit applied to the first 15 million gallons of annual production for a producer whose capacity did not exceed 60 million gallons [34].

The corn ethanol process has been criticized on several grounds. Environmental critics claim that the process is energy negative in that more carbon-based energy is used to grow and convert the corn into ethanol than is produced through the process [47, 45]. Public choice critics claim that the ethanol subsidies may be a result of seeking rents and lobbying [51, 55] or that production must receive subsidies to become sufficiently economically attractive (a point discussed in this paper). Other critics claim that diverting corn from food or feed consumption has caused an increase in food prices and price variability [36, 39, 40] and that ethanol subsidies have other effects on social welfare [31]. A year following the lapse of these subsidies, about one quarter of Nebraska's ethanol plants were in idle status [42]. It is possible that reduction in ethanol policy was a contributing factor.

In this paper, real option analysis is used to assess the optimal operating strategy for an ethanol production facility from management's perspective. In addition, the firm's decision of when to optimally enter the business of ethanol production is also analyzed. The model aims to realistically capture the flexibility inherent in the full life of the project through the ability to switch production on and off. There is a cost associated with switching production which means management faces a "wait and see" period before making a decision to change production. This resulting decision is non-trivial and must be modelled as a stochastic optimal control problem.

This real option method of modelling resource project management decisions was introduced by [28] in a seminal paper which considered the problem of optimally starting and stopping production to maximize the profits of a natural resource project. The optimal entry and exit from investment projects was also considered by [32] in another classical real option paper.

Our paper also investigates the effects of increased price correlation between ethanol and corn resulting from the diversion of corn from feedstock to fuel [36, 39, 40]. Investigating correlation is expedient because it follows from straightforward economic arguments reducible to a single parameter. Earlier work in this area has focused on changes in correlation over time. [37] suggest it may have increased; [49] present evidence it may have decreased. In either scenario, investigating its impact on pricing and operating decisions is important. Further, the effects of policy subsidy on project value are also investigated.

This paper represents a direct extension of the analysis in [37] and uses similar methods to those presented in [49]. [37] use a bootstrap Monte Carlo analysis to estimate the value of an ethanol production facility modeled as a strip of exchange options. Our paper expands the analysis in [37] to capture the rational operator's optimal strategy which hinges on the "wait and see" phenomenon. [49] investigate the effects of ethanol policy on prices and the firm's decision to enter into and divest itself from the business on an infinite time horizon. This paper adds to their analysis by (1) using a finite time horizon for entry into the project in keeping with for instance a private equity firm's finite investment horizon requirements; and (2) investigating its subsequent optimal operation once initiated. The effect of ethanol policies on prices are investigated using increases in a simple ethanol-corn correlation metric designed to capture the increased linkage between the two markets.

Section 2.2 assembles a framework including price models, parameters for management flexibility and rules for optimal operation. Section 2.3 illustrates concepts and heuristic results from similar closed-form models while Section 2.4 contains the numerical results from the full analysis. Finally, Section 2.5 presents insights and conclusions from the investigation.

## 2.2 Assembling the Model

Firms have the flexibility to begin or defer projects given current economic and price environments, a flexibility not captured by net present value (NPV) or discounted cash flow (DCF) analyses, as described in [33].

After entering into an ethanol project, management has the ability to pause and resume production given price conditions and their profits. This enables management to capitalize on the upside profits while mitigating the downside losses. Again a simple NPV DCF analysis would fail to capture the true value of flexibility given uncertain (stochastic) future prices.

The goal of this paper is to examine how ethanol policy affects producers business entry and subsequent facility operation decisions given price conditions, subsidy expectations, and the remaining facility life.

To develop a model, the following inputs are required:

- 1. Equations representing the plant economics including capitalized costs to construct the facility, costs to pause and resume production, and instantaneous profit as a function of ethanol and corn prices; and
- 2. A stochastic model for corn and ethanol prices including econometric analysis of the relevant parameters.

Throughout this paper, all currency units are United States dollars (USD); all liquid volume units are gallons (1 US gal = 3.785 L); all solid volume units are bushels (1 US bushel = 0.0352 m<sup>3</sup>); all weight units are in tons (1 short ton = 2000 pounds = 907.185 kg); and all interest rates are percent per year and appropriate to USD deposits.

#### 2.2.1 A Model for the Plant

The plant produces ethanol (priced in USD/gallon) from corn (priced in USD/bushel).

#### Reaction models and instantaneous running profits

The running profit from the corn-ethanol crush spread is developed on a per bushel per year (\$/bushel-year). Our analysis uses the standard reaction from [26] for the popular dry grind process of producing ethanol

$$\operatorname{corn} \to \operatorname{ethanol} + \operatorname{byproducts},$$
 (2.1)

which implies the profit function

$$f(L_t, C_t) = \kappa(L_t - K) + \omega A_t - C_t, \qquad (2.2)$$

The net running cost K may further be decomposed into the fixed running cost p less any government volumetric subsidy s,

$$K = p - s. \tag{2.3}$$

 $C_t$  is the price of corn per bushel;  $L_t$  is the price of ethanol per gallon; and  $A_t$  is the price of byproduct distillers dried grains in dollars per ton. The process produces 17 lbs of distillers dried grains per bushel of corn and consequently  $\omega = 17/2000.^1$  The conversion factor,  $\kappa = 2.8$ , represents how many gallons of ethanol are produced per bushel of corn; taken from [26] and is consistent with the CME Group's references on ethanol crush spreads [29]. A subsidy of \$0.10/gallon was used along with a fixed running cost of \$0.68/gallon for facilities with nameplate capacities of 40,000,000 gallons/year [48].

The analysis is simplified by considering two stochastic factors, ethanol and corn, independently; while accounting for each additional factor with affine terms. This yields a simple instantaneous running profit function

$$f_1(L_t, C_t) = \kappa (L_t - K_1) - C_t \tag{2.4}$$

on a per bushel consumed per year basis. Average distillers dried grains,  $A_t$ , is one constituent of the parameter  $K_1$ 

$$K_1 = p - \frac{\omega}{\kappa} \bar{A}_t - s. \tag{2.5}$$

While production is idle, [48] estimated that fixed running costs are roughly 1% of capitalized cost per gallon of capacity, B, or roughly 20% of fixed running cost while in production. Our analysis takes the average of these two fixed running cost estimates. While production is halted there is no subsidy since no ethanol is being produced. The profit function while off is

$$f_0(L_t, C_t) = -\kappa K_0 \tag{2.6}$$

where

$$K_0 = \left(\frac{0.20p + 0.01B}{2}\right) \tag{2.7}$$

is the midpoint of the two possible estimates of  $K_0$ .

#### Switching and Capitalized Construction Costs

For a medium-sized facility (40,000,000 gallons/year) [49] estimated a capitalized cost of 1.40/gallon is required to construct a turn-key facility from a green field (while maintenance costs are included in the fixed running costs K). The medium-sized

 $<sup>^{1}</sup>$ There are 2000 lbs in a ton and distillers dried grain prices are quoted in USD/ton



Figure 2.1: Historical ethanol-corn price series from Jan/02-Dec/11.

facility is taken as the representative model which also qualifies for the small ethanol producer subsidy being less than 60,000,000 gallons in capacity. Costs to resume production from an idle state are estimated by [48] to be 10% of capitalized cost per gallon of capacity; costs to pause production from an active state are estimated to be 5% of capitalized cost; finally the liquidation value at the end of facility life is estimated to be 10% of capitalized cost.

#### 2.2.2 Models of the Prices

Ethanol and corn prices are modeled as stochastic geometric Brownian motion (GBM) processes in this analysis. Despite some well-known drawbacks, GBM is very popular in mathematical finance and financial economics due to its simplicity and desirable properties for modeling financial time series (e.g. nonnegativity, volatility proportional to price level, etc.). The historical price series from Dec/02–Jan/11 is shown in Figure 2.1.

A GBM random process  $X_t$  follows the stochastic differential equation (SDE):

$$dX_t = \mu X_t dt + \sigma X_t dW_t,$$

where  $\mu$  is its drift (average rate of continuously compounded growth) and  $\sigma$  is its volatility. The differential increment of Brownian motion  $dW_t$  corresponding to the

interval between t and t+dt is drawn from the normal random variable with zero mean and variance dt, independent of other such draws on non-overlapping time intervals.

Although intuitively one might expect commodity prices to be mean-reverting to some equilibrium price, it is generally hard to observe statistically. Accordingly, our model choice is reasonable since statistical tests on the time series in [37] rejected mean-reversion and seasonality. It was found however that the data exhibit serial autocorrelation. The effects of autocorrelation in the drift of the lagged process was found to be statistically zero in [49] and hence serial correlation is ignored in our paper's analysis; the time series were also subjected to augmented Dickey-Fuller tests which found weak evidence against the presence of unit roots and hence the time series can be treated as stationary. This also allows the use of well-developed theory of Markov processes and Ito calculus in the analysis that follows.

The logarithm of a GBM process  $\ln X_t$  follows an even simpler constant volatility arithmetic Brownian motion (ABM) process

$$d\ln X_t = \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma dW_t$$

The econometric parameters are estimated by ordinary least-squares regression. The differenced ABM series  $\Delta \ln X_t = \ln \frac{X_t}{X_{t-1}}$  has representation

$$\Delta \ln X_t = \left(\mu - \frac{1}{2}\sigma^2\right)\Delta t + \sigma\sqrt{\Delta t}\xi.$$

where  $\Delta t = t_i - t_{i-1}$  and  $\xi \sim N(0, 1)$ . Thus the parameters may now be estimated via

$$\Delta \ln X_t = \beta_0 + \epsilon \xi,$$

where the constant term is the drift of the series and the volatility is read directly from the root mean squared error of the innovation  $\epsilon\xi$ 

$$\beta_0 = \left(\mu - \frac{1}{2}\sigma^2\right)\Delta t,$$
  

$$RMSE = \sqrt{\epsilon} = \sqrt{\Delta t}\sigma.$$

Estimates of the correlation,  $\rho$ , between two time series are obtained via the sample correlation of the residuals.

Prices for the no. 2 Omaha, Nebraska yellow corn used to underpin the standard CME contract were obtained from the US department of agriculture feed grains

Parameter estimate	Value	t-test
$\hat{\mu}$	0	$P\left(\frac{\hat{\mu}-\mu}{s.e.} >  t  \mid \mu = 0\right) = 0.409$
$\hat{\sigma}$	0.156	-
â	0	$P\left(\frac{\hat{a}-a}{s.e.} >  t    a = 0\right) = 0.202$
$\hat{b}$	0.123	-
$\hat{ ho}$	0.105	_

Table 2.1: Parameter estimation results.

database [52]. Average rack price freight on board Ethanol prices were obtained from the Nebraska Energy Office [43]. Nebraska data was selected to reflect the size of Nebraska in US corn and ethanol markets and to be reflective of national prices [41]. To be consistent with [37], 10 years of monthly historical price data was used spanning the period between Jan/02–Dec/11.

For simplicity, the relatively small inflation adjustment for prices over this 10 year period were ignored. Note that inflation enters into the price dynamics via the drift, the specification of which does not affect the estimates for volatility and correlation.

Ethanol and corn are modeled by correlated GBMs with SDEs

$$dL_t = \mu L_t dt + \sigma L_t dW_{1t}, \qquad (2.8)$$

$$dC_t = aC_t dt + bC_t dW_{2t}, (2.9)$$

$$\operatorname{Corr}[W_{1t}, W_{2t}] = \rho.$$
 (2.10)

The drifts of both ethanol and corn did not reject the null hypothesis of zero drift at the 95% confidence interval. The annualized results are summarized in Table 2.1.

The estimate for the average distillers dried grains price  $\bar{A}_t$  was estimated by regressing the time series against a constant. The result was  $\hat{A}_t = 115.6$  with a standard error of the estimator (*s.e.*) of 3.6. At the 95% student-*t* percentile with 119 degrees of freedom,  $t_{0.975,119} = 1.9801$ , the confidence interval is  $\bar{A}_t \in [108.4, 122.8]$ .

#### 2.2.3 The Real Option Model

Now a model is developed for the optimal operating strategy and expected earnings of the plant. All earnings are discounted using an annualized interest rate of r = 8%which aims to capture the credit risk associated with ethanol projects [48]. We note that it is possible to "hedge" this real option with futures contracts in which case rshould be set to the risk free rate. There may be incentives to hedge or in fact not to hedge at all depending on the market dynamics and management's risk appetite.

Variable	Description	Value	Source
В	Capitalized cost of construc-	\$1.40/gallon	[48, 49]
	tion		
Q	Liquidation value at end of	0.1B = 0.14/gallon	[48]
	life		
$D_{01}$	Cost to switch production on	0.1B = 0.14/gallon	[48]
$D_{10}$	Cost to switch production off	0.05B = 0.07/gallon	[48]
r	Discount rate	8% per annum	[49]
p	Fixed running cost	\$0.68/gallon	[48]
s	Subsidy	0.10/gallon	[35]
$\bar{A}_t$	Average price of distiller's	115.58/ton	[52]
	dried grains per ton		
$\kappa$	Gallons of ethanol produced	2.8 gallons/bushel	[26]
	per bushel of corn		
ω	Tons of distillers dried grains	$\frac{17}{2000}$ tons/bushel	[26]
	produced per bushel of corn	2000	
$K_1$	Net running costs while in	0.23/gallon	[48]
	production		
$K_0$	Net running costs while idle	0.07/gallon	[48]
$f_1(L_t, C_t)$	Running profits while in pro-	$\kappa(L_t - K_1) - C_t = 2.8(L_t +$	
	duction	$(0.12) - C_t$ /bushel-year	
$f_0(L_t, C_t)$	Running profits (losses) while	$-K_0 = -0.07$ /bushel-year	
	production is idle		

Table 2.2: Real option model parameters.

We discuss this briefly in the appendix.

The plant has two operating modes or states: 1, denoting "on" or in production, and 0, denoting "off" or production temporarily suspended. The instantaneous running profit while on is given by  $f_1$ ; while off by  $f_0$ . The cost of switching production back on after being temporarily suspended is  $D_{01}$  and the cost of switching production off from an active state is  $D_{10}$ .

The capitalized cost of construction of the facility is given by B and its liquidation value at the end of its normal useful life is Q. All parameter and function values are listed in Table 2.2.

The total expected earnings over the life of the project is given by the value function  $V_i$  where  $i = \{1, 0\}$ 

$$V_{i}(l,c,t) = \sup_{\tau,u} E\left[\int_{t}^{T} e^{-r(s-t)} f_{I_{s}}(L_{s},C_{s}) ds + \sum_{k=1}^{n} e^{-r(\tau_{k}-t)} D_{u_{k-1},u_{k}} \middle| (L_{t},C_{t},u_{0}) = (l,c,i) \right]$$
(2.11)

The pair  $(\tau, u)$  is the control that the manager has over the facility in his ability to toggle production on and off. It consists of a set of switching times  $\tau_k$  and states to be switched into  $u_k$  with  $I_t = u_k, t \in [\tau_k, \tau_{k+1})$ . Thus  $\tau_k$  is an increasing set of switching times with  $\tau_k \in [t, T]$  and  $\tau_k < \tau_{k+1}$ .

From the dynamic programming principle, it is known that

$$V_i(l,c,t) = \sup_{\tau} E\left[\int_t^{\tau} e^{-r(s-t)} f_i(L_s,C_s) ds + e^{-r(\tau-t)} \left\{ V_j(L_\tau,C_\tau,\tau) - D_{ij} \right\} \middle| (L_t,C_t) = (l,c) \right]$$
(2.12)

where  $\tau$  is the optimal time switch production on from off  $(0 \to 1)$  or off from on  $(1 \to 0)$ .

This problem can be reduced to a question of finding the optimal price boundaries for ethanol and corn  $(L_t, C_t)$  at which to switch production. Now the problem is to solve for the sets of prices at which the operator should:

- continue production if production is currently on,  $H_1$ ;
- pause production if the state is currently on,  $S_{10}$ ;
- keep production halted if the state is currently idle,  $H_0$ ; and
- resume production if the state is currently idle,  $S_{01}$ .

Thus given the production is in state i at time t only one of two decisions is possible. (1) If it is optimal to keep production in its current state, then by Ito's lemma the value function evolves by the partial differential equation (PDE) on  $H_i$ 

$$\frac{\partial V_i}{\partial t} + \mathcal{L}[V_i] + f_i(l,c,t) - rV_i = 0, \qquad (2.13)$$

where  $\mathcal{L}$  is the generator of the joint processes  $(L_t, C_t)$ 

$$\mathcal{L} = \mu l \frac{\partial}{\partial l} + ac \frac{\partial}{\partial c} + \frac{1}{2} \sigma^2 l^2 \frac{\partial^2}{\partial l^2} + \rho \sigma b l c \frac{\partial^2}{\partial l \partial c} + \frac{1}{2} b^2 c^2 \frac{\partial^2}{\partial c^2}.$$
 (2.14)

Similarly (2) if it is optimal to switch (if the value of the *i*-state were to fall below the *j*-state less switching costs  $V_i(l, c, t) \leq V_j(l, c, t) - D_{ij}$ ) then immediately

$$V_i(l, c, t) = V_j(l, c, t) - D_{ij}$$
(2.15)

on  $S_{ij}$  and the operator switches to receive the profits in state  $j, f_j$ .

This leads to the set of free boundary PDEs for the optimal switching problem. The free boundary  $\partial H_i$  gives the optimal set of prices at which to toggle production. In order to solve the PDE, the moving free boundary must be determined; it is not known a priori. Along the free boundary, there is continuity of the value functions and its first spatial derivatives, the so-called "high contact" principle [27]. By writing the free boundary problem in complementary form below (noting that either the PDE holds or the constraint is saturated), it is no longer necessary to track the free boundary as the equation is extended to the whole space.

$$\max\left[\frac{\partial V_1}{\partial t} + \mathcal{L}[V_1] + f_1(l,c,t) - rV_1, \ (V_0 - D_{10}) - V_1\right] = 0, \qquad (2.16)$$

$$\max\left[\frac{\partial V_0}{\partial t} + \mathcal{L}[V_0] + f_0(l,c,t) - rV_0, \ (V_1 - D_{01}) - V_0\right] = 0$$
(2.17)

with final conditions  $V_1(l, c, T) = V_0(l, c, T) = Q$ .

These equations may be solved numerically using methods similar to those described in [54]. The PDE is solved using standard centred finite differences for accuracy along with an implicit Crank-Nicholson time stepping discretization. With the parameters considered (i.e.  $\rho \ll 1$ ), this leads to a stable monotone scheme. However, as  $\rho$  becomes larger, care must be taken with the cross derivative difference to ensure the *M*-matrix property is conserved in order for the scheme to be stable. The complimentarity condition for the optimal switching is enforced using a fixed point value iteration method. Conceptually, the technique is similar to projected successive over-relaxation [30] and can be accelerated with multigrid or Krylov methods. Each system  $V_1, V_2$  is iterated simultaneously until convergence. For additional information on optimal switching problems and stochastic calculus, see [25, 27, 46, 44].

Suppose the firm has a lease over a finite time horizon on the green field site on which they plan to build the production facility. If prices are particularly unfavourable, it would be naive to immediately enter into the project. A rational investor that seeks to maximize his expected earnings P should wait at least until the expected earnings of the optimally managed facility exceed the capital cost of investment. This is analogous to an American call option on the facility struck at B with payoff (max[ $V_1(l, c, t), V_0(l, c, t)$ ] - B)<sup>+</sup> over the remaining horizon T - t. (Note that  $X^+ = \max(X, 0)$ ). The free boundary problem for this option (following a similar dynamic programming optimal stopping argument) is

$$\max\left[\frac{\partial P}{\partial t} + \mathcal{L}[P] - rP, \ (\max\{V_1(l,c,t), V_0(l,c,t)\} - B) - P\right] = 0$$
(2.18)

with final condition  $P(l, c, T) = \max[0, \max\{V_1(l, c, T), V_0(l, c, T)\} - B]$ . Again, this

is reduced to finding a set on which it is optimal to wait, H, and set at which it is optimal to enter into the investment, S. The free boundary between these two sets is the set of prices at which it is optimal to make the decision. See [53] for additional details. Generally, one would not build the plant if it was expected to be idled immediately after construction and so we can drop  $V_0$  in Equation 2.18 (e.g.  $P(l, c, T) = \max[0, V_1(l, c, t) - B])$ . This model assumes that the construction is immediate. In realty, there is some delay between when the project is initiated and the facility is complete and ready for operation. To model this, accurately an extra time variable would be needed in the PDE to track the time to completion of the facility resulting in an "ultraparabolic" 4-dimensional PDE (l, c, t) and time to completion).

As the green field project is quite expensive to initiate relative to its salvage value upon abandonment, the option to abandon adds little value and for financially reasonable parameters does not materially alter the decision to enter the investment. A thorough argument is presented in 2.6.

### 2.3 Lessons from Exchange Options

In this section, two simplifications of the above model are presented to predict the effects of increased correlation on the complete model.

#### 2.3.1 A Running Margrabe Exchange Option

Assume that switching costs and fixed running costs are both zero. This makes it possible to find an analytic solution for the expected earnings of the facility. If switching costs are zero, the problem reduces to the simple PDE

$$\frac{\partial V}{\partial t} + \mathcal{L}[V] + (\kappa l - c)^+ - rV = 0, \qquad (2.19)$$

where  $V_1 = V_0 = V$ . This is the running payoff analogue of the classical Margrabe European exchange option [38].

The solution to this problem follows from the Feynman-Kac representation theorem

$$V(l,c,t) = E\left[\int_{t}^{T} e^{-r(s-t)} \left(\kappa L_{s} - C_{s}\right)^{+} ds \left| \left(L_{t}, C_{t}\right) = (l,c) \right].$$
 (2.20)

After some reflection, it is apparent Equation 2.20 is similar to a running Margrabe exchange option or a Black-Scholes call on ethanol struck at the corn price. Following



Figure 2.2:  $V/V_{ref}$  as a function of  $C_t$  is approximately semilinear. All parameters are as in Table 2.2 except K = 0,  $D_{01} = D_{10} = 0$ , and  $\kappa = 1$ .

the Black-Scholes analogy, Equation 2.20 is reduced to

$$V(l,c,t) = \int_{t}^{T} e^{-r(s-t)} \left[ \kappa l e^{\mu(s-t)} \Phi(d_{+}) - c e^{a(s-t)} \Phi(d_{-}) \right] ds$$
(2.21)

where

$$\nu^{2} = \sigma^{2} - 2\rho\sigma b + b^{2}, \qquad (2.22)$$

$$d_{+} = \frac{\ln \frac{\kappa t e^{\mu(s-t)}}{c e^{a(s-t)}}}{\nu \sqrt{s-t}} + \frac{1}{2} \nu \sqrt{s-t}, \qquad (2.23)$$

$$d_{-} = \frac{\ln \frac{\kappa l e^{\mu(s-t)}}{c e^{a(s-t)}}}{\nu \sqrt{s-t}} - \frac{1}{2} \nu \sqrt{s-t}.$$
(2.24)

From (2.21) to (2.24) above, it apparent that  $\nu$  is decreasing in  $\rho$ . Since this is akin to a Black-Scholes option, its value is accordingly decreasing in  $\rho$ . Similarly it is approximately semilinear in c, the operating cost, deep into or out of the money. This is illustrated in Figure 2.2 for a generic-parameter option in the risk neutral measure (where  $\kappa = 1$ ,  $L_t = 1$ ). Therefore, as a rough approximation, V can be considered almost semilinear and decreasing in K.

The value of the facility is also strongly linked to its achievable yield of ethanol per bushel of corn. As before, the value is almost linear in  $\kappa$ .

Figure 2.3 shows the percent decrease in V at the money as a function of increasing



Figure 2.3:  $V/V_{ref}$  as a function of  $\rho$  is decreasing. Parameters are as in Table 2.2 except K = 0,  $D_{01} = D_{10} = 0$ .

 $\rho$  normalized by the reference value  $V_{ref} = V(\rho = 0)$ .

#### 2.3.2 An Infinite Horizon Model

Using a clever dimensional reduction to obtain coupled differential equations, [49] were able to solve an infinite time horizon problem in closed form. Changing notation to that used in the current paper, their solution can be represented as the system of nonlinear equations

$$v_0(z) = A z^{\lambda_-}, \qquad (2.25)$$

$$v_1(z) = Bz^{\lambda_+} + \frac{z}{r-\mu} - \frac{1}{r-a},$$
 (2.26)

$$\lambda_{\pm} = \left(\frac{1}{2} - \frac{\mu - a}{\nu^2}\right) \pm \sqrt{\left(\frac{\mu - a}{\nu^2} - \frac{1}{2}\right)^2 + \frac{2(r - a)}{\nu^2}},$$
 (2.27)

where  $V(l, c) = cv\left(\frac{l}{c}\right) = cv(z), f_0(l, c) = 0$  and  $f_1(l, c) = l - c$ .

The remaining four unknowns A and B, and  $z_{01}$  and  $z_{10}$ —which represent the z at which production should be switched on or off respectively—derive from continuity of the value functions and the smooth-pasting optimality condition (i.e.  $1^{st}$  derivatives) at the switching boundary which constitutes a system of four nonlinear equations in four unknowns.



Figure 2.4: Baseline results for  $v_0$  and  $v_1$ . Note the value is increasing in l/c and the presence of a hysteresis zone in the value functions. Parameters are as in Table 2.2 except  $T = \infty$ .

The value function for the parameters calculated is shown in Figure 2.4. The switching boundaries  $z_{01}, z_{10}$  as a function of  $\rho$  are shown in Figure 2.5.

As can be seen from the figures, the effect of increasing  $\rho$  tightens the "waitand-see" gap resulting in shorter periods of operation before making the decision to switch. In particular, the plant manager is less optimistic about prices rebounding in making the decision to switch production off. This is accompanied by decreased value and potentially riskier cash flows since production is started and stopped more often.

A technical term describing this gap phenomenon is hysteresis. It represents a "sticky" region where it is not definitively optimal to be in either state (on or off) but rather to remain operating as is. Once prices reach the switching boundaries  $S_{01}$  and  $S_{10}$ , it is definitively optimal to be in either the on or off state respectively and switching occurs as required. Stated precisely, the hysteresis zone is given by the set  $H_0 \cap H_1$  which is also equivalent to  $H_1 \setminus S_{10} = H_0 \setminus S_{01}$ .

# 2.4 Numerical Results

The analysis begins with a retrospective look at the profits that would have been realized by the model facility given historical prices from Jan/02 to Dec/11. As a baseline, 10 year model values at the Jan/02 price of  $L_t =$ \$0.94/gallon for ethanol



Figure 2.5: Switching boundaries  $z_{01}$  and  $z_{10}$  as a function of  $\rho$  where z = l/c. Note that as  $\rho$  increases, the boundaries at which production is started  $z_{01}$  and stopped  $z_{10}$  converge indicating reduced "optimism" in prices rebounding. Parameters are as in Table 2.2 except  $T = \infty$ .

Baseline result	Description	Value
$V_1(0.94, 1.90, 0)$	Model income value started "on"	\$2.10/bushel-year
$V_0(0.94, 1.90, 0)$	Model income value started "off"	\$1.77/bushel-year
P(0.94, 1.90, 0)	Model plant value after construction	\$0.84/bushel-year
$ V_1 _{realized time series}$	Retrospective realized income	\$10.73/bushel-year
$V_0 _{realized time series}$	Retrospective realized income	\$10.27/bushel-year

Table 2.3: Expected income value as Jan/02 and retrospective historical realized income during the period Jan/02 through Dec/11.

and  $C_t = \$1.90$ /bushel for corn are listed in Table 2.3 ignoring the value of liquidating the plant at the end of its life (i.e. Q = 0). As before,  $V_i(l, c, t)$  refers to the expected value of income generated by the facility from time t, given production begins in state i, with  $L_t = l$  and  $C_t = c$ .

The actual profits given the past 10 year time series from Jan/02-Dec/11 realized from optimal operation are also recorded in Table 2.3, noted as  $V_i|_{realized time series}$ . The higher than expected realized profits do not reflect negatively on the model's validity but rather represent one of many possible realized outcomes from the stochastic model.

The retrospective plant operating status from Jan/02-Dec/11 as determined by



Figure 2.6: Baseline income valuation results for  $V_1$  (solid) and  $V_0$  (dashed) for given levels of l. Note that l and c are the initial ethanol and corn prices given the 10 year period respectively. The y-axis is the income value. Parameters are as in Table 2.2.

following the optimal operating scheme indicates that the facility should always be in production regardless of price conditions (see Figure 2.1). The results suggest that the ethanol subsidy policy may be higher than necessary to ensure NPV positivity and may in fact be reduced with minimal effects on producers.

#### 2.4.1 Baseline Value

The baseline valuation results are shown in Figures 2.6 which include the liquidation value at the end of facility life on a per bushel basis.

The baseline switching  $(S_{01}, S_{10})$  and continuation sets  $(H_0, H_1)$  are shown in Figure 2.7.

#### 2.4.2 Effects of Increased Correlation on Value

As expected from the Margrabe option results presented in Section 2.3, increasing correlation significantly reduces option value. There is evidence for increased correlation in recent years. Figure 2.8 shows the rolling correlation over the previous 3 year period on the price series data from Jan/02–Dec/11 calculated using the correlation of the log monthly returns.

Figure 2.9 shows the percent loss in income value as the correlation  $\rho$  increases,



Figure 2.7: Baseline switching,  $S_{ij}$ , and continuation sets,  $C_{ij}$ . Parameters are as in Table 2.2.



Figure 2.8: The 3 year rolling correlation,  $\rho$ , over the 7 year period Jan/04–Dec/11 from the 10 year monthly price data, Jan/02–Dec/11. There is evidence of increased correlation in recent years which may be related to increased production and demand in corn ethanol.



Figure 2.9:  $V_1(l,c)/V_{1,ref}$  and  $V_0(l,c)/V_{0,ref}$  versus  $\rho$ . The initial ethanol l and corn c prices are (l,c) = (0.94, 1.90) and the reference value  $V_{i,ref}$  is taken at  $\rho = 0$ . The value is decreasing in  $\rho$ . All other parameters are as in Table 2.2.

shown for 10 years of income without liquidation given the Jan/02 average monthly prices of (l, c) = (0.94, 1.90) with  $V_{i,ref} = V_i(\rho = 0)$ . It is no coincidence that  $V_0$ appears more sensitive to changes in  $\rho$  at the chosen reference point: The point is close to  $\partial S_{10}$  and therefore more sensitive to the decision to "turn on"  $(0 \to 1)$  rather than "stay on"  $(1 \to 1)$ .

Pushing  $\rho$  away from zero correlation results in changes in  $\pm 50\%$  of income option value. The concavity of the graph indicates that the option is very sensitive to  $\rho$ .

Figure 2.10 shows  $\frac{\partial V_1}{\partial \rho}$  evaluated at the estimated value of  $\rho = 0.105$  along with the switching boundaries overlayed on the plot. (The result for  $\frac{\partial V_0}{\partial \rho}$  is very similar.)

The effects of increasing  $\rho$  are strongest near the switching regions (i.e. in the hysteresis zone). The most significant losses in the hysteresis zone are near -0.50. Thus in these price regions for a 10% increase in  $\rho$ , there is a loss of nearly \$0.05 of value following the Taylor approximation,  $V(\rho_0 + \Delta \rho) = V(\rho_0) + \frac{dV}{d\rho}\Delta \rho + O(\Delta \rho^2)$ . Outside the hysteresis zone, all partials become equal since  $V_i = V_j - D_{ij}$  only differs by an additive constant. The hysteresis zone is the result of uncertainty as to which decision or operating status is optimal. Intuitively, the option value would be most sensitive to changes in variance (via correlation) in this uncertainty or hysteresis zone which is observed in Figure 2.10.

The loss in value associated with increasing  $\rho$  becomes more persistent and pro-



Figure 2.10: The sensitivity  $\frac{\partial V_1}{\partial \rho}$  as a contour plot. Note that  $V_1$  is most sensitive to changes in  $\rho$  near the hysteresis zone. All parameters are as in Table 2.2.

nounced as T - t increases. This is a financially intuitive result since there is more time for the losses related to  $\rho$  accrue. But more importantly, the longer increased correlation persists the more damaging the effect becomes. To illustrate, Figure 2.11 shows the relevant option "Greek,"  $\frac{\partial^2 V_1}{\partial t \partial \rho}$ , taken at Jan/02 price levels (0.94, 1.90) (note that similar results hold for  $V_0$ ).

#### 2.4.3 Effects of Subsidy Policy on Value

The loss in value from removing the subsidy is shown in Figure 2.12. As the subsidy s is lowered, the facility is expected to lose value at a near linear rate when deep in the money and at a lower rate as the value moves further out of the money. This behaviour is illustrated in Figure 2.12 which plots  $V_i(s)$  at (l, c) = (0.94, 1.90). It can also be seen from the plot that, across all  $s \in [0, 10]$  ¢/gallon, the point (l, c) remains in the hysteresis zone. The distance between the two values remains less than the switching cost  $V_1(l, c) - V_0(l, c) < D_{01}$ , a characteristic feature of the hysteresis zone.

#### 2.4.4 NPV Positivity and the Value of Waiting to Invest

Since both increasing  $\rho$  and decreasing the subsidy have the effect of reducing the value of the income option, it is natural to expect that the value of waiting to invest, P, is also reduced. If the value is reduced, it is expected that the optimal price



Figure 2.11: The second-order sensitivity  $\frac{\partial^2 V_1}{\partial t \partial \rho}$  as a function of t. Note that  $V_0$  becomes more sensitive to changes in  $\rho$  as t increases. All parameters are as in Table 2.2.



Figure 2.12:  $V_1(l,c)$  and  $V_0(l,c)$  as a function of s. The value is increasing in s. All other parameters are as in Table 2.2.



Figure 2.13: NPV positivity regions at  $\rho = 0.9$  and  $\rho = -0.9$ . The boundaries denote the area over which projects are NPV positive. It is largest when  $\rho$  approaches -1and smallest when  $\rho$  approaches +1. The area bounded between the two regions indicates how the boundary (NPV positive set) decreases as  $\rho$  increases which implies that fewer projects are NPV positive. All other parameters are as in Table 2.2.

levels to begin the project,  $\partial S$ , should be closer to the NPV positive region<sup>2</sup>. This reflects the lowered optimism of entry into the investment. This is illustrated with exaggeration in Figures 2.13, 2.14 and 2.15.

As is apparent from Figure 2.15, the subsidy policy results in some otherwise economically unattractive projects being initiated. The net effect of this is to reduce the productive activity of firms contemplating entry into the investment project.

#### 2.4.5 Retrospective Analysis without Subsidy

The investigation closes with a retrospective look at the performance of the optimal operating schedule without a subsidy; this gives an indication of the kind of performance one might expect in the future given many ethanol subsidies have been discontinued. Due to high realized ethanol prices, even a non-subsidized facility has a productive run and is nearly always in operation.

Contrast that operating result with Figure 2.16 which indicates when the facility is operating at a profit, 1, or at a loss, 0. That is, Figure 2.16 is a graph of  $1_{f_1(l,c)>0}$ 

<sup>&</sup>lt;sup>2</sup>The NPV positive region is given by the set at t = 0 { $(l, c) : (\max[V_1, V_0] - B) > 0$ } over 10 years including liquidation proceeds.



Figure 2.14: The entry boundary  $\partial S$  as  $\rho$  increases. Note that the region over which we wait to invest, H, decreases as  $\rho$  increases. Compared with Figure 2.13, the distance between deciding to invest and the region of NPV positivity shrinks as  $\rho$  increases. All other parameters are as in Table 2.2.



Figure 2.15: The entry set boundary  $\partial S$  as  $K_1$  increases. Note that the region over which we wait to invest, H, increases as  $K_1$  increases. All other parameters are as in Table 2.2.



Figure 2.16:  $1_{f_1(l,c)>0}$  with and without subsidy. Note that despite that there are moments when the facility is operating at a loss, the presence of switching costs filters the operating signal to be almost always on.

with and without subsidy. In the absence of a subsidy, it is still optimal to always remain in operation given the historical time series; despite the fact that on several occasions the profits become negative. The presence of switching costs acts like a low pass filter on the zero-cost switching signal and accordingly switching occurs less frequently.

Historical information regarding the operating status of about 215 plants across the United States is available from [42]. Figure 2.17 shows the number of plants that were historically idled in the 5 years 2008–2012. There is an increase in idled plants consistent with the decrease in profitability observed in 2011–2012 apparent in Figures 2.16 and 2.1. The decrease in profitability is not only due to a tightening in the crush spread but also the loss of subsidy in 2012 which partly explains the sudden rise in idle plants observed in 2012. Note that the tightening of the spreads and loss in profitability does not immediately equate in plants idling which is consistent with our model predictions. This is due to the cost of ramping down production which acts as a filter inducing managers to continue operating until prices revert or crush spreads become too tight or negative.

It may appear that the facility is profitable even in the absence of subsidy, but part of the story is missing. Figure 2.18 indicates when it is optimal to enter into the investment over the 10 year horizon. As can be seen, for most of the time it is



Figure 2.17: Historical count of idled plants in the 5 years from 2008–2012 inclusive of a sample population with approximately 215 plants. Note the increase in idled plants beginning 2011 and sharp rise in 2012. Source: [42].

in fact not optimal to initiate the project even though the retrospective operating status advises to be in production. This means that while the operator of an existing facility would produce from it, the resulting profits would not be so large as to entice the development of a new facility. It is in fact optimal to wait nearly 2 years before initiating the project even in spite of low corn prices and the otherwise continuous production signal. In addition, it is apparent that the presence of the subsidy does not greatly influence the historical decision to enter into the project.

The assumption that the facility is able to easily market and sell its distillers dried grains may not always hold. An investigation of the historical operating status given  $\bar{A}_t = 0$  is shown in Figure 2.19. The upper indicator assumes the state is initially on,  $V_1, i = 1$ ; the lower status assumes it begins in the off state  $V_0, i = 0$ . The investigation shows that the economic viability of an existing facility is sensitive to its ability to market its byproducts in addition to ethanol.

#### 2.4.6 Future Risk Profile

In the next few figures, the distribution of profits, 95% value-at-risk (VaR) and conditional value-at-risk (CVaR) are provided followed by an investigation of the amount



Figure 2.18: Retrospective decision status whether to enter into the investment with and without subsidy. Note that it is optimal to wait nearly 2 years before initiating the projects. All other parameters are as in Table 2.2.



Figure 2.19: Retrospective operating status of a facility with no marketable grains,  $\bar{A}_t = 0$ . Note that it is only optimal to be in production for roughly 60% of the time over the 10 year period. All other parameters are as in Table 2.2.



Figure 2.20: Monte Carlo simulations of Z given i = 1 following the optimal operating strategy.  $(L_t, C_t) = (2.49, 6.02)$  and all parameters are as in Table 2.2 with 100,000 simulations.

of time spent idle and operating at a loss. The  $VaR_{\alpha}$  of a project at given confidence level  $\alpha \in (0,1)$  is the smallest number  $\gamma$  such that the probability that the loss  $\Gamma$  exceeds  $\gamma$  is at most  $(1 - \alpha)$ . The  $CVaR_{\alpha}$  is the expectation of this tail, i.e.  $CVaR_{\alpha} = E[X|X \leq VaR_{\alpha}]$ . The subsidy is taken to be zero, s = 0, for the investigation as it aims to investigate the cash flows going forward on a 10 year horizon.

The probability density function (PDF) of income assuming the project is immediately started,

$$Z = \int_{t}^{T} e^{-r(s-t)} f_{I_s} ds - \sum_{k=0}^{T} e^{-r(\tau_k - t)} D_{u_{k-1}, u_k}, \qquad (2.28)$$

is shown in Figures 2.20 and 2.21. The investigation is performed at the Dec/11 price  $(L_t, C_t) = (2.49, 6.02).$ 

The large peaks in the distributions of incomes indicate that many simulated project outcomes remain idle for extended periods of time.

The simulated cumulative distribution function (CDF) of income and capitalized costs assuming the site is available on a 10 year horizon given the operator must first



Figure 2.21: Monte Carlo simulations of Z given i = 0 following the optimal operating strategy.  $(L_t, C_t) = (2.49, 6.02)$  and all parameters are as in Table 2.2 with 100,000 simulations.

decide to initiate the project at optimal time  $\tau$ ,

$$M = \left\{ -e^{-r(\tau-t)}B + \int_{\tau}^{T} e^{-r(s-t)} f_{I_s} ds - \sum_{k=0}^{T} e^{-r(\tau_k-t)} D_{u_{k-1},u_k} + Q e^{-r(T-t)} \right\} \mathbf{1}_{\{\tau < T\}},$$
(2.29)

is shown in Figure 2.22. The jump in the CDF  $Prob(M \leq m)$  at zero indicates that a large number of simulated outcomes where projects are never optimally initiated. The large point mass at zero in Figures 2.20–2.22 show that many projects wait a very long time to begin or are in fact never initiated.

The section concludes by investigating how increased correlation affects the following factors:  $VaR_{0.05}$ ,  $CVaR_{0.05}$ , fraction of time spent idle  $t_{idle}$ , the fraction of time spent operating at a loss  $t_{op\ loss}$ , and the fraction time spent waiting to enter into the project  $\tau/T$ . These are summarized in Figures 2.23 and 2.24 given the project is initiated optimally from a green field site.

It can be observed from Figures 2.23 and 2.24 that, as  $\rho$  increases, the value at risk of the project typically decreases. However, the project value also decreases supporting our earlier assertion that the optimal operating strategy becomes "less optimistic" in that the investor waits longer to enter. The amount of time spent idle or operating at a loss tends to decrease. This is expected since the investor has already waited until prices were more favourable before initially entering into



Figure 2.22: Experimental CDF of M following the optimal operating strategy. Note the large point mass of projects which are never started or begin very late in the cycle.  $(L_t, C_t) = (2.49, 6.02)$  and all parameters are as in Table 2.2 with 100,000 simulations.



Figure 2.23: Left: Expected value of investment P. Right: 5% VaR and CVaR of investment as a function of  $\rho$ .  $(L_t, C_t) = (2.49, 6.02)$  and all parameters are as in Table 2.2 with 50,000 simulations. Error bars indicate 3 standard deviations of the estimator.



Figure 2.24: Time spent idle, operating at a loss and waiting for entry as a function of  $\rho$ .  $(L_t, C_t) = (2.49, 6.02)$  and all parameters are as in Table 2.2 with 50,000 simulations. Error bars indicate 3 standard deviations of the estimator.

the project. Increasing  $\rho$  reduces the standard distribution of Z and M (reduced volatility). There is more certainty in the project, but this comes at a cost to option value.

# 2.5 Discussion and Conclusion

Our paper investigated the economic viability of a corn ethanol production facility using real option models. The results indicate that the viability of the project is sensitive to changes in correlation and subsidy policy along with the ability to market its byproducts.

#### Correlation

The investigations with the Margrabe exchange options showed that the option can lose over 70% of its value as the correlation increases from uncorrelated to nearly perfectly correlated,  $\rho \approx 0.9$  (Figure 2.3). Further the complete model showed that given the deep in the money initial price at Jan/02, the facility can lose over 50% of its value as the prices become more correlated (Figure 2.9). The contour plot of  $\frac{\partial V}{\partial \rho}$  (Figure 2.10) showed that in the hysteresis zone, the facility is most sensitive to changes in correlation.

Our investigations using the infinite time horizon model indicated that as the

correlation increased, the size of the hysteresis zone shrank (Figure 2.5). This may indicate more certainty in the income cash flows but also indicates lowered expectation for value or prices rebounding favourably for the operator (Figure 2.23). Additionally, our risk profile analysis indicated that in most cases, as correlation increases, the fraction of time spent waiting to start the project increases resulting in lowered productivity (Figure 2.24).

From our investigation it is clear that, as correlation increases, the number of projects that are economically viable decrease. That is, the sets of initial prices for which the project is NPV positive shrinks as the prices become more correlated (Figures 2.13–2.14). Thus fewer projects may be NPV positive, and hence not initiated, at any given time and price environment. Perhaps counterintuitively, the optimal price trigger at which to enter the project is in fact lowered as correlation increases but again this reflects lowered expectations for the project. The value of waiting to invest is reduced since the optimal entry price trigger is moving closer to the region at which it is first NPV positive.

The risk profile investigation also yielded additional insight about the viability of the project. In particular, many of the projects are not NPV positive if the entry decision is made suboptimally. The PDF of potential realized profits shows that there is a large mass of risk-adjusted realizations that do not exceed the initial capitalized costs of construction. However if the option to enter the project is exercised optimally, the risk of losses is greatly reduced (Figures 2.20–2.22).

#### Subsidy Policy

As the Margrabe exchange option predicted, the value of the facility is semilinearly decreasing in s, the subsidy policy. Thus, for example, when the investment is deep in the money, the value of subsidy has a term proportional to s(T-t) in the absence of discounting. When the subsidy is removed, our investigation showed that the number of projects which were economically viable was reduced (Figures 2.12 and 2.15). This was evidenced by a reduction in the set of prices for which the project was NPV positive.

Our retrospective analysis revealed an interesting fact about the optimal operating strategy for the facility: The subsidy had a minimal effect on the operating decisions regarding when to pause/resume production and to enter into the investment. With or without the subsidy, the decisions were nearly identical. Thus tax dollars are subsidizing a project that may in any case have been economically attractive, and investment capital is misappropriated from other possible projects. These numerical results indicate that the recent idling of many ethanol plants in 2013 may be the result of market factors as opposed to subsidy policy.

On the other hand, the subsidy may be successful in inducing ethanol production investment where none would otherwise exist. Although without the subsidy the facility would have historically been in production, the subsidy also reduces the operating risk. This has the effect of smoothing the distribution of income over the life of the project, reducing the presence of the distributional spike of simulated outcomes which are never initiated. Thus a primary effect of the subsidy, and arguably a main goal, is to ameliorate the apparent risk profile of entering into the ethanol business; rather than to increase the value of the project or to influence operating decisions.

#### Efficiency of the Facility

In the retrospective analysis, our paper showed that the success of the facility is contingent on its ability to market and sell its byproduct grains. It is possible that the facility may have difficulty collecting and marketing its distillers dried grain byproducts due to factors including its proximity to principle markets, its ability to collect and store the byproducts, and the grade or quality of the distillers dried grain byproducts. All of these factors will affect the price the operator can get and subsequently the value of the facility is strongly linked to the firm's ability to market its byproducts. In particular, the retrospective analysis showed that the facility would only be in production approximately 60% of the time if it were unable to market its grain.

Our investigation with simple Margrabe options showed that the loss in value is approximately semilinear in  $\kappa$ . Thus facility yield is also a key component to success for an ethanol facility; particularly in the presence of high corn prices.

#### Conclusion

Our paper provided an in-depth investigation of the retrospective and future economic viability of a typical North American corn ethanol production facility. It investigated the effects of ethanol policy manifested as increased price correlation due to increased demand for corn ethanol, as well as the direct effects of the subsidy on firms' operating decisions. Our results show that the future viability of these facilities without the subsidy is still positive although with the subsidy, the effects of these risk factors are greatly reduced.

# 2.6 Appendix A: Abandonment and Hedging

#### Abandonment

In the above analysis the option to abandon was not considered. In this appendix we present a formalism for incorporating the option to abandon. We show that this omission is not material, at least in the parameter regimes considered in the current paper. Empirically abandonments are rarely observed in reality compared to the frequency of idling [42].

First, we observe that the option to abandon the facility can be considered an effective floor on the income of the facility. Consider for example an idle facility in the presence of very unfavourable ethanol and corn prices. It has the option to either idle at a loss for the foreseeable future or cut its losses and abandon, assuming the salvage value exceeds the expected accrued running costs or potential profits over the remaining facility life.

A facility can be abandoned from idle in which case the operator gets a salvage value F, or it can (in principle at least) be abandoned from the running state in which case a cost D somewhat less than  $D_{10}$  will be incurred. The total expected earnings over the life of the facility is

$$V_{i}(l,c,t) = \sup_{\tau,u,\theta} E\left[\int_{t}^{\theta} e^{-r(s-t)} f_{I_{s}}(L_{s},C_{s}) ds + \sum_{k=1}^{n} e^{-r(\tau_{k}-t)} D_{u_{k-1},u_{k}} + 1_{\{\theta < T\}} e^{-r(\theta-t)} (F - 1_{\{u_{n}=1\}}D) + 1_{\{\theta \neq T\}} e^{-r(T-t)} Q \left| (L_{t},C_{t},u_{0}) = (l,c,i) \right]$$

where all notation is as previously defined. Here  $\theta$  is the optimal time to abandon whereupon the abandonment value is received. Given the facility is not abandoned before T, i.e.  $\theta \notin [t, T)$ , the salvage value Q is received at the end of the lease.

Dynamic programming reduces the problem to that of finding  $\tau$ 

$$V_{i}(l,c,t) = \sup_{\tau} E\left[\int_{t}^{\tau} e^{-r(s-t)} f_{i}(L_{s},C_{s})ds + e^{-r(\tau-t)} \max\left\{\left(V_{j}(L_{\tau},C_{\tau},\tau) - D_{ij}\right), F - 1_{\{i=1\}}D\right\} \middle| (L_{t},C_{t}) = (l,c)\right]$$

where  $\tau < T$ . The associated free boundary system is



Figure 2.25:  $V_1(l, c, t)$  with and without the option to abandon as a function of the spread  $\kappa l - c$ . Near the lower limit value of NPV positive entry,  $V_1 = B$ , the difference is small. All parameters are as in Table 2.2 along with an abandonment value of F = 0.5Q and  $D = 0.75D_{10}$ .

$$\max\left[\frac{\partial V_1}{\partial t} + L[V_1] + f_1(l,c,t) - rV_1, \max\left\{(V_0 - D_{10}) - V_1, (F - D) - V_1\right\}\right] = 0,$$
$$\max\left[\frac{\partial V_0}{\partial t} + L[V_0] + f_0(l,c,t) - rV_0, \max\left\{(V_1 - D_{01}) - V_0, F - V_0\right\}\right] = 0$$

with final conditions  $V_1(l, c, T) = V_0(l, c, T) = Q$ . Note that Q need not be the same as F, and in general will be larger, as Q incorporates the fact that the facility at the end of the lease may potentially be renovated and then continue to operate as a going concern. Accordingly it may have more value than just the scrapping and liquidation of its constituent parts.

The capitalized construction cost is much larger than the abandonment value, B > F, and thus the option to abandon does not materially affect the decision point to enter  $\partial S$ . In particular, since  $P \ge 0$  and F - B < 0, the decision to enter is never made at a point where abandonment would have occurred as per Equation 2.18. For the parameters considered, at the lowest bound where entry to the investment may be considered (i.e. where  $V_1 = B$ ), the difference between the values  $V_1(l, c, t)$  with and without abandonment is very small. Figure 2.25 numerically illustrates this feature.

#### The Possibility of Hedging

In some cases it may be desirable to hedge the real option associated with the ethanol facility. This could be achieved by trading in the front month future contract for example as a proxy for spot ethanol and corn prices. These contracts trade on the Chicago Mercantile Exchange [29]. An advantage of hedging is that the project value becomes "certain" in that all the market risk can be hedged away. A disadvantage is that the rate of return on the investment r is reduced to the risk free rate.

If the owner is very risk averse, he may hedge the facility income. On the other hand, a risk-loving investor in search of higher returns may opt to leave the project unhedged. A large agricultural, energy, or investment firm might be sufficiently diversified that hedging the option is not necessary. Some private equity or alternative investment funds may use this option in combination with other energy trades as part of a strategic fund. In practice, management may choose the middle ground, partially hedging some of the risk.

We briefly point out how this affects the option pricing free boundary PDE. Say  $F_L(t,T)$  and  $F_C(t,T)$  are the ethanol and corn future contract prices at time t expiring at T. If there is no cost of carry or convenience yield, the forward/future price of X is given by

$$F_X(t,T) = Xe^{r(T-t)}$$

where r the risk free rate is constant. If X has risk neutral dynamics  $dX = rXdt + \eta XdW_t$ , the dynamics of  $F_X(t,T)$  is then

$$dF_X(t,T) = \eta F_X(t,T)dW_t.$$

The governing free boundary PDE can then be derived for corn and ethanol

$$\max\left[\frac{\partial V_i}{\partial t} + \mathcal{L}[V_i] + f_i(l,c,t) - rV_i, \ (V_j - D_{ij}) - V_i\right] = 0$$

where now

$$\mathcal{L} = \frac{1}{2} F_L^2 \sigma^2 \frac{\partial^2}{\partial F_L^2} + \rho \sigma b F_L F_C \frac{\partial^2}{\partial F_L \partial F_C} + \frac{1}{2} F_C^2 b^2 \frac{\partial^2}{\partial F_C^2}.$$

Note that the form of the equation effectively reduces to physical measure case since  $a = \mu = 0$  (however here r is the risk free rate).

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# Chapter 3

# Real Options with Regulatory Policy Uncertainty

#### Chapter Summary:

Energy Finance as a field is particularly bedeviled by regulatory uncertainty. This is notably the case for the real option analysis of long-lived energy infrastructure. How can one decide optimal build times on a 50 year project horizon when regulations regarding pricing and costs change on a much shorter time scale? In this paper we present a quantitative framework for modelling and interpreting regulatory changes for energy real options as a Poisson jump process, in a context where other relevant prices follow diffusion processes. We illustrate this conceptual framework with a case study involving the US corn ethanol market for which subsidy levels have experienced frequent changes. Subsidy levels have an easily quantified impact on operations and profitability, making this a nice arena to introduce ideas which might later be extended to less easily quantified regulatory changes. Numerical techniques are presented to solve the resulting partial integro differential variational inequalities. These solution techniques are deployed to solve instructive numerical examples, and conclusions for public policy are drawn.

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## **3.1** Introduction

All large energy and natural resource projects are subject to government policy or regulation of some kind. These regulations are intended to achieve public policy goals and their effects should be taken into account by firms planning to enter into energy or resource investments. Energy and resource projects often have long project horizons and operating life spans on the order of decades. Consider the example of a firm deciding to enter into a 50 year energy production investment. Policy in terms of taxation, environmental regulations and other laws may materially affect project cash flows. These policies have been known to change at various time scales. Some policy amendments are well broadcast and announced while others are not. Although policy changes may appear "predictable" in the short term, forecasting onto a 50 year project horizon renders the policy changes apparently random, and hence requiring models of *policy uncertainty*.

**Policy uncertainty** is characterized by changes in taxation, legal and other regulatory policies that affect a business' operations and profitability. The uncertainty derives from the inability to predict policy in the long term; uncertainty about forthcoming policy or announcements of policy changes; or sudden and abrupt changes in policy. Some anecdotal examples of policy uncertainty in energy and resource markets from recent North American news headlines follow:

ONTARIO LOOKS SET TO CUT GREEN ENERGY SUBSIDIES: Solar rates expected to be cut substantially. Industry has six weeks to provide input. [87]

ONTARIO DROPS PLAN FOR TRANSCANADA POWER PLANT: Ontario cancels planned TransCanada power plant with province to discuss compensation with TransCanada. Costs may exceed \$1 billion CAD and affect off peak pricing. [80, 86]

IVANHOE 'SURPRISED' BY NEW MONGOLIAN WINDFALL TAX: Mongolia sets surprise windfall tax on (among possibly others) Ivanhoe's Oyu Tolgoi mine of 68% when gold hits \$500 per ounce. [62]

This does not by any means represent an exhaustive list. Attempts have been made to quantify and measure policy uncertainty (e.g. [58]). In [58] and [72] the authors also note that policy uncertainty can make firms hesitate or delay to enter into long term projects as they wait for more policy certainty before making decisions. This has caught the eye of Canadian and American macroeconomic policy makers noting both that firms appear to accumulate cash and hesitate to make business decisions amid regulatory uncertainty [71, 91].

In this paper we present a quantitative framework for modeling and interpreting regulatory changes for energy real options as a jump diffusion process, in a context where other relevant prices follow pure diffusion processes. Policy uncertainty by its nature is very difficult to hedge, at best, and leads to market incompleteness even if the remaining underlying prices could otherwise be traded. This real option method of modeling resource project management decisions was introduced by [61] in a seminal paper that considered the problem of optimally starting and stopping production to maximize the profits of a natural resource project. The optimal entry and exit from investment projects was also considered by [66] in another classical real option paper. A collection of illustrative real option papers can be found in [67].

In particular, we consider a firm contemplating the option to invest in an ethanol from corn production plant. We build on the analysis of our past work [79] which intended to quantify the impact (both intended and unintended consequences) of ethanol policy on production. This current work adds the complication of policy uncertainty deriving from a volumetric production tax subsidy which has changed several times over the past 35 years. We aim to understand the effects of ethanol policy uncertainty on production from the producer's perspective. An example of the application of real option analysis to understand the effects of windfall taxes on mining operations can be found in [89]. A complementary and interesting analysis on policy uncertainty and real options can be found in [72]. The authors of [72]use empirical data to determine how regulatory uncertainty in American electricity markets affects start up and shut down decisions for power plants; their evidence supports the anecdotal claims mentioned above that uncertainty leads management to defer decision making. Our real option model sets out to design a framework to quantitatively model this added uncertainty and capture its effects on decision making.

## 3.1.1 Corn Ethanol Production and Subsidy Policy

The ethanol market in the US is large, estimated at 13.3 billion gallons produced in 2012 by over 209 plants [88]. Efforts to promote US energy independence and initiatives to obtain fuel from environmentally friendly sources have led to the subsidization of the production of ethanol biofuel from corn. Subsidies have historically been provided to ethanol producers by means of a volumetric ethanol excise tax credit for blenders and a small ethanol producer tax credit. The subsidy amount has changed from \$0.40/gallon at its introduction in 1978 (Energy Tax Act) and been adjusted several times until its final level \$0.45/gallon in the 2008 Farm Bill followed by termination (by non-renewal) at the end of 2012 [68, 70]. Table 3.1 shows the history of ethanol subsidy policy changes and amendments since its inception.

A year following the lapse of many of the energy subsidies, about one quarter

Act	Year	Subsidy (\$/gallon)
Energy Tax Act	1978	0.40
Surface Transportation Assistance Act of 1982	1983	0.50
Tax Reform Act	1984	0.60
Omnibus Budget Reconciliation Act	1990	0.54
1998 policy adjustment effective 2001	2001	0.53
1998 policy adjustment effective 2003	2003	0.52
Extension of policy with adjustment	2005	0.51
Farm Bill	2008	0.45
Expiration of tax credit	2012	_

Table 3.1: Historical ethanol subsidies. Source: [70]

of Nebraska's ethanol plants were in idle status [83]. The loss of the subsidy was a possible contributing factor to the shut downs as [77] note that without subsidies ethanol plants may lose their economic viability.

## 3.1.2 Outline

Our paper uses a crush spread analysis to value a facility which produces ethanol from corn using a real options analysis following our framework in [79]. The outline is as follows: Section 3.2 specifies the plant characteristics, management decisions, and associated costs and profits. Section 3.3 derives the stochastic optimal control problem for the optimal plant operating rule. Section 3.4 illustrates the numerical results. Finally Section 3.5 draws conclusions about policy uncertainty and its effects on ethanol production, closing off with some policy recommendations.

# 3.2 The Real Option Model

Management contemplating the decision to invest in an ethanol production plant has the flexibility to enter or defer the project given price conditions and expected future profitability [67]. After initiating and building the ethanol plant, management again has the flexibility to switch production on (1) and off (0) given prevailing economic conditions. The goal of this paper is to examine how ethanol price and policy uncertainty affects a producer's business entry and subsequent operating decisions given price conditions, subsidy policy expectations, and the remaining project life.

Following our analysis [79], throughout this paper all currency is in United States dollars (USD); liquid volume is in gallons; solid volume is in bushels; weight is in

tons; and interest is percent per year appropriate to USD deposits continuously compounded.

## 3.2.1 Plant Specification and Operating Costs

The following costs are scaled in terms of gallon of production capacity per year and were estimated by [90]. The model is based on our detailed ethanol real option analysis in [79].

The capitalized construction cost B is estimated at \$1.40/gallon for a "typical" sized facility with nameplate capacity of 40,000,000 gallons/year. The plant salvage value Q is estimated at 10% of capitalized cost. The switching cost  $D_{01}$  to resume production from an idle state is estimated at 10% of capitalized cost per gallon of annual production capacity. Similarly, the switching cost  $D_{10}$  to pause production from an active operating state is estimated at 5% of capitalized cost per gallon of annual production capacity.

## 3.2.2 Running Profits

The plant produces ethanol  $L_t$  (priced in USD/gallon) from corn  $C_t$  (priced in USD/bushel). The running profit from the corn ethanol crush spread is developed in [79] on a per bushel per year basis assuming the popular dry grind process for producing ethanol [59].

$$\operatorname{corn} \to \operatorname{ethanol} + \operatorname{by-products}$$
(3.1)

The profit function while operating,  $f_1$ , is given by

$$f_1(L_t, C_t, Z_t) = \kappa (L_t + Z_t - K_1) - C_t \tag{3.2}$$

where  $Z_t$  is the government volumetric subsidy (USD/gallon). The conversion factor  $\kappa = 2.8$  is the yield in terms of gallons of ethanol produced per bushel of corn [59] and is consistent with the CME Group's references on trading ethanol crush spreads [63].

The net running cost while on can be decomposed in terms of the fixed running cost p of \$0.68/gallon, less the average by-product distillers dried grains G (USD/ton) produced per bushel of corn [77, 79, 90]

$$K_1 = p - \frac{\omega}{\kappa} G. \tag{3.3}$$

The process produces 17 lbs of by-product per bushel and hence the yield factor  $\omega = 17/2000^{1}$ .

While production is idle, [90] estimated that fixed running costs  $K_0$  are roughly 1% of capitalized construction costs per gallon of production capacity or 20% of fixed running cost while in production (note that, while idle, no ethanol is produced and consequently no subsidy is applied). The profit function while off,  $f_0$ , is

$$f_0(L_t, C_t, Z_t) = -\kappa K_0 \tag{3.4}$$

where the midpoint between the two estimates is used [79]

$$K_0 = \frac{0.01B + 0.20p}{2}.$$
(3.5)

Finally, the interest rate r is taken to be a target return of 8% per annum continuously compounded to account for the risk associated with the ethanol project cash flows [79, 90]. Our analysis uses only the physical measure for the stochastic assets. We note however that the price risk associated with corn and ethanol can be hedged using futures and the arbitrage free return can be determined by assuming that the jumps are not correlated with the market following an argument popularized in [81].

## **3.2.3** Stochastic Price Models

Following our analysis in [79], ethanol  $L_t$  and corn  $C_t$  are modelled by a joint geometric Brownian motion (GBM) diffusion

$$dL_t = \mu L_t dt + \sigma L_t dW_{1t} \tag{3.6}$$

$$dC_t = aC_t dt + bC_t dW_{2t} aga{3.7}$$

$$\operatorname{Corr}[W_{1t}, W_{2t}] = \rho \tag{3.8}$$

where  $(W_{1t}, W_{2t})$  is a 2-dimensional Brownian motion defined on a filtered probability space  $(\Omega, \mathcal{F}_t, P)$  which satisfies the usual conditions [84].

The econometric model parameters are estimated by ordinary least-squares regression of the log time series  $\ln \frac{X_t}{X_{t-1}}$  using the 10 year monthly historical price series from Dec/02-Jan/11 capitalizing on earlier work in [79]. Prices for no. 2 yellow corn Omaha, NE underlying the CME corn futures contract were obtained from [92]. Average rack prices freight on board for ethanol were obtained from [82]. The correlation

<sup>&</sup>lt;sup>1</sup>There are 2000 lbs in a ton.

Parameter estimate	Value	t-test
$\hat{\mu}$	0	$P\left(\frac{\hat{\mu}-\mu}{s.e.} > t \mid \mu = 0\right) = 0.409$
$\hat{\sigma}$	0.156	-
$\hat{a}$	0	$P\left(\frac{\hat{a}-a}{s.e.} > t \mid \mu = 0\right) = 0.202$
$\hat{b}$	0.123	-
$\hat{ ho}$	0.105	-
$\hat{G}$	\$115.6	$G \in [108.4, 122.8]^a$

Table 3.2: Regression estimation results. <sup>a</sup> based on 95% confidence interval Student-t with 119 degrees of freedom.

estimate  $\rho$  was obtained via the sample correlation of the residuals. Parameter estimation results are in Table 3.2. Note that both ethanol and corn were found not to reject the null hypothesis of zero drift the 95% confidence interval. The estimate for the average distillers dried grains price  $\hat{G}$  was estimated by regressing the time series against a constant.

The stochastic subsidy  $Z_t$  is modeled as a pure Poisson arrival time jump process with arrival rate  $\lambda$  with jumps of size J.

$$dZ_t = (J - Z_{t-})dN_t (3.9)$$

where  $dN_t$ , defined on the probability space, is a continuous-time counting process  $\{N_t, t \ge 0\}$  that counts the number of jumps over time dt and

$$dN_t = \begin{cases} 1 & \text{with probability } \lambda dt \\ 0 & \text{otherwise.} \end{cases}$$
(3.10)

It is assumed in our model that J and  $N_t$  are independent of each other, and independent of  $W_{1t}$  and  $W_{2t}$  (which are correlated by  $\rho$ ).

The times between jumps  $t_i - t_{i-1}$  are seen to be quite well modelled by independently exponentially distributed Poisson arrivals (see Figure 3.1). The jumps J are assumed to be drawn from a lognormal distribution with parameters  $LogN(\alpha, \beta^2)$ . The parameters are estimated via maximum likelihood using the data in Table 3.1. The estimation results are summarized in Table 3.3.

Goodness of Fit of Subsidy Model The sample set for the subsidy policy is small (8 observations) and requires a test of the goodness of fit. By our model choice, the time between arrivals  $\Delta t$  of subsidy changes is exponentially distributed

Parameter estimate	Estimator	Value	95% confidence interval
$\hat{\lambda}$	$\left(\frac{1}{n}\sum_{i=1}^{n}t_{i}-t_{i-1}\right)^{-1}$	0.24	[0.10, 0.42]
$\hat{\alpha}$	$\frac{1}{n}\sum_{i=1}^{n}\ln x_i$	-0.69	[-0.79, -0.58]
$\hat{\beta}^2$	$\frac{1}{n-1}\sum_{i=1}^{n}(\ln x_i - \hat{\alpha})^2$	$0.015^{a}$	[0.0066, 0.062]

Table 3.3: Maximum likelihood estimation results. <sup>a</sup> Corrected unbiased estimator.

with parameter  $\lambda$   $(Exp(\lambda))$  and the series  $\frac{\ln Z_t - \hat{\alpha}}{\hat{\beta}}$  is a Student's *t*-distribution with n = 8 - 1 = 7 degrees of freedom  $(t_7)$  since  $\ln Z_t \sim N(\alpha, \beta^2)$ . The plots of the estimated theoretical cumulative distribution functions (CDFs) versus the empirical distributions are included in Figure 3.1 along with the QQ plots. By visual inspection, both data appear to be reasonably suited to the proposed subsidy model.

Lilliefors tests (a nonparametric variant of the Kolmogorov-Smirnoff test) were applied to test for normality in the log subsidy series and exponentiality in the subsidy arrival times using MATLAB's lilliefors.m function. Both samples accepted the null hypothesis of normality and exponentiality at the 5% significance level. This statistical evidence further supports our proposed model. We note however that this result is based on a small sample size.

## 3.2.4 Policy Uncertainty "at its Worst"

Since the policy uncertainty cannot be hedged and is presumably not strongly correlated with any market assets, there is cause for concern in terms of how to price this ethanol real option. Not only is there risk in the randomness of the process, but there is an added complexity of risk in the choice of model since it is truly uncertain, so-called "Knightian" uncertainty. To account for this model risk, uncertainty around the jump process parameters is included.

There are several possible ways to deal with model uncertainty and market incompleteness including: (1) cautiously deploying assumptions to simplify the problem; (2) utility indifference pricing with model uncertainty [75, 78]; and (3) best/worst case pricing (similar to the idea of good deal bounds and super-replication) [57]. Our analysis follows alternative (3) due to its financial intuition, transparency, and lack of subjectivity around economic aversion parameters or choice utility functions associated with utility-based pricing (which produce a subjective "personal price"). There is a connection between (2) and (3) however in that as the risk aversion parameter tends to infinity, the utility indifference price tends to the worst-case price. Management buying into an ethanol project can be considered "long" the real option. The



Figure 3.1: The empirical CDF (solid black) vs the theoretical CDF (grey dashed) of the time between arrivals  $\Delta t \sim Exp(\hat{\lambda})$  (upper left). The QQ plot of the time between arrivals (upper right). The empirical CDF (solid black) vs the theoretical CDF (grey dashed) of the normalized subsidy series  $\frac{\ln Z_t - \hat{\alpha}}{\hat{\beta}} \sim t_7$  (lower left). The QQ plot of the subsidy series (lower right).

worst case price is what a strongly risk averse buyer may consider when purchasing an option.

Management contemplating investment in an ethanol project may ask the question: Given the uncertainty around subsidy policy over the past 35 years, what is the expected case and worst case project value? To answer this question, the reference policy uncertainty distribution is adjusted within the following heuristically determined parameter bounds to form best and worst case bounds for the project value.

**Bounds on**  $\alpha$  Suppose management assumes  $VaR_{05}$  style bounds on  $\alpha$ .<sup>2</sup> In order to choose a lower bound for  $\alpha$ , management chooses a parameter  $\alpha_{min}$  such that the probability of observing a subsidy level J lower than the lowest historical subsidy  $Z_{min} = 0.40$  is 95%, i.e.  $P(J < Z_{min}) = 0.95$ . For a lognormal distribution with variance  $\beta^2 = 0.015$ ,  $\alpha_{min} = -1.118$ . An upper bound can be chosen as  $\alpha_{max}$  such that the probability of observing a lower subsidy J than the historical maximum  $Z_{max} = 0.60$  is also less than 5%, i.e.  $P(J < Z_{max}) = 0.05$ . In this case, the upper bound is  $\alpha_{max} = -0.309.^3$ 

**Bounds on**  $\lambda$  Similarly, the average arrival time of subsidy changes is bounded by infinity (i.e. no changes at all) where  $\lambda_{min} = 0$ . Reasoning that the US Farm Bill is the primary means by which ethanol subsidy policies are amended and that a new omnibus bill is passed every 5 years or so,  $\lambda_{max}$  can be chosen such that the probability of observing at least one jump in a 5 year cycle is at least 95%. Thus management seeks  $\lambda_{max}$  such that  $P(k = 0; \lambda_{max}, t = 5) \leq 0.05$  (i.e. the probability of observing zero jumps is at most 5%) where the probability of exactly k jumps occurring over t is  $P(k; \lambda, t) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}$ . This is given by  $e^{-\lambda_{max}5} \leq 0.05 \Rightarrow \lambda_{max} = \frac{\ln(0.05)}{5}$  or  $\lambda_{max} = 0.60$ .

The Best and Worst Case Bounds The best and worst case bounds can be summarized by the following:

$$\alpha \in [\alpha_{min}, \alpha_{max}] = [-1.118, -0.309]$$
 (3.11)

$$\lambda \in [\lambda_{\min}, \lambda_{\max}] = [0, 0.60]. \tag{3.12}$$

 $<sup>^{2}</sup>$ We note that management could use another technique to choose bounds such as the 95% confidence intervals on the mean estimate for example in Table 3.3.

<sup>&</sup>lt;sup>3</sup>We note that these bounds were chosen heuristically based on ethanol policy history and with reference to political precedent of the subsidy level. They do not represent a rigourous mathematical treatment of the small sample population time series.

# 3.3 The Stochastic Control Problem

In this section, we develop the jump diffusion counterpart of our model in [79] which leads to a system of interconnected obstacle problems, i.e. partial integro differential (PID) variational inequalities.

The total expected earnings  $V_i$  over the life of the project is given by the sum of its profits, plus the sum of any switching costs incurred over its operating life

$$V_{i}(l,c,z,t) = \sup_{\tau,u} E\left[\int_{t}^{T} e^{-r(s-t)} f_{I_{s}}(L_{s},C_{s},Z_{s}) ds + \sum_{k=1}^{n} e^{-r(\tau_{k}-t)} D_{u_{k-1},u_{k}} \right] (L_{t},C_{t},Z_{t},u_{0}) = (l,c,z,i)$$
(3.13)

The pair  $(\tau, u)$  is the control that the manager has over the facility in his ability to toggle production on and off. It consists of a set of switching times  $\tau_k$  and states to be switched into  $u_k$  with  $I_t = u_k, t \in [\tau_k, \tau_{k+1})$ . Thus  $\tau_k$  is an increasing set of switching times with  $\tau_k \in [t, T]$  and  $\tau_k < \tau_{k+1}$  given the initial operating state  $u_0 = i$ .

If management assumes a worst case pricing scenario for the policy parameters  $(\lambda, \alpha)$ , then

$$V_{i}(l,c,z,t) = \sup_{\tau,u} \inf_{\lambda,\alpha} E\left[\int_{t}^{T} e^{-r(s-t)} f_{I_{s}}(L_{s},C_{s},Z_{s}) ds + \sum_{k=1}^{n} e^{-r(\tau_{k}-t)} D_{u_{k-1},u_{k}} \right] (L_{t},C_{t},Z_{t},u_{0}) = (l,c,z,i)$$
(3.14)

where  $\lambda \in [\lambda_{min}, \lambda_{max}]$  and  $\alpha \in [\alpha_{min}, \alpha_{max}]$ . The limits on  $\lambda$  and  $\alpha$  prevent the optimization argument from growing unbounded and becoming singular [85]. The controls  $(u, \tau, \alpha, \lambda)$  come from an admissible set of non-anticipating controls (i.e.  $\mathcal{F}_t$ -measurable and Markovian).

## 3.3.1 An Intuition Building 1-dimensional Simplified Model

To make the full model exposition easier and to develop intuition, consider for the time being a simplified 1-dimensional approximation of the spread less fixed running costs

$$X_t = \kappa L_t - C_t - K \tag{3.15}$$

$$dX_t = adt + bdW_t \tag{3.16}$$

where a and b are naively chosen to fit the model. To further simplify the process, assume now that  $Z_t$  has normally distributed jumps such that

$$dZ_t = JdN_t \tag{3.17}$$

where  $J \sim N(\alpha, \beta^2)$ . The two  $(X_t + Z_t)$  can be combined into a jump diffusion process  $Y_t$ 

$$dY_t = adt + bdW_t + JdN_t \tag{3.18}$$

with solution

$$Y_t = Y_0 + at + bW_t + \sum_{k=1}^{N_t} J_k$$
(3.19)

where  $\sum_{k=1}^{n} J_k \sim N(n\alpha, n\beta^2)$ .

The expected income of the facility over its lifespan is

$$V_{i}(y,t) = \sup_{\tau,u} \inf_{\lambda,\alpha} E\left[\int_{t}^{T} e^{-r(s-t)} f_{I_{s}}(Y_{s}) ds + \sum_{k=1}^{n} e^{-r(\tau_{k}-t)} D_{u_{k-1},u_{k}} \middle| (Y_{t},u_{0}) = (y,i) \right]$$
(3.20)

By application of dynamic programming (see [60] or [85]) for optimal switching problems, the value function can be written as

$$V_i(y,t) = \sup_{\tau} \inf_{\lambda,\alpha} E\left[\int_t^{\tau} e^{-r(s-t)} f_i(Y_s) ds + e^{-r(\tau-t)} \left\{ V_j(Y_{\tau},\tau) - D_{ij} \right\}\right]$$
(3.21)

where  $i, j \in \{0, 1\}$  and  $\tau$  is the first time it is optimal to switch production regimes. Now the problem consists of finding the optimal sets of prices and times to either

- hold production in its current state i, denoting this continuation or (hold) set as  $H_i$ , or
- switch production into the other state j, denoting this switching set as  $S_{ij}$ .

By another application of dynamic programming and Ito's lemma for jump diffusions, this equation leads to a coupled system of free boundary PID equations (PIDEs). The free boundary problem can be written in complementary form by noting that either it is optimal to switch and  $V_i = V_j - D_{ij}$  or it is optimal to hold and  $V_i$  satisfies a PIDE subject to  $V_i \ge V_j - D_{ij}$ . Thus the equation extends on the whole space easing the need to track the switching boundary as a PID variational inequality (see [84] for an excellent reference on controlled jump diffusions). Thus the system of equations may be expressed as

$$\max\left[\underbrace{\frac{\partial V_i}{\partial t} + \mathcal{L}[V_i] + \inf_{\lambda,\alpha} \mathcal{I}[V_i] + f_i - rV_i}_{H_i}, \underbrace{(V_j - D_{ij}) - V_i}_{S_{ij}}\right] = 0.$$
(3.22)

where the spatial differential part of the generator is

$$\mathcal{L}[V] = a\frac{\partial V}{\partial y} + \frac{1}{2}b^2\frac{\partial^2 V}{\partial y^2}$$
(3.23)

and the integro part is

$$\mathcal{I}[V] = \lambda(E[V(y+J)] - V(y)). \tag{3.24}$$

The expectation E is taken with respect to a normal  $N(\alpha, \beta^2)$  kernel  $g_N$ 

$$E[V(y+J)] = \int_{-\infty}^{\infty} V(y+J)g_N(J)dJ.$$
 (3.25)

**Theorem 1** (Worst Case Price). The minimal optimal control is given by

$$\alpha = \alpha_{min}, \quad \lambda = \begin{cases} \lambda_{min} & \text{if } E[V(y+J)] - V(y) \ge 0, \\ \lambda_{max} & \text{if } E[V(y+J)] - V(y) < 0 \end{cases}$$
(3.26)

Theorem 2 (Best Case Price). The maximal optimal control is given by

$$\alpha = \alpha_{max}, \quad \lambda = \begin{cases} \lambda_{max} & \text{if } E[V(y+J)] - V(y) \ge 0, \\ \lambda_{min} & \text{if } E[V(y+J)] - V(y) < 0 \end{cases}$$
(3.27)

**Theorem 3** (Worst and Best Case Price if  $\alpha = 0$ ). The minimal optimal control is given by

$$\lambda = \lambda_{min},\tag{3.28}$$

and the maximal optimal control is given by

$$\lambda = \lambda_{max},\tag{3.29}$$

if  $\alpha = 0$  for all y.

See Appendix B for proofs of the above.

An interpretation of the maximal (respectively minimal) optimal control is as follows: (1) If the expected value post-jump E[V(y+J)] is better than its current value V(y), assume that the jump arrives as (in)frequently as possible  $1/\lambda_{max} (1/\lambda_{min})$ . (2) Assume that the jumps are in general as (un)favourable as possible  $\alpha_{max} (\alpha_{min})$ .

### Lessons from Merton

In the simplification where (1) the policy parameters  $(\lambda, \alpha)$  are constant and (2) switching costs  $D_{ij}$  are zero, the problem reduces to a PIDE which yields the option price

$$\frac{\partial V}{\partial t} + a\frac{\partial V}{\partial y} + \frac{1}{2}b^2\frac{\partial^2 V}{\partial y^2} + \lambda(E[V(y+J)] - V(y)) - rV + f(y) = 0$$
(3.30)

where  $f(y) = y^{+} = \max(y, 0)$ .

Using the Feynman-Kac Formula [84] and following Merton's classical paper on jump diffusions [81], the solution to the PIDE is

$$V(y,t) = E\left[\int_{t}^{T} e^{-r(s-t)} f(Y_s) ds \,\middle|\, Y_t = y\right].$$
(3.31)

**Theorem 4** (Constant Coefficient Option Price). The option price V(y,t) satisfies

$$V(y,t) = \sum_{n=0}^{\infty} \int_{t}^{T} e^{-\lambda(s-t)} \frac{\lambda^{n}(s-t)^{n}}{n!} e^{-r(s-t)} \left( A_{s,n} \Phi(d) + \frac{B_{s,n}}{\sqrt{2\pi}} e^{-\frac{d^{2}}{2}} \right) ds \qquad (3.32)$$

where  $A_{s,n} = y + a(s-t) + n\alpha$ ,  $B_{s,n}^2 = b^2(s-t) + n\beta^2$ ,  $d = A_{s,n}/B_{s,n}$  and  $\Phi(x)$  is the standard normal cumulative distribution function.

See Appendix B for the derivation of the governing PIDE and option price.

## 3.3.2 The Complete Problem

Return now to the stochastic control problem for the real option

$$V_{i}(l,c,z,t) = \sup_{\tau,u} \inf_{\lambda,\alpha} E\left[\int_{t}^{T} e^{-r(s-t)} f_{I_{s}}(L_{s},C_{s},Z_{s}) ds + \sum_{k=1}^{n} e^{-r(\tau_{k}-t)} D_{u_{k-1},u_{k}} \right] (L_{t},C_{t},Z_{t},u_{0}) = (l,c,z,i)$$
(3.33)

where  $\lambda \in [\lambda_{min}, \lambda_{max}]$  and  $\alpha \in [\alpha_{min}, \alpha_{max}]$ . We follow a similar argument as before using dynamic programming reducing the switching problem to a single decision  $\tau$ 

$$V_i(l, c, z, t) = \sup_{\tau} \inf_{\lambda, \alpha} E\left[\int_t^{\tau} e^{-r(s-t)} f_i(L_s, C_s, Z_s) ds + e^{-r(\tau-t)} \{V_j(L_\tau, C_\tau, Z_\tau, \tau) - D_{ij}\}\right].$$
(3.34)

Using Ito's lemma for jump diffusions and noting as in [60, 85, 93] that the problem can be written in complementary form as a variational inequality

$$\max\left[\underbrace{\frac{\partial V_i}{\partial t} + \mathcal{L}[V_i] + \inf_{\lambda,\alpha} \mathcal{I}[V_i] + f_i - rV_i}_{H_i}, \underbrace{(V_j - D_{ij}) - V_i}_{S_{ij}}\right] = 0.$$
(3.35)

where the spatial differential part of the generator is

$$\mathcal{L}[V] = \mu l \frac{\partial V}{\partial l} + ac \frac{\partial V}{\partial c} + \frac{1}{2} \sigma^2 l^2 \frac{\partial^2 V}{\partial l^2} + \rho \sigma l bc \frac{\partial^2 V}{\partial l \partial c} + \frac{1}{2} b^2 c^2 \frac{\partial^2 V}{\partial c^2}$$
(3.36)

and the integro part is

$$\mathcal{I}[V] = \lambda(E[V(l,c,J)] - V(l,c,z)).$$
(3.37)

**Theorem 5** (Worst Case Price). The minimal optimal control is given by

$$\alpha = \alpha_{min}, \quad \lambda = \begin{cases} \lambda_{min} & \text{if } E[V(l,c,J)] - V(l,c,z) \ge 0, \\ \lambda_{max} & \text{if } E[V(l,c,J)] - V(l,c,z) < 0 \end{cases}$$
(3.38)

**Theorem 6** (Best Case Price). The maximal optimal control is given by

$$\alpha = \alpha_{max}, \quad \lambda = \begin{cases} \lambda_{max} & \text{if } E[V(l,c,J)] - V(l,c,z) \ge 0, \\ \lambda_{min} & \text{if } E[V(l,c,J)] - V(l,c,z) < 0 \end{cases}$$
(3.39)

See Appendix B for proofs of the above.

## 3.3.3 The Decision to Enter

Management's optimal decision time to enter into the business  $\tau$  maximizes the expected value

$$V(l, c, z, t) = \sup_{\tau} \inf_{\lambda, \alpha} E\left[e^{-r(\tau - t)} \max\{V_1, V_0\}(L_{\tau}, C_{\tau}, Z_{\tau}, \tau) - B \right| (L_t, C_t, Z_t) = (l, c, z)\right]$$
(3.40)

and is a classical "American" style exercise call option. By dynamic programming, the optimal stopping problem satisfies the PID variational inequality

$$\max\left[\underbrace{\frac{\partial V}{\partial t} + \mathcal{L}[V] + \inf_{\lambda,\alpha} \mathcal{I}[V] - rV}_{H}, \underbrace{(\max(V_1, V_0) - B) - V}_{S}\right] = 0.$$
(3.41)

This completes the jump diffusion analogue of [79] and represents the optimal entry strategy for investment into a corn-ethanol biofuel production plant.

# **3.4** Numerical Results

This section begins with a numerical investigation of the behaviour of the constant coefficient analytical model. The section then proceeds with an investigation of the effects of policy uncertainty on the 1-dimensional model including (i) the loss in value and (ii) the effects on switching decisions (which is also a proxy investigation of the effects on the entry decision). Finally, the section concludes with an investigation of the change in value between the full model with both policy uncertainty and model certainty or uncertainty.



Figure 3.2: The option value V(y,t) at various levels of  $\alpha$  (expected jump level) given standard parameters of  $\lambda = 1$  (Poisson arrival rate of jumps),  $\beta = 1$  (volatility of jump distribution), a = 0 and b = 1 (drift and volatility of diffusion), r = 0.01 (discount rate), and T - t = 1 (remaining option tenor).

## 3.4.1 The Constant Coefficient Model

Consider V(y, t) in Equation 3.32. Its behaviour is monotone increasing in y. Figure 3.2 shows that the function is increasing in  $\alpha$ . This is as expected since if the jumps tend to be more positive ( $\alpha > 0$ ), the spread tends to jump non-locally to a higher value of y (recall the option is monotone increasing in y), and vice versa if  $\alpha$  tends to be more negative.

Figure 3.3 indicates V is an increasing function of  $\lambda$  (although it is generally insensitive to  $\lambda$ ). This makes sense intuitively since as the frequency of jumps increases, more volatility is added to the option in terms of  $B_{s,n}$ , and Black-Scholes style options are increasing functions in volatility.

Figure 3.4 shows that V is sensitive to  $\lambda$  when there is an expected direction with the jumps (i.e.  $\alpha \neq 0$ ).



Figure 3.3: The option value V(y,t) at various levels of  $\lambda$  (Poisson arrival rate of jumps) given standard parameters of  $\alpha = 0$  (expected jump level),  $\beta = 1$  (volatility of jump distribution), a = 0 and b = 1 (drift and volatility of diffusion), r = 0.01 (discount rate), and T - t = 1 (remaining option tenor).



Figure 3.4: The option value V(y,t) at various levels of  $\lambda$  (Poisson arrival rate of jumps) and  $\alpha$  (expected jump level) given standard parameters of  $\beta = 1$  (volatility of jump distribution), a = 0 and b = 1 (drift and volatility of diffusion), r = 0.01 (discount rate), and T - t = 1 (remaining option tenor). On the left,  $\alpha = 1$  and on the right  $\alpha = -1$ .

**Impact on Value** The parameters  $\lambda$  and  $\alpha$  can be interpreted as measures of how infrequently policy changes occur and where management expects the subsidy to level move to, respectively. If the subsidy is expected to move up in value  $\alpha > 0$ , the jumps make the project more favourable. The opposite occurs if  $\alpha < 0$ : The future policy outlook is negative, and the project/option loses value.

As  $\lambda$  increases, policy changes occur more frequently which adds project/option value by means of the increased volatility associated with each jump. As the option to switch production off mitigates downside jumps on value V, the upside value of the jump volatility disproportionately increases the option's value. Figure 3.3 also reveals that the option is very insensitive to  $\lambda$  when there is no expected "directionality" in the jumps, i.e. when  $\alpha = 0$ .

## 3.4.2 The 1-dimensional Model

We now turn to an investigation of the effects of model uncertainty for a risk averse investor into the real option ethanol project. In this analysis,  $f_1(y) = y$  and  $f_0(y) = 0$ while  $D_{01} = 0.2$  and  $D_{10} = 0.1$ .

Figure 3.5 shows the project valuation results for the expected price with policy uncertainty, best and worst case prices given policy uncertainty where  $\alpha = 0$  is fixed and  $\lambda \in [0, 1]$ . The underlay shows the switching boundaries  $S_{ij}$  in y. Figure 3.6 shows the same information as Figure 3.5 but in this case there is model uncertainty  $\alpha \in [-0.2, 0.2]$  with expected parameter  $\alpha = 0$ .

Impact on Value The gap between the best and worst case prices can be significantly large if  $\alpha$  is allowed to vary as indicated in Figure 3.6; otherwise the difference is small (Figure 3.5) as expected from our results with the constant coefficient model. Since this function is convex, the integral operator is single-signed and the parameter  $\lambda$  assumes either  $\lambda_{min}$  in the worst case or  $\lambda_{max}$  in the best case when  $\alpha = 0$  in the example in Figure 3.5 by Jensen's inequality. The constant coefficient expected case model will always be bounded by the best and worst case project prices. In these examples, the expected case is nearer to the worst case since  $\lambda = 0.1$  is closer to  $\lambda_{min} = 0$  than  $\lambda_{max} = 1$ .

**Impact on Switching Decision** Although the effects are not very pronounced on the 1 year time horizon, model uncertainty has an impact on switching decisions. The lower charts in Figures 3.5 and 3.6 represent the switching boundaries



Figure 3.5: The option value V(y,t) at an "expected case" of  $\lambda = 0.1$  (Poisson arrival rate of jumps) and  $(\lambda_{min}, \lambda_{max}) = (0, 1)$  (parameter boundaries),  $\alpha = 0$  and  $\beta^2 = 0.1$  (mean and variance of jump distribution), a = 0 and b = 1 (drift and volatility of diffusion), r = 0.01 (disount rate), and T - t = 1 (option tenor). Switching costs are  $D_{01} = 0.2$  and  $D_{10} = 0.1$ .



Figure 3.6: The option value V(y,t) at an "expected case" of  $\lambda = 0.1$  (Poisson arrival rate of jumps) and  $\alpha = 0$  (expected mean jump size), but where  $\lambda \in [0,1]$  and  $\alpha \in [-0.2, 0.2]$  (parameter boundaries). The remaining parameters are  $\beta^2 = 0.1$  (variance of jump distribution), a = 0 and b = 1 (drift and volatility of diffusion), r = 0.01 (disount rate), and T - t = 1 (option tenor). Switching costs are  $D_{01} = 0.2$  and  $D_{10} = 0.1$ .

- $S_{01} = \{y : V_0(y,0) = V_1(y,0) D_{01}\}$ , the set of prices where the operating status is optimally switched on from idle, and
- $S_{10} = \{y : V_1(y,0) = V_0(y,0) D_{10}\}$ , the set of prices where the operating status is optimally switched off from running.

It can be seen that in the...

- ...worst case scenario: The operator switches production on later than in the expected case (i.e. at  $y > y^*$  if  $y^*$  is where the operator would switch production on in the expected case). Similarly, the operator switches production off earlier compared to the expected case (i.e. at  $y < y^*$  if  $y^*$  is where the operator would switch production off in the expected case).
- ...best case scenario: The operator switches production on earlier and switches production off later compared to the expected case.

In the example where  $\alpha = 0$  is fixed, the differences in switching boundaries between the best, worst and expected cases are almost negligible (Figure 3.5). However in the other example where  $-0.2 \leq \alpha \leq 0.2$  can vary, the differences in switching boundaries between the best, worst and expected cases can deviate a great deal. Thus it is not so much when management thinks a change in policy might occur (i.e.  $\lambda$ -driven) but rather how management expects that policy to change with respect to its current policy conditions—that is,  $\alpha$ -driven.

## 3.4.3 The Complete Model

This section concludes with a numerical investigation of the ethanol plant value in the presence or absence of policy uncertainty and model uncertainty. The ethanol plant is assumed to have a 10 year investment horizon, T - t = 10.

#### With and without Policy Uncertainty

We compare the real option project valuation of the ethanol plant in two cases where:

- Management ignores the uncertainty in the ethanol subsidy policy and assumes  $Z_t = Z$  (constant) to take its Jan/2011 value (Table 3.1),
  - in this case,  $f_1(L_s, C_s, Z) = \kappa (L_s K_1 + Z) C_s$  where Z = \$0.45/gallon is constant (also  $\lambda = 0$ ); and



Figure 3.7:  $V(L_t, C_t, Z, t)$  without policy uncertainty vs  $V(L_t, C_t, Z_t, t)$  with policy uncertainty. Parameters (from Tables 3.2 and 3.3) are  $\mu = 0$  and  $\sigma = 0.156$  (drift and volatility of ethanol), a = 0 and b = 0.123 (drift and volatility of corn), Z = 0.45without policy uncertainty and  $Z_t = 0.45$ ,  $\lambda = 0.24$ ,  $\alpha = -0.69$  and  $\beta^2 = 0.015$ (arrival rate, mean and variance of jumps) with policy uncertainty.

• Management considers the uncertainty in the ethanol subsidy policy with known parameters (model certainty) and assumes the model parameters in Table 3.3 subject to the initial subsidy level being its Jan/2011 value as above,

- in this case,  $f_1(L_s, C_s, Z_s) = \kappa(L_s - K_1 + Z_s) - C_s$  where  $Z_t =$ \$0.45/gallon.

Figure 3.7 shows the value functions at various levels of  $C_t$  in the presence and absence of policy uncertainty. Figure 3.8 shows the switching boundaries in both cases.

Impact of Policy Uncertainty on Value As inferred from our 1-dimensional analysis in Section 3.3.1, policy uncertainty adds more value to the real option due to two distinct factors: (1) Given  $Z_t = 0.45 < 0.51 = e^{\alpha + \frac{1}{2}\beta^2} = E[J]$ , it is likely that the subsidy policy will jump to a higher level giving the option more value in the presence of policy uncertainty. (2) The extra volatility provided by the jump process adds volatility value to the option. The downside of policy switches on an ethanol plant can be mitigated by switching production off, while the upside value is



Figure 3.8: The switching boundaries  $\partial S_{01}$  and  $\partial S_{10}$  in the presence and absence of policy uncertainty. Parameters (from Tables 3.2 and 3.3) are  $\mu = 0$  and  $\sigma = 0.156$  (drift and volatility of ethanol), a = 0 and b = 0.123 (drift and volatility of corn), Z = 0.45 without policy uncertainty and  $Z_t = 0.45$ ,  $\lambda = 0.24$ ,  $\alpha = -0.69$  and  $\beta^2 = 0.015$  (arrival rate, mean and variance of jumps) with policy uncertainty.

maintained by keeping (or switching) production on when prices favourably allow for it. The capitalized cost of construction on a per bushel basis  $\kappa B$  is also included in Figure 3.8.

Impact of Policy Uncertainty on Switching Decisions The boundary at which production is switched on from an idle state is  $\partial S_{01}$  and the boundary at which production is turned off from a running state is  $\partial S_{10}$ . In this case, the initial subsidy level  $Z_t$  is less than the long run average  $E[J] = e^{\alpha + \frac{1}{2}\beta^2}$ ,  $Z_t = 0.45 < 0.51 = e^{-0.69 + \frac{1}{2}0.015}$ . Thus, the operator generally waits longer before turning production off, due to a positive outlook that the subsidy might jump up to its long term average. Similarly, the operator generally turns production on sooner in hope that the subsidy might again jump to its (higher) long run average. More precisely, given a point (c, l) on  $\partial S_{01}$  in the absence of policy uncertainty, if  $(c, l^*)$  is on  $\partial S_{01}^*$  in the presence of policy uncertainty, then  $l^* < l$  (respectively  $l^* > l$ ) when production is shut down earlier (later).

Changes in z shift value and switching decisions up or down non-locally as  $Z_t$  jumps. The general direction of the jumps is illustrated in Figure 3.8 by the arrow  $Z_t \xrightarrow{J} Z_{t+dt}$ .

It should be noted that if management were expecting the subsidy to jump to a lower level, the opposite situation as described above would occur. Management would switch production off earlier and turn production on later for fear that the subsidy might fall.

### Policy Uncertainty with Model Uncertainty

In the likely event that the distribution and parameters of the regulatory uncertainty process are unknown, management may choose a worst case valuation for the ethanol plant project value. The assumed boundaries for policy change arrival rate are  $\lambda \in [0, 0.60]$  and expected mean subsidy policy  $\alpha \in [-1.118, -0.309]$ .

Figure 3.9 illustrates the worst case value compared to the expected case given by the model parameters in Tables 3.2 and 3.3. The switching boundaries are illustrated in Figure 3.10 comparing the worst case operating decisions to the expected case.

For completeness, Figure 3.11 shows the envelope of best case, worst case and expected project values in the presence of policy and model uncertainty. The bounds can be quite large between the best and worst project values even for "seemingly small" parameter boundaries. The switching boundaries are illustrated in Figure 3.12 comparing the best case operating decisions to the expected case.



Figure 3.9:  $V(L_t, C_t, Z_t, t)$  vs  $\inf_{\lambda,\alpha} V(L_t, C_t, Z_t, t)$  with policy (and model) uncertainty. Constant parameters (from Tables 3.2 and 3.3) are  $\mu = 0$  and  $\sigma = 0.156$ (drift and volatility of ethanol), a = 0 and b = 0.123 (drift and volatility of corn), and  $Z_t = 0.45$ ,  $\lambda = 0.24$ ,  $\alpha = -0.69$  and  $\beta^2 = 0.015$  (arrival rate, mean and variance of jumps). Non-constant parameters for model uncertainty are  $\alpha \in [-1.118, -0.309]$ and  $\lambda \in [0, 0.60]$ .



Figure 3.10: The switching boundaries  $\partial S_{01}$  and  $\partial S_{10}$  in the presence of policy uncertainty and model uncertainty in the worst case. Constant parameters (from Tables 3.2 and 3.3) are  $\mu = 0$  and  $\sigma = 0.156$  (drift and volatility of ethanol), a = 0 and b = 0.123 (drift and volatility of corn), and  $Z_t = 0.45$ ,  $\lambda = 0.24$ ,  $\alpha = -0.69$  and  $\beta^2 = 0.015$  (arrival rate, mean and variance of jumps). Non-constant parameters for model uncertainty are  $\alpha \in [-1.118, -0.309]$  and  $\lambda \in [0, 0.60]$ .



Figure 3.11:  $V_1(L_t, C_t, Z_t, t)$  vs  $\inf_{\lambda,\alpha} V_1(L_t, C_t, Z_t, t)$  vs  $\sup_{\lambda,\alpha} V_1(L_t, C_t, Z_t, t)$  with policy uncertainty. Constant parameters (from Tables 3.2 and 3.3) are  $\mu = 0$  and  $\sigma = 0.156$  (drift and volatility of ethanol), a = 0 and b = 0.123 (drift and volatility of corn), and  $Z_t = 0.45$ ,  $\lambda = 0.24$ ,  $\alpha = -0.69$  and  $\beta^2 = 0.015$  (arrival rate, mean and variance of jumps). Non-constant parameters for model uncertainty are  $\alpha \in$ [-1.118, -0.309] and  $\lambda \in [0, 0.60]$ .



Figure 3.12: The switching boundaries  $\partial S_{01}$  and  $\partial S_{10}$  in the presence of policy uncertainty and model uncertainty in the best case. Constant parameters (from Tables 3.2 and 3.3) are  $\mu = 0$  and  $\sigma = 0.156$  (drift and volatility of ethanol), a = 0 and b = 0.123 (drift and volatility of corn), and  $Z_t = 0.45$ ,  $\lambda = 0.24$ ,  $\alpha = -0.69$  and  $\beta^2 = 0.015$  (arrival rate, mean and variance of jumps). Non-constant parameters for model uncertainty are  $\alpha \in [-1.118, -0.309]$  and  $\lambda \in [0, 0.60]$ .

Impact of Worst Case Model Uncertainty on Value The worst case real option ethanol plant value represents a lower bound in project value. Figure 3.9 also includes the capitalized cost of construction on a per bushel of capacity basis  $\kappa B$ . As expected, fewer projects are net present value positive in the worst case project value compared to the expected case. That is, given the two sets of prices at a time t the set of prices that are "Net Present Value (NPV) positive" for entering into the project are

$$NPV = \{(l,c) : \max(V_1, V_0) - B > 0\} \text{ and } NPV^* = \{(l,c) : \inf_{\lambda,\alpha} \max(V_1, V_0) - B > 0\},$$
(3.42)

then

$$NPV^* \subseteq NPV \tag{3.43}$$

This means that fewer investments are entered into during times of high policy uncertainty if management is risk averse.

The converse to the above statement is that firms that are risk-loving may prefer a project with more policy uncertainty. These types of options tend to increase in value with volatility which may be more appealing to investors with a higher risk appetite. Further, risk-loving investors may tend to weight the better case model parameters higher than the worse case model parameters again resulting in higher valuations.

In certain cases, the integral operator may be  $\mathcal{I}[V] = E[V(l, c, J)] - V(l, c, 0.45) > 0$  and accordingly  $\lambda = \lambda_{min} = 0$  in the minimization. This is similar to the case with zero policy uncertainty. Thus, the worst case option value may at times approach the option value in the absence of policy uncertainty.

Impact of Worst Case Model Uncertainty on Operating Decisions The possible subsidy outcomes in the worst case scenario have a much more negative outlook than the expected case. Thus in the worst case scenario, the optimal strategy tends to be conservative when making switching decisions (Figure 3.10). The net result is that management switches production on much later and switches production off much earlier compared to the expected case operating strategy.

**Comments on the Best Case Model** Figure 3.11 shows that the gap between the best and worst case prices can be quite large. This is an artifact of the stochastic optimization problem that leads to very large arbitrage free price good deal bounds in practice with financial derivatives. Similar to before, management switches production on earlier and switches production off later compared to the expected case operating strategy (Figure 3.12).

## 3.5 Conclusions

The goal of our paper is to develop a quantitative model for managing and pricing regulatory risk. The accomplishments and overall theme of our paper are summarized in what follows.

## 3.5.1 Summary

Our paper laid out several research goals to contribute to the existing real options literature and the less developed body of research in policy uncertainty.

We presented a real option model to attempt to quantitatively model policy uncertainty using a jump diffusion process. This model allows for the valuation of long term energy projects in the presence of policy uncertainty. For a corn-ethanol case study (following [79]), we presented a real option model involving both standard price uncertainty modelled using a simplified one dimensional jump diffusion process for the relevant price spread and stochastic subsidy. We followed this with a more sophisticated multivariate model which independently modeled both the input and the output price. In addition, this model included the impact of policy uncertainty using a randomly fluctuating subsidy level. This fluctuating subsidy was quantified using a pure jump process. Given that there may be model uncertainty for the subsidy policy process, our proposed model includes a "worst case" (modelled using a VaR level) policy uncertainty scenario which allows the project investor to quantify and manage his worst case regulatory downside risk. This work allowed us to draw some general conclusions with policy level implications, as summarized and described in the next section.

## **3.5.2** Policy Conclusions

We outline the policy effects and numerical conclusions from our analysis in Section 3.4.

**Policy Uncertainty** In the case of policy certainty versus uncertainty, for the convex (or "long vol") real options considered here, the effects of policy uncertainty

always increase the value of the option when there is no directionality in the subsidy jumps.

More generally, the effects of policy uncertainty may be positive or negative for the project valuation. For example, if the subsidy is currently low and the future subsidy level is expected to be higher, the possibility of a jump in policy increases the overall value of the option. The opposite holds when the subsidy is high and the future subsidy is expected to be lower than today.

**Model Uncertainty** Typically, the effect of ambiguity in policy uncertainty models on project valuation is negative: A strongly risk averse manager taking a long position in the option should price the project using the worst case of possible parameters.

The optimal operating strategy in terms of the sets of prices, times, and subsidy levels to switch production vary based on the scenario. The strategy however generally obeys the following rules: (1) If the scenario is a worst case (respectively best case), then production is switched off earlier (later) compared to the constant parameter expected case, and production is switched on later (earlier) compared to the expected case. This represents an pessimistic (optimistic) outlook on regulatory policy changes. (2) If the scenario is a constant parameter case with policy uncertainty, then production is switched on earlier (later) if the current policy regime is lower (higher) than the expected long run trend. Similarly production is switched off later (earlier) if the current policy regime is lower (higher) than the expected long run trend.

The anecdotal evidence that suggests businesses delay investment longer in periods of high policy uncertainty is seen to be consistent with our model, supporting those claims [58, 72, 91]. In particular, given the tendency is generally to delay during periods of policy uncertainty suggests that investors use pessimistic model outlooks when making investment decisions. Given that fewer projects were net present value positive in the model uncertainty case versus the policy uncertainty with known parameters case, our model supports the claim that fewer investments are entered into during periods of high policy uncertainty.

## 3.5.3 Possible Extensions

The lognormal distribution for the policy subsidy jump process was chosen for several reasons: (1) subsidies cannot become negative; (2) model familiarity since geometric Brownian motion itself leads to a lognormal distribution and Merton's seminal jump



Figure 3.13: The probability distribution functions dP(J) of the jumps J of the expected case LogN(-0.69, 0.015), worst case LogN(-1.118, 0.015), best case LogN(-0.309, 0.015), and a referece case LogN(-0.7, 0.1) highlight the skew.

diffusion paper [81]; (3) analytical tractability; and (4) its second moments exist. The distribution however has large positive skew with a fat tail. This choice of distribution can lead to results which are relatively indifferent toward downside risk in the subsidy process, as the probability of observing very low subsidies is much smaller than the probability of observing very high subsidies. For reference, plots of the expected, worst and best case subsidy jump probability distribution functions are shown in Figure 3.13 along with a reference case to better illustrate the positive skew and fat tail.

To improve the model, more classes of jump distributions or non-constant Poisson arrival rates could be considered for future work. Another possible improvement to the expected subsidy jump model would be to incorporate management's views on the probability of possible policy outcomes or cases, each with an associated probability determined by management (an idea motivated by [76] but here simplified). This is both easier to justify to industry practitioners and greatly simplifies the analysis as it effectively removes the continuous variable J and replaces it with a discrete variable  $J_i$ . This reduces the dimensionality of the PID variational inequality system, which greatly reduces the computational time by reducing the problem to solving discrete weighted probabilities for each outcome  $J_i$ . For completeness, the integro operator would be replaced with  $\mathcal{I}[V] = \lambda(\sum_i V_i P_i - V)$  and a PID variational inequality solved for each outcome *i* with associated value function  $V_i$  and management probability estimate  $P_i$ . This method can be particularly helpful in situations where little historical time series information is available regarding policy uncertainty. The modeler can defer to management's views and experience.

# **3.6** Appendix A: Numerical Method

A brief exposition of the numerical method used to solve this PID variational inequality system is presented below. We refer the reader to [64, 69, 73, 84] for a more detailed analysis of the finite difference solutions to stochastic control problems and PIDEs.

The general PID variational inequality is of the form

$$\max\left[\frac{\partial V}{\partial t} + \mathcal{L}[V] + \mathcal{I}[V] + f - rV, \quad h - V\right] = 0.$$

where the differential operator is (occasionally suppressing any l, c, z dependence of  $\mu, \sigma, a, b$ )

$$\mathcal{L}[V] = \mu \frac{\partial V}{\partial l} + a \frac{\partial V}{\partial c} + \frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial l^2} + \rho \sigma b \frac{\partial^2 V}{\partial l \partial c} + \frac{1}{2} b^2 \frac{\partial^2 V}{\partial c^2}$$

and the integro operator is

$$\mathcal{I}[V] = \lambda(E[V(l, c, J)] - V(l, c, z))$$

and the constraint is

$$h = V_u - D_u$$

The numerical solution is obtained via finite differences at grid points  $V(l_i, c_j, z_p, t_k) = V_{i,j,p}^k$  usually using second order centred differences except possibly at the boundary conditions. The stencils are chosen to ensure the discretization matrix retains the

M-matrix property for stability [74]. The grid points are

$$t_k = t_0 + k\Delta t$$
$$l_i = l_0 + i\Delta l$$
$$c_j = c_0 + j\Delta c$$
$$z_p = z_0 + p\Delta z$$

where the increments  $\Delta$  need not necessarily be uniform. Divided differences are used to approximate the derivatives. Two are shown below for reference

$$\frac{\partial V}{\partial l} \approx \frac{V_{i+1,j,p}^k - V_{i-1,j,p}^k}{2\Delta l}$$
$$\frac{\partial V}{\partial t} \approx \frac{V_{i,j,p}^{k+1} - V_{i,j,p}^k}{\Delta t}$$

The integral E[V(l, c, J)] is simply truncated and approximated along a grid as well

$$E[V(l,c,J)] \approx \int_0^{J_{max}} V(l,c,J) P(J) dJ \approx \sum_{p=0}^P V_{i,j,p}^k g(z_p) \Delta z$$

where the expectation is truncated by a point  $J_{max} = z_P$  at which the error in the approximation is small. Note any kind of quadrature rule can be used along with non-uniform grid spacing besides the rule shown above.

A fitted scheme is used to write out a system of equations for  $V_{i,j,p}^k$  at the grid points

$$\frac{V^{k+1} - V^k}{\Delta t} + \theta L V^{k+1} + (1 - \theta) L V^k + \phi I V^{k+1} + (1 - \phi) I V^k + f \le 0$$

where L is the differentiation matrix associated with the partial differential operator  $\mathcal{L}$  including the source term -rV and I is the integration matrix associated with the integro operator  $\mathcal{I}$ . The parameters  $\theta$  and  $\phi$  blend averages of the discretized PIDE at time steps k and k+1 (e.g.  $\theta = 0$  is fully implicit and  $\theta = \frac{1}{2}$  yields a Crank-Nicholson scheme). A small abuse of notation  $V^k$  refers to the entire collection of grid points i, j, p at time step k. The running profit function at all grid points is simply f. The differentiation matrix L tends to be stiff whereas the integration matrix I tends to be non-stiff allowing for the use of IMEX style time marching schemes.<sup>4</sup> A fully implicit scheme can be used in order to have a L-stable method. When the correlation  $\rho$  is

<sup>&</sup>lt;sup>4</sup>We note that using a Crank-Nicholson scheme in both L and I appeared to deliver good results.

small, centred differences may deliver a stable *M*-matrix. As  $\rho$  grows, however, care must be taken to choose the stencils for the cross derivative term (e.g. 7-point stencils [74]). For nonuniform grid points, one-sided differences may be required for the first order derivatives to maintain stability [74].

For reference, L can be considered a tensor that operates on a square  $V_{i,j}$  at all p. In tensor notation, at the interior points L is for example

$$L_{i,j,i,j} = -\frac{2}{\Delta l^2} \frac{1}{2} \sigma_{i,j}^2 - \frac{2}{\Delta c^2} \frac{1}{2} b_{i,j}^2 - r$$

$$L_{i,j,i,j-1} = -\frac{1}{2\Delta c} a_{i,j} + \frac{1}{\Delta c^2} \frac{1}{2} b_{i,j}^2$$

$$L_{i,j,i-1,j-1} = \frac{1}{4\Delta l\Delta c} \rho \sigma_{i,j} b_{i,j}$$

where  $L_{i,j,i+i^*,j+j^*} = 0$  if  $|i^*|, |j^*| \ge 2$ . Conditions must be applied along the boundary (e.g. linearity at far field). The integration matrix I is applied to a column  $V_{i,j,p}$  across all p at a point (i, j), like a matrix in p constant across all i, j. For example,

$$I_{p,p} = \lambda \left[ \frac{1}{2} g(z_p)(z_{p+1} - z_p) - 1 \right]$$
$$I_{p,q} = \lambda \frac{1}{2} g(z_q)(z_{q+1} - z_{q-1}).$$

using a trapezoidal quadrature rule.

The system is solved subject to a known final condition V(l, c, z, T) = Q(l, c, z)(being a backward Kolmogorov type equation). If there is no salvage value at the end of the facility life  $V_{i,j,p}^{K} = Q_{i,j,p} = 0$  (where  $T = t_0 + K\Delta t$ ) but in general the salvage value should satisfy some inequalities around the switching costs  $D_{ij}$ .

This is a complementary problem

$$MV^{k} - b \le 0, \quad h \le V^{k}, \quad (MV^{k} - b)^{T} (V^{k} - h) = 0$$

where superscript T denotes the matrix transpose. The matrix M is an aggregation of the integration and differentiation matrix pre-multipliers of  $V^k$  while b is a vector of collected knowns at time k (from k + 1). This matrix system is then solved using an value iteration fixed point method similar to projected successive over-relaxation. Several iterative schemes for non-linear control problems are described in [56, 65, 64, 69, 73, 84, 93]. The method is consistent following a Taylor series and Riemann sum definition of the integral argument. If fully implicit methods are used, the discretization is stable if M is itself an M-matrix as M-matrices have
the monotone property. Following [64] this discretization converges to the viscosity solution of the HJB PID variational inequality.

## 3.7 Appendix B: Optimal Control

The intuition behind the proofs of the theorems in Section 3.3 are presented in this appendix.

## Regarding the 1-dimensional Model Optimal Stochastic Control 3.3.1

Proof of Theorems 2 and 2. Consider the optimization with respect to  $\lambda$ 

$$\inf_{\lambda_{\min} \le \lambda \le \lambda_{\max}} \mathcal{I}[V].$$

Due to the boundedness of  $\lambda$ , this problem is nonsingular. Since  $\mathcal{I}[V] = \lambda(E[V(y + J)] - V(y))$  is linear in  $\lambda$ , it achieves its critical values at the endpoints  $[\lambda_{min}, \lambda_{max}]$  and the optimal  $\lambda$  satisfies

$$\lambda = \begin{cases} \lambda_{min} & \text{if } E[V(y+J)] - V(y) \ge 0, \\ \lambda_{max} & \text{if } E[V(y+J)] - V(y) < 0. \end{cases}$$

Turning now to the optimization with respect to  $\alpha$ ,

$$\inf_{\alpha_{\min} \le \alpha \le \alpha_{\max}} \lambda(E[V(y+J)] - V(y)) \Rightarrow \inf_{\alpha} E[V(y+J)]$$

where we drop the  $\alpha$  bounds for notational brevity. The expectation can be written as

$$\inf_{\alpha} E[V(y+J)] = \inf_{\alpha} \int_{-\infty}^{\infty} V(y+J)g_N(J)dJ, \quad g_N \text{ is the normal kernel } N(\alpha,\beta^2)$$
$$= \int_{-\infty}^{\infty} \inf_{\alpha} \{V(y+\alpha+z)\}g_z^*(z)dz, \quad g_N^* \text{ is the kernel } N(0,\beta^2)$$
$$= \int_{-\infty}^{\infty} V(y+\alpha_{min}+z)g_z^*(z)dz$$

if V(y) is monotone increasing in y which is true of the class of profit functions f(y) considered in this analysis. (This result follows from the Feynman-Kac or Green's

formula for V(y) given f(y) is monotone increasing.)

A similar argument applies for deriving the maximal optimal control (Theorem 2) but applied in the opposite direction.

Summarizing, the *worst case* project value is given by the minimal optimal control and the *best case* is given by the maximal optimal control subject to certain regularity conditions on V and f (namely monotonicity).

Proof of Theorem 3. Note that the integro operator  $\mathcal{I}$  is single-signed almost everywhere if f is such that V(y) is convex and  $\alpha = 0$ . The justification follows from Jensen's inequality  $V(E[y+J]) \leq E[V(y+J)]$  and that  $E[y+J] = y + \alpha = y$ . Thus

$$\begin{split} E[V(y+J)] - V(E[y+J]) &= \\ E[V(y+J)] - V(y) &= \frac{1}{\lambda} \mathcal{I}[V] \geq 0 \end{split}$$

and accordingly  $\lambda = \lambda_{min}$  for all y (and vice versa for the maximal control).

#### Regarding the Constant Coefficient Option Price 3.3.1

Proof of Theorem 4. For a function  $u(Y_t = y, t)$ , applying Itos lemma for jump diffusions results in

$$u(Y_T, T) - u(y, t) = \int_t^T b \frac{\partial u}{\partial y} dW_s + \int_t^T \left( \frac{\partial u}{\partial t} + a \frac{\partial u}{\partial y} + \frac{1}{2} b^2 \frac{\partial^2 u}{\partial y^2} \right) ds + \int_t^T \left[ u(Y_s + J, s) - u(Y_s, s) \right] dN_t.$$

Taking the expectation causes the Ito integral to become zero (since  $E[\int_t^T u dW_s | \mathcal{F}_t] = 0$  for smooth functions u). The expectation of the jump term becomes

$$E\left[\int_{t}^{T}\left[u(Y_{s}+J,s)-u(Y_{s},s)\right]dN_{t}\right] = \int_{t}^{T}E_{J}\left[u(Y_{s}+J,s)-u(Y_{s},s)\right]\lambda ds$$

since the Poisson arrivals  $dN_t$  and Brownian motion  $dW_t$  are independent, and  $dN_t = 1$  with probability  $\lambda ds$  or 0 otherwise. Here  $E_J$  denotes an expectation with respect to J only (recall  $J \perp W_t$ ).

When  $u(\cdot, T) = 0$ , and the jumps J and Brownian motion are independent, the

expectation is

$$E[u(Y_T, T) - u(y, t)] = -u(y, t) =$$

$$E\left[\int_t^T \left(\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial y} + \frac{1}{2}b^2\frac{\partial^2 u}{\partial y^2} + \lambda(E_J[u(Y_s + J, s)] - u(Y_s, s))\right)ds\right]$$

If u(y,t) satisfies the nonhomogeneous PIDE

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial y} + \frac{1}{2}b^2\frac{\partial^2 u}{\partial y^2} + \lambda(E_J[u(y+J,t)] - u(y,t)) = -f(y),$$

the solution has the probabilistic (Feynman-Kac) representation

$$u(y,t) = E\left[\int_{t}^{T} f(Y_s)ds \middle| Y_t = y\right]$$

The discounted value function  $V(Y_s, s) = e^{-r(s-t)}u(Y_s, s)$  satisfies the PIDE of Theorem 4 and has probabilistic representation

$$V(y,t) = E\left[\int_t^T e^{-r(s-t)} f(Y_s) ds \middle| Y_t = y\right].$$

The key to solving this expectation is to condition Y on n, the number of jumps so far, denoted  $Y_{s,n}|n$ . Note that the probability of observing n Poisson jumps over a time period s - t is  $P(n, s - t) = e^{-\lambda(s-t)} \frac{\lambda^n(s-t)^n}{n!}$ . Thus

$$V = E\left(E\left[\int_{t}^{T} f(Y_{s,n})ds \middle| n\right]\right)$$
  
=  $\sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \int_{t}^{T} e^{-\lambda(s-t)} \frac{\lambda^{n}(s-t)^{n}}{n!} e^{-r(s-t)} y_{s,n}^{+} \frac{1}{\sqrt{2\pi B_{s,n}^{2}}} e^{-\frac{(y_{s,n}-A_{s,n})^{2}}{2B_{s,n}^{2}}} ds dy_{s,n}$ 

where  $A_{s,n} = y + a(s-t) + n\alpha$  and  $B_{s,n}^2 = b^2(s-t) + n\beta^2$ .

$$V = \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \int_{t}^{T} e^{-\lambda(s-t)} \frac{\lambda^{n}(s-t)^{n}}{n!} e^{-r(s-t)} (A_{s,n} + B_{s,n}z)^{+} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^{2}}{2}} ds dz$$
$$= \sum_{n=0}^{\infty} \int_{t}^{T} \int_{-d}^{\infty} e^{-\lambda(s-t)} \frac{\lambda^{n}(s-t)^{n}}{n!} e^{-r(s-t)} (A_{s,n} + B_{s,n}z) \frac{1}{\sqrt{2\pi}} e^{-\frac{z^{2}}{2}} dz ds$$

where  $d = A_{s,n}/B_{s,n}$ . Changing variables x = -z and flipping the limits of integration

yields V =

$$\sum_{n=0}^{\infty} \int_{t}^{T} e^{-\lambda(s-t)} \frac{\lambda^{n}(s-t)^{n}}{n!} e^{-r(s-t)} \left( \int_{-\infty}^{d} A_{s,n} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^{2}}{2}} dx - \int_{-\infty}^{d} B_{s,n} x \frac{1}{\sqrt{2\pi}} e^{-\frac{x^{2}}{2}} dx \right) ds$$

and so

$$V(y,t) = \sum_{n=0}^{\infty} \int_{t}^{T} e^{-\lambda(s-t)} \frac{\lambda^{n}(s-t)^{n}}{n!} e^{-r(s-t)} \left( A_{s,n} \Phi(d) + \frac{B_{s,n}}{\sqrt{2\pi}} e^{-\frac{d^{2}}{2}} \right) ds$$

where  $\Phi(x)$  is the standard normal cumulative distribution function.

#### Regarding the Complete Stochastic Control Problem 3.3.2

Proof of Theorems 5 and 6. The argument for obtaining the optimal  $\lambda$  is identical to the 1-dimensional case. Determining the optimal  $\alpha$  is similar to the previous case, but slightly more delicate. Again, it rests on the monotonicity of f. Recall

$$f_1(l,c,z) = \kappa(l+z-K_1) - c, \quad f_0(l,c,z) = -\kappa K_0$$

and thus  $f_1$  is monotone increasing in z and  $f_0$  is unaffected by z. By the Feynman-Kac representation for  $V_1$  in Equation 3.34,  $V_1$  is monotone increasing in z. Similarly  $V_0$ , via the free boundary condition  $V_0 = V_1 - D_{01}$  in Equation 3.35, is monotone increasing in z by virtue of the boundary condition and regularity results along the free boundary [84, 85]. Now it remains to show that the expectation has a minimum

$$\inf_{\alpha} E[V(l,c,J)] = \inf_{\alpha} \int_{0}^{\infty} V(l,c,J) g_{LN}(J) dJ, \ g_{LN} \text{ is the lognormal kernel } LogN(\alpha,\beta^{2})$$
$$= \int_{0}^{\infty} \inf_{\alpha} \{V(l,c,xe^{\alpha})\} g_{LN}^{*}(x) dx, \ g_{LN} \text{ is the kernel } LogN(0,\beta^{2})$$
$$= \int_{0}^{\infty} V(l,c,xe^{\alpha_{min}}) g_{LN}^{*}(x) dx$$

Summarizing, the PID variational inequality yields the *worst case* project value (minimal optimal control) when

$$\alpha = \alpha_{min}, \quad \lambda = \begin{cases} \lambda_{min} & \text{if } E[V(l,c,J)] - V(l,c,z) \ge 0, \\ \lambda_{max} & \text{if } E[V(l,c,J)] - V(l,c,z) < 0 \end{cases}$$

and following a similar argument as above yields the best case value (maximal optimal

control) when

$$\alpha = \alpha_{max}, \quad \lambda = \begin{cases} \lambda_{max} & \text{if } E[V(l,c,J)] - V(l,c,z) \ge 0, \\ \lambda_{min} & \text{if } E[V(l,c,J)] - V(l,c,z) < 0 \end{cases}$$

$$\square$$

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## Chapter 4

# Optimal Hedging in Illiquid Markets

#### Chapter Summary:

In a complete market with zero market frictions, classical finance theory states that there is a unique no arbitrage price for every derivative contract. However, all markets are incomplete to some degree, if only because transaction costs and market impact costs apply for all trades. We propose a model of market impact and transaction costs to reflect order book liquidity. Using this model, we illustrate how liquidity impacts derivative value when hedging with the asset under the proposed market frictions. We develop a mathematical formulation to compute bid and ask prices along with optimal hedging strategies for market makers in OTC derivative markets using a utility indifference framework. By numerically solving the associated Hamilton Jacobi Bellman equations, we develop some market intuition from the resulting prices and hedge ratios.

Working Paper: Christian Maxwell and Matt Davison, Optimal Hedging in Illiquid Markets, 2014.

## 4.1 Introduction

In a complete market with model certainty, a complete set of spanning assets, and zero market frictions, classical finance theory states that there is a unique no arbitrage price for every derivative contract [107]. In reality, nearly all markets are incomplete to some extent. This is not necessarily a negative characteristic of financial markets: Market incompleteness makes the world go 'round. If all risks were known and hedgeable, there would be little incentive to do any trading at all since market makers could not justify charging spreads, nor could buy side investors hope to make any profit from instruments they deem mispriced. For an insightful treatment and overview of market incompleteness, see [124].

A market maker in an incomplete over the counter (OTC) market seeks to establish reasonable bid and ask prices for derivative securities f based on a marketed asset  $S_t$ . In a Black-Scholes complete market setting, there is a unique price which is the discounted expected future payoff under a martingale measure Q,  $p_{BS} = E_Q[e^{-r(T-t)}f(S_T)]$ . With market incompleteness there exists a range of possible equivalent martingale measures Q, and one candidate set of bid and ask prices are the no-arbitrage bounds

$$\left(\inf_{Q\in\mathcal{Q}} E_Q[e^{-r(T-t)}f(S_T)], \sup_{Q\in\mathcal{Q}} E_Q[e^{-r(T-t)}f(S_T)]\right)$$
(4.1)

inside of which no arbitrage is guaranteed. These bounds are generally too large to act as useful bid-ask prices as observed in our earlier work [117], and noted by many authors, e.g. [124].

Another alternative is quadratic or variance hedging where the market maker seeks to minimize his expected hedging error. Strategies can be local such as described in [94] or global where all strategies over a time interval are considered and an efficient frontier of execution prices is developed as outlined in [121].

Another promising strategy, and the one considered in this paper, is to embed the pricing and hedging problem within a portfolio optimization problem. We seek a global strategy where the market maker tries to optimize the utility of his terminal wealth with and without the derivative security, otherwise known as utility indifference pricing [98, 101, 109]. He may hedge with several underlying assets in the utility maximization. The utility ask price is the amount of cash the market maker would need to accept at the outset to make him indifferent between his optimal terminal wealth short the option and without the option, or the amount a trader may be willing to pay to buy the derivative; and vice versa for the bid price. If his optimal terminal utility value function is U, then with an initial endowment  $B_t$  the utility indifference ask price  $p_a$  is

$$E[U(B_T)] = E[U(B_T - f(S_T) + p_a e^{r(T-t)})].$$

The utility indifference bid and ask prices form a possible set of bid-ask prices for the derivative security

$$(p_b, p_a)$$

We prefer the utility indifference framework as it has a sound economic interpretation as a personal replacement price at which the agent is indifferent between his feasible alternative investments. We are aware that the "personal" nature of these prices, the need to consider holistic portfolios (including initial endowment), and the specifications of a utility function and statistical probability measure to value any projects have all been identified as possible shortcomings with this approach [99].

We consider the problem of hedging and pricing derivatives in incomplete markets caused by market frictions or transaction costs. In a case where a market maker is hedging a large option position, it is possible his hedging strategy may consume deep into the order book and incur large bid ask spreads. If there is market illiquidity, there may not only be a temporary price impact incurred from market orders going deep into the order book, but a permanent price impact as well. This creates a sort of feedback effect from hedging which almost always moves the value of the underlying in a direction unfavourable to the derivative holder (as will be shown later). Additionally, there is almost always some fixed cost associated with executing any trade (including at the very least, the value of the trader's time spent considering and executing the trade decision). This is the case for all market making: The trader must consider the trade-off between execution risk and market risk when deciding at what rate to hedge his position or manage his inventory. Shares are traded by means of an order book. Market makers post volumes of shares they are willing to sell (ask) or buy (bid). When, say, a portfolio manager sends a market order to buy, he will be transferred shares at the best ask price. If his order amount is sufficiently large, it may exhaust all the shares available at the best ask price. His order is then cleared at the second best ask price (which will be slightly more expensive) and so on as his order "walks through the book." It is also possible to trade more "slowly" at smaller amounts via limit orders. A limit buy order placed at X dollars will only execute if the best bid price is less than or equal to X. It is possible that this trade may not be executed if the bid never reaches X, or if not enough volume is available at X, then the order may only be partially filled. This is the trade-off between execution and market risk. (An example order book is shown in Figure 4.1.)

Some early work on transaction cost modeling can be found in [113, 126]. These are locally optimal models which consider proportional transaction costs (i.e. modeling the bid-ask spread). The globally optimal utility indifference hedge in the presence of proportional transaction costs was first introduced by [109] and then improved by [101]. Our transaction cost model considers market impact from large trades and is similar to the models presented in [95, 101, 104, 108, 116], which consist mostly of

Volume	Bid	Ask	Volume
100	14.99	15.01	150
200	14.98	15.02	250
350	14.97	15.03	400
500	14.96	15.04	600

Figure 4.1: A typical limit order book with bid-ask spreadss and a mid-price of \$15.00. At each price level there is a volume of shares available. This amount is not cumulative (e.g. up to an ask price level of \$15.02, there are 400 shares available for sale).

portfolio optimization and optimal liquidation applications. We have translated these models into the utility indifference pricing setting.

We will present a very general framework to model the depth, spread and liquidity of the order book using simple functions meant to capture the temporary and permanent price impacts based on the intensity of trading, along with possibly a fixed cost associated with each trade. The first contribution to the literature of this paper is to present a general framework that can be easily adapted to various market impact and transaction cost models which can transition between continuous trading rate controls and impulse control market orders. Our remaining contributions are to incorporate this framework into the utility indifference hedging and pricing approach and to present a numerical finite difference scheme to solve the associated equations. We note our framework is sufficiently general that it can be applied not only to hedging OTC derivatives but more daily aspects of market making such as inventory management or inventory quantile risk for example.

The problem was originally motivated by market makers seeking to price Asian style options in the presence of transaction costs, market impact or illiquidity. Asian options are typically sold OTC and accordingly useful bid-ask prices need to be established, and may be preferred in illiquid markets since average prices are more stable than market prices. Another possible situation is if the market impact is introduced by the hedger making large trades in an already relatively liquid market such as an accelerated share repurchase agreement. In these Asian style equity derivatives, the hedger may have to trade over short time horizons (days to weeks) and buy a significant amount of shares relative to the number of shares outstanding (market impact) [111].

We discovered however that even simple European options in the presence of market impact yield a sufficiently rich example against which to study our model. This is the primary example of our paper which is outlined as follows: Section 4.2 presents our model of market impact and utility indifference pricing; Section 4.3 presents the utility indifference pricing equations for a simple European call option; Section 4.4 contains our numerical investigation where the results which include some very interesting intuition are also discussed; and finally in Section 4.5 we discuss other possible applications of this model and extensions.

The Asian option application is discussed in Appendix A (4.6) along with our numerical scheme in Appendix B (4.7).

## 4.2 The Model

This section begins with an exposition of the market impact model for the trader with a large position in the underlying. The trader's utility objective is introduced and then we consider the utility indifference pricing problem. Following a dynamic programming argument, the Hamilton Jacobi Bellman (HJB) partial differential quasi variational inequality (PDQVI) is derived.

#### 4.2.1 Market Impact Model with Transaction Costs

We present a model of the market where the agent's hedging decisions have an impact on the market. The model aims to capture the effects of the trading rate  $\dot{x}_t$  on the stock price  $S_t$ . There is no non-trivial hedging strategy for option pricing even with much simpler transaction costs [123].

We aim to present a model that can transition smoothly from a continuous to a discrete trading framework. While continuous trading is only an approximation to reality, as trades are generally performed discretely, the continuous trading model has the advantage of being able to straightforwardly map intuitive feedback laws into the price impact. If any sort of fixed cost is incurred per trade, continuous trading becomes obsolete, since at each instantaneous trade, a nonproportional fixed cost is incurred. These accrued fixed costs become unbounded.

The derivative contract is hedged by holding a portfolio of the underlying shares. The total number of shares held at time t is  $X_t$  and the trading rate may be denoted as  $dX_t = \dot{x}_t dt$ , so

$$X_T = X_0 + \int_0^t \dot{x}_t dt.$$
 (4.2)

By definition,  $X_{0-} = 0$  (i.e. we begin with zero shares) and  $X_t \in \mathbb{R}$  (i.e. the trader can take long or short positions in the share).

The stock price is assumed to follow a geometric Brownian motion (GBM) with drift that is affected by the trading rate

$$dS_t = (\mu + g(\dot{x}_t))S_t dt + \sigma S_t dW_t \tag{4.3}$$

where  $dW_t$  is an increment of a standard Brownian motion satisfying the usual conditions on a filtered probability space  $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$  [119]. (Of course, every traders' actions have an effect on the stock price, which is another contributor to the apparent randomness observed in market prices.) In the literature, g is the permanent price impact function [95] which should be chosen so as to not admit round trip quasiarbitrage [104, 106, 110]. Thus in what follows we choose g to be symmetric such that

$$\oint_0^T g(\dot{x}_t) dt = 0, \quad \text{if } X_T = X_0$$

for any closed loop (i.e. round-trip trade). The presence of multiple agents in the market place all seeking profit precludes the prospect of round trip quasi-arbitrage. An intuitive form satisfying this condition is

$$g(\dot{x}_t) = \gamma \dot{x}_t. \tag{4.4}$$

Another advantage of this form is that it can also model discrete trading with the choice  $\dot{x}_t = \lambda \delta(t)$  where  $\delta$  is the Dirac delta function and  $\lambda = X_t - X_{t-}$  is the net change in position. At these discrete trade times, a change in the price occurs of

$$S_t = S_{t-} e^{\gamma \lambda}. \tag{4.5}$$

We note briefly that manipulating prices *favourably* is not possible. Round trip arbitrage trades are not profitable since the money earned pushing the price favourably will generally be lost while attempting to take profits closing out the position. Further if other market players became aware of a single player taking on a large enough position to attempt the manipulate prices, they would trade against the player with the large position. In this way, favourable manipulation is also generally not possible for option positions. After writing a put option, it may appear desirable to run up the share price so the put becomes worthless by purchasing a large position. This one-side view overlooks that the put holder may attempt the opposite trade and that other players in the market would trade against a player attempting to take on such a large position.

While trading, a temporary price impact of  $h(\dot{x}_t)$  is experienced by the stock price. This price impact reflects the bid-ask spread on the exchange order book. We assume a form that is symmetric<sup>1</sup> and simple

$$h(\dot{x}_t) = (1 + \eta \tanh(k\dot{x}_t)) \tag{4.6}$$

rather than the more traditional power forms (e.g.  $h(\dot{x}_t) = \operatorname{sgn}(\dot{x})|\dot{x}|^k$ ) as suggested in [96]. Here k > 0 is a constant that reflects the rate at which continuous trading approaches the market order bid-ask spread or in other words, the depth of the order book. Small trading rates do not consume deep into (walk through) the order book and can be interpreted as slower limit orders executed nearer the mid price. As the trade rate become instantaneous and discrete, the temporary price impact function approaches the standard proportional trading cost form as in [101]

$$\lim_{\dot{x}_t \to \pm \infty} h(\dot{x}_t) = (1 + \eta \operatorname{sgn}(\dot{x}_t))$$

which is not the case with power forms which become unbounded. (We drop the k above since k > 0.)

Slower trading rates minimize the temporary market impact, but leave the agent exposed to more market risk. Faster trading rates quickly walk through the order book, and therefore incur a higher spread (i.e. execution/liquidity risk), but reduce the market risk. This is illustrated in Figure 4.2.

The trader hedging the option position also holds cash in a risk-free bank account  $B_t$  earning a constant interest rate r such that

$$dB_t = rB_t dt - \dot{x}_t S_t h(\dot{x}_t) dt.$$
(4.7)

<sup>&</sup>lt;sup>1</sup>The assumption of symmetry simplifies the problem although it may not be reflective of reality. For some market players it is easier to buy, and for others to sell. Thus the form h could depend explicitly on the sign of  $\dot{x}_t$ , e.g.  $h(\dot{x}_t) = (1 + \xi(\dot{x}_t))$  where  $\xi(x) = \eta_+ 1_{x \ge 0} - \eta_- 1_{x < 0}$ . The parameters  $(\eta_+, \eta_-)$  could then be chosen to reflect the relative ease/difficulty of buying relative to selling.

Volume	Bid	Ask	Volume	
100	14.99	15.01	0	
200	14.98	15.02	200	
350	14.97	15.03	400	
500	14.96	15.04	600	

Figure 4.2: The order book of Figure 4.1 after a market order to buy 200 shares. The order has walked into the second layer of ask prices. A limit order with a smaller lot could be placed nearer to the mid-price but would take time to fill. Limit orders rely on the volatility of bid-ask prices in a sideway market to execute and may suffer from execution risk in a trending market.

#### 4.2.2 The Trader's Utility Objective

The trader is aiming to maximize the utility of his terminal wealth at the end of the time horizon T. In the classical Merton portfolio optimization problem [118], the agent invests a proportion y of his total wealth in the stock  $S_t$  and the remainder 1 - y in a risk free account  $B_t$ . His wealth process  $R_t$  evolves according to the SDE

$$dR_t = yR_t \frac{dS_t}{S_t} + (1-y)R_t \frac{dB_t}{B_t}$$

If the stock is a GBM with zero permanent or temporary price impact, using the SDEs above for  $dS_t$  and  $dB_t$ , the wealth process is simply

$$dR_t = (y\mu + (1-y)r)R_t dt + y\sigma R_t dW_t.$$

The wealth process is self-financing and the trader chooses a portfolio strategy to optimize his terminal utility of wealth. The value function v is

$$v(R,t) = \sup_{y} E[U(R_T)|\mathcal{F}_t]$$

where  $R_t = R$ . By the dynamic programming principle and Ito's lemma, the value function satisfies the HJB equation

$$\partial_t v + \sup_y \left[ (y\mu + (1-y)r)R\partial_R v + y^2 \sigma^2 R^2 \partial_{RR} v \right] = 0.$$

Returning to our trader hedging an option in the presence of market impact, he may choose to hold an option position that pays  $f(S_T)$  at expiry. His terminal wealth with n units of the option position is given by  $R_{n,T}$ 

$$R_{n,T} = B_T + c(X_T, S_T, nf(S_T))$$
(4.8)

where c(X, S, nf(S)) is the cost to liquidate the X shares of stock S net of receiving his payout nf(S) from holding the options. (The number of contracts n can be in  $\mathbb{R}$ or restricted to market contract conventions.) For example, if he holds long one call struck at K which expires in the money, his net liquidation position consists of X + 1shares with  $B_T - K$  cash.<sup>2</sup> His portfolio optimization problem in the absence of an option position is simply

$$R_{0,T} = B_T + c(X_T, S_T, 0). (4.9)$$

Note if  $\mu > r$ , it may be desirable to invest in the stock regardless of whether or not the trader holds an option position.

It is important that if the trader's net stock position at expiry is zero, there should be no liquidation costs, i.e.

$$c(X_T = 0, S_T, 0) = 0 (4.10)$$

when no contracts are held; and

$$c(X_T = X^*, S_T, nf(S_T)) = 0 (4.11)$$

where  $X^*$  is our stock position after fulfilling our obligations under the option contract. For example  $X^* = n$  if we hold n put options which expire in the money and we are fully hedged. This condition ensures our pricing rule stands on solid foundations as will be shown later.

The trader chooses a utility function  $U(\cdot)$  satisfying the Inada conditions. (The most important of which for us are that U is smooth, concave, and strictly increasing.) He then considers the Merton style portfolio optimization problem with and without the option position. Call v the value function from the portfolio optimization problem. Then

$$v_n(B_t, X_t, S_t, t) = v_n(B, X, S, t) = \sup_{\dot{x}_t} E[U(R_{n,T})|\mathcal{F}_t]$$
(4.12)

is the portfolio optimization where the trader holds n options with initial endowment  $B_t = B$ , position  $X_t = X$ , and share price  $S_t = S$ . Similarly his portfolio problem in

<sup>&</sup>lt;sup>2</sup>If he is hedged perfectly (no market frictions) then  $X_T = -1$  and  $B_T - K = 0$ .

the absence of an option position is

$$v_0(B, X, S, t) = \sup_{\dot{x}_t} E[U(R_{0,T})|\mathcal{F}_t].$$
(4.13)

The indifference price for the trader is the initial cash amount where he is indifferent between entering the option position and simply optimizing his portfolio as is<sup>3</sup> and has been proposed as early as [109] for models with trading costs and again in [101]. See [98] for an excellent reference on indifference pricing.

The utility indifference bid price  $p_n^b$  (i.e. the price a trader would be willing to pay) for n options is given by

$$v_n(B - p_n^b, 0, S, t) = v_0(B, 0, S, t)$$
(4.14)

where the trader initially holds zero shares. Similarly the indifference ask price  $p_n^a$  is given by

$$v_{-n}(B + p_n^a, 0, S, t) = v_0(B, 0, S, t).$$
(4.15)

It can be inferred from the above that  $p_n^b = -p_{-n}^a$ .

The pricing pair  $[p_n^b, p_n^a]$  form a good deal bound for the trader and accordingly induces a risk measure for pricing [112, 124]. Further, if  $p^{sup}$  and  $p^{sub}$  represent the super and sub-hedging prices respectively (the no arbitrage bounds of Equation 4.1), then

$$p^{sub} \le p_n^b \le p_n^a \le p^{sup} \tag{4.16}$$

with equivalence when markets are complete [98, 124].

#### Convergence to the Risk Neutral Price

The condition that there be no liquidation costs when our net stock position is zero ensures that the utility indifference price reduces to the risk neutral price in the absence of trading frictions. Following [101], suppose there is a replicating portfolio  $\dot{x}_{BS}$  (the Fundamental Theorem of Asset Pricing guarantees this under certain mild conditions [107]) that results in  $X_{BS,T} = -X^*$  and  $B_{BS,T} = -K$  where  $X^*$  is our stock position after fulfilling our obligations under the contract (i.e. the Black-Scholes  $\Delta$ ) and K is the cash we obtain on termination. For example,  $K = K^* 1_{\{K^* > S\}}$  for the case of a call option struck at  $K^*$ . Here  $\mathbf{1}_{\{A\}}$  denotes the indicator variable that

<sup>&</sup>lt;sup>3</sup>There is no need for the investment horizon to terminate at T coinciding with the option expiry. Nor is the trader restricted to only trading in the underlying security of the derivative. These two assumptions, however, greatly simplify the problem.

event A has occurred. Suppose  $-p_{BS}$  is the minimum endowment required to obtain the replicating strategy given  $X_t = 0$ , and accordingly  $p_{BS}$  is the Black-Scholes (risk neutral) price. Suppose further that an arbitrary trading strategy  $\dot{x}$  is linear and can be decomposed as  $\dot{x} = \dot{x}_{BS} + \dot{x}_q$ , attainable given  $B = -p_{BS} + B_q$  and X = 0. Then without loss of generality consider the case of holding one option

$$v_{1}(B_{q} - p_{BS}, 0, S, t) = \sup_{\dot{x}} E[U(R_{1,T})]$$

$$= \sup_{\dot{x}} E[U(B_{T} + c(X_{T}, S_{T}, f(S_{T})))]$$

$$= \sup_{\dot{x}_{q}} E[U(B_{q,T} + B_{BS,T} + c(X_{q,T} + X_{BS,T}, S_{T}, f(S_{T})))]$$

$$= \sup_{\dot{x}_{q}} E[U(B_{q,T} - K\mathbf{1}_{\{S_{T} > K\}} + c(X_{q,T} + X^{*}, S_{T}, f(S_{T})))]$$

$$= \sup_{\dot{x}_{q}} E[U(B_{q,T} + c(X_{q,T}, S_{T}, 0))]$$

$$= v_{0}(B_{q}, 0, S, t)$$
(4.17)

which coincides with our definition of the utility bid price  $p_n^b = p_{BS}$ . By multiplying the relevant coefficients by -1, we could follow the same logic used above to show that  $p_n^a = p_{BS}$ .

#### 4.2.3 The HJB Equation

Following the definition of the value functions  $v_n$  and  $v_0$ , we use a dynamic programming argument to derive the HJB PDQVIs for the portfolio optimization problem associated with the indifference price. Since the partial differential equations for both problems are equivalent except for the terminal conditions, in a slight abuse of notation we use v to describe both variants.

**Theorem 7.** By the dynamic programming principle, the HJB equation for the portfolio optimization problem is given by

$$\partial_t v + \sup_{\dot{x}_t \in A} \left\{ \partial_B v \left( rB - \dot{x}_t Sh(\dot{x}_t) \right) + \dot{x}_t \partial_X v + g(\dot{x}_t) S \partial_S v \right\} + \mathcal{L}[v] = 0$$
(4.18)

where  $\mathcal{L} = \mu S \partial_S + \frac{1}{2} \sigma^2 S^2 \partial_{SS}$  and A is the set of non-anticipating (Markovian) admissible controls.

The final form of the governing HJB equation rests largely on the specifications of the market impact model and whether or not there are fixed costs for trading. We derive the HJB equation for several special cases of the model presented in Section **Fixed short term price impact** If the short term price impact function  $h(\dot{x})$  is fixed at

$$h(\dot{x}) = (1 + \eta \operatorname{sgn}(\dot{x}))$$
 (4.19)

then the HJB equation becomes an HJB PDQVI in the limit that the set of admissible controls becomes unbounded  $A \rightarrow [-\infty, \infty]$  [122].

If  $\dot{x}_{min} \leq \dot{x} \leq \dot{x}_{max}$ , then following similar arguments as in [101] the HJB equation becomes

$$\partial_t v + \sup_{\dot{x}_{min} \le \dot{x}_t \le \dot{x}_{max}} \dot{x}_t \left\{ -S(1+\eta \operatorname{sgn}(\dot{x}_t))\partial_B v + \partial_X v + \gamma S \partial_S v \right\} + rB \partial_B v + \mathcal{L}[v] = 0.$$
(4.20)

**Theorem 8.** The optimal control is given by considering the following three cases:

- $-S(1+\eta)\partial_B v + \partial_X v + \gamma S \partial_S v \ge 0$  and  $-S(1-\eta)\partial_B v + \partial_X v + \gamma S \partial_S v > 0$ , then the agent buys at  $\dot{x}_t = \dot{x}_{max}$ ;
- $-S(1+\eta)\partial_B v + \partial_X v + \gamma S \partial_S v < 0$  and  $-S(1-\eta)\partial_B v + \partial_X v + \gamma S \partial_S v \leq 0$ , then the agent sells at  $\dot{x}_t = \dot{x}_{min}$ ; and
- $-S(1+\eta)\partial_B v + \partial_X v + \gamma S \partial_S v \leq 0$  and  $-S(1-\eta)\partial_B v + \partial_X v + \gamma S \partial_S v \geq 0$ , then the agent does nothing and  $\dot{x}_t = 0$ .

Note that the 4th permutation of inequalities:

•  $-S(1+\eta)\partial_B v + \partial_X v + \gamma S \partial_S v > 0$  and  $-S(1-\eta)\partial_B v + \partial_X v + \gamma S \partial_S v < 0$ 

implies  $2\eta \partial_B v < 0$  which is not possible as v is increasing in B ( $\partial_B v \ge 0$ ) and  $\eta$  is nonnegative.

Note that the block of terms maximized in Equation 4.20 can be grouped as  $\mathcal{H}$  (intuitively "H" for "HJB")

$$\mathcal{H}[v] = \dot{x}_t \left[ -S(1 + \eta \operatorname{sgn}(\dot{x}_t))\partial_B v + \partial_X v + \gamma S \partial_S v \right]$$

which gives the change in portfolio value arising from continuously trading shares at rate  $\dot{x}_t$ . Then the first case corresponds to a situation where the incremental value of the portfolio is increased by buying more shares. This incremental increase is maximized by buying at the maximum rate  $\dot{x}_{max}dt$ . The increase in value by adding to our share position  $\partial_X v$  overcomes the cost of purchasing shares in addition to paying the ask spread  $-S(1+\eta)\partial_B v$  along with any permanent price impact effects  $\gamma S \partial_S v$ . In the second case, selling at the bid price increases the value of the portfolio  $-S(1-\eta)\partial_B v$  and overcomes any loss in value by lowering our share position  $\partial_X v$ and driving down the stock price  $\gamma S \partial_S v$ . The third case is a situation where either buying or selling decreases the overall portfolio value so it is optimal to do neither (i.e.  $\dot{x}_t = 0$ ).

There is another possible case where

$$\partial_B v = \partial_X v = \partial_S v = 0.$$

In this situation any possible control  $\dot{x}_t$  can satisfy the conditions above for optimality and therefore the optimal control becomes nonunique. By choosing any control at that instant, the state space is pushed into a new region where a unique optimal control is defined. The nonuniqueness can be removed by noting a financially intuitive condition: If no net benefit occurs from trading, it is not worth the trader's time and effort to make a trade (i.e.  $\dot{x}_t = 0$ ).

In the case that the trader can make impulse trades (i.e.  $\dot{x} \to \pm \infty$ ), then the above equations reduce to an HJB PDQVI. See [120] for a helpful reference on impulse control problems.

**Theorem 9.** The optimal trading policy follows from the HJB PDQVI

$$\max\left\{\underbrace{-S(1+\eta)\partial_{B}v + \partial_{X}v + \gamma S\partial_{S}v}_{buy}, \underbrace{\partial_{t}v + rB\partial_{B}v + \mathcal{L}[v]}_{wait}, \underbrace{-(-S(1-\eta)\partial_{B}v + \partial_{X}v + \gamma S\partial_{S}v)}_{sell}\right\} = 0 \quad (4.21)$$

As before, there are three regions of space (B, S, X): One region where it is optimal to buy shares via an impulse trade to push the state space vector onto the border of the wait region; a second region where it is optimal to sell via an impulse trade back to the border of the wait region; the wait region where the agent does not need to rebalance his portfolio at all (until the diffusion vector (B, S, X) exits the wait region) [101].

Following [101], if the solution at the wait boundary is known  $v^*(B_w, S_w, X_w, t)$ , the function in the buy or sell regions can be determined. At a point (B, S, X) outside of the wait region (for example the buy region), the HJB PDQVI is governed by a



Figure 4.3: In the buy region, we execute a trade of  $\lambda$  to bring us back to the border of the wait region.

first-order PDE  $\dot{x}_t(-S(1+\eta)\partial_B v + \partial_X v + \gamma S \partial_S v)$ . Taking the total derivative

$$\frac{dv}{dt} = \frac{dB}{dt}\partial_B v + \frac{dx}{dt}\partial_X v + \frac{dS}{dt}\partial_S v = 0$$

and solving yields

$$v(B - \lambda Se^{\gamma\lambda}(1+\eta), Se^{\gamma\lambda}, X+\lambda, t) = v^*(B_w, S_w, X_w, t)$$
(4.22)

where  $\dot{x}_t = \lambda \delta(t)$  and  $\lambda = X_w - X$ . That is, an impulse trade of  $\lambda$  is made at time t. In other words, at any (B, S, X) there is a characteristic impulse jump to  $(B_w, S_w, X_w)$ , or  $(B, S, X) \xrightarrow{\lambda} (B_w, S_w, X_w)$ . The intuition here is that, within the buy/sell region, the solution v derives directly from the solution  $v^*$  along the boundary of the wait region at some characteristic point.<sup>4</sup> This is illustrated in Figure 4.3.

**Continuous short term price impact** If the short term price impact h is the continuous function previously described in Section 4.2, then the control problem is

$$\partial_t v + \sup_{\dot{x}_t \in A} \dot{x}_t \left\{ -S(1 + \eta \tanh(k\dot{x}_t))\partial_B v + \partial_X v + \gamma S \partial_S v \right\} + rB \partial_B v + \mathcal{L}[v] = 0.$$
(4.23)

**Theorem 10.** If the set of admissible controls is unbounded, then the optimal control

<sup>&</sup>lt;sup>4</sup>If the boundary surface can be written as a function  $X_w = f(B_w, S_w)$ , then along with  $\lambda$ , any point in space (B, S, X) can be spanned along these characteristic impulses.

is as follows:

$$\max\left\{\underbrace{-S(1+\eta)\partial_{B}v + \partial_{X}v + \gamma S\partial_{S}v}_{buy}, \underbrace{\partial_{t}v + \mathcal{H}[v] + rB\partial_{B}v + \mathcal{L}[v]}_{wait}, \underbrace{-\left(-S(1-\eta)\partial_{B}v + \partial_{X}v + \gamma S\partial_{S}v\right)}_{sell}\right\} = 0. \quad (4.24)$$

The operator  $\mathcal{H}$  is defined as

$$\mathcal{H}[v] = \dot{x} \left\{ -S(1 + \eta \tanh(k\dot{x}))\partial_B v + \partial_X v + \gamma S \partial_S v \right\}$$
(4.25)

where  $\dot{x}$  is given by

$$\mathcal{H}'[v] = 0 \tag{4.26}$$

where  $\mathcal{H}'$  denotes differentiation with respect to  $\dot{x}$ .

**Remark 4.2.1.** The optimal trade rate while in the continuous trading region is satisfies the equation

$$\mathcal{H}'[v] = 0$$
  
$$\tanh(k\dot{x}) + \dot{x}\tanh(k\dot{x})' = \frac{\partial_X v + \gamma S \partial_S v - S \partial_B v}{S \eta \partial_B v}$$

and has a solution which is satisfied within the continuous trading region. Although

$$\max_{\dot{x}} \tanh(k\dot{x}) + \dot{x} \tanh(k\dot{x})' = A \approx 1.2 > 1$$
$$\min_{\dot{x}} \tanh(k\dot{x}) + \dot{x} \tanh(k\dot{x})' = -A \approx -1.2, -1$$

(which occurs where  $\dot{x} > 0$  and  $\dot{x} < 0$  respectively) is bounded, within the continuous trading region  $-S(1+\eta)\partial_B v + \partial_X v + \gamma S\partial_S v \leq 0$  we have

$$\frac{\partial_X v + \gamma S \partial_S v - S \partial_B v}{S \eta \partial_B v} \leq 1.$$
$$\frac{\partial_X v + \gamma S \partial_S v - S \partial_B v}{S \eta \partial_B v} \geq -1.$$

Thus, there exists a solution  $\dot{x}$ .

Moreover, there may exist two solutions to the problem of Equations 4.24–4.26, particularly approaching the impulse region  $\frac{\partial_X v + \gamma S \partial_S v - S \partial_B v}{S_\eta \partial_B v} \rightarrow 1$ . The optimal  $\dot{x}$  then

follows from the second derivative condition

$$\mathcal{H}'' = -2\tanh(k\dot{x})' - \dot{x}\tanh(k\dot{x})'' < 0.$$

**Remark 4.2.2.** It is interesting that the sign of second derivative is independent of  $\eta$ . This is because the group  $\mathcal{H}$  is almost linear in x except for one term

$$\mathcal{H}[v] = -\eta \dot{x} \tanh(k\dot{x}) + \dot{x} \left\{ -S\partial_B v + \partial_X v + \gamma S\partial_S v \right\}$$

Thus in the above all the curvature derives from the first term only. The  $\eta$  factors out when considering the optimal trading rate for the points where multiple solutions exist to the first derivative condition  $\mathcal{H}'[v] = 0$ .

Furthermore, there exists a point at which it is no longer most economical to trade continuously but rather to make impulse trades. Consider for example a point in the buy region where it is not optimal to make an impulse trade  $0 < \dot{x} < \infty$ . By the inequalities above, there is an  $\epsilon = \tanh(k\dot{x})$ 

$$-S(1+\eta)\partial_B v + \partial_X v + \gamma S \partial_S v < 0$$
  
$$-S(1+\epsilon\eta)\partial_B v + \partial_X v + \gamma S \partial_S v \geq 0 = \frac{1}{\dot{x}} \mathcal{H}[v]$$

where  $\dot{x}$  optimizes  $\mathcal{H}$ . (A similar argument holds where  $-\infty < \dot{x} < 0$ .) After removing the constant terms  $-S\partial_B v + \partial_X v + \gamma S\partial_S v$ , scaling out the factor  $\eta \partial_B v$  considered along with the second derivative condition, we see that the point at which it becomes optimal to impulse trade is dependent on k (the rate at which we consume into the order book—i.e. depth) rather than  $\eta$  (the spread in the order book).

Where there are fixed costs associated with each trade If there is a fixed cost D > 0 associated with each infinitesimal trade, then it is no longer optimal to trade continuously. The trader must make impulse trades leading to a PDQVI. Accordingly, the number of shares held becomes a jump process affected by an impulse control strategy. Following [116], the trader must choose an optimal sequence of impulse controls  $\dot{x} = (\tau_n, \lambda_n)$  where  $\tau_n$  is an increasing sequence of stopping (trading) times and  $\lambda_n$  are the amount of shares to be bought or sold at each time. The set of admissible controls is non-anticipating (Markovian),  $\mathcal{F}_{\tau_n}$ -measurable and is such that the value function is bounded from below (e.g. finite number of trades, position limits, etc.).

This control sequence can be interpreted as  $\dot{x}_t = \sum_n \lambda_n \delta(t - \tau_n)$ , thus the state

variable  $X_t$  becomes

$$X_t = X_{\tau_n}, \ \tau_n \le t < \tau_{n+1}$$
 (4.27)

$$X_t = X_{\tau_n} + \lambda_{n+1}, \quad t = \tau_{n+1}.$$
(4.28)

Specifically,  $dX_t = 0$  except at times  $t = \tau_n$ .

As stated in Section 4.2, the permanent price impact after a trade occurs at  $t=\tau_n$  is

$$S_{\tau_n} = S_{\tau_n} - e^{\gamma \lambda_n}$$

and evolves by the regular GBM SDE  $dS_t = \mu S_t dt + \sigma S_t dW_t$  otherwise, so

$$S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \gamma \sum_n \lambda_n + \sigma W_t}.$$
(4.29)

The cash holdings evolve by the process

$$dB_t = rB_t dt, \quad \tau_n \le t < \tau_{n+1} \tag{4.30}$$

$$B_t = B_{t-} - \lambda_{n+1} S_t e^{\gamma \lambda_{n+1}} (1 + \eta \operatorname{sgn}(\lambda_{n+1})) - D, \quad t = \tau_{n+1}.$$
(4.31)

The value function  $v_n$  is as defined before

$$v_n(B, X, S, t) = \sup_{\dot{x}} E[U(R_{n,T})|\mathcal{F}_t].$$

Following a dynamic programming argument as in [116],

$$v_n(B, X, S, t) = \sup_{\tau, \lambda} E[v_n(B_\tau, X_\tau, S_\tau, \tau) | \mathcal{F}_t]$$
(4.32)

which leads to a HJB PDQVI.

**Theorem 11.** The value function v and optimal control satisfy

$$\max\left\{\underbrace{\partial_t v + rB\partial_B v + \mathcal{L}[v]}_{wait}, \quad \underbrace{\mathcal{H}[v] - v}_{trade}\right\} = 0 \tag{4.33}$$

where

$$\mathcal{H}[v] = \sup_{\lambda} v(B - \lambda S e^{\gamma \lambda} (1 + \eta \operatorname{sgn}(\lambda)) - D, X + \lambda, S e^{\gamma \lambda}, t).$$
(4.34)

As in the earlier model iterations (Theorems 8–10), there are three regions: A trade region where it is optimal to buy  $(\lambda > 0)$ ; a second trade region where it is

optimal to sell ( $\lambda < 0$ ); and a final "wait" region where it is optimal to make no trades at all ( $\lambda = 0$ ) and the PDE evolves under the uncontrolled diffusion  $dS_t$ . Intuitively, the singular control problem described earlier is a special case of the impulse control problem in that the limit as  $D \rightarrow 0$ , Equation 4.33 reduces to 4.21.

The trade (buy/sell) and no-trade/wait regions are given by the sets

$$trade = \{ (B, S, X, t) : v(B, S, X, t) = \mathcal{H}[v(B, S, X, t)] \}$$
(4.35)

$$wait = \{ (B, S, X, t) : v(B, S, X, t) > \mathcal{H}[v(B, S, X, t)] \}.$$
(4.36)

In [116], it is shown that the control set is non-empty along with the wait region. Uniqueness and continuity of the solution are also verified. Given the presence of the permanent price impact, it is possible that net wealth can increase after executing some transactions, however the solution is still shown to be bounded [116]. Boundedness of our solution follows from the imposition of position limits on the trader.

#### 4.2.4 Under Exponential Utility

Assume the trader has an exponential utility with risk aversion parameter  $\beta$ 

$$U(R) = 1 - e^{-\beta R}.$$
(4.37)

The parameter  $\beta$  is called the risk aversion because as  $\beta$  grows, the downside risk is weighted much more heavily than the upside gains, hence more aversion to any residual risk. As  $\beta \to 0+$ , the agent tends to weight downside losses and upside gains equally as  $U(R) \sim \beta R$ . With the choice of exponential utility, the value function is

$$v_n(B, X, S, t) = 1 - \inf_{\dot{x}} E\left[ e^{\beta B_T - \beta c(X_T, S_T, nf(S_T))} \middle| \mathcal{F}_t \right].$$
(4.38)

Since  $B_T$  has the integral form

$$B_T = Be^{r(T-t)} - \int_t^T e^{r(T-s)} \dot{x}_s S_s h(\dot{x}_s) ds, \qquad (4.39)$$

following a similar argument to [101] the value function can be rewritten as

$$v_n(B, X, S, t) = 1 - e^{-\beta B e^{r(T-t)}} G_n(X, S, t)$$
(4.40)

where  $G_n$  is convex decreasing in (X, S).

The indifference bid and ask prices for n units of the option are given by the nonlinear pricing rule

$$p_n^b = -\frac{e^{-r(T-t)}}{\beta} \ln\left(\frac{G_n(0,S,t)}{G_0(0,S,t)}\right)$$
(4.41)

$$p_n^a = \frac{e^{-r(T-t)}}{\beta} \ln\left(\frac{G_{-n}(0,S,t)}{G_0(0,S,t)}\right).$$
(4.42)

Note that these prices are independent of initial wealth B.

The HJB equation for v becomes

$$\partial_t G + \inf_{\dot{x}_t \in A} \left\{ \beta e^{r(T-t)} \dot{x}_t Sh(\dot{x}_t) G + \dot{x}_t \partial_X G + g(\dot{x}_t) S \partial_S G \right\} + \mathcal{L}[G] = 0.$$
(4.43)

This choice of utility has the advantage of reducing the dimensionality of the HJB equation by one (removing  $\partial_B$ ). In fact, the initial wealth *B* factors out entirely thereby eliminating the pricing dependence on initial cash. Depending on the agent's opinion, this may be a more attractive feature. Further, by using the impulse control with fixed trading costs, the differential dimensionality is reduced yet again (effectively removing  $\partial_X$  from the differential equation yielding a relatively stable parabolic PDE rather than a numerically more dangerous hyperbolic PDE).

#### 4.2.5 The Simplified Deterministic Problem

To build some intuition around when trade decisions are made, we will take a slight departure to consider the deterministic optimal control equivalent to the model above. We consider the case of optimizing terminal wealth utility in the absence of holding any derivative contracts. The stock price is assumed to follow an ordinary differential equation

$$\frac{dS_t}{dt} = (\mu + \gamma \dot{x}_t)S_t \tag{4.44}$$

and so  $S_T = S_t e^{\mu(T-t) + \gamma \int_t^T \dot{x}_s ds}$ .

We observe some facts about the deterministic control problem: If there is no uncertainty, all our future decisions are known with certainty today. There is no value in waiting or deferring decisions. The fixed trading cost D precludes any intermediate trading between today and the end horizon. It would be suboptimal to accrue any extra trading costs D since we know what the price and portfolio outcomes will be at T. If we do in fact trade today (time t), our known gains must overcome any liquidation costs at terminal time T. Under exponential utility as before, the optimal control is

$$v(B, X, S, t) = \sup_{\dot{x}} 1 - \exp\left[-\beta(B_T + c(X_T, S_T))\right].$$

By the assumptions above, the solution is

$$v(B, X, S, t) = \begin{cases} 1 - \exp\left[-\beta(Be^{r(T-t)} + c(-X, S_T))\right] = \text{no trade, if it is optimal not to trade} \\ 1 - \exp\left[-\beta(Be^{r(T-t)} + c(-(X+\lambda), S_T)) + c(\lambda, S)e^{r(T-t)}\right] = \text{trade, otherwise} \end{cases}$$

$$(4.45)$$

where  $\lambda$  maximizes the value of trading. At time t, the value function and associated strategy is

$$v(B, X, S, t) = \max(\text{trade}, \text{no trade}).$$

If it optimal to trade, then

$$1 - \exp\left[-\beta(Be^{r(T-t)} + c(-X, S_T))\right] < 1 - \exp\left[-\beta(Be^{r(T-t)} + c(-(X+\lambda), S_T)) + c(\lambda, S)e^{r(T-t)}\right] c(-X, S_T) < c(-(X+\lambda), S_T)) + c(\lambda, S)e^{r(T-t)}.$$

Using the temporary and permanent price impact and fixed cost model from earlier,

$$X(1+\eta\operatorname{sgn}(-X))Se^{\mu(T-t)-\gamma X} - D1_{X\neq 0} < (X+\lambda)(1+\eta\operatorname{sgn}(-(X+\lambda)))Se^{\mu(T-t)-\gamma X} - D1_{\lambda\neq -X} - \lambda(1+\eta\operatorname{sgn}(\lambda))Se^{\gamma\lambda+r(T-t)} - De^{r(T-t)}$$

noting that  $S_T = Se^{\mu(T-t)-\gamma(X+\lambda)+\gamma\lambda} = Se^{\mu(T-t)-\gamma X}$  reflects the cumulative permanent trade impact if any trading occurs at t.

We now consider some simplifications to arrive at a special case which will use to build some intuition. We also use this special case as a benchmark against which to test our stochastic HJB numerical solutions. Assume  $\gamma = r = 0$ . Then the simplified problem reduces to

$$X(1+\eta \operatorname{sgn}(-X))Se^{\mu(T-t)} - D1_{X\neq 0} < (X+\lambda)(1+\eta \operatorname{sgn}(-(X+\lambda)))Se^{\mu(T-t)} - D1_{\lambda\neq -X} - \lambda(1+\eta \operatorname{sgn}(\lambda))S - D.$$

It may be challenging to glean much information from the above, so consider the assumption that markets are very liquid and  $\eta \ll 1$ 

$$XSe^{\mu(T-t)} - D1_{X \neq 0} < (X+\lambda)Se^{\mu(T-t)} - D1_{\lambda \neq -X} - \lambda S - D$$
  
-  $D1_{X \neq 0} < \lambda S \left( e^{\mu(T-t)} - 1 \right) - D1_{\lambda \neq -X} - D.$ 

Two cases can occur: (1)  $X \neq 0$  and so

$$D1_{\lambda \neq -X} \le \lambda S \left( e^{\mu(T-t)} - 1 \right) \tag{4.46}$$

or (2) X = 0 (noting that, if a trade in fact occurs,  $\lambda \neq -X$ ) and so

$$2D \le \lambda S \left( e^{\mu(T-t)} - 1 \right).$$
 (4.47)

That is, we must overcome two trades 2D without the counterbalance of liquidating the portfolio D when  $X \neq 0$ ; otherwise we need only overcome the cost of trading D from our expected gains  $\lambda S(e^{r(T-t)} - 1)$ .

To determine the optimal  $\lambda$ , consider the case  $\beta \ll 1$  and accordingly

$$1 - e^{-\beta y} = 1 - (1 + -\beta y + O(\beta^2)) \approx \beta y.$$

Including all our earlier simplifying assumptions,  $\lambda$  satisfies

$$\max_{\lambda} \beta \left( (X+\lambda)Se^{\mu(T-t)} - D\mathbf{1}_{\lambda \neq -X} - \lambda S - D \right).$$
(4.48)

Again, there are two cases: (1) If  $\lambda \neq -X$ , then ignoring  $\beta$  and collecting terms

$$\max_{\lambda} \lambda S \left( e^{\mu(T-t)} - 1 \right) + X S e^{\mu(T-t)} - 2D$$

which is linear in  $\lambda$  and so the maximum occurs at the endpoint

$$\lambda = X_{max} - X \tag{4.49}$$

where  $X_{max}$  is the maximum allowed long position.<sup>5</sup> (2) It is also possible that  $\lambda = -X$  where the earlier argument (4.48) reduces to simply XS - D. In this case,

<sup>&</sup>lt;sup>5</sup>Here it is assumed that  $\mu > 0$  otherwise there would be no incentive at all to trade in the stock.

it must be that

$$XS - D \geq (X_{max} - X)S(e^{\mu(T-t)} - 1) + XSe^{\mu(T-t)} - 2D$$
  

$$XS - D \geq X_{max}S(e^{\mu(T-t)} - 1) + XS - 2D$$
  

$$0 \geq X_{max}S(e^{\mu(T-t)} - 1) - D.$$
(4.50)

That is, given our best terminal wealth after trading, we are still worse off than having never traded.

This exercise allows us to develop some excellent intuition:

- If our risk aversion is small  $\beta \ll 1$ , our optimal strategy is to buy as many shares as possible.
- If the stock price is low, it may not be worth it to buy any additional shares given the expected return may not overcome our additional fixed costs associated with the trade.
- If we start in a position where we are already long the share, even with a high stock price, the additional gains from taking a maximal long position may not overcome the additional fixed trading costs.
- There is a tendency toward having a net zero position at the end horizon T since there is zero transaction cost associated with closing out that position. We must overcome the cost of two fixed costs when starting from a position of X = 0. This is as opposed to when X ≠ 0, where there is only one extra fixed cost since a sunk costs already exists at T due to closing out the position.

These phenomena are illustrated in Figure 4.4.

## 4.3 A Call Option Example

As a first case study, we consider the optimal replication strategy for a European call option in the presence of market impact. At expiry T, a call option struck at K pays  $V(S_T,T) = (S_T - K)^+$ . In the complete (Black-Scholes) market model, the governing



Figure 4.4: The optimal deterministic portfolio strategy. The colour map represents the trade size  $\lambda$ . There are three regions: One where it is optimal not to trade; one where it is optimal to trade to zero position; and a final where it is optimal to trade to the maximum position  $X_{max} = 1$ . Here  $\eta = 0.02, D = 1.0, \mu = 0.30, T - t = 1.0$ . We observe similar results in our finite difference code as we let  $\sigma, \beta \to 0$ .

PDE is

$$\partial_t V + rS \partial_S V + \frac{1}{2} \sigma^2 S^2 \partial_{SS} V - rV = 0$$

$$V(S,T) = (S_T - K)^+$$

$$\lim_{S \to 0} V(S,t) = 0$$

$$\lim_{S \to \infty} \partial_S V(S,t) = 1.$$
(4.51)

The solution of this is

$$V(S,t) = SN(d_{+}) - Ke^{-r(T-t)}N(d_{-})$$
  

$$d_{\pm} = \frac{\ln\left(\frac{S}{K}\right) + \left(r \pm \frac{1}{2}\sigma^{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$
  

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^{2}}{2}} dy.$$
(4.52)

In the Black-Scholes setting, the optimal hedging strategy is the so-called  $\Delta$  hedge, where at any time one should hold  $\Delta = \partial_S V$  shares of the stock. In our notation, this is  $\lambda(X = 0, S) = \partial_S V = N(d_+)$ . When there is market impact as described in Section 4.2, we refer to our indifference model. The value function under exponential utility satisfies

$$v_n(B, X, S, t) = \sup_{\dot{x} \in A} E\left[1 - e^{-\beta R_{n,T}} \middle| \mathcal{F}_t\right] = 1 - \inf_{\dot{x}} E\left[e^{-\beta R_{n,T}} \middle| \mathcal{F}_t\right]$$

Following a dynamic programming argument, we derive the HJB equation for the personal valuation of the call option under exponential utility

$$v_{n}(B, X, S, t) = 1 - \inf_{\dot{x}} E\left[ e^{-\beta B e^{r(T-t)} + \beta \int_{t}^{T} e^{r(T-s)} \dot{x}_{s} S_{s} h(\dot{x}_{s}) ds - \beta c(X_{T}, S_{T}, nf(S_{T}))} \middle| \mathcal{F}_{t} \right]$$
  
=  $1 - e^{-\beta B e^{r(T-t)}} G(X, S, t)$ 

$$G(X, S, t) = \inf_{\lambda, \tau} E\left[ e^{\beta e^{r(T-\tau)} (\lambda(1+\eta \operatorname{sgn}(\lambda))S_{\tau} e^{\gamma\lambda} + D)} G(X_{\tau}, S_{\tau}, \tau) \middle| \mathcal{F}_t \right].$$
(4.53)

The associated HJB equation to the dynamic programming problem follows from Ito's lemma.

**Theorem 12.** The optimal terminal portfolio utility and trading strategy satisfy

$$\min\left\{\partial_t G + \mathcal{L}[G], \quad \mathcal{H}[G] - G\right\} = 0 \tag{4.54}$$

where the operator  $\mathcal{H}$  is given by

$$\mathcal{H}[G] = \inf_{\lambda} e^{\beta e^{r(T-t)}(\lambda(1+\eta \operatorname{sgn}(\lambda))Se^{\gamma\lambda} + D)} G(X+\lambda, Se^{\gamma\lambda}, t).$$
(4.55)

The final condition is given by the cost to liquidate the portfolio at T

$$G_n(X_T, S_T, T) = \exp\left[-\beta c(X_T, S_T, nf(S_T))\right].$$

In the case of holding (shorting) n options, when the option expires in the money, we receive (sell) n stocks and pay (receive) the strike nK. We must close out our position of X + n shares ( $\lambda = -(X + n)$ ). Thus, the final condition is

$$G_{n}(X, S, T) = \begin{cases} \exp\left[-\beta \left(X(1+\eta \operatorname{sgn}(-X))Se^{-\gamma X} - D1_{\{X\neq 0\}}\right)\right] & \text{if } S \leq K \\ \exp\left[-\beta \left((X+n)(1+\eta \operatorname{sgn}[-(X+n)])Se^{-\gamma (X+n)} - D1_{\{X\neq -n\}} - nK\right)\right] & \text{if } S > K. \end{cases}$$

$$(4.56)$$

The associated boundary conditions are

$$\lim_{S \to 0} \partial_t G_n(X, S, t) = 0 \tag{4.57}$$

$$\lim_{S \to \infty} \frac{\partial_S G_n(X, S, t)}{G_n(X, S, t)} = \beta(X + n)(1 + \operatorname{sgn}(-(X + n)))e^{-\gamma(X + n)}$$
(4.58)

For completeness, the optimization problem in the absence of holding the option is given by

$$\min\left\{\partial_t G_0 + \mathcal{L}[G_0], \quad \mathcal{H}[G_0] - G_0\right\} = 0$$

where the operator  $\mathcal{H}$  is given by

$$\mathcal{H}[G_0] = \inf_{\lambda} e^{\beta e^{r(T-t)}(\lambda(1+\eta \operatorname{sgn}(\lambda))Se^{\gamma\lambda} + D)} G_0(X+\lambda, Se^{\gamma\lambda}, t)$$

subject to final condition

$$G_0(X, S, T) = \exp\left[-\beta \left[X(1+\eta \operatorname{sgn}(-X))Se^{-\gamma X} - D1_{\{X\neq 0\}}\right]\right]$$
(4.59)

with similar boundary conditions as above (i.e. n = 0).

The indifference bid and ask prices for n units of the call option are given by

$$p_n^b = -\frac{e^{-r(T-t)}}{\beta} \ln\left(\frac{G_n(0, S, t)}{G_0(0, S, t)}\right)$$
$$p_n^a = \frac{e^{-r(T-t)}}{\beta} \ln\left(\frac{G_{-n}(0, S, t)}{G_0(0, S, t)}\right).$$

## 4.4 Results

The numerical investigation is split into three subsections: The first is a thorough numerical investigation into the effects of transaction costs and liquidity on the hedging and pricing of a European call. The second subsection illustrates how this model yields some financially intuitive but atypical results. The third subsection discusses the results in more depth and explains the effects in financial terms. We summarize the pricing results in Table 4.1.

Associated Figure	Parameter Investigated	Bid	Ask
4.6	$\beta = 0.5$	0.34	0.45
	$\beta = 2.0$	0.26	0.47
4.8	D = 0.05	0.32	0.47
	D = 0.50	0.11	0.89
4.10	$\eta = 0.01$	0.29	0.53
	$\eta = 0.05$	0.19	0.70
4.12	$\gamma = 0.01$	0.22	0.68
	$\gamma = 0.05$	0.12	0.85

Table 4.1: At the money bid and ask prices (\$) for various parameter regimes. The Black-Scholes price is \$0.42.

## 4.4.1 The Effects of Temporary and Permanent Price Impact and Transaction Costs

We begin by investigating the effects of fixed transaction costs D (e.g. brokerage costs), temporary price impact  $\eta$  (e.g. bid-ask spread, order book depth and liquidity), and permanent price impact  $\gamma$  (e.g. feedback effects). Unless stated otherwise, the risk aversion parameter is  $\beta = 1.0$ . The risk free interest rate is r = 1% per annum on dollar deposits. The expected return and volatility of the stock are  $\mu = 1.5\%$  and  $\sigma = 20\%$  annualized respectively. The option tenor and investment time horizon are T - t = 1.

**Exponential Utility and Risk Aversion**  $\beta$  We begin with a numerical illustration of the effects of risk aversion  $\beta$  on the price and hedging strategy, often referred to as  $\Delta$ . In our notation,  $\Delta = \lambda$  at X = 0. The results are displayed in Figures 4.5-4.6.

**Fixed Costs** D We begin the investigation of the market impact parameters with a numerical illustration of the effects of fixed transaction costs D on the price and hedging strategy. The results are displayed in Figures 4.7–4.8.

**Bid-Ask Spread**  $\eta$  We follow with a numerical illustration of the effects of bid-ask spread or temporary price impact  $\eta$  on the price and hedging strategy. The results are displayed in Figures 4.9–4.10.

**Permanent Impact**  $\gamma$  We follow with a numerical illustration of the effects of liquidity or permanent price impact  $\gamma$  on the price and hedging strategy. The results



Figure 4.5: The optimal hedging strategy for a long and short European call. The solid black line is the Black-Scholes strategy (delta  $\Delta$ ). The solid grey (dashed black) line is with risk aversion  $\beta = 0.5$  ( $\beta = 2.0$ ). The fixed cost is D = 0.05 while the temporary and permanent price impacts are zero ( $\eta = \gamma = 0$ ). The remaining parameters are T - t = 1,  $\mu = 1.5\%$ ,  $\sigma = 20\%$ , and r = 1%.

are displayed in Figures 4.11–4.12.

Incentives to Break up the Order Some very interesting behaviour occurs as the magnitude of the permanent price impact  $\gamma$  increases relative to the fixed costs D and spread  $\eta$ . There is an incentive to break up the hedging order into several smaller subtrades. This phenomenon is illustrated in Figures 4.13–4.14. As S grows, the fully long hedge first moves to one half (1/2) then, as S increases further, to one third (1/3).

#### 4.4.2 Discussion

We explain and discuss the effects of each parameter in what follows. First, note the strategies in Figures 4.5–4.14 show the optimal trading/hedging strategy  $\lambda$  as a function of S along the line X = 0 at t. The policy  $\lambda$  is dependent on both S and X as is apparent in Figure 4.4. Thus for example in Figure 4.7, the strategy does not suffer from "pin risk" where any time the stock is greater than the strike K we fully hedge and as soon as S dips below K, we fully divest the share. This would result in very high transaction costs. The strategy is contingent on the entire space



Figure 4.6: The Black-Scholes and utility indifference bid and ask prices for the European call. The solid black line is the Black-Scholes price. The solid grey (dashed black) line is with risk aversion  $\beta = 0.5$  ( $\beta = 2.0$ ). The fixed cost is D = 0.05 with temporary impact (bid-ask spread)  $\eta = 0.005$ . The permanent price impact is zero ( $\gamma = 0$ ). The remaining parameters are T - t = 1,  $\mu = 1.5\%$ ,  $\sigma = 20\%$ , and r = 1%.

(X, S). Thus once S exceeds K (say  $S = K + \epsilon$ ) and we have fully hedged  $\lambda = \Delta = 1$  (assuming we are short the call), we are at a new state X = 1. The trading strategy  $\lambda$  at  $S = K - \epsilon$  and X = 1 is not to sell the stock  $\lambda = -1$  but rather to do nothing at all  $\lambda = 0$ . We wait until we are deeper out of the money before accruing the extra trading cost. This eliminates the "pin risk" or rapidly repeated rebalancing.

Exponential Utility and Risk Aversion  $\beta$  In all the hedging strategies (Figures 4.5–4.14), it is always optimal to hedge sooner (i.e. at lower S) due to the asymmetry of the payoff. When short the option, our downside risk is unlimited, whereas long the option, our downside risk is limited. The choice of exponential utility  $U(x) = 1 - e^{-\beta x}$  weights downside losses more heavily than upside gains. Hence, the higher residual downside risk at terminal wealth from the short call is more likely to be hedged away. This effect is illustrated clearly in Figure 4.5. As risk aversion  $\beta$  grows (c.f. the  $\beta = 0.5$  result to the corresponding  $\beta = 2.0$  result), the hedging strategy becomes more symmetric between the long and short call positions. In fact, we tend to hedge more closely to the Black-Scholes hedge regardless of the extra trading costs we are accruing. This is because the Black-Scholes  $\Delta$  eliminates all the residual risk with



Figure 4.7: The optimal hedging strategy for a long and short European call. The solid black line is the Black-Scholes strategy (delta  $\Delta$ ). The grey solid (black dashed) line is with fixed cost D = 0.05 (D = 0.50). The temporary and permanent price impacts are zero ( $\eta = \gamma = 0$ ). The remaining parameters are T - t = 1,  $\mu = 1.5\%$ ,  $\sigma = 20\%$ , and r = 1%.

the option, hence the higher the risk aversion, the closer we tend to track the Black-Scholes hedge.

This result is also reflected in the pricing (Figure 4.6). The extra transaction costs accrued by tracking the Black-Scholes hedge more closely (which relies on continuous, i.e. frequent, trading) result in larger bid-ask spreads in the utility indifference price as  $\beta$ , and hence the risk aversion, increases.

**Fixed Costs** D The effect of fixed transaction costs is to reduce the number of rehedges. As apparent in Figure 4.7, the fixed costs D filter out some of the early and late stage intermediate hedges relative to the Black-Scholes  $\Delta$ . It appears like a step function moving into and away from at the money S = K = \$5. In the relatively low cost case that D = 0.05, there is an intermediate area at the money where the optimal hedge follows the Black-Scholes  $\Delta$ . As we move deeper into (out of) the money with the long call position, we become more certain of how the option will expire. Because there is an additional transaction cost D at expiry if  $X \neq -1$  ( $X \neq 0$ ) when we are in (out of) the money, it becomes optimal to just become fully short (divested from) the stock. In the intermediate zone where the optimal trade tracks the  $\Delta$ , the


Figure 4.8: The Black-Scholes and utility indifference bid and ask prices for the European call. The solid black line is the Black-Scholes price. The grey solid (black dashed) line is with fixed cost D = 0.05 (D = 0.50). The temporary and permanent price impacts are zero ( $\eta = \gamma = 0$ ). The remaining parameters are T - t = 1,  $\mu = 1.5\%$ ,  $\sigma = 20\%$ , and r = 1%.

exposure to market risk outweighs the potential (double) fixed trading costs, so the trader makes an intermediate hedge immediately to avoid exposing himself to further market risk.

At the high cost value of D = 0.50, the optimal hedging strategy becomes a binary step function. The costs associated with additional intermediate trading far outweigh any foreseeable market risk (volatility) and so the trader either goes fully short in the stock when in the money, or fully divested from the stock when out of the money (i.e. no position X = 0).

The behaviour is also as expected in Figure 4.8, which shows the price as a function of S. Deep out of the money, the option approaches zero. There is no need to buy/sell any stock to hedge as it is known the stock will expire worthless and thus no fixed costs are accrued. Deep into the money, the story is similar. The spread between the Black-Scholes price approaches D as the optimal hedge indicates that one should go fully long/short the stock to cover the near certain obligations at expiry. Since only one trade occurs (the settlement precludes any additional trades or fixed costs from being incurred), the differences in price approach the singular fixed cost.

The "bulge" in the spread between the Black-Scholes and indifference prices at



Figure 4.9: The optimal hedging strategy for a long and short European call. The solid black line is the Black-Scholes strategy (delta  $\Delta$ ). The solid grey (black dashed) line is with temporary impact  $\eta = 0.01$  ( $\eta = 0.05$ ). The permanent price impacts is zero ( $\gamma = 0$ ) with fixed cost D = 0.05. The remaining parameters are T - t = 1,  $\mu = 1.5\%$ ,  $\sigma = 20\%$ , and r = 1%.

the money follow from the uncertainty. In the case where D is small and intermediate trading occurs, two fixed costs are accrued. However, the expected gains in the upside of the option cancel out some of the accrued fixed costs on average and so the spread does not necessarily approach 2D. Similarly as D grows large and the optimal hedge becomes a binary long/short or zero strategy, there is also a bulge. This is caused by the tracking error in the hedge when the option expires in the wrong direction from the hedge. Additional trading costs are accrued to rebalance the hedge at expiry.

**Bid-ask Spread**  $\eta$  The effects of  $\eta$  are more subtle (Figure 4.9). Since  $\eta$  is a temporary impact, in some ways increasing the spread cost is akin to increasing D. Thus as  $\eta$  increases, the optimal hedge approaches the binary decision.

On the other hand, since the costs scale with the stock price, there is a possibility and an incentive to minimize the hedging costs. In the long call hedge, we make our money from shorting the call. We make the most money from our hedge by shorting at the highest price and hence there is a longer delay between when we hedge the long call compared to the short call case or the case where  $\eta = 0$  but with larger D. We cannot affect the magnitude of the fixed costs but we can affect the magnitude



Figure 4.10: The Black-Scholes and utility indifference bid and ask prices for the European call. The solid black line is the Black-Scholes price. The solid grey (black dashed) line is with temporary impact  $\eta = 0.01$  ( $\eta = 0.05$ ). The permanent price impacts is zero ( $\gamma = 0$ ) with fixed cost D = 0.05. The remaining parameters are  $T - t = 1, \mu = 1.5\%, \sigma = 20\%$ , and r = 1%.

of the spread costs. Similarly with the short call, we spend our option premium by hedging the stock long. It is less costly to acquire the stock due to the transaction spreads when it is cheaper. Hence, we tend to hedge earlier compared to the long call hedge. In both the long or short call case, there is an incentive to wait before hedging compared to the Black-Scholes hedge. For example if the short call is out of the money, hedging by buying the stock with a spread premium to only close it out in a sideway market guarantees a loss. Hence it is expedient to delay before making any hedging decisions while out of the money.

In terms of pricing (Figure 4.10), the spread between the Black-Scholes price and the indifference price grows with S since the transaction costs are proportional to the stock price, unlike with increased D which tends to a constant spread.

**Permanent Impact**  $\gamma$  The effects of permanent impact  $\gamma$  are more complex (Figure 4.11). There still exists a band where it is optimal to track the Black-Scholes hedge  $\Delta$ . In fact in the short call position, it is optimal to begin hedging earlier as  $\gamma$  grows since the effect of hedging tends to push the stock against your favour deeper into the money, creating yet more downside risk. There is a tendency to track the Black-



Figure 4.11: The optimal hedging strategy for a long and short European call. The solid black line is the Black-Scholes strategy (delta  $\Delta$ ). The solid grey (black dashed) line is with permanent impact  $\gamma = 0.01$  ( $\gamma = 0.05$ ). The temporary price impacts is zero ( $\eta = 0$ ) with fixed cost D = 0.20. The remaining parameters are T - t = 1,  $\mu = 1.5\%$ ,  $\sigma = 20\%$ , and r = 1%.

Scholes hedge longer however since that is a smaller trade than going fully long the stock. The larger the trade, the more likely the option is to be pushed into the money and so there is an incentive to reduce the magnitude of the hedge. A similar strategy occurs in the long call hedge. As  $\gamma$  increases, there is a tendency to delay hedging as it is likely to push the stock out of the money. Further, there is a band where it is optimal to track the Black-Scholes hedge and engage in some intermediate trading when there is large  $\gamma$ , unlike the small  $\gamma$  case. The effects of  $\gamma$  compared to the fixed cost D tend to make intermediate hedging not as appealing when  $\gamma$  is smaller. Again, we tend to hedge later and differently between the long-short call position cases due to the asymmetry of the exponential utility in terms of how much more strongly it penalizes downside risk.

The effects of permanent impact on price are similar to the effects of temporary impact (Figure 4.12) in that both scale with the stock price. The spread is asymmetric since the scaling factor  $e^{\gamma\lambda}$  is asymmetric in  $\lambda$ .

Minimum Impact Trading: Breaking up the Order A very interesting phenomenon occurs as  $\gamma$  grows in relation to the fixed cost D and spread  $\eta$ . We see



Figure 4.12: The Black-Scholes and utility indifference bid and ask prices for the European call. The solid black line is the Black-Scholes price. The solid grey (black dashed) line is with permanent impact  $\gamma = 0.01$  ( $\gamma = 0.05$ ). The temporary price impacts is zero ( $\eta = 0$ ) with fixed cost D = 0.20. The remaining parameters are  $T - t = 1, \mu = 1.5\%, \sigma = 20\%$ , and r = 1%.

that an unusual hedging strategy arises from Figures 4.13–4.14. There exists a point at which we appear to break away from the Black-Scholes hedge and approach some midpoint hedge  $|\lambda(X = 0, S)| = 1/2$ . The reality is slightly different. The option is still being fully hedged, i.e.  $|\Delta| = 1$ , but it is now optimal to break the hedge into two subtrades to minimize the costs of the permanent price impact.

Recall from Figure 4.4 that the policy  $\lambda$  is a function of both X and S (and t). Thus in the 1/2 case deep in the money while short the call,  $\lambda(X = 0, S) = \frac{1}{2}$  while  $\lambda(X = \frac{1}{2}, S) = \frac{1}{2}$ . The net effect of this strategy is to hedge fully to X = 1 by  $X \Rightarrow 0 \to \frac{1}{2} \to 1$ . Similarly when  $\lambda(0, S) = \frac{1}{3}$ , then  $\lambda(\frac{1}{3}, S) = \lambda(\frac{2}{3}, S) = \frac{1}{3}$ . The combined effect of this strategy is to hedge to X = 1 via three suborders  $X \Rightarrow 0 \to \frac{1}{3} \to \frac{2}{3} \to 1$ . An apparent geometric symmetry arises from this phenomenon.

To see why this is optimal, consider the accrued costs from trading. Say we are deep in the money on a short call and wish to reach a position of X = 1 from initially holding zero shares X = 0. Our accrued trading costs are

$$-\lambda(1+\eta \operatorname{sgn}(\lambda))Se^{\gamma\lambda}-D$$



Figure 4.13: The optimal hedging strategy for a long and short European call. The solid black line is the Black-Scholes strategy (delta  $\Delta$ ). The solid grey (black dashed) line is with fixed cost D = 0.10 (D = 0.05). The temporary price impacts is zero ( $\eta = 0$ ) with permanent impact  $\gamma = 0.05$ . The remaining parameters are T - t = 1,  $\mu = 1.5\%$ ,  $\sigma = 20\%$ , and r = 1%.

where  $\lambda = 1$ . If the permanent price impact costs begin to outweigh the additional fixed costs D, there is an incentive to break the order into two or more possible suborders. This could be accomplished by sending the suborders serially to the exchange, or in parallel via multiple desks or brokers. The optimal subtrades minimize the accrued trading costs

$$\min_{\lambda_i,N} \sum_{i=1}^{N} \left[ \lambda_i (1+\eta \operatorname{sgn}(\lambda_i)) S e^{\gamma \sum_{j=1}^{i} \lambda_j} - D \right].$$
(4.60)

In the single trade, we incur one fixed cost and  $\lambda = 1$ . With two trades, we incur two fixed costs while the trades are  $0 < \lambda_1 < 1$  and  $\lambda_2 = 1 - \lambda_1$ . If it optimal to make three trades, then  $0 < \lambda_1 < 1$ ,  $0 < \lambda_2 < 1 - \lambda_1$  and  $\lambda_3 = 1 - \lambda_1 - \lambda_2$ . This problem looks nearly symmetric and so one would expect the optimal strategy is to divide the trades evenly (e.g. 1/2, 1/3, 1/4, etc.). That is,

$$\lambda_i \approx \frac{1}{N}.\tag{4.61}$$

We verify this numerically for some typical parameters in the two and three subtrade



Figure 4.14: The Black-Scholes and utility indifference bid and ask prices for the European call. The solid black line is the Black-Scholes price. The solid grey (black dashed) line is with fixed cost D = 0.10 (D = 0.10). The temporary price impacts is zero ( $\eta = 0$ ) with permanent impact  $\gamma = 0.05$ . The remaining parameters are  $T - t = 1, \mu = 1.5\%, \sigma = 20\%$ , and r = 1%.

cases in Figures 4.15 and 4.16.

In effect, a point exists where the permanent impact savings of splitting up the order overcome the fixed costs associated with each order. We then move from one trade to two subtrades to three subtrades and so on. This is why the hedging strategy in Figure 4.13 has a step function appearance.

## 4.5 Conclusion

In this paper, we presented a framework to determine bid-ask prices for hedging OTC equity derivatives with market frictions (e.g. transaction costs, bid-ask spreads, illiquidity, etc.). The functional forms of the temporary and permanent impacts and fixed costs can be customized according to the market scenario in which the trader will be hedging. Further, the methodology can accommodate other forms of aversion or hedging targets such as quadratic hedging, which may be more popular in industrial applications.

We obtained some intuitive results from the bid-ask spread, along with some intriguing hedging strategies following from the permanent price impact. Our model



Figure 4.15: The total trading costs from two subtrades along with the optimal trade (1/2) taken at typical parameters with stock price S = 15, fixed costs D = 0.05, spread  $\eta = 0$  and permanent impact  $\gamma = 0.05$ .



Figure 4.16: The total trading costs from three subtrades along with the optimal trade (1/3) taken at typical parameters with stock price S = 15, fixed costs D = 0.05, spread  $\eta = 0$  and permanent impact  $\gamma = 0.05$ .

recovers many of the strategies one may expect to employ in a situation with liquidity risk. The fixed costs and spreads induce us to limit our total number of trades, which creates a playoff between residual risk and accrued transaction costs. Further, as the liquidity dries up, our model lets us know when it is optimal to begin breaking down our hedging decisions into smaller subtrades to avoid moving the market too unfavourably against our derivative position.

Areas for Further Investigation, i.e. price manipulation We would like to investigate whether the presence of permanent price impact may allow the market maker to manipulate the price and to what extent price manipulation may be possible with share repurchase agreement style options. This investigation could be completed by pricing the option with and without  $\gamma$  and examining the extent to which the price changes. Is favourable price manipulation possible with certain product structures in the presence of permanent price impact?

We identify the study of Asian options as another area of research interest. Although the volume or time weighted average prices (VWAP or TWAP) are less affected by the temporary price impact and fixed trading costs, it is still affected by the permanent price impact. The average is unaffected immediately but eventually the effects are felt as time progresses. If a trade of size  $\lambda$  is executed at time  $\tau$ , then

$$Z_T = S_0 + \int_0^{\tau} S_t dt + \int_{\tau}^T e^{\gamma \lambda} S_t dt$$
$$dZ_t = S_t dt \mathbf{1}_{\{\tau \neq t\}} + S_t e^{\gamma \lambda} dt \mathbf{1}_{\{\tau = t\}}$$

and so  $\lim_{t\to\tau^-} Z_t = Z_{\tau}$ . Accordingly our trading decisions may impact the average nonlinearly and there may be interesting strategies to "manipulate" the average. Of course, it is much harder to manipulate the average in this setting than to manipulate the spot price.

Applications in Stressed Markets Our market impact model has useful applications, especially when hedging in distressed markets. For example, during the financial crisis of 2007, the 1987 market crash or the 1997 Asian financial crisis, liquidity in the markets dried up dramatically. This resulted in large bid-ask spreads and transaction costs. Derivative positions in equity and credit instruments for large institutions however still had to be hedged. Our model provides a globally optimal hedging strategy during times of market stress and illiquidity.

At other times, it is possible a single institution might take a derivative position so

large that their hedging strategies end up moving the market, such as the infamous "London Whale" [114] or, worse still, a combination of a market crash and large positions as with Long Term Capital Management [115]. It is possible a derivative may be written on an underlying that was originally liquid but later becomes illiquid. Perhaps it becomes undesirable after some regulatory or market change like an on the run Government of Canada bond losing liquidity after a new debt issue. Our model allows us to account for illiquidity in terms of bid-ask spreads and permanent price impact, which again is very useful for situations such as these.

Rather than building the impact parameters  $(\eta, \gamma, D)$  on a single current snapshot in time, to reflect future liquidity risk, one might use a "term structure of illiquidity." This could be based on an index, futures, or analyst estimates. The parameters could be time varying and stochastic such as the Hull-White model, fitted to some forward curve or deterministic and fitted similarly to a local volatility style model based on some liquidity index.

Applications in Market Microstructure It is possible to choose impact functions (particularly the bid-ask spread  $\eta$ ) that better reflect the order book structure. The magnitude of the bid-ask spread could grow with the size of the order to reflect how market order consume deeper into the order book. A buy order for example first consumes all the order lots at the best price level (nearest the mid price), then all the order at the next best price level and so on. Thus the larger the order the larger the average (effective) weighted bid-ask spread.

**Inventory Risk Management and Optimal Liquidation** Our framework is sufficiently general that by adjusting the utility function and goal from say terminal wealth, it can be incorporated into a more general market making framework such as managing inventory risk or optimal liquidation.

Extensions to American Style Exercise It is possible to price American style options within our framework. In the model with no fixed trading costs, it is a simple question of adding an additional obstacle besides the hyperbolic exercise obstacle in Section 4.2. The pricing equation is similar to [102]. For the impulse control model with fixed costs, there is a new constraint. In the case of a put option, after exercising n puts, we then have zero positions in any put options. We can use the same notation as before, and state the exercise constraint

$$v_n(B, X, S, t) \ge v_0(B + nK, X - n, S, t)$$
(4.62)

since no trade occurs in the transfer of the stocks from the settlement of the call (from the perspective of the agent). Thus the HJB equation for holding n options becomes

$$\max\left\{\partial_t v_n + rB\partial_B v_n + \mathcal{L}[v_n], \mathcal{H}[v_n] - v_n, \mathcal{I}[v_n] - v_n\right\} = 0$$
(4.63)

where

$$\mathcal{I}[v_n] = v_0(B + nK, X - n, S, t).$$
(4.64)

We note it is possible to also exercise the option while simultaneously making a trade. In this case,  $\mathcal{I}$  becomes

$$\mathcal{I}[v_n] = \sup_{\lambda} v_0(B + nK + \lambda(1 + \eta \operatorname{sgn}(\lambda))Se^{\gamma\lambda}, X - n + \lambda, Se^{\gamma\lambda}, t)$$
(4.65)

or alternatively  $\mathcal{I}[v_n] = \mathcal{H}[v_0(B + nK, X - n, S, t)].$ 

Multiple Assets and Opportunity Cost Lastly we suggest the relatively simple extension of multi-asset options, which is a straightforward extension of the HJB equation. Additionally, we believe a sound pricing framework should better account for opportunity cost in the indifference price. We suggest for future research the consideration that the agent in his portfolio optimization without the option may also be allowed to hedge with some sort of market index ETF. This would possibly cause the agent to be less willing to buy or sell the option since his opportunity cost of investing with the stock and ETF alone becomes higher. It is also possible the agent may wish to use the ETF in his option hedge as it may be correlated to the stock and have lower transaction costs due to the general liquidity advantage of ETFs over some single name stocks.

## 4.6 Appendix A: Asian Options with Market Impact

Options based on averaging are referred to as "Asian." The advantage of Asian options in the context of equity derivatives with market impact is that the average over time is harder to manipulate than a single price (at expiry for example). The volatility of the average is lower than the volatility of the stock, which also results in a lower up front option premium. Basing the payout on the average price has appeal to firms with continuous predictable exposure to the underlying (i.e. a firm buying back a large amount of shares). Asian options may be based on TWAP or VWAP and we denote both generally by  $\bar{S}$ .

As a case study, we consider the optimal replication strategy for an average strike Asian call, which at maturity T pays

$$f(S_T, \bar{S}_T) = (S_T - \bar{S}_T)^+ \tag{4.66}$$

to the buyer. In this structure, the holder may possibly have an incentive to drive the share price down early and run it back up later, if liquidity and residual risk conditions allow.

### 4.6.1 Asian Options in the Risk Neutral Framework

In the risk neutral framework with no market frictions, the pricing PDE for an Asian option is a 2-dimensional (plus time) hyperbolic PDE. To simplify the problem, we assume the option is calculated on the arithmetic TWAP  $\bar{S}$  where

$$\bar{S}_T = \frac{1}{T} \int_0^T S_t dt.$$
 (4.67)

The option price p is

$$p = E[f(S_T, \bar{S}_T) | \mathcal{F}_t]$$
(4.68)

with initial conditions  $S_t = S$  and  $\bar{S}_t = \bar{S} = S$ . It is easier however to consider the state variable  $Z_T$  where

$$Z_T = \int_0^T S_t dt \tag{4.69}$$

and hence  $\bar{S}_T = \frac{1}{T}Z_T$ .

By Ito's lemma and the Feynman-Kac theorem, the pricing PDE with corresponding terminal and boundary conditions are

$$\partial_t p + S \partial_Z p + rS \partial_S p + \frac{1}{2} \sigma^2 S^2 \partial_{SS} p - rp = 0$$

$$V(S, Z, T) = f\left(S, \frac{Z}{T}\right) = \left(S_T - \frac{Z}{T}\right)^+$$

$$\lim_{S \to 0} V(S, Z, t) = 0$$

$$\lim_{S \to \infty} \partial_{SS} V(S, Z, t) = 0$$

$$\lim_{Z \to \infty} V(S, Z, t) = 0$$
(4.70)

This problem admits a change of variables reduction of the form

$$y = \frac{S}{Z}.$$
(4.71)

The final condition becomes

$$Z\left(\frac{S}{Z} - \frac{1}{T}\right)^{+} = Zf\left(y, \frac{1}{T}\right)$$
(4.72)

and hence we write a new function

$$p(S, Z, t) = Zq(y, t) \tag{4.73}$$

The PDE is reduced to a single spatial variable (plus time) and becomes

$$\partial_t q + y(r-y)\partial_y q + \frac{1}{2}\sigma^2 y^2 \partial_{yy} q + (y-r)q = 0, \quad q(y,T) = \left(y - \frac{1}{T}\right)^+.$$
 (4.74)

### 4.6.2 Asian Options in the Indifference Framework

In the presence of transaction costs, we resort to a utility-based framework. The utility valuation HJB equation can become cumbersome as it is in 4 variables (plus time)

$$\partial_t v + \sup_{\dot{x}_t \in A} \left\{ \partial_B v \left( rB - \dot{x}_t Sh(\dot{x}_t) \right) + \dot{x}_t \partial_X v + g(\dot{x}_t) S \partial_S v \right\} + S \partial_Z v + \mathcal{L}[v] = 0.$$
(4.75)

This HJB equation is both nonlinear and hyperbolic. Accordingly, numerical solutions should be very careful to avoid instability. We can reduce the PDE by one variable B by using exponential utility. Further, when trading incurs fixed costs, the trading strategy reduces to an impulse problem which reduces the differential component in X. At this point, the differential component is in only 2 dimensions (Z and S) and admits a greatly simplified numerical solution.

The value function under exponential utility satisfies

$$v_n(B, X, S, Z, t) = \sup_{\dot{x} \in A} E\left[1 - e^{-\beta R_{n,T}} \middle| \mathcal{F}_t\right] = 1 - \inf_{\dot{x}} E\left[e^{-\beta R_{n,T}} \middle| \mathcal{F}_t\right].$$
 (4.76)

Following a dynamic programming argument, we derive the HJB equation for the

personal valuation of an Asian option under exponential utility

$$v_n(B, X, S, Z, t) = 1 - \inf_{\dot{x}} E\left[ e^{-\beta B e^{r(T-t)} + \beta \int_t^T e^{r(T-s)} \dot{x}_s S_s h(\dot{x}_s) ds - \beta c(X_T, S_T, nf(S_T, Z_T))} \middle| \mathcal{F}_t \right]$$
  
=  $1 - e^{-\beta B e^{r(T-t)}} G(X, S, Z, t)$ 

$$G(X, S, Z, t) = \inf_{\lambda, \tau} E\left[ e^{\beta e^{r(T-\tau)}(\lambda(1+\eta \operatorname{sgn}(\lambda))S_{\tau}e^{\gamma\lambda}+D)}G(X_{\tau}, S_{\tau}, Z_{\tau}, \tau) \middle| \mathcal{F}_t \right].$$
(4.77)

The associated HJB equation to the dynamic programming problem follows from Ito's lemma.

**Theorem 13.** The optimal terminal portfolio utility and trading strategy satisfy

$$\min\left\{\underbrace{\partial_t G + S\partial_Z G + \mathcal{L}[G]}_{wait}, \quad \underbrace{\mathcal{H}[G] - G}_{trade}\right\} = 0 \tag{4.78}$$

where the operator  $\mathcal{H}$  is given by

$$\mathcal{H}[G] = \inf_{\lambda} e^{\beta e^{r(T-t)}(\lambda(1+\eta \operatorname{sgn}(\lambda))Se^{\gamma\lambda} + D)} G(X+\lambda, Se^{\gamma\lambda}, Z, t).$$
(4.79)

The final condition is given by the cost to liquidate the portfolio at T

$$G_n(X_T, S_T, Z_T, T) = \exp\left[-\beta c(X_T, S_T, Z_T, nf(S_T, Z_T))\right].$$

In the case of the Asian call described earlier, this is given by

$$G_{n}(X, S, Z, T) = \begin{cases} \exp\left[-\beta \left(X(1+\eta \operatorname{sgn}(-X))Se^{-\gamma X} - D1_{\{X\neq 0\}}\right)\right] & \text{if } S \leq \frac{Z}{T} \\ \exp\left[-\beta \left((X+n)(1+\eta \operatorname{sgn}[-(X+n)])Se^{-\gamma (X+n)} - D1_{\{X\neq -n\}} - n\frac{Z}{T}\right)\right] & \text{if } S > \frac{Z}{T} \end{cases}$$

$$(4.80)$$

where we recall the average strike payoff (Equation 4.66).

The boundary conditions are

$$\lim_{S \to 0} \partial_t G_n(X, S, Z, t) = 0 \tag{4.81}$$

$$\lim_{S \to \infty} \frac{\partial_S G_n(X, S, Z, t)}{G_n(X, S, Z, t)} = \beta(X+n)(1 + \operatorname{sgn}(-(X+n)))e^{-\gamma(X+n)}$$
(4.82)

$$\lim_{Z \to \infty} G(X, S, Z, t) = G_0(X, S, t)$$

$$(4.83)$$

where  $G_0$  is the utility value function with zero options. Since the call is so far out of the money, its value is near zero.

For completeness, the optimization problem in the absence of holding the Asian option is independent of Z and given by

$$\min\left\{\partial_t G_0 + \mathcal{L}[G_0], \quad \mathcal{H}[G_0] - G_0\right\} = 0$$

with similar terminal and boundary conditions as in Section 4.3.

The indifference bid and ask prices for n units of the Asian option are given by

$$p_n^b = -\frac{e^{-r(T-t)}}{\beta} \ln\left(\frac{G_n(0, S, S, t)}{G_0(0, S, t)}\right)$$
(4.84)

$$p_n^a = \frac{e^{-r(T-t)}}{\beta} \ln\left(\frac{G_{-n}(0, S, S, t)}{G_0(0, S, t)}\right)$$
(4.85)

since  $Z_t = \bar{S}_t = S$ .

## 4.7 Appendix B: Numerical Method

We utilize an implicit finite difference scheme to solve the PDE component within the wait region, along with an explicit projected successive over-relaxation (PSOR) technique to determine the optimal impulse trade region. Particularly, an IMEX scheme is used to solve the HJB PDQVI. The method used is very similar to the numerical scheme used by the authors to solve a multidimensional impulse control problem in a real option context [117]. We refer the reader to [103, 105] for a more detailed analysis of the finite difference solutions to stochastic control problems in finance.

### 4.7.1 The Main HJB Equation

The PD QVI is of the form

$$\min\left\{\partial_t G + \mathcal{L}[G], \quad \mathcal{H}[G] - G\right\} = 0 \tag{4.86}$$

where the differential operator  $\mathcal{L}$  is in S only

$$\mathcal{L} = \mu S \partial_S + \frac{1}{2} \sigma^2 S^2 \partial_{SS} \tag{4.87}$$

so X can be interpreted like a parameter.

The constraint  $\mathcal{H}$  is itself an optimization problem

$$\mathcal{H}[G] = \inf_{\lambda} e^{\beta e^{r(T-t)}(\lambda(1+\operatorname{sgn}(\lambda))Se^{\gamma\lambda} + D)} G(X+\lambda, Se^{\gamma\lambda}, t)$$
(4.88)

which adds significant non-locality to the problem. However, the nonlocality does not affect the solution within the wait region (the partial differential component) since it is distinct, non-empty, and evolves under the uncontrolled diffusion as discussed in Section 4.2 and [101, 116].

The numerical solution is obtained via finite differences at (possibly nonuniform) grid points  $G(X_i, S_j, t_k) = G_{i,j}^k$  using second order centred differences except at the boundary conditions where one-sided differences are used. To retain the *M*-matrix property, one-sided differences may occasionally be used. The grid is truncated between  $[S_{min}, S_{max}]$  and  $[X_{min}, X_{max}]$  where  $X_{min}, X_{max}$  can be interpreted financially as the trader's position limits. The grid points increase such that  $S_{j+1} > S_j$  and  $S_0 = S_{min}$ , as goes for  $X_i$ . The derivatives are approximated by divided differences; two are shown below for reference

$$\partial_S G \approx \frac{G_{i,j+1}^k - G_{i,j-1}^k}{S_{j+1} - S_{j-1}}$$
(4.89)

$$\partial_t G \approx \frac{G_{i,j}^{k+1} - G_{i,j}^k}{t_{k+1} - t_k}.$$
 (4.90)

Using a stable implicit time stepping scheme, the finite difference method leads to a matrix system of equations

$$\frac{G_i^{k+1} - G_i^k}{t_{k+1} - t_k} + LG_i^k \ge 0, \quad \forall i$$
(4.91)

where L is the differentiation matrix associated with the partial differential operator  $\mathcal{L}$  and  $G_i^k$  refers to the whole vector of values G in  $S_j$  at a given k, i. It is possible to take differently fitted schemes (e.g. Crank Nicholson or explicit).

The trading constraint  $\mathcal{H}$  is nonstiff and can be handled explicitly using the function vector  $G^{k+1}$  at all points i, j from a time step earlier than k. The constraint in discrete form is

$$\mathcal{H}[G_{i,j}^k] = \min_{\lambda = X_{i^*} - X_i} \exp[\beta e^{r(T - t_k)} (\lambda(1 + \eta \operatorname{sgn}(\lambda)) S_j e^{\gamma \lambda} + D)] G(X_i + \lambda, S_j e^{\gamma \lambda}, t_{k+1})$$
(4.92)

where  $G(X_i + \lambda, S_j e^{\gamma \lambda}, t_{k+1})$  is approximated by interpolation. Note that  $X_{i^*} = X_i + \lambda$ and

$$G(X_i + \lambda, S_j e^{\gamma \lambda}, t_{k+1}) = \theta G_{i^*, S_{j_*}}^{k+1} + (1 - \theta) G_{i^*, S_{j^*}}^{k+1}$$
(4.93)

where  $S_{j_*} \leq S_j e^{\gamma \lambda} \leq S_{j^*}$  are two adjacent nodes and  $\theta$  is the appropriate interpolating weight. The constraint  $\mathcal{H}[G]$  is maximized by a brute force search in order to guarantee a global optimum is achieved along the grid points  $X_i$ . This is required for convergence of the solution.

In the limit that  $\max_i |X_{i+1} - X_i|$ ,  $\max_j |S_{j+1} - S_j|$ ,  $\max_k |t_{k+1} - t_k| \to 0$ , a Taylor series argument shows that this formulation is consistent.

By using an explicit formulation for the constraint (that is, using  $G^{k+1}$ ), we can treat it as an arbitrary function h(X, S). It is possible to use a penalization technique to enforce the constraint [105], however we use the PSOR technique and cast the matrix system as a complementary problem [127]:

$$MG^{k} - b \ge 0, \quad G^{k} \le h, \quad \left(MG^{k} - b\right)^{T} \left(G^{k} - h\right) = 0$$

$$(4.94)$$

where superscript T denotes the matrix transpose. The matrix M is an aggregation of the differentiation and boundary condition matrix pre-multipliers of  $G^k$  while bis a vector of collected solution values known at time k (from k + 1). This matrix system is then solved using a fixed point value iteration (projected successive overrelaxation). Several iterative schemes for non-linear control problems are described in [97, 100, 103, 105, 120, 127]. Following the criteria in [105, 127], our scheme is consistent, monotone (given uniform grid points) and stable (*L*-stability from the implicit time marching scheme). Therefore the discrete approximation converges to the (viscosity) solution of the HJB equation.

### 4.7.2 The Asian HJB Equation

The Asian option HJB equation is of the form

$$\min\left\{\partial_t G + S \partial_Z G + \mathcal{L}[G], \quad \mathcal{H}[G] - G\right\} = 0 \tag{4.95}$$

which has a multidimensional differential component. The solution is approximated along a grid via finite differences  $G(X_i, S_j, Z_c, t_k) = G_{i,j,c}^k$ . We name the differential operator in Z as

$$\mathcal{L}_Z = S\partial_G \tag{4.96}$$

which has an associated differential matrix  $L_{Z,j}$  which operates on  $G_{i,j}^k$  the vector of values G in all  $Z_c$  at time  $t_k$  and  $S_j$ . We suggest a stable and consistent scheme using operator splitting

$$\partial_t G + \mathcal{L}[G] \ge 0, \quad \partial_t G + \mathcal{L}_Z[G] \ge 0$$
(4.97)

as illustrated in [103]. This reduces the multidimensional problem into a series of one dimensional problems with familiar vector-matrix techniques. The matrix L is as defined before but operates on a vector of values  $G_{i,c}^k$  at constant  $X_i, Z_c$  in all  $S_j$ . The scheme is simple and robust but incurs a splitting error as a cost

$$\frac{G_{i,c}^{k+1} - G_{i,c}^{k+1/2}}{t_{k+1} - t_k} + LG_{i,c}^{k+1/2} \ge 0, \quad \forall i, c$$

$$(4.98)$$

$$\frac{G_{i,j}^{k+1/2} - G_{i,j}^k}{t_{k+1} - t_k} + L_{Z,i}G_{i,j}^k \ge 0, \quad \forall i, j.$$

$$(4.99)$$

For numerical stability and simplicity, we suggest a simple upwinding scheme

$$S_j \partial_Z G(X_i, S_j, Z_c, t_k) \approx S_j \frac{G_{i,j,c+1}^k - G_{i,j,c}^k}{z_{c+1} - z_c}.$$
 (4.100)

At each step in the operator splitting scheme, we apply the PSOR algorithm as before to enforce the constraint  $\mathcal{H}[G_{i,j,c}^k]$  which is defined as before using  $G^{k+1}$  following [125]. In effect,

$$M_S G_{i,c}^{k+1/2} - b^{k+1} \ge 0, \quad G_{i,c}^{k+1/2} \le h, \quad \left(M G_{i,c}^{k+1/2} - b\right)^T \left(G_{i,c}^{k+1/2} - h\right) = 0 \quad (4.101)$$

$$M_Z G_{i,j}^k - b^{k+1/2} \ge 0, \quad G_{i,j}^k \le h, \quad \left(M G_{i,j}^k - b\right)^T \left(G_{i,j}^k - h\right) = 0$$

$$(4.102)$$

where  $M_S, M_Z$  are the aggregated matrix premultipliers from the  $L, L_{Z,j}$  steps with associated boundary conditions and  $b^{k+1}, b^{k+1/2}$  are the collected knowns at time k.

We note it is possible to use symmetric operator (Strang) splitting

$$\frac{G_{i,j}^{k+1} - G_{i,j}^{k+3/4}}{\frac{1}{2}(t_{k+1} - t_k)} + L_{Z,j}G_{i,j}^{k+3/4} \ge 0, \quad \forall i, j$$
(4.103)

$$\frac{G_{i,c}^{k+3/4} - G_{i,c}^{k+1/4}}{t_{k+1} - t_k} + LG_{i,c}^{k+1/4} \ge 0, \quad \forall i, c$$

$$(4.104)$$

$$\frac{G_{i,j}^{k+1/4} - G_{i,j}^k}{\frac{1}{2}(t_{k+1} - t_k)} + L_{Z,j}G_{i,j}^k \ge 0, \quad \forall i, j$$
(4.105)

and to update the constraint  $\mathcal{H}[G]$  at each fractional step to achieve added accuracy nearly  $O(\Delta t^2)$  versus  $O(\Delta t)$  [125].

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# Chapter 5 Conclusion

Stochastic optimal control is a powerful tool. We have seen how it can provide insight, guidance and solutions to many problems encountered in finance and engineering. Our applications have shown how many complex problems in real life can be modeled by a simple equation

value today = expected future profits + terminal project value

given our information today

or, in integral form,

$$V(x,0) = E\left[\int_{0}^{T} f(X_{t},t)dt + g(X_{T}) \middle| X_{0} = x\right].$$

Decisions must be made and values computed in the face of future uncertainty from multiple sources (e.g. price uncertainty, regulatory uncertainty, execution risk, etc.). Whether one is

- seeking a profit maximizing operating and entry strategy for a biofuel production plant in Chapter 2,
- attempting to account for regulatory and taxation uncertainty in an energy project in Chapter 3, or
- trading to minimize the market impact and transaction costs of the hedging strategy for an OTC equity derivative in Chapter 4,

the tools are identical. In these seemingly disparate examples (particularly Chapters 2–3 contrasted with Chapter 4), the key to solving these problems is stochastic control and dynamic programming.

In each article (Chapters 2–4) the strategy was the same:

- 1. Identify a problem of interest in engineering or finance,
- 2. Generate a mathematical model of the income or payoffs using stochastic processes,
- 3. State the associated HJB equation for the problem using stochastic control and the dynamic programming principle,
- 4. Where possible seek closed-form solutions or more generally employ robust finite difference methods to reach the solution, and
- 5. Analyze and comment on the results.

## 5.1 Contributions to the Literature

In Chapter 2, we presented a detailed real option model to value the entry decision and optimal operating strategy for an ethanol biofuel facility. In addition to the numerical solutions, we also derived analytical solutions to the switching problem. We investigated the effects of stronger correlation between corn and ethanol possibly resulting from increased firm competition in the corn-ethanol market. Given our optimal operating strategy, we considered the retrospective value of an ethanol project given the historical prices of corn and ethanol. Finally, we investigated the effects of the subsidy policy on the future profitability of ethanol projects along with its risk profile via the profit distribution, value at risk, and expected shortfall via Monte Carlo analysis. We derived interesting policy conclusions from our very complete and in-depth green energy case study.

In Chapter 3, we presented a novel framework for studying and quantifying regulatory uncertainty and policy risk. We took as a case study the ethanol production plant of Chapter 2. Looking back at the past 35 years of ethanol subsidy history and noting how frequently it had changed, we developed a stochastic jump process to model the subsidy level. Although in the near term, regulatory changes may appear predictable, when planning for a long term energy infrastructure project, the inability to forecast regulations 50 years out renders it apparently random. This is a topic of much research interest at present, and our model is one of the only applications we are aware of that treats regulatory uncertainty in this stochastic PIDE framework. This technique may allow firms to better understand their future regulatory risk exposure and to value long-lived projects in infrastructure, energy, resources and elsewhere. We developed a stochastic control PIDE model that accounts for model uncertainty in the regulatory stochastic process and other economic considerations such as the proper choice of discount rate. We also discussed alternative methods of addressing model uncertainty and, as a possible remedy, developed a worst case valuation scenario. We presented a detailed numerical method to solve the HJB PID QVI system along with analytical solutions to the jump diffusion switching model. Based on our results, we developed policy conclusions and investigated the effects of uncertainty on project value and the firm's entry and operational decisions. We also use our model to support anecdotal and empirical evidence that increased uncertainty may result in firms delaying investment.

In Chapter 4, we presented a general but novel framework for hedging equity derivatives in the presence of market impact. We used a utility indifference approach to develop bid and ask prices. Our framework can incorporate general market impact models and can transition mathematically consistently from continuous to impulse trading controls. We developed the HJB equations associated with several different kinds of market impact structure subject to fairly general utility functions. We verified our model framework was consistent with risk neutral pricing to ensure our bid and ask prices stood on a sound theoretical footing. For our analysis, we chose the most general case of market impact that included fixed transaction costs in addition to any temporary or permanent price impact and then specialized our pricing results to the case of exponential utility. In addition, we presented a convergent finite difference method to calculate solutions. Our model solution produced prices and hedging strategies that balanced the trade-off between execution and market risk. We also discovered some interesting phenomena resulting from the permanent price impact including incentives to split the trade orders to reduce market impact costs.

## 5.2 Future Work

My thesis (and the associated tools) naturally builds on itself as the chapters progress. This is because each subsequent article was an extension of the previous. In Chapter 2, we identified regulatory and subsidy policy uncertainty as a primary area of future research. This was the subject of our second article in Chapter 3. In the second article, we identified the bandwidth of the best and worst case prices as being too large and that perhaps utility indifference may be a better tool to address model uncertainty. In our next article in Chapter 4, already aware of the shortcomings of the superhedging and subhedging prices, we employed a utility indifference approach to generate the bid-ask prices. In Chapter 4, we also identify areas of future research for ourselves and interested readers.

Our model in Chapter 3 can be extended on several fronts. To improve the model, more classes of jump distributions or non-constant (in fact, possibly stochastic) Poisson arrival rates could be considered for future work. Another possible improvement to the expected subsidy jump model would be to incorporate management's views on the probability of possible policy outcomes or cases, each with an associated probability determined by management. Beyond the worst case pricing scenario, we could consider the whole space of equivalent martingale measures and seek a pricing measure following the ambiguity aversion methods  $v(x, 0) = \sup_{\alpha} \inf_{Q} E[U(X_T, T) + \kappa h(Q|P)]$ where U is the utility of terminal wealth  $X_T$ , P is the estimated jump diffusion measure, Q is another possible measure from our uncertain parameter bounds, and h is a penalty function that penalizes choices Q different from P, with associated ambiguity aversion parameter  $\kappa$ . We note that the worst case measure and pricing equation associated with our method follows from the aversion parameter approaching zero  $\kappa = 0$  and the risk neutral utility function U(X) = X. This would be an application of utility based pricing. Further still, one could try to hedge the policy risk factor with some sort of correlated asset using a utility indifference pricing approach. Different possible hedging targets could be chosen such as exponential utility or a global mean variance technique.

In Chapter 4, we identified many possible extensions and additional applications of our model. One could investigate whether the presence of permanent price impact may allow the market maker to manipulate the derivative price and to what extent price manipulation may be mitigated with Asian style options. Our market impact model can be used when hedging in distressed markets like, for example, during the financial crisis of 2007 where liquidity in the markets dried up dramatically. Rather than building the impact parameters on a single current snapshot in time, to reflect future liquidity risk, one might use a "term structure of illiquidity." This could be based on an index, futures, or analyst estimates and the parameters could be time varying and possibly stochastic. The model can also be applied in daily market making such as market microstructure models. It is possible to choose impact functions that better reflect the limit order book structure on exchanges or electronic brokerdealer networks. Our framework is sufficiently general that it can be incorporated into a more general market making framework such as managing inventory risk or optimal liquidation. It is possible to price American style options within our framework and we presented the associated HJB equations for future research. Lastly we suggest the relatively simple extension of multi-asset options, which is a straightforward extension of the HJB equation. Additionally, we believe a sound pricing framework should better account for opportunity cost in the indifference price. We suggest for future research the consideration that the agent in his portfolio optimization may also be allowed to hedge with some sort of market index ETF.

## EDUCATION

PhD in Applied Mathematics: Financial Mathematics University of Western Ontario	2015
BASc in Engineering Science: Aerospace Engineering University of Toronto	2008
Work and Volunteer History	
<b>Quantitative Associate</b> BMO Capital Markets Toronto, ON	2014–Present
Summer Associate CIBC Risk Management Toronto, ON	2013
<b>Lecturer</b> Western University London, ON	2012
<b>Senior Analyst</b> Ernst & Young LLP Toronto, ON	2009–2010
<b>Incident Manager</b> Virgin Mobile Toronto, ON	2008
<b>Volunteer</b> University of Toronto Centre for International Health (CIH) Kep, Cambodia	2008
<b>Program Coordinator</b> University of Toronto, Science Outreach Toronto, ON	2007
<b>Camp Counselor</b> University of Toronto, Science Outreach Toronto, ON	2006

### EXTRACURRICULAR ACTIVITIES

SIAM Student Chapter Executive	2013
PRMIA Toronto Risk Management Case Competition	2012
University of Western Ontario Student Development Centre Peer Advisor	2010 - 2012
TEDXUTIHP Volunteer and Event Photographer	2011
University of Toronto Health and Human Rights Conference Volunteer	2009 - 2010
Scarborough Sharks Hockey Volunteer	2008 - 2009
U of T Centre for International Health Cambodia Project	2008
Chair of the Engineering Science Club	2007 - 2008

### Relevant Works and Papers

- PhD Thesis: Applications of Stochastic Control to Energy Real Options and Market Illiquidity. 2014.
- Working Paper: Maxwell, C and Davison, M. Optimal Hedging in Illiquid Markets. 2014. Working Paper.
- Accepted Article: Maxwell, C and Davison, M. Real Options with Regulatory Policy Uncertainty. 2014. Fields Special Volume.
- Published Article: Maxwell, C and Davison, M. Using Real Option Analysis to Quantify Ethanol Policy Impact on the Firm's Entry into and Optimal Operation of Corn Ethanol Facilities. 2013. Energy Economics.
- BASc Thesis: The Propulsive Performance of Ramrockets at Lift-off.

#### INTERESTS

Banjo  $\cdot$  Piano  $\cdot$  Drumming  $\cdot$  DJing  $\cdot$  Photography  $\cdot$  Hockey