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Kinematics Of Spatial Mechanisms

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KINEMATICS OF SPATIAL MECHANISMS

by

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ABSTRACT

Large scale endeavor of research on spatial mechanisms started about half a century ago. Although significant research work has been done, the underlying framework of the theory of spatial mechanisms still appears weak. In this thesis, a theoretical foundation for the kinematic analysis and design of spatial mechanisms and robots has been developed.

In the last twenty years, the *matrix method* and the *spherical trigonometry method* have emerged as the most efficient ones among approximately ten other different methods for the kinematic analysis of spatial mechanisms. In this thesis a new method, the *vector algebraic method*, has been introduced. In comparison with the two methods, the proposed method has shown advantages on its efficiency, uniformity and simplicity.

The goal of this thesis is to enhance the education, research and application of spatial mechanisms.

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TO MY PARENTS, BROTHERS AND SISTER.

此論文謹獻給
亮清、如英
玉華、濤々、浩々.

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NOMENCLATURE

$\alpha_{i,i\pm1}$	the twist angle between the pair axes of i and $i\pm1$.
θ_i	angular variable, which could be rotary or helix-angular variable.
Θ_i	output angle, the first angular variable to be determined.
ψ_i	auxiliary, the angular variable to be eliminated from simultaneous displacement equations.
$\mathbf{a}_i$	usually represents unit axial vector of constrained axis.
$\mathbf{q}_i$	usually represents unit link vector.
$S_i\,\mathbf{a}_i$	S_i is the offset corresponding to axial vector $\mathbf{a}_i$.
$p_i\,\mathbf{q}_i$	p_i is the length of link i.
$\mathbf{I}$	the input vector, the sum of those vectors in the loop of a mechanism that are given or known at the beginning.
$\mathbf{J}$	the output vector, the sum of the constant-magnitude vectors in the loop of a mechanism that can be expressed as function of the output angle.
$\mathbf{K}$	the sum of the input and output vectors, $\mathbf{K}=\mathbf{I}+\mathbf{J}$.
$\mathbf{L}$	the auxiliary vector, the sum of those unknown constant-magnitude vectors between the adjacent angular variables of the auxiliary that can be expressed as function of the auxiliary (angle).
$\mathbf{F}$	the floating vector; Dissecting a closed-loop into two serial linkages, we obtain a ground linkage and a floating linkage. The sum of all the vectors of the floating linkage is the floating vector.
λ	small lambda, which is the total number of the angular variables of the basic pair variables of a mechanism or robot.
Λ	big lambda, defined as $\Lambda=\lambda-\delta$, where δ is an adjusting parameter which equals either one or zero.
ε	the degree of complexity of mechanism or robot, $\varepsilon=\lambda+\Lambda$.

GENERAL REVIEW

Linkage-mechanisms are a common sight in practical engineering. They can be generally divided into two groups: one is planar linkage-mechanisms, the other the spatial linkage-mechanisms. Human beings began to design and use linkage-mechanisms long ago.

Compared to spatial mechanisms, planar mechanisms are much easier to analyze and design; therefore, many senior scientists have done a great deal of systematic research work on planar mechanisms. The vast majority of linkage mechanisms in engineering are planar mechanisms. The theory of the kinematics of planar mechanisms is now very mature, and has been widely taught at the undergraduate level in engineering schools.

In comparison with planar mechanisms, spatial mechanisms have their distinct attributes. They can trace curves and transmit required motions and forces in 3-dimensional space that are difficult or impossible for planar mechanisms to accomplish. Though more and more useful spatial mechanisms are invented and applied in engineering, it is a pity, that the education on the theory of the kinematics of mechanisms and machines is basically still confined in 2-dimensional space. Less than ten universities in the world have ever offered courses specifically devoted to the kinematics of spatial mechanisms, and most of them are only at introductory level.

The graphic method, an old-fashioned method for the analysis of mechanisms, had always been the favorite method for designers and scholars before the computer age. In spite of its inaccuracy, it is visually intuitive and can be used to solve most of the problems in the analysis, synthesis and design of planar mechanisms. However, it becomes virtually impotent for tackling spatial mechanisms. Alternatively, the analytical method (or algebraic method) depends on obtaining numerical values from complicated algebraic expressions, it is natural that this method had drawn little attention until the difficulty of computation was overcome.

With the widespread application of digital computers since the '50s, the ancient theory of mechanisms gained new vitality, and the extensive field of investigation, design and application of spatial mechanisms was unfolded.

Following the well known pioneering works of the Russian scholar Dimentberg [12] since 1948, a series of new analytical methods were appearing. In references [12,13] Dimentberg successfully obtained analytical solutions for some 4-link and 5-link spatial mechanisms using *screw algebra* and *dual numbers* (or dual vectors).

In 1947 and 1952 Dobrovolskii [14,15] investigated the *method of spherical representation* in the theory of spatial mechanisms.

In 1955 Denavit and Hartenberg published their milestone paper [10], in which a clear description of the motion of spatial linkages containing lower-pairs by *matrices* was proposed for the first time.

In 1962 Zhang [98] completed his milestone thesis in which substantial advances were made and the *direction cosine matrix method* was developed. In the same year Chace [6] creatively explored the application of *vector mathematics* for kinematic analysis of some spatial mechanisms.

In 1963 Yang [82] studied the application of *quaternion algebra* and *dual numbers* to the analysis of spatial mechanisms.

In 1964 Uicker, Denavit and Hartenberg [74] developed an *iterative method* based on matrices for the analysis of spatial mechanisms. This is actually a *numerical method*, which does not yield closed-form solutions.

In 1966 Suh [68] analyzed some 4-link spatial mechanisms using a combination of *vector and matrices* to describe displacements. In the same year, Ho [38] analyzed spatial 4-bar linkage by the *tensor method*, Beggs published his beautifully written book [4] which significantly contributed to the application of *matrices* in the analysis of 3-link and 4-link spatial mechanisms.

In 1968 Wallace [75] analyzed two 5-link spatial mechanisms (i.e. the generalized Tracta and Clemens couplings) using the *geometric-configuration method*.

In 1969 Yang [84] investigated the R_0-$CRCR$ mechanism using (3×3) *matrices with dual-number elements*. A major event in 1969 was the emergence of Joseph Duffy [17] who began to tackle spatial mechanisms using *spherical trigonometry*.

In 1970 Yuan [93] derived the solution of the R_0-$RCCR$ mechanism by the method of *line geometry and line coordinates*.

In 1971 Yuan, Freudenstein and Woo [94] developed the *screw coordinates* approach. In the same year Torfason and Crossley [71] exploited the simulation of mechanical movements by *electronic analog*; and two years later Torfason and Sharma [73] solved the displacement problem of the R_0-$RGRR$ mechanism by the *method of generated surfaces*.

In 1973 Keen and Duffy [41] presented the *matrix and vector technique*.

In 1975 Litvin [51] studied the technique to simplify the *matrix method* by dissembling kinematic chain. The same idea can also be seen in Refs.[75-77] (Wallace and Freudenstein).

In 1976 Hiroshi Makino [53] proposed the *rotation transformation tensor method* for the analysis of manipulators, and this method was further developed by Xie and Zheng ([78-80] 1979-82) on the analysis of spatial mechanisms.

In 1978 Hunt published his book [40], which is good in terms of its treatise on some fundamentals and geometrical features of spatial mechanisms. It is helpful for the understanding of the motions of spatial mechanisms in general; however, this book does not help the derivation of the displacement equations of spatial mechanisms.

The major events in 1980 were the publication of two books on spatial mechanisms. The first one was Zhang's [106] which, largely based on his M.Sc. thesis [98] completed in 1962, systematically introduced *direction cosine matrix method*. The second one was Duffy's [28] which summarized his significant research on spatial mechanisms using the *spherical trigonometry method* since 1969.

In 1981 Liu [52] studied spatial mechanisms by the *method of dual vector rotation*, which led to the publication of the book [111] by Zhu and Liu in 1986.

The major event in 1982 was also the publication of two books. One was Angeles' book [3], which made a significant contribution on probing the rationale behind the methods of analysis, synthesis and optimization of linkages. Another one was Yu's book [87], which was a comprehensive treatise on the application of *vector calculus* and of the *theory of screws* to classical mechanics in general and to the analysis of spatial mechanisms in particular. The book [87] yielded papers [88-92], which significantly contributed to the application of the vectors in the analysis of spatial mechanisms.

In 1984 Sandor and Kohli [64] analyzed some three-link spatial mechanisms containing higher pairs using *differential constraint method*. In the same year Lee completed his M.Sc. thesis [46] in which a new *vector theory* was developed, largely based on Duffy's spherical trigonometry method in addition to the traditional *vector analysis theory* and *dual-number algebra*. It was Ref.[46] that led to the conquering of the famous 7R problem in 1986 by Lee and Liang and the work was published in 1988 [49].

In 1986 Zhou [107] began to investigate the application of *vector mathematics* on the analysis of spatial mechanisms.

In 1987 Hiroshi Markino, Xie and Zheng published their book [54], in which the *rotation transformation tensor method* was systematically introduced.

In 1990 Raghavan and Roth [59,60] solved the 6R manipulator problem (or 7R problem of spatial mechanism) using *matrices*; and two years later Kohli and Osvatic [42,43] tackled the same problem using *matrices* from a slightly different approach. Papers [42,43,59,60] have contributed to the development of the *matrix method*.

In 1991 Huang published his book [39], in which Duffy's *spherical trigonometry method* was adopted.

The latest important event was the publication of the milestone book [29], *Modern Kinematics — Developments in the Last Forty Years*, which included a chapter entitled *Spatial Linkages: Analysis and Synthesis* (pp.137-230) and a comprehensive listing of the published (English) literatures on spatial linkages.

The common goal of all the analytical methods is to establish the relations of the known input motional parameter, structural parameters and the unknown motional variables of linkage kinematic loop, and then to derive the input-output (algebraic) equations.

Of all the kinematicians in the field of spatial mechanisms, the following scholars, in my opinion, have made the most significant contributions to the *methodological* research on the kinematics of *spatial* mechanisms and robots.

- Prof. F. M. Dimentberg, Institute for Machine Design, Moscow, Russian;
- Prof. J. Duffy, University of Florida, U.S.A.;
- Prof. J. Denavit and Prof. R.S. Hartenberg, Northwestern Univ., U.S.A.;
- Prof. Q.X. Zhang, Beijing Univ. of Aeronautics and Astronautics, China;
- Prof. F. Freudenstein, Columbia University, U.S.A.

Prof. Dimentberg was a pioneer in tackling the kinematics of spatial mechanisms using analytical approaches. His significant works [12,13] unfolded the new and exciting research field which soon attracted the attention of many kinematicians.

Prof. Duffy developed *spherical trigonometry method* [17-28], which is unique and mature, and has played a leading role in conquering most of the difficult kinematic problems of spatial mechanisms in the last twenty years.

Prof. Denavit and Prof. Hartenberg developed a symbolic notation [10], the *D-H notation*, which enabled the description of the kinematic properties of lower-pair mechanisms using matrices. This laid the basis for studying lower-pair mechanisms by means of matrix algebra. They demonstrated that a series of matrix multiplication could lead to a set of trigonometric algebraic equations, which could be further used to derive the input-output displacement equations. In recent years, the *matrix method*, sometimes called *D-H method* or *homogeneous matrix transformation method*, has been introduced in most of the robotics books. In my opinion none of these literatures can give the *D-H method* enough integrity as being a mature method. Prof. Denavit and Prof. Hartenberg have just provided a basis for dealing with spatial mechanisms. The basis is important; however, the systematic special techniques for tackling various kinds of spatial mechanisms developed by Zhang [98-106], Litvin [51], Raghavan and Roth [59,60], and Kohli and Osvatic [42,43], etc. are also indispensable. Put plainly, if you just read D-H's papers and any one of the robotics books, except for some simple mechanisms or the robots with special geometric conditions, you would still find it very difficult to analyze most of the kinematic problems of spatial mechanisms and robots with general geometries by the means of matrix transformation.

Prof. Zhang creatively developed the *direction cosine matrix method* [98-106], (or *matrix method* for short), which involves a series of techniques that are systematic and mature for analyzing various kinds of spatial mechanisms. It is fair to say that the *matrix method* was first proposed by Prof. Denavit and Prof. Hartenberg in 1955, and then further developed by Prof. Zhang and several other scholars.

Prof. Freudenstein is the one of the greatest educators in the area of mechanical design in general and in the field of the kinematics of mechanisms in particular. There are nearly 200 Ph.D. offspring in his academic family tree, and there is at least one-fifth generation descendant researching and teaching kinematics ([29] Erdman, Page 2). A large number of the members in this family tree are outstanding scholars. In the last four decades, Prof. Freudenstein's words of wisdom have inspired almost all the kinematicians in the field of spatial mechanisms.

Of all the published books on spatial mechanisms, Prof. Zhang's [106] and Prof. Duffy's [28] can be ranked as the best and the second-best, respectively. Prof. Zhang and Prof. Duffy are such scholars whose effort has endowed the theory of the kinematics of spatial mechanisms with much integrity. The contents of the two books are substantial and comprehensive. They clearly demonstrate that the theory of the kinematics of spatial mechanisms can stand up as an independent subject of knowledge.

Of all the methods listed above, the *matrix method* and Duffy's *spherical trigonometry method* have been widely acknowledged as the most efficient ones.

The publication of Makino, Xie and Zheng's book [54] in 1987 symbolized the maturation of the *rotation transformation tensor method* (RTT method). Essentially, the RTT method can also be classified as matrix method. To a certain extent, the RTT method has shown some advantages over the traditional matrix method, for the RTT method uses not only matrices but also tensors and vectors, which simplifies the algebraic expressions. However, the techniques of the RTT method are very akin to those in Zhang's book [106], and the simplification on algebraic expression is also limited. Therefore, methodologically, the RTT method is not distinct enough to stand on its own as a unique method.

All other approaches using *quaternion algebra*, *dual-number quaternion*, *line-coordinates*, *screw coordinates*, *dual-vector rotation*, *tensor*, *screw theory*, *geometric configuration method*, *differential constraint method*, *electronic analog*, *the method of generated surfaces*, (2×2) *dual matrices* and (3×3) *matrices with dual-number elements* are unlikely to generate anything exciting. They are usually *ad hoc* and become cumbersome in one way or another. In my opinion, there is not much room left for these approaches to develop.

Why is the teaching of the kinematics of mechanisms is still basically confined to 2-dimensional space? The reasons are due to the complexity of the structure and motion of spatial mechanisms, as well as the complexity of the methods being available for tackling them.

The *matrix method* is *conceptually* elegant and *conceptually* easy to teach without analyzing anything specific. This is why it was soon adopted by many scholars after it came to the scene in 1955, and in the last twenty years, it has been widely introduced in robotics books. However, when it comes to analyzing any specific problem of spatial mechanisms or robots, the matrix method often requires laborious algebraic manipulation, and moreover, the process usually involves extraneous work and is error prone (see the conclusion of chapt.9). Therefore, in reality, to teach the *matrix method* in depth is difficult. This is why only the basic ideas of the matrix method has been introduced in most of the robotics books. As to the techniques on how to analyze various kinds of complex spatial mechanisms and robots, the authors of these books have left them out.

In the last 20 years, the *spherical trigonometry method* has caught the spotlight in the race to conquer many difficult kinematic problems of spatial mechanisms. Though it has long been recognized as a powerful method, the users of this method have always been few. This is largely due to the formal complexity of its algebraic symbolic system, whose physical implication is not easy to perceive. And the rationale behind the techniques for establishing the algebraic equations being used for deriving the input-output equations is not easy to comprehend.

The *vector mathematics*, however, holds great potential as being a natural and succinct language for describing the kinematics of spatial mechanisms. "Almost every quantity involved in kinematics is a vector or magnitude of a vector. Angular quantities are exceptions, but all orders of their derivatives are vectors," ([8] Chace, Page 1).

In this thesis, the *vector algebraic method* has been systematically introduced. This is a new method for the kinematic analysis of spatial mechanisms.

The characteristics of the *vector algebraic method* are as follows,

- The analysis steps and expressions are *standardized*;

  Comparing the different sections of a same chapter, and comparing the different chapters with each other, you will find that the analysis steps and the algebraic expressions are identical. You may get the impression that there are a lot of repetitions in the thesis. However, please be aware that this is exactly the manifestation of the advantage of its uniformity of the *vector algebraic method*.

- The analysis steps and expressions are *simple*.

  Representing the position of a point or the direction of a line in space, a single bold face letter, for instance **a** or **b**, will do. It is clear that in terms of simplicity no other mathematics tool can beat the vector in this respect. Moreover, the standardized approach is carefully developed in such a way that the required algebraic manipulation has been kept to its minimum.

The mechanisms analyzed in this thesis are of course just a small portion of many different kinds of spatial mechanisms. Since the purpose of the thesis is to introduce the theory of spatial mechanisms and the analytical method, in most cases, I intentionally choose to analyze the mechanisms with the geometries of their structure in general forms. By appropriately adjusting the geometries of a mechanism, such as the length of the links and the relative direction of the pairing axes, we may obtain many special cases. This is one of the most fascinating aspects of spatial mechanisms. In fact, the vast majority of the practical spatial mechanisms being used in engineering are special cases of mechanisms. However, once you know how to kinematically analyze the mechanisms with general geometries, you will know how to handle their special cases, for it is easier.

In recent years, the rapid development of computer graphics technology has given the kinematicians and mechanical engineers unparalleled convenience in analyzing, synthesizing and designing spatial mechanisms. We are living in 3-dimensional space; why should we leave out the teaching of the theory of spatial mechanisms, and confine ourselves on 2-dimensional mechanisms? I believe that sooner or later the theory of the kinematics of spatial mechanisms will be widely taught in engineering schools, for this is not only desirable but also feasible.

Following is a summary of the related original work in this thesis:

- Developed a theoretical foundation and the *vector algebraic method* for spatial mechanisms;
- Introduced more than 50 new concepts and terminologies concerning spatial mechanisms;
- Amended and Re-defined 11 existing important concepts and terminologies;
- Introduced 17 propositions and theorems;
- Introduced the star product operation, *, simplifying the analysis of mechanisms;
- Classified spatial mechanisms on the bases of their kinematic features;
- Reformulated and analyzed the vector tetrahedron equations, and exploited their applications;
- Analyzed more than 40 various representative spatial mechanisms and robots.

CHAPTER 1. FUNDAMENTALS OF SPATIAL MECHANISMS

1.1. Introduction

This chapter lays a foundation for the theory of spatial mechanisms. It covers the following areas: kinematic pairs are analyzed and divided into definite and indefinite pairs; all practical and typical kinematic input pairs used in a single-loop spatial mechanism (SSM) are classified using a new symbolic system; four basic groups of SSMs are defined: and a series of new concepts and terminologies concerning spatial mechanisms are introduced.

1.2. Kinematic Pairs

Kinematics is the mathematical theory of pure motion, irrespective of the causes that generate the motion. A *kinematic pair* is two contacting bodies permitting a constrained motion of one body relative to the other. The bodies, the elements of kinematic pairs, are connected together either by surface contact, forming a *Lower Pair*, or point or line contact, forming a *Higher Pair*. In this thesis *body* refers to a rigid-body, or a body that can be reasonably approximated as a rigid-body. Although the bodies shown on Fig. 1.1 form a kinematic pair, we can also consider that the kinematic pair is only constituted by the portions of the bodies participating in the contact with each other. as shown in Fig. 1.1 by the (small) dashed circle.

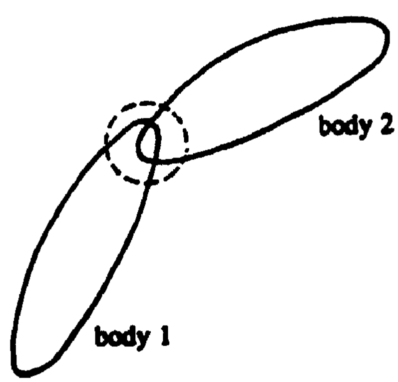


Fig. 1.1

Definition. *Contact Point-Set*.

Let s_1 and s_2 represent all points on the surfaces of body 1 and body 2, respectively. Let $s_{\xi 1} \subset s_1$ and $s_{\xi 2} \subset s_2$. If for any point of $s_{\xi 1}$ (or $s_{\xi 2}$), there is at least one point of $s_{\xi 2}$ (or $s_{\xi 1}$), such that the two points can be in contact with each other during the constrained relative motion of the two bodies, then $s_{\xi 1}$ and $s_{\xi 2}$ are the *contact point-sets* of pair ξ. Specifically, $s_{\xi i}$ $(i = 1, 2)$ is the *contact point-set* of pair ξ on body i. A contact point-set can be either a single point, a line or a surface.

Definition. *Basic Contact Surfaces.*

There are three basic contact surfaces that are important in practical engineering: the *rotary*, the *translational* and the *helical surface*.

(1). Given a fixed straight line (reference) and a curve that are coplanar, let the curve rotate about the fixed straight line. The surface obtained is a *rotary* or *R-surface*.

(2). Given a fixed straight line (reference) and a second straight line parallel to it, let the second straight line move in space while it remains parallel to the fixed straight line. The surface obtained is a *translational* or a *T-surface*;

(3). Given a fixed straight line and a curve segment that are coplanar, let the curve segment simultaneously rotate about and slide along the straight line such that the rotation angle θ and the sliding distance x relates as $x = k\theta$, where k is a constant. The surface obtained is a *helical* or *H-surface* and p $(= 2\pi k)$ is its *pitch*.

The word *curve* in the above definition also includes straight lines. Two curves are coplanar, if a plane exists such that any point of the two curves is also a point of the plane.

R-surface has two important special cases: the surface of a sphere or S-surface and the surface of a cylinder or C-surface. The T-surface has one important special case, the plane surface or E-surface.

Definition. *Basic Contacts.*

Single point, *line*, and *surface contact* between two bodies are defined as *basic contacts*. These contacts can be obtained as follows:

Single point contact:
- Contact of a surface with a surface;
- Contact of a surface with a line;
- Contact of a line with a line.

Line contact:
- Contact of a surface with a surface;
- Contact of a surface with a line;
- Contact of a line with a line.

Surface contact:
- Contact of a surface with a surface.

Surface contact is the best for avoiding large stress concentrations. In practice only surface contacts exist. However, if we assume that the *bodies* are *rigid*, then if the area of the contact region of two bodies is *small* enough, the contact region can be considered as a (single) point. Similarly, if the contact region of two bodies is a strip which is *thin* enough, the contact region can be considered as a line. How *small* is small enough, how *thin* is thin enough, are difficult to specify quantitatively. However, it is generally easy to make a reasonable judgement intuitively for practical purposes.

Let $s_{\xi 1}$ be the contact point-set of pair ξ on body 1. If $s_{\xi 1}$ is a surface whose area is very small compared to its physical size and motion range, then $s_{\xi 1}$ can be regarded as a single point. The single point can then be replaced or simulated by the surface of a small sphere whose radius approaches zero. This is the *small sphere approximation* of $s_{\xi 1}$.

Definition. *Basic Contact Lines*.

The *circular arc*, the *straight line* and the *helical line* are the *basic contact lines*, denoted as the *R-line*, the *T-line* and the *H-line*, respectively.

Definition. *Pair-Axis*.

Let $s_{\xi 1}$ and $s_{\xi 2}$ be the contact point-sets of pair ξ on bodies 1 and 2, respectively. If

(1). $s_{\xi 1}$ is a subset of a basic contact line or a basic contact surface; and

(2). $s_{\xi 1}$ is kinematically equivalent to its corresponding basic lines or surfaces when $s_{\xi 1}$ is acting with $s_{\xi 2}$,

then the corresponding (reference) straight line of the basic line or surface is the *pair-axis* (of $s_{\xi 1}$ or body 1).

Specifically we have *rotary*, *translational* and *helical axis*, denoted as *R*, *T* and *H-axis* respectively. The positions of rotary and helical axes relative to their corresponding surfaces or lines are uniquely defined. while the position of a translational axis relative to its surface may not necessarily be unique, for any straight line which is parallel to the reference straight line and itself is fixed to the T-surface, can be chosen as a T-axis.

It is worth noting that if $s_{\xi 1}$ is a subset of and kinematically equivalent to an E-surface, then, $s_{\xi 1}$ may have up to three independent pair-axes, i.e. two (non-parallel) T-axes and one R-axis. The T-axes are parallel to and the R-axis is perpendicular to the plane, see Fig. 1.2. If $s_{\xi 1}$ is a subset of and kinematically equivalent to an S-surface, then $s_{\xi 1}$ may have up to three independent (intersecting) R-axes, because the existence of a pair-axis is determined by the *two* bodies or the two contact point-sets of a kinematic pair.

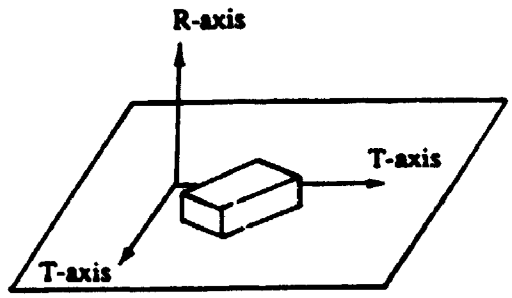


Fig. 1.2

It is also worth noting that if $s_{\xi 1}$ is a subset of and kinematically equivalent to a straight line, then the straight line itself is not only a T-axis but also an R-axis. Any other straight line which is fixed to body 1 and parallel to the straight line of the contact point-set can also be selected as a T-axis.

Let unit vector $\mathbf{a}_i$ represents the direction of a pair-axis, then, $\mathbf{a}_i$ is the *axis vector*. It is useful to distinguish three types of axis vectors,

$$\text{axis vector } \mathbf{a}_i = \begin{cases} \mathbf{a}_{Ri} & \text{- rotary axis vector;} \\ \mathbf{a}_{Ti} & \text{- translational axis vector;} \\ \mathbf{a}_{Hi} & \text{- helical axis vector;} \end{cases}$$

Definition. *I-axis*.

An *I-axis* (of a pair) is the common "instantaneous" line of the two bodies such that the relative instantaneous motion of the two bodies is either a rotation, a translation or a helical (or screw) motion along this line.

There are three types of I-axes: rotary, translational and helical. The position and direction of an I-axis relative to the two bodies of a kinematic pair can be either fixed or variable. A kinematic pair must have at least one and can have no more than five independent I-axes at any instant.

Let unit vector $\mathbf{a}_i$ represent the direction of an I-axis; then, $\mathbf{a}_i$ is an *I-axis vector*. It is useful to distinguish three types of I-axis vectors,

$$\text{I-axis vector } \mathbf{a}_i \rightarrow \begin{cases} \mathbf{a}_{Ri} & \text{- rotary I-axis vector;} \\ \mathbf{a}_{Ti} & \text{- translational I-axis vector;} \\ \mathbf{a}_{Hi} & \text{- helical I-axis vector;} \end{cases}$$

Definition. *Ball-point.*

A *ball-point* is the common point of three intersecting rotary I-axes of two bodies. The position of a ball-point relative to the bodies can be either fixed or variable. Two bodies may have either none or only one ball-point at any instant, depending on the constraint condition between them.

Definition. *R-point.*

Let $s_{\xi 1}$ and $s_{\xi 2}$ be the contact point-sets of pair ξ on bodies 1 and 2, respectively. If (1). $s_{\xi 1}$ is a subset of an S-surface; and (2). $s_{\xi 1}$ is kinematically equivalent to the S-surface when $s_{\xi 1}$ is interacting with $s_{\xi 2}$, then, the c ·· er of the sphere (of the S-surface) is an *R-point* (of pair ξ on body 1).

Here are several examples for ball-points and R-points: if two bodies make a point contact with each other, the contact point is a ball-point. If a spherical surface makes a line or surface contact with another body, the center of the sphere is not only an R-point but also a ball-point. However, a spherical center, or the center of a sphere, may not necessarily be a ball-point. This is illustrated on Fig. 1.3, where the contact point of the sphere and the plane is a ball-point, whereas the center of the sphere is an R-point instead of a ball-point. If contact point-set $s_{\xi 1}$ is a single point (or simulated as a single point), then this point is both a ball-point and an R-point, due to the *small sphere approximation*.

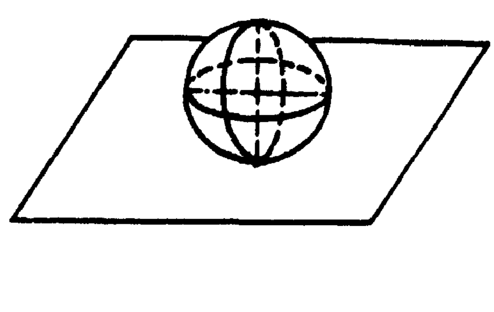


Fig.1.3

Definition. *Link.*

A *link* can be defined by either of the following two ways:

(1). A *link* is a body connecting two other bodies. i.e. if two bodies are connected together by contacting the same intermediate body, then the middle body is the common *link* of the *two* bodies.

(2). A *link* is a body connecting two pairs, i.e. if two pairs are connected together by a body, then this body is the common *link* of the *two* pairs.

A link is defined in terms of either three bodies or two pairs as illustrated on Fig. 1.4, where body 2 is the common *link* of bodies 1 and 3 and the common *link* of pairs ξ and η.

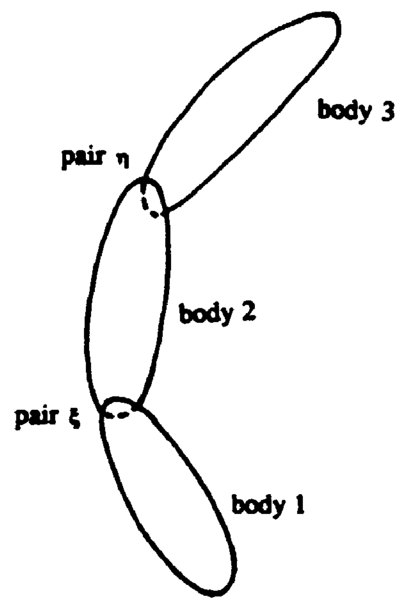


Fig. 1.4

Definition. *Definite* and *Indefinite Links*.

Given two pairs ξ and η connected by link 2, and $s_{\xi 2}$ and $s_{\eta 2}$ are respectively the two *contact point-sets* of pair ξ and pair η on link 2. If both $s_{\xi 2}$ and $s_{\eta 2}$ are single points or subsets of basic line or basic surface, then the *distance* between the two pairs ξ and η is *definite*, otherwise the *distance* is *undefined*. Accordingly, a *link* of two pairs can be either a *definite* or an *indefinite link*.

Definition. *Central Line* (of a definite Link) — *Link Length*.

Let body 2 be a *definite link* of pairs ξ and η with *contact point-sets* $s_{\xi 2}$ and $s_{\eta 2}$, namely, each of $s_{\xi 2}$ and $s_{\eta 2}$ is a single point or a subset of a basic line or basic surface, then the *central line* of link 2 is defined as the shortest line between either

- the two fixed ball-points of the two pairs or
- the two R-points of the two pairs or
- a fixed ball-point and an R-point of the two pairs or
- a fixed ball-point and a pair-axis of the two pairs or
- an R-point and a pair-axis of the two pairs or
- the two pair-axes of the two pairs.

The *length* of a definite link is the length of its central line.

The central line is a straight line. Examples are shown in Fig. 1.5. Points m and n of Fig. 1.5(a) are the two fixed ball-points or R-points of the two pairs connected by the body, then the *central line* is the straight line connecting points m and n. Fig. 1.5(b) and Fig 1.5(c) are self evident.

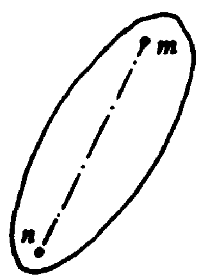


Fig. 1.5(a)

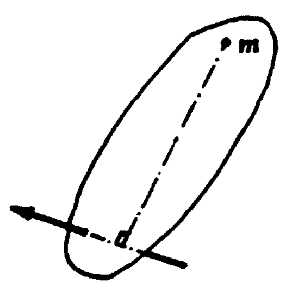


Fig. 1.5(b)

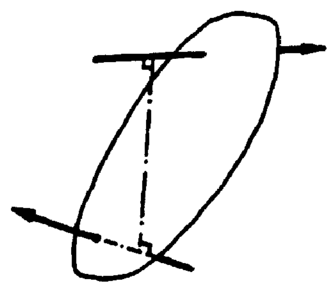


Fig. 1.5(c)

Definition. *Diameter* of a Kinematic Pair.

Given a pair ξ composed of two contacting bodies 1 and 2, let $s_{\xi 1}$ and $s_{\xi 2}$ be the contact point-sets of pair ξ. If $s_{\xi 1}$ (also $s_{\xi 2}$) is a subset of a basic line or basic surface, and if $s_{\xi 1}$ (also $s_{\xi 2}$) is kinematically equivalent to its corresponding basic line or basic surface when $s_{\xi 1}$ (also $s_{\xi 2}$) is interacting with $s_{\xi 2}$ (also $s_{\xi 1}$), then the *diameter* of pair ξ is the shortest distance between either

- the two fixed ball-points of $s_{\xi 1}$ and $s_{\xi 2}$ or
- the two R-points of $s_{\xi 1}$ and $s_{\xi 2}$ or
- a fixed ball-point and an R-point of $s_{\xi 1}$ and $s_{\xi 2}$ or
- a fixed ball-point and a pair-axis of $s_{\xi 1}$ and $s_{\xi 2}$ or
- an R-point and a pair-axis of $s_{\xi 1}$ and $s_{\xi 2}$ or

- the two pair-axes of $s_{\xi 1}$ and $s_{\xi 2}$.

The *diameter* of a kinematic pair can be either constant or variable.

Definition. *Definite Pair* and *Indefinite Pair*.

If the diameter of a kinematic pair is constant, then the kinematic pair is a *definite pair*, otherwise it is an *indefinite pair*.

It is not difficult to perceive that a *definite link* may connect either

- two definite pairs; or
- one definite and one indefinite pairs; or
- two indefinite pairs.

We can also say something similarly for *indefinite link*, *definite* and *indefinite pairs*.

If a T-axis is a pair-axis of a definite link, although its position in general is not unique, however, once it is specified the length of the definite link is fixed accordingly.

If body 2 is a indefinite link of pair ξ and pair η with contact point-sets $s_{\xi 2}$ and $s_{\eta 2}$, then at least one of $s_{\xi 2}$ and $s_{\eta 2}$ has none of a fixed ball-point, an R-point and a pair-axis. Therefore, one can not define the length of this link. Furthermore, the length of an indefinite link is generally variable.

A pair of two surfaces making surface contact can only be a definite pair. Indefinite pairs must involve either line or point contacts. There are many practical examples for indefinite pairs in cam mechanisms.

Definition. *Generalized Kinematic Pair*.

Let body 2 be a definite link connecting to bodies 1 and 3 with pairs ξ and η respectively. If the length of the central line of body 2 is zero, and both the diameters of the pairs ξ and η are also zero, then bodies 1, 2 and 3 are defined as a *generalized kinematic pair*. Moreover, let body 3 be a definite link connected to bodies 2 and 4 with pairs η and ζ, respectively. If the length of the central line of body 3 is zero, and the diameter of pair ζ is also zero, then bodies 1, 2, 3 and 4 are defined as a *generalized kinematic pair*. The word *pair* in the above two cases actually refers to the first and the last body.

A *generalized kinematic pair* which involves either more than four bodies serially connected or any indefinite link or indefinite pair is trivial. Three examples of generalized pairs are shown in Fig. 1.6(a-c), which can be denoted as (RP), (RR) and (RRR), respectively. The (RP) pair of Fig. 1.6(a) can also be denoted as (PR). The (RR) pair of Fig. 1.6(b) is kinematically equivalent to the Torus (pair) shown in Fig. 1.7. The former pair has a bigger motion range, is easier to design and it is more heavy duty; naturally it is more widely used in engineering as compared to the latter. If $a_i \cdot a_{i+1} = 0$ $(i = 1, 2, 3)$ in Fig. 1.6(b), the well known Hook joint is obtained. The (RRR) pair of Fig. 1.6(c) is kinematically equivalent to a spherical pair. It is worth noting that the center of a spherical pair is both a ball-point and an R-point, whereas the center or the common intersecting point of the three rotary axes of the (RRR) pair is a ball-point rather than an R-point of the two links a and b, as shown in Fig. 1.6(c).

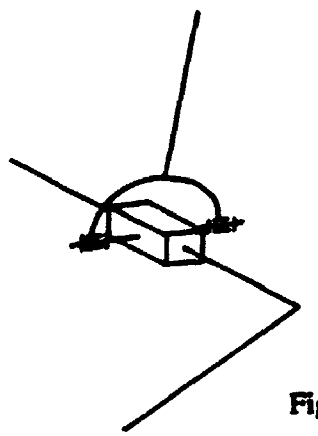


Fig. 1.6(a)

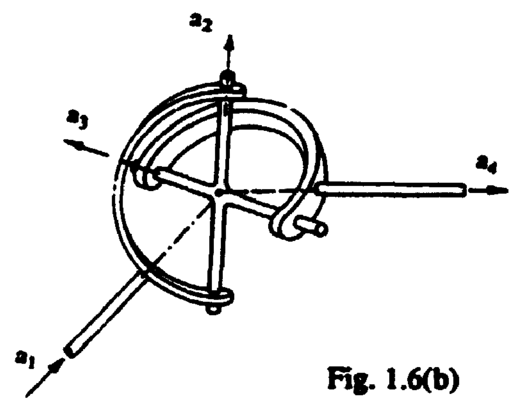


Fig. 1.6(b)

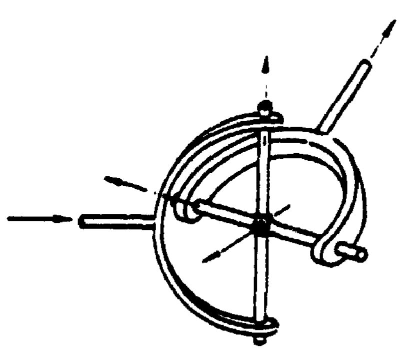


Fig. 1.6(c)

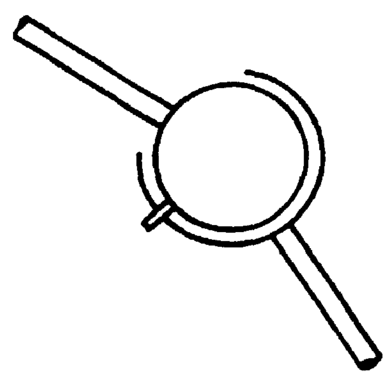


Fig. 1.7

We now introduce some notations. The *pair type*, or the type of a kinematic pair, is a three-digit number $(N_R N_T N_H)$ representing the numbers of possible independent rotary, translational and helical (screw) motion of one body relative to another of the pair as N_R, N_T and N_H, respectively. The *Degree of Freedom of a Kinematic Pair* is the number of independent coordinates needed to specify the relative position and orientation of the two bodies of a pair, i.e. it is the sum of the numbers of a pair's independent rotary, translational and helical freedoms. i.e. $(N_R + N_T + N_H)$.

All typical definite kinematic pairs are listed in Table 1.1. This table was developed on the basis of Table 2 of [36](Harrisberger, 1965). The eight pairs mentioned in [36] which physically do not exist or are not typical are not included here in Table 1.1. Pair T_H (Torus-helix) is also excluded, for it is a generalized kinematic pair. However, we introduced an additional kinematic pair, a round-bar making point contact with a round-bar and we adopted "B_B" as its symbol.

The three pairs, R, P and H pairs, of class I (in Table 1.1) are the most important kinematic pairs. Each of the other pairs of class II to V in Table 1.1 are kinematically equivalent to a combination of R, P and H pairs. For example, a C pair is equivalent to two co-axial pairs, i.e. one R and one P pair; An S pair is equivalent to three (axially) intersecting R pairs; etc.

Proposition 1.1. *Definite Pairs.*

The necessary and sufficient condition for a kinematic pair of $DoF \geq 2$ to be a definite pair is that it is kinematically equivalent to a combination of R, P and H pairs.

1.3. Mechanisms

Definition. *Serial Linkage.*

A *serial linkage* is a system of bodies $\{B_i\}$ $(i = 1 - n,\ n \geq 2)$ interconnected in such a way that except for two bodies, which contact only one body each, every body of $\{B_i\}$ contacts two and only two bodies, see Fig. 1.8.

Definition. *Single-loop Linkage.*

A *single-loop linkage* is a system of bodies $\{B_i\}$ $(i = 1 - n,\ n \geq 2)$ interconnected in such a way that each of the bodies connects two and only two kinematic pairs, see Fig. 1.9.

Definition. *Linkage.*

A *linkage* is a system of bodies $\{B_i\}$ $(i = 1 - n,\ n \geq 2)$ interconnected in such a way that for any two bodies of the system, $B_\alpha, B_\beta \in \{B_i\}$, there is a subsystem $\{B_j'\} \subset \{B_i\}$, $(j = 1 - m,\ 2 \leq m \leq n)$ such that

(1). $B_\alpha, B_\beta \in \{B_j'\}$; and

(2). $\{B_j'\}$ $(j = 1 - m,\ 2 \leq m \leq n)$ is a *serial linkage*.

Table 1.1. Kinematic pairs

Class	DoF	Class symbol	Type ($N_R N_T N_H$)	Type symbol	Name
I	1	p_1	100	R	Revolute
			010	P	Prism
			001	H	Helix
II	2	p_2	110	C	Cylinder
			200	T	Torus
III	3	p_3	300	S	Sphere
			120	E	Plane
			210	S_{SC}	Sphere-slotted cylinder
			201	S_{SH}	Sphere-slotted helix
IV	4	p_4	310	S_G	Sphere-groove
			220	C_P	Cylinder-plane
			301	S_{GH}	Sphere-grooved helix
V	5	p_5	320	S_P	Sphere-plane
			320	B_B	Bar-bar

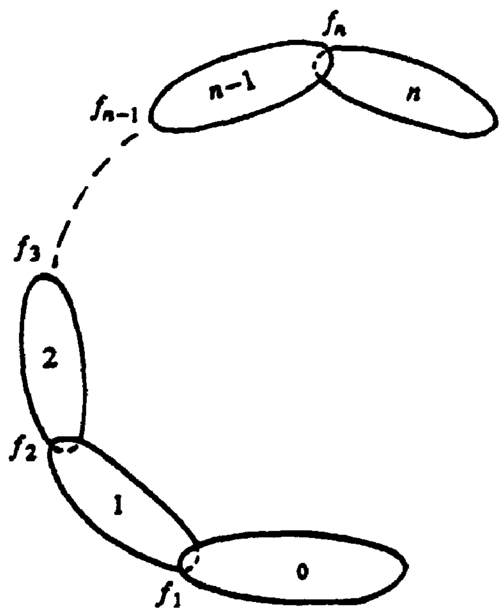


Fig. 1.8

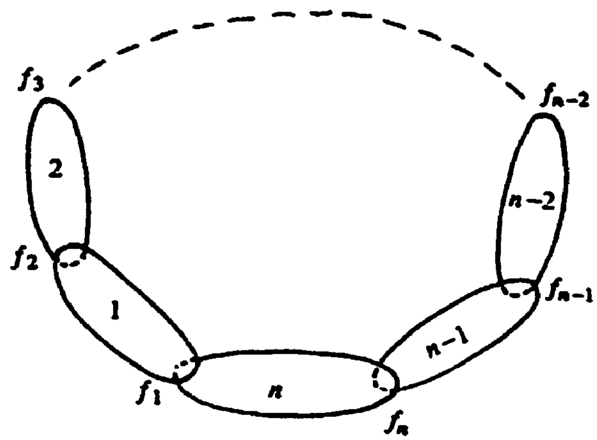


Fig. 1.9

It is clear that *serial linkage* and *single-loop linkage* are special cases of *linkage*.

Definition: *Mechanism*.

A mechanism is a linkage device that

- physically, it is a system of *bodies* connected together by *pairs* or a system of *pairs* connected together by *bodies*;
- functionally, it can be used to generate required motion and force.

In general, a mechanism is either a *planar* or a *spatial mechanism*. Given a mechanism, if a plane exists such that the distance from any point of the moving bodies of the mechanism to the plane is constant, then the mechanism is a *planar mechanism*; otherwise, it is a *spatial mechanism*. If a point o exists such that the distance from any point of the moving bodies of a mechanism to the point o is constant, then the mechanism is a *spherical mechanism* and point o is its center. Spherical mechanisms are spatial mechanisms.

Definition. The *Dimension* of a Mechanism.

If the position of any point of the mechanism can be specified by k $(1 \le k \le 3)$ independent coordinates $\{q_i\}$ $(i = 1 - k)$ that are independent of the mechanism, then the *dimension* of the mechanism is k.

A planar mechanism could be either one or two-dimension mechanism; a spatial mechanism could be either two or three-dimension mechanism. For instance, a spherical mechanism is a spatial mechanism, though its dimension is two instead of three.

Definition. *Definite* and *Indefinite Mechanism*.

If a mechanism contains an indefinite link or an indefinite pair, it is an ***indefinite mechanism***, otherwise it is a ***definite mechanism***.

Proposition 1.2.

The displacement equations of any definite mechanism can be expressed by algebraic equations, whereas for indefinite mechanisms this is usually impossible.

Note a body performing unconstrained spatial motion has six degrees of freedom, namely, translations in three non-coplanar directions and rotations about three mutually perpendicular axes. Therefore

Theorem 1.1. *End-effector DoF*.

Given a serial linkage composed of $n+1$ bodies, i.e. $\{B_i\}$ $(i=0-n,\ n\geq 1)$, as shown in Fig. 1.8. Denote $f=\sum f_j$, where $f_j=(N_k^j+N_f^j+N_h^j)$ $(j=1-n)$ is the DoF of pair j. Let D_n be the DoF of body n relative to body 0. It follows:

(1). If $f\geq 6$, then, $1\leq D_n\leq 6$;

(2). If $f\leq 6$, then, $1\leq D_n\leq f$.

Most industrial robots are serial linkages with one end fixed to the ground and the other end acting as a "hand", called *end-effector*, moving in space. The proof for theorem 1.1 is easy and it is omitted here.

Definition. *DoF of a Linkage*.

Given a linkage composed of $n+1$ bodies, $\{B_i\}$ $(i=0-n,\ n\geq 1)$, and m pairs. The *total DoF of the pairs of the linkage* is $f=\sum_{j=1}^{m} f_j$, where f_j $(j=1-m)$ is the DoF of pair j. *The DoF of the linkage*, F, is the number of the independent coordinates needed to specify the relative position and orientation of all bodies of the linkage.

The lower and upper limit of the number of pairs, m, for a linkage for $n+1$ $(n\geq 1)$ bodies is $n-1\leq m\leq 2n$. The proof is straightforward and we do not display it here.

Theorem 1.2. *About f & F*.

The relationship between the *total DoF of the pairs of a linkage* (i.e. f) and the *DoF of the linkage* (i.e. F) is as follows:

(1). For a serial linkage: $F = f$;

(2). For a single-loop linkage:

(a). If $f \geq 7$, then, $f - 6 \leq F \leq f - 1$;

(b). If $2 \leq f \leq 6$, then, $0 \leq F \leq f - 1$.

Proof :

(1). For a serial linkage, $F = f$ and $f \geq 1$ are self-evident;

(2). For a single-loop linkage $f \geq 7$ means that there are f DoF to contribute to the position and orientation of body n, see Fig. 1.8. Since body n can only have a maximum DoF of six, of the f (≥ 7) DoF there can only be a maximum of six independent DoF that completely determine the position and orientation of body n. Accordingly, there will be at least $f - 6$ DoF of the serial linkage whose existence is independent of body n or whose existence do not affect body n. Now let's fix body n to the ground, i.e. let bodies n and 0 become one body, this means that the DoF of body n becomes zero, namely a maximum of six DoF of the pairs of the (original) serial linkage are eliminated by body n, thus there will be at least $f - 6$ DoF left in the (newly-obtained) single-loop linkage, i.e. $F \geq f - 6$; Again, from theorem 1.1, body n must have at least one DoF relative to body 0. This means there will be at least one DoF of f which contributes to or completely determines the position and orientation of body n. Obviously after body n is fixed to body 0, there can be a maximum of $f - 1$ DoF left in the (newly-obtained) single-loop linkage, i.e. $F \leq f - 1$. Therefore, $f - 6 \leq F \leq f - 1$. The proof for the case of $2 \leq f \leq 6$ can be similarly performed.

A body performing either unconstrained planar or spherical motion has only three degrees of freedom. For planar motion, they are translations in two non-parallel directions and rotation about an axis perpendicular to the plane of motion; For spherical motion, they are rotations about three mutually perpendicular axes passing through the center of the sphere. Note if the three rotary axes of the spherical motion is non-coplanar instead of mutually perpendicular, the domain of spherical motion will cover only part of a sphere. Moreover, the resultant of three rotations is a rotation, hence, any spherical motion is a rotation, whose axis, being passing through the center of the sphere, could be either fixed or instantaneous. Although theorems 1.1 and 1.2 are applicable to planar and spherical linkages, we have the following three theorems which better describe the two groups of linkages.

Theorem 1.3. *End-effector DoF for Planar and Spherical Linkages*

Given a planar or spherical serial linkage composed of $n+1$ bodies, i.e. $\{B_i\}$ ($i=0-n$, $n \geq 1$). as shown in Fig. 1.8. Denote $f = \sum f_j$, where f_j ($j=1-n$) is the DoF of pair j. Let D_n be the DoF of body n relative to body 0. It follows:

(1). If $f \geq 3$, then, $1 \leq D_n \leq 3$;

(2). If $f \leq 3$, then, $1 \leq D_n \leq f$.

It is clear that only rotary and translational freedom can generate planar motion, whereas helical freedom always renders non-planar motion. Specifically, *only* the rotary freedom whose axis is perpendicular to the plane of motion and the translational freedom whose axis is coplanar with the plane of motion contribute to the planar motion.

Theorem 1.4. *About f & F for Planar Linkages.*

The relationship between the *total DoF of the pairs of a planar linkage* (i.e. f) and the *DoF of the planar linkage* (i.e. F) is as follows:

(1). For a planar *serial* linkage: $F = f$;

(2). For a planar *single-loop* linkage, let us denote

$$
\begin{aligned}
F &= F_1 + F_0 \\
f &= f_1 + f_0 \\
f_1 &= r + t \\
f_0 &= r_0 + t_0
\end{aligned}
$$

where F_1 is the number of the planar DoF of the linkage, F_0 is the number of the DoF of the linkage which renders motion deviating from the plane of motion; r and r_0 are the numbers of the rotary axes perpendicular to and not perpendicular to the plane of motion, respectively, whereas t and t_0 are the numbers of the translational axes coplanar with and perpendicular to the plane of motion, respectively; Now the implications of f_1 and f_0 are obvious: they are the numbers of the DoF of the *pairs* that contribute to the planar motion and contribute to the motion deviating from the plane of motion, respectively. We have conclusions:

(a). If $f_1 \geq 4$, then, $f_1 - 3 \leq F_1 \leq f_1 - 1$;

(b). If $f_1 \leq 3$, then, $0 \leq F_1 \leq f_1 - 1$;

(c). If either of the following two conditions is true, then $F_0 = 0$; otherwise $F_0 \neq 0$, which renders freedom deviating from the plane of motion.

- Condition 1: $r_0 \leq 1$, $t_0 \leq 1$;
- Condition 2: $r_0 \geq 2$ and there is no coaxial couple among the r_0 rotary freedoms.

It is clear that only the rotary freedom whose axis passes through the center of a spherical linkage contributes to the spherical motion of the spherical linkage, whereas translational freedom may exist and helical freedom can never exist in the *pairs* of a spherical linkage, for the latter always renders non-spherical motion.

Theorem 1.5. *About f & F for Spherical Linkages*.

The relationship between the *total DoF of the pairs of a spherical linkage* (i.e. f) and the *DoF of the spherical linkage* (i.e. F) is as follows:

(1). For a spherical *serial* linkage: $F = f$, and all the pairs must be R pairs;

(2). For a spherical *single-loop* linkage, let us denote

$$F = F_1 + F_0$$
$$f = r + t + 2s$$

where F_1 is the number of the spherical DoF of the spherical linkage, and F_0 is the number of the DoF of the spherical linkage that renders non-spherical motion; r is the number of the rotary freedoms whose axes pass through the center of the spherical linkage; t is the number of the translational freedom of the *pairs* of the spherical linkage; and finally, s is the number of the *spherical pair* of the spherical linkage. We have the conclusions:

(a). If $r \geq 4$, then, $F_1 = r - 3$;

(b). If $2 \leq r \leq 3$, then, $F_1 = 0$;

(c). If $s \leq 1$, then $F_0 = 0$; otherwise, $F_0 \neq 0$, which renders non-spherical motion.

Although, theoretically, the translational freedom whose axis passes through the center of the spherical linkage does not affect the spherical motion of the spherical linkage, it is a good practice to let $t \leq 1$, in order to strengthen the function of the linkage.

The validation of Theorem 1.3 is quite obvious. The proofs for Theorems 1.4 and 1.5 can be performed in a way similar to the proof of Theorem 1.2.

1.4. The Input Kinematic Pairs

The actuator of the input kinematic pair of an SSM controls a certain number of rotary, translational and helical motions of the input pair. The following notations will now be introduced. A three-digit number $(\bar{N}_R\bar{N}_T\bar{N}_H)$, defined as the *3-Digit Input Control Number*, will be used to specify the numbers of independent rotary, translational and helical motions of the input pair controlled by the actuator, $\bar{N}_R$, $\bar{N}_T$ and $\bar{N}_H$, respectively. The sum of $\bar{N}_R$, $\bar{N}_T$ and $\bar{N}_H$ is the *Controlled DoF* (of a kinematic pair). The *3-Digit Number of Active DoF* of a pair is the three-digit number $(\tilde{N}_R\tilde{N}_T\tilde{N}_H)$ such that $\tilde{N}_R=N_R-\bar{N}_R$, $\tilde{N}_T=N_T-\bar{N}_T$ and $\tilde{N}_H=N_H-\bar{N}_H$. The sum of $\tilde{N}_R$, $\tilde{N}_T$ and $\tilde{N}_H$ is the *Active DoF* of a pair. The combination of the *3-digit number of active DoF* and the *3-digit input control number* fully specifies the input pair. It is the *Resultant Input Type* of the input pair and denoted as $X_{(\tilde{N}_R\tilde{N}_T\tilde{N}_H)(\bar{N}_R\bar{N}_T\bar{N}_H)}$.

Table 1.2. Classification of input kinematic pairs

(4)	(5) $(\tilde{N}_R\tilde{N}_T\tilde{N}_H)$ \ (3) \ (1)	0	1			2				3				4			
	(2) $(\bar{N}_R\bar{N}_T\bar{N}_H)$	000	100	010	001	110	200	101	020	210	120	201	300	310	220	301	320
0	000	..	R	P	H	C	T	T_H	..	S_{SC}	E	S_{SH}	S	S_G	C_P	S_{GH}	S_P, B_B
1	100	R	T	C	T_H	S_{SC}	S	S_{SH}	E	S_G	C_P	S_{GH}	..	..	S_P, B_B		
	010	P	C	..	..	E	S_{SC}	..	..	C_P	..	..	S_G	S_P, B_B			
	001	H	T_H	..	..	..	S_{SH}	..	..	..	..	..	S_{GH}				
2	110	C	S_{SC}	E	..	C_P	S_G	..	..	S_P, B_B	..						
	200	T	S	S_{SC}	S_{SH}	S_G	..	S_{SH}	C_P	..	S_P, B_B						
	101	T_H	S_{SH}	..	..	..	S_{GH}										
	020	..	E	..	..	..	C_P										
3	210	S_{SC}	S_G	C_P	..	S_P, B_B	..										
	120	E	C_P	..	..	..	S_P, B_B										
	201	S_{SH}	S_{GH}	..	..												
	300	S	..	S_G	S_{GH}												
4	310	S_G	..	S_P, B_B													
	220	C_P	S_P, B_B														
	301	S_{GH}															
5	320	S_P, B_B															

Remarks:
(1) Controlled DoF ;
(2) The 3-Digit Input Control Number ;
(3) Input Pair ;
(4) Active DoF ;
(5) The 3-Digit Number of Active DoF .

For example, $R_{(000)(100)}$, $C_{(100)(010)}$, $C_{(010)(100)}$ or $S_{G(300)(010)}$, etc. represent different input pairs. The 3-digit input control number may be omitted and the notation of $X_{(\tilde{N}_R\tilde{N}_T\tilde{N}_H)}$ or $X_{(\tilde{N}_R+\tilde{N}_T+\tilde{N}_H)}$ can be used to represent general cases. For instance, X_{100} represents an input that has a 3-digit number of active DoF of (100), its 3-digit input control number could be any one of the thirteen cases corresponding to $(\tilde{N}_R \tilde{N}_T \tilde{N}_H) = (100)$ as shown in Table 1.2. x_1 and x_2 symbolically represent those input pairs whose active DoF are 1 and 2, respectively, while the symbol $x_{i,j}$ denotes an input pair where the subscript i is the active and j is the controlled DoF of the pair.

Table 1.2 shows the classification of the input kinematic pairs. The empty spaces in Table 1.2 imply that either no such kinematic pair exists or that these input pairs are not typical or can not be expressed by any pairs of Table 1.1. An example is given below for those input pairs which can not be expressed by any pairs of Table 1.1.

Let Fig. 1.1 represent an input kinematic pair. Link 2 is connected to the kinematic chain and link 1 is either a *fixed* or a *moving* body, and its position and orientation relative to the frame (ground) can be determined.

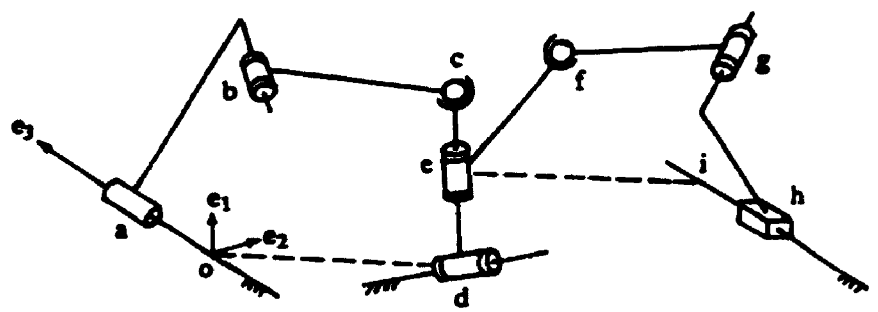


Fig. 1.10

Multi-loop spatial mechanisms are composed of two or more single-loop spatial mechanisms. For example, the mechanism shown in Fig. 1.10 is a two-loop spatial mechanism. The input and output-pairs of the first loop (*o-a-b-c-d-o*) are *a* and *d* and the input and output-pairs of the second loop (*d-e-f-g-h-i-d*) are *d* and *h* respectively. The first loop can have either of the following two ***Resultant Input Types:*** $C_{(010)(100)}$ or $C_{(100)(010)}$. In the first case, the loop is $C_{(010)(100)}$–***RSR*** (or C_{010}–***RSR***) and in the second case it is $C_{(100)(010)}$–***RSR*** (or C_{100}–***RSR***). The second loop is $R_{(000)(100)}$–***RSRP*** (or R_{000}–***RSRP*** or R_0–***RSRP***).

However, if we study another two-loop spatial mechanism shown in Fig. 1.11(a), we will find that the second loop can not simply be (*d-c-e-f-g-h-i-d*) and denoted using the conventional notation as ***RSRSRP***. First of all, the total number of the active DoF of the pairs in ***RSRSRP*** is ten, which exceeds the limit for an independent single-loop spatial mechanism which is free of idle degrees of freedom, according to theorem 1.1. Secondly, the first two pairs ***RS*** of ***RSRSRP***, which determine the motion of the kinematic chain of (*e-f-g-h*), are not a unique group of pairs. As a matter of fact, we can also consider the mechanism as the combination of the first loop and loop (*o-a-b-e-f-g-h-i-o*). In this case, the second loop can be denoted conventionally as

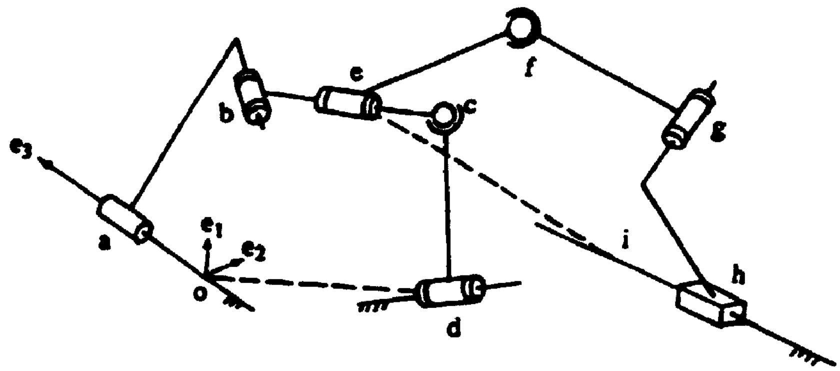


Fig. 1.11(a)

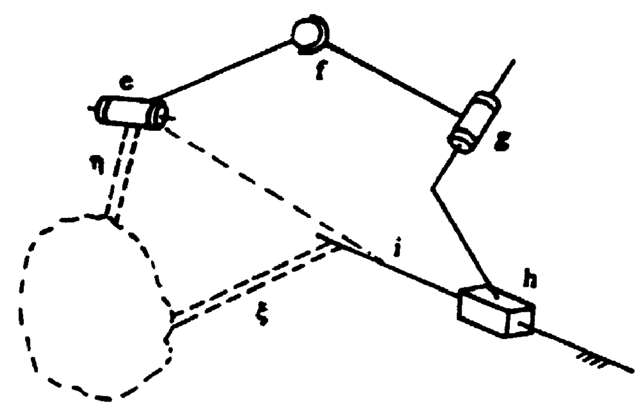


Fig. 1.11(b)

CRRSRP. Apparently the first two pairs *CR* of *CRRSRP* and the first two pairs *RS* of *RSRSRP* are kinematically equivalent as far as the remaining kinematic chain, (*e-f-g-h*) i.e. *RSRP*, is concerned, thus uncertainty emerges. Obviously, it is desirable to introduce a new way to define and describe precisely the second loop of the mechanism of Fig. 1.11(a). According to the structure of the mechanism, we can see that the first loop *(o-a-b-c-d-o)* as a whole can be reasonably regarded as the "input-pair" for loop *(e-f-g-h-i-e)*. This "input-pair" can not be expressed by using any specific kinematic pair listed in Table 1.1. However, it can be replaced by an imaginary kinematic pair with two links (link ξ and link η) as shown in Fig. 1.11(b). Link ξ could either be a fixed or a moving body, as long as its position and orientation can be determined at any instant. The relative position and orientation of link η to link ξ are completely controlled by the "actuator" of the "input-pair". We introduce X_0 to represent this "input-pair" and consider the second loop as *(e-f-g-h-i-e)* and represent it symbolically as X_0-*RSRP*. The advantage of defining and representing the second loop in such a way is that, not considering the first loop, the structure of the remaining kinematic chain *(e-f-g-h)* is such that the possible maximum number of assembly configuration can be completely determined by itself (i.e. for X_0-$RSRP$) and it is entirely independent of the first kinematic loop.

Not all input pairs listed in Table 1.2 have the same frequency of application. The most frequently used input pair is $R_{(000)(100)}$ (i.e. $R_{0,1}$ or R_0).

The symbols $X_{(\tilde{N}_R\tilde{N}_T\tilde{N}_H)}$ and X_i $(i=0-5)$ are quite general. They cover any number of resultant input pairs whose 3-digit number of active DoF are equal to $(\tilde{N}_R\tilde{N}_T\tilde{N}_H)$ and whose active DoF are equal to i, respectively. For instance, X_0 covers any number of resultant input pairs of zero active DoF which can not be expressed by a specific kinematic pair. In addition, it also covers some of those input pairs which are expressed by kinematic pairs such as R_0, P_0 and C_0, et al.

1.5. Basic SSM Groups

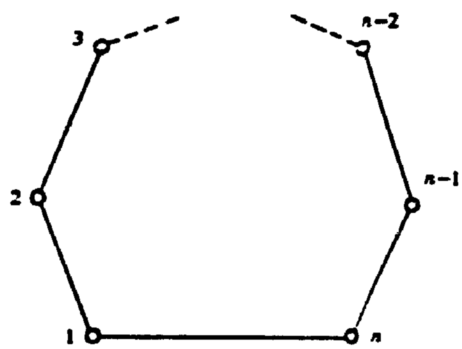


Fig. 1.12

Fig. 1.12 represents the abstract model of an n-pair single-loop spatial mechanism. The total active DoF of the pairs of the mechanism is

$$f = \sum f_i = \sum_{i=1}^{n} (\tilde{N}_R^i + \tilde{N}_T^i + \tilde{N}_H^i)\,. \qquad (2 \le n \le 7) \qquad (1.1)$$

where the superscript i denotes the sequential position of the pair in the kinematic loop. It is possible that two or more input pairs actuate a single kinematic loop. Therefore, theoretically the number of pairs in the chain, n, can be any integer number equal to or greater than two, as long as $1 \le f \le 6$. However, we focus only on the basic groups of mechanisms defined by the following conditions:

(a) Only one pair of the kinematic loop is actuated. Consequently, $2 \le n \le 7$. The actuated pair is the input pair of the loop and the loop is composed of the pairs listed in Table 1.1.

(b) The input pair, without any loss of generality, is denoted as the first pair of the loop. The second, third, etc. pairs are labeled consecutively along the kinematic chain, and the last pair is *fixed* to the ground. The first pair could be a *moving* pair as long as the position and orientation of at least one of its elements relative to the frame can be determined.

It is apparent that condition (b) confines the input pair to be adjacent to the pa·· fixed to the ground. On the basis of condition (a) and (b), we can define four basic groups of SSMs as follows.

Definition. *Regular Mechanism (i.e. RM).*

An SSM which complies with the following three conditions is a regular mechanism:

(1) Conditions (a) and (b) above;

(2) $2 \le n \le 7$. $f = 6$;

(3) $\sum_{i=1}^{n} \tilde{N}_H^i = 0$.

Definition. *Over-constrained Regular Mechanism (i.e. ORM).*

An SSM which complies with the following three conditions is an over-constrained regular mechanism:

(1) Conditions (a) and (b) above;

(2) $2 \le n \le 6$. $1 \le f \le 5$;

(3) $\sum_{i=1}^{n} \tilde{N}_H^i = 0$.

It is useful to introduce the symbol of O_iRM, where $i = (6-f)$ is the number of over-constraints of the kinematic loop. Since $1 \le f \le 5$. hence $1 \le i = (6-f) \le 5$.

Definition. *Helical Mechanism (i.e. HM).*

An SSM which complies with the following three conditions is a helical mechanism:

(1) Conditions (a) and (b) above;

(2) $2 \le n \le 7$. $f = 6$;

(3) $1 \le \sum_{i=1}^{n} \tilde{N}_H^i \le 6$.

Similarly, we can introduce the symbol of H_jM, where $j = \sum_{i=1}^{n} \tilde{N}_H^i$ is the total number of the active helical freedoms of kinematic loop.

Definition. *Over-constrained Helical Mechanism (i.e. OHM).*

An SSM which complies with the following three conditions is an over-constrained helical mechanism:

(1) Conditions (a) and (b) above;

(2) $2 \le n \le 6$, $1 \le f \le 5$;

(3) $1 \le \sum_{i=1}^{n} \tilde{N}_H^i \le 5$.

Now there is no difficulty to understand the implication of O_iH_jM, where $1 \le i = (6-f) \le 5$, $j = \sum_{i=1}^{n} \tilde{N}_H^i$ and $(2 \le i+j \le 6)$.

Proposition 1.3. *Kinematic Features of Spatial Mechanisms.*

The differences between the four basic groups of mechanisms on the basis of their kinematic features are as follows:

RM
- Generally these mechanisms are capable of motion;
- The input-output displacement equations can always be expressed by polynomials.

ORM
- Generally these are structures and can not move. Under special geometric conditions they may become movable mechanisms;
- The input-output displacement (or structure) equations can always be expressed by polynomials.

HM
- Generally these mechanisms are capable of motion;
- Generally they are more difficult to analyze than their corresponding RM, derived by assuming that all the pitches of the helical freedoms of the HM equal zero. Most input-output displacement equations corresponding to helical angular freedoms can not be transformed into polynomials.

OHM
- Generally these are immovable structures. Under special geometric conditions they may become movable mechanisms;
- Generally they are more difficult to analyze than their corresponding ORM, derived by assuming that all the pitches of the helical freedoms of the OHM equal zero. Most input-output displacement equations corresponding to helical angular freedoms can not be transformed into polynomials.

The rationale for classifying SSMs into four basic groups is based on whether or not a mechanism is over constrained and whether a mechanism involves a helical freedom or not. A helical pair (H) is a very special pair. When the pitch of its screw motion is equal to zero it becomes a revolute pair (R); when the pitch of its screw is equal to infinity, it becomes a prismatic pair (P).

Because of conditions (a) and (b) above, for the pairs of any single loop of the four basic groups of spatial mechanisms the following is true:

$$(\tilde{N}_R \tilde{N}_T \tilde{N}_H) = \begin{cases} (N_R N_T N_H) - (\bar{N}_R \bar{N}_T \bar{N}_H), & \text{for an input pair;} \\ (N_R N_T N_H), & \text{for any other pair.} \end{cases}$$

1.6. Central Vector Polygon

If the total active DoF of all the pairs of a mechanism is equal to or greater than seven, i.e. $f = \sum_{i=1}^{n} (\tilde{N}_R^i + \tilde{N}_T^i + \tilde{N}_H^i) \geq 7$, then the mechanism will have at least $(f-6)$ idle degrees of freedom (IDoF).

Definition. *Idle Degree of Freedom (IDoF).*

The *idle degree of freedom* of a single-loop mechanism is the unconstrained degree of freedom, in addition to the DoF controlled by the actuator of the input pair, between the two contacting bodies of any pair in the mechanism. The word *unconstrained* implies that the DoF is independant of the effect of the actuator.

Definition. *The Number of IDoF.*

The number of IDoF of a single-loop mechanism is the number of the independent constraints, in addition to the number of DoF controlled by the actuator of the input pair, that is required to fully specify the relative position and orientation of all the bodies of the mechanism.

In the following we will distinguish several kinds of IDoF. But first we need to introduce the concept of the *central vector polygon* of an SSM.

Definition. *The Central Vector Polygon.*

For any *definite* single-loop mechanism, along the central lines of its links, the pair-axes and the diameters of the pairs, we can construct a spatial polygon which is the *central vector polygon* of the mechanism.

Central vector polygon is a concept of vital importance in the theory of mechanisms. It exposes the very essence of the structural and kinematical features of spatial mechanisms. In fact, the kinematic analyses of spatial mechanisms and serial robots is, in most cases, simply the analyses of the configurations of their central vector polygons.

Definition. *IDoF-1* and *IDoF-2*.

If an IDoF does not affect the configuration of the central vector polygon of a mechanism, the IDoF is the *IDoF of the first kind.* If an IDoF affects the configuration of the central vector polygon of a mechanism, the IDoF is then the *IDoF of the second kind.* For short, they are denoted as *IDoF-1* and *IDoF-2*, respectively.

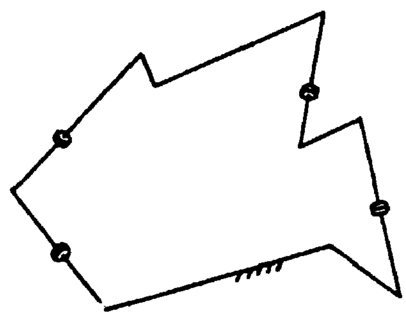


Fig. 1.13

From the diagram of an arbitrary spatial polygon as shown in Fig. 1.13 we can easily see that an IDoF-2 will make either the direction or the length or both of at least two edges of the spatial polygon undeterminable. On the contrary, an IDoF-1 does not alter the configuration of the spatial polygon of the mechanism. The best known mechanism with an IDoF-1 is R_0-SSR (and its variant mechanism R_0-RSS). The IDoF of this mechanism can produce a rotation of the coupler about the axis joining the two S pairs, but this has no effect on the configuration of the spatial polygon of the mechanism. The IDoF-1 is more common than the IDoF-2. Generally an IDoF-2 renders the mechanism useless. However, some useful mechanisms with IDoF-2 may exist, provided that the influence of the IDoF upon the mechanism is negligible.

We can easily further distinguish three kinds of *IDoF-2* s.

Definition. *R-IDoF-2*, *T-IDoF-2* and *H-IDoF-2*.

If an IDoF-2 affects only the direction of the edges of the central vector polygon of the mechanism, it is an *rotary IDoF of the second kind* and denoted as *R-IDoF-2*; If an IDoF-2 affects only the length of the edges of the central vector polygon of the mechanism, it is the *translational IDoF of the second kind* and denoted as *T-IDoF-2* ; If an IDoF simultaneously affects the direction θ and the length x of the edges of the vector polygon of the mechanism and $x = k\theta$, the IDoF is then an *helical IDoF of the second kind* and denoted as *H-IDoF-2* , where k is a constant.

We do not consider the SSMs of $f \geq 7$ as an independent basic group of mechanisms. The reasons are as follows: (a) the IDoF-1 of a mechanism can always be eliminated, i.e. for any mechanism with an IDoF-1, we can always find a corresponding mechanism from the four basic groups such that the motions of the central vector polygons of the two mechanisms are exactly the same; for instance, replacing any one of the two S pairs of R_0–SSR by a Hook pair, the IDoF-1 of R_0–SSR is eliminated. (b) If a mechanism with an IDoF-2 is a useful mechanism, its mission can always be performed satisfactorily by a mechanism from the four basic groups; (c) There are too many SSMs of $f \geq 7$ to list. We can only say that the number of the "mechanisms" in this group is infinite; however, a vast maiority of them can not stand firmly, i.e. they are just *loose* linkages.

1.7. Symbolic Representation of SSMs

Given, for instance, an input pair X_{100} and four pairs: one P, one C and two R pairs, we can obtain 12 different RMs by positioning the four pairs differently, i.e.

$$\begin{cases} X_{100}\text{–}RRPC & X_{100}\text{–}RPCR & X_{100}\text{–}PRRC \\ X_{100}\text{–}RRCP & X_{100}\text{–}RCPR & X_{100}\text{–}CRRP \\ X_{100}\text{–}RPRC & X_{100}\text{–}PRCR & X_{100}\text{–}PCRR \\ X_{100}\text{–}RCRP & X_{100}\text{–}CRPR & X_{100}\text{–}CPRR \end{cases} \tag{1.2}$$

Without considering the relative position of the four pairs, we can denote the 12 mechanisms of (1.2) as

$$X_{100}\text{–}2R\text{–}C\text{–}P \tag{1.3}$$

(1.3) can also be replaced by any one of the following five expressions,

$$\begin{cases} X_{100}\text{–}2R\text{–}P\text{–}C & X_{100}\text{–}P\text{–}C\text{–}2R & X_{100}\text{–}P\text{–}2R\text{–}C \\ X_{100}\text{–}C\text{–}2R\text{–}P & X_{100}\text{–}C\text{–}P\text{–}2R & \end{cases} \tag{1.3a}$$

The difference between (1.2) and (1.3) is obvious. There is no dash between the different pairs in (1.2), indicating that the relative sequential position of the pairs in the kinematic loop is specified. The dashes between the different type of pairs in (1.3) imply that the relative position of these pairs are not specified.

Definition. *Variant Mechanism*.

Every two mechanisms of (1.2) are *variant mechanisms* of each other; each of the mechanisms of (1.2) has a total of 11 *variant mechanisms*; the expression of (1.3), X_{100}–$2R$–C–P, represents a total of 12 *variant mechanisms*.

If a single-loop spatial mechanism contains one of each of the three types of pairs of Class I (see Table 1.1), then *another* pair, and only one, can be chosen from Class II or III in order not to violate the total DoF constraint. $f \le 6$, therefore,

Without counting the input pair, any SSM of the four basic groups (RM, ORM, HM and OHM) must be composed of less than or equal to 4 different types of pairs (listed in Table 1).

Let Γ_i $(i = 1 - 4)$ represent four different types of pairs, and let their numbers in a kinematic loop be α, β, γ and κ, respectively, then the number of variant mechanisms for four possible situations can be obtained easily by combinatorial analysis.

$$\begin{cases} X_{(\hat{N}_R \hat{N}_T \hat{N}_H)} - \alpha\Gamma_1 & \rightarrow\ m = C_\alpha^\alpha = 1 \\ X_{(\hat{N}_R \hat{N}_T \hat{N}_H)} - \alpha\Gamma_1 - \beta\Gamma_2 & \rightarrow\ m = C_{\alpha+\beta}^\alpha\, C_\beta^\beta \\ X_{(\hat{N}_R \hat{N}_T \hat{N}_H)} - \alpha\Gamma_1 - \beta\Gamma_2 - \gamma\Gamma_3 & \rightarrow\ m = C_{\alpha+\beta+\gamma}^\alpha\, C_{\beta+\gamma}^\beta\, C_\gamma^\gamma \\ X_{(\hat{N}_R \hat{N}_T \hat{N}_H)} - \alpha\Gamma_1 - \beta\Gamma_2 - \gamma\Gamma_3 - \kappa\Gamma_4 & \rightarrow\ m = C_{\alpha+\beta+\gamma+\kappa}^\alpha\, C_{\beta+\gamma+\kappa}^\beta\, C_{\gamma+\kappa}^\gamma\, C_\kappa^\kappa \end{cases} \tag{1.4}$$

where $1 \le (\alpha+\beta+\gamma+\kappa) \le 6$, (due to $1 \le f \le 6$), and

$$C_p^q = \frac{p!}{q!(p-q)!} \tag{1.5}$$

Although the formulae of (1.4) are conceptually easy to comprehend, they can be further simplified by using (1.5):

$$\begin{cases} X_{(\hat{N}_R \hat{N}_T \hat{N}_H)} - \alpha\Gamma_1 & \rightarrow\ m = 1 \\ X_{(\hat{N}_R \hat{N}_T \hat{N}_H)} - \alpha\Gamma_1 - \beta\Gamma_2 & \rightarrow\ m = (\alpha+\beta)!/(\alpha!\,\beta!) \\ X_{(\hat{N}_R \hat{N}_T \hat{N}_H)} - \alpha\Gamma_1 - \beta\Gamma_2 - \gamma\Gamma_3 & \rightarrow\ m = (\alpha+\beta+\gamma)!/(\alpha!\,\beta!\,\gamma!) \\ X_{(\hat{N}_R \hat{N}_T \hat{N}_H)} - \alpha\Gamma_1 - \beta\Gamma_2 - \gamma\Gamma_3 - \kappa\Gamma_4 & \rightarrow\ m = (\alpha+\beta+\gamma+\kappa)!/(\alpha!\,\beta!\,\gamma!\,\kappa!) \end{cases} \tag{1.6}$$

As an example, the fact that the expression of (1.3) represents a total of 12 variant mechanisms can also be derived directly from the third formula of (1.4) or (1.6):

$$\begin{aligned} m &= (\alpha+\beta+\gamma)!/(\alpha!\,\beta!\,\gamma!) \\ &= (2+1+1)!/(2!\,1!\,1!) = 4!/2! = 12 \end{aligned}$$

CHAPTER 2. VECTOR ALGEBRAIC METHOD

2.1. Introduction

Vector mathematics was developed in 1881 by Gibbs [30], based on Hamiltion and Grassmann's Quaternion [34,35,31](1843-44), in response to the need for a natural and succinct language for describing the problems of science and engineering.

"It seems inevitable that kinematic analysis should be pursued by vector methods. Almost every quantity involved in kinematics is a vector or magnitude of a vector. Angular quantities are exceptions, but all orders of their derivatives are vectors. Most kinematic problems can be formulated as single or simultaneous vector equations, and these equations can usually be solved through use of vector operations."[8](Chace, Page 1, 1964).

It is generally considered that Chace ([6-9] 1962-65) is the representative in the early stage who adopted the tool of vector mathematics to perform the analysis of spatial mechanisms.

The development of vector method had completely stagnated for almost twenty years since 1964. Although the vector approach proposed by Chace has been recognized as a method, it has not been popularized in practical use and has even been widely considered not to be an effective method as compared to other methods. The reasons are twofold. The first reason is that the way Chace dealt with vectors did not highlight the succinctness of vector expression and operation. In his approach, vectors were expressed in terms of spherical coordinates, which inevitably complicates the vector expression and geometrical visualization. The second reason is that Chace had been able to analyze only some simple mechanisms.

The structural simplicity of the mechanisms analyzed by Chace can be measured by the values of *Big* Λ of these mechanisms: $\Lambda \leq 2$, and generally the bigger the Λ ($0 \leq \Lambda \leq 5$), the more complex the mechanism. The *Big* Λ will be discussed in Chapter 14.

It is necessary to point out that in spite of the limitations the work of M.A. Chace is still very significant in the history of applying vector mathematics to the analysis of spatial linkage mechanisms.

Early in the eighties, Yu ([87-91] 1982-83) broke out the stagnant state in the application of vector mathematics to the analysis of spatial mechanisms. He obtained vectorial solutions for several 4-link spatial mechanisms of $\Lambda = 2$ and for one 4-link spatial mechanism (i.e. R_0-SCR) of $\Lambda=3$, and his approach is different from Chace's [6-9]. The main advancements of the works [87-91] as compared to [6-9](Chace) are the abandoning of the use of spherical coordinates and that the pure vector operation was emphasized. However, Yu advocated the use of pure vector expressions, instead of incorporating any trigonometry algebra and expression in the procedure of analysis; this hindered his approach from analyzing the more complex mechanisms.

In 1986, Zhou [109] presented an approach using vector mathematics to tackle the more complex spatial mechanisms of $\Lambda\leq4$ and the approach is different from [6-9, 87-91]. The crucial factors that rendered the improvement of [109] as compared to [87-91](Yu) are the giving up of pure vector expression and the employment of a proper combination of vector and trigonometry expressions.

In 1987, Lee and Liang [47-49] proposed an approach called vector theory which was developed on the basis of spherical trigonometry method ([28] Duffy, 1980) and dual-number quaternion algebra method ([83] Yang and Freudenstein, 1964). Papers [47,49] analyzed the general forms of RMs (i.e. Regular Mechanisms) whose $\Lambda=5$ such as R_0-$5R$-P and R_0-$6R$ mechanisms. Paper [49] marked a new stage that the input-output displacement equations free of extraneous roots for all the RMs can be derived.

Essentially, there are three kinds of methods for the displacement analysis of SLMs. They are geometric (or graphic) methods, numerical methods and algebraic methods. Formally, however, there have been about ten different methods or theories developed following the well known pioneering work of Dimentberg since 1948.

Of the three essential kinds of methods, algebraic methods (sometimes called *analytical methods*) are the most powerful and majority of the presently available methods belong to this group. The common objective of the various algebraic methods is to establish the relations of the given input motional parameters, structural parameters and the unknown motional variables of linkage kinematic loop, and then to deduce the input-output displacement equations. In this way, one can not only get the result with excellent accuracy, but can also associate the problems of design, optimization and synthesis of spatial linkage mechanisms.

The Vector algebraic method (VAM) presented in this chapter is developed on the basis of [12], [19] and [20]. The core contents and the attributes of this method are as follows:

- Several important vectors related to the loops of spatial linkage mechanisms are introduced; thus the vector loop equation is no longer merely the summation of every individual vector in the loop of a mechanism, and the internal structural relationship of all the vectors in a kinematic loop becomes clear;

- Standardized analysis steps and expressions are developed, and the kinematic analysis for mechanisms becomes an easy routine procedure which can be expressed in a very compact form.

Although it is narrated on the basis of analyzing RM (i.e. Regular Mechanisms) and ORM (i.e. Over-constrained Regular Mechanisms), the main ideas of VAM are also applicable to the analyses of HM (i.e. Helical Mechanisms) and OHM (i.e. Over-constrained Helical Mechanisms).

2.2. The input-output displacement equation

Definition. *Pair Variable.*

For any *definite pair*, its *pair variables* are the variables corresponding to its *pair type* $(N_R N_T N_H)$. There are three kinds of pair variables, i.e. $\{\theta_R, x_T, \theta_H\}$ or $\{\theta_R, x_T, x_H\}$, where θ_H and x_H are linearly correlated.

θ_R and θ_H are *angular variables* (of kinematic pairs); whereas x_T and x_H are *rectilinear variables*. Specifically, θ_R is *rotary variable*, θ_H is *helix-angular variable*, x_T is *translational variable* and x_T is *helix-rectilinear variable*.

Definition. *Degree of Angular Freedom.*

The sum of N_R and N_H is the *degree of angular freedom* of a kinematic pair. It is denoted as $N_A = N_R + N_H$. From the angular freedom point of view, we have four different kinds of pairs:

(1) 0-angular-freedom pair, $N_A = 0$;

(2) 1-angular-freedom pair, $N_A = 1$;

(3) 2-angular-freedom pair, $N_A = 2$;

(4) 3-angular-freedom pair, $N_A = 3$ or 4;

Definition. *Degree of Active Angular Freedom.*

The sum of the active rotary freedoms $\tilde{N}_R$ and the active helical freedoms $\tilde{N}_H$ is the *degree of active angular freedom* of a kinematic pair. It is denoted as $\tilde{N}_A = \tilde{N}_R + \tilde{N}_H$. From the *active* angular-freedom point of view, similarly, we also have four different kinds of pairs.

Definition. *Unconstrained Rotary Freedom.*

If the direction of the axis of a rotary freedom of a kinematic pair can be arbitrary chosen, then the rotary freedom is defined as an *unconstrained rotary freedom*. Of all the definite pairs shown in Table 1.1, the five pairs S, S_G, S_{GH}, S_P

and B_b (whose $N_R = 3$) possess ***unconstrained rotary freedoms***. It is worth noting that ***unconstrained rotary freedom*** always occurs triplely in a kinematic pair.

Definition. *Constrained Axis*.

If the axis of a freedom of a kinematic pair is fixed to either one of the two contacting bodies. then the axis is a ***constrained axis*** of the kinematic pair. Except the ***uncontained rotary axes***, all other axes of kinematic pairs are ***constrained axes***.

The axes of all ***generalized pairs*** are constrained axes. Although the directions of the three axes of the pair (RRR) is not fixed to the two bodies of both ends, each of them is fixed to its corresponding two contacting bodies.

Let unit vectors $\mathbf{q}_i$ $(i=1-n)$ be representing the directions of the ***central lines*** of the links of a kinematic loop. We call $\mathbf{q}_i$ a ***link vector***, a unit vector attached to link i and associated with the link length p_i. Generally $\mathbf{q}_i$ $(1 \le i \le n-1)$ is directed from the first pair along the kinematic chain to the last pair.

Fig. 2.1 represents an abstract model of a kinematic pair. If the pair in Fig. 2.1 is a 0-angular-freedom pair, then, once the direction and orientation of link i is given, the direction and orientation of link $i+1$ is also determined; If the pair in Fig. 2.1 is a j-angular-freedom pair $(j=1,2,3)$, then, once the direction and orientation of link i is given, the direction and orientation of link $i+1$ can also be determined in terms of j angular variables.

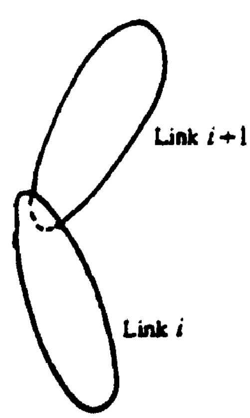


Fig. 2.1

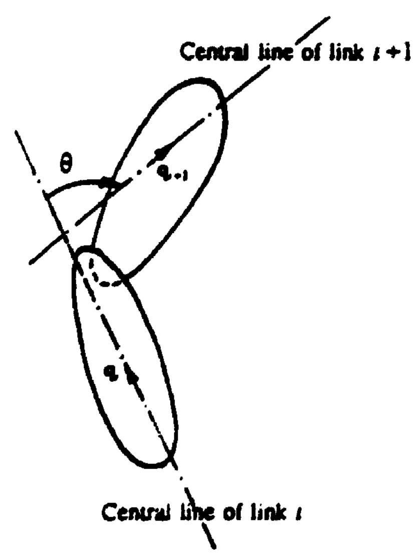


Fig. 2.1(a)

Suppose Fig. 2.1 is the pair of a single-loop mechanism that contains *unconstrained rotary freedoms*. In this case, it is quite awkward to specify the relative direction and orientation of link $i+1$ to link i in terms of three angular variables, for the number of the feasible set of the three angles adopted to represent the relative direction and orientation of the two links is infinite.

As a matter of fact, once the pair variables corresponding to other pairs of the mechanism are determined, the relative direction and orientation of the two links of the pair containing *unconstrained rotary freedoms* are also specified and can be determined very easily. Though, it is useful to introduce only one angle θ describing the relative direction (angle) of the two link vectors q_i and q_{i+1}, as shown in Fig. 2.1(a).

Definition. *Basic Variables* (of Mechanisms)

Given a single-loop spatial linkage mechanism, denoting as $\varpi = (\bar{N}_R \bar{N}_T \bar{N}_H)$ the *3-digit input control number*, and

- assigning to each *constrained* rotary and helical axis one angular variable;
- assigning to each *constrained* translational axis one (scalar) variable;
- assigning to each pair containing a triple of *unconstrained rotary axes* one angular variable,

then, we obtain a set of angular variables $\{\tilde{\theta}_i\}$ and a set of translational variables $\{\tilde{x}_i\}$ along the kinematic chain as follows,

$$\{\tilde{\theta}_i\} = \{\tilde{\theta}_1, \tilde{\theta}_2, \cdots, \tilde{\theta}_\lambda\}, \quad (0 \le \lambda \le 6) \tag{2.1}$$

$$\{\tilde{x}_i\} = \{\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_t\}, \quad (0 \le t \le 6) \tag{2.2}$$

where $(1 \le \lambda + t \le 6)$. The variables $\{\tilde{\theta}_i, \tilde{x}_i\}$ of (2.1) and (2.2) are *basic pair variables* of the mechanism, or *basic variables* for short.

It is worth noting that a *basic pair variable* may not necessarily be a *pair variable* and vice versa. For instance, Spherical pair has three *pair variables* corresponding to its three rotary freedoms, however, it has only one *basic pair variable*, which is the angle specifying the *relative* direction (instead of orientation) of the two contacting pair-elements.

The angular variables of (2.1) can be divided into three kinds:

$$\{\tilde{\theta}_i\} \rightarrow \begin{cases} \tilde{\theta}_{Ri} & - \text{ Rotary variable;} \\ \tilde{\theta}_{Hi} & - \text{ Helix-angular variable;} \\ \tilde{\theta}_{Ui} & - \text{ rotary variable corresponding to} \\ & \quad \text{the triple of Unconstrained rotary freedoms.} \end{cases} \tag{2.3}$$

Definition. *Displacement Equation*.

Any algebraic equation,

$$f(\{\tilde{\theta}_{i_0}\}, \{\tilde{x}_{i_0}\}, \varpi) = 0$$

$$(\text{ where } \{\tilde{\theta}_{i_0}\} \subset \{\tilde{\theta}_i\}, \quad \{\tilde{x}_{i_0}\} \subset \{\tilde{x}_i\}),$$

derived from the mechanism which includes at least two unknown basic variables, is a ***displacement equation*** of the mechanism.

Definition. *I/O Equation*.

The algebraic equations,

$$f(\tilde{\theta}_i, \varpi) = 0 \qquad \tilde{\theta}_i \in \{\tilde{\theta}_i\}$$

$$f(\tilde{x}_i, \varpi) = 0 \qquad \tilde{x}_i \in \{\tilde{x}_i\}.$$

derived from the mechanism and each of them includes only one unknown basic variable, are defined as the ***input-output displacement equations*** of the mechanism, or ***I/O equations*** for short.

An I/O equation is a special case of displacement equation.

Definition. *Small* λ.

For any spatial mechanism of the four basic groups {*RM*, *ORM*, *HM*, *OHM* }, its ***Small*** λ is defined as the total number of the ***angular*** variables of its ***basic pair variables***

Small λ is a very useful parameter. It can facilitate the understanding of kinematic features of spatial mechanisms.

2.3. On the loop of spatial mechanisms

Fig. 2.2 represents the abstract model of an n-pair $(2 \le n \le 7)$ single-loop spatial mechanism.

Without any loss of generality the input pair is designated as the first pair of the loop, the second, third, etc. pairs are labeled in a consecutive manner along the kinematic chain, and the last pair is fixed to the ground.

Definition. *Central Vector Loop Equation*.

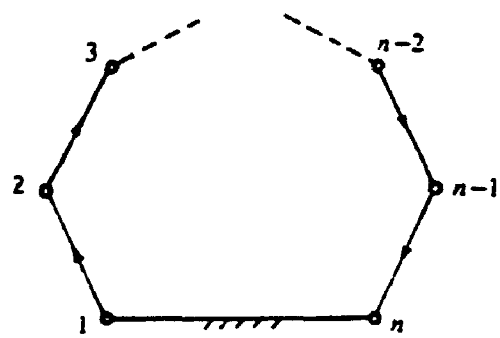


Fig. 2.2

For any single-loop spatial mechanism, the sum of all the vectors corresponding to every edges of its *central vector polygon* is the *central vector loop equation* of the mechanism. It can be called *vector loop equation* or *loop equation*.

Definition. *Input Vector.* (i.e. $\mathbf{I}$)

In the course of deriving the I/O equation the *3-digit input control number* $\mathbf{\bar{N}}=(\bar{N}_R\bar{N}_T\bar{N}_H)$ can always be assumed as known invariable. Then the sum of all those known vectors of the kinematic loop (or the known edges of the vector polygon) is the *input vector* of the kinematic loop, which is denoted by $\mathbf{I}$.

For a given mechanism, suppose all the individual known vectors of the kinematic loop are $\{u_1\mathbf{u}_1, \cdots, u_l\mathbf{u}_l\}$, where $\{\mathbf{u}_i\}$ $(i=1-l)$ are known unit vectors and $\{u_i\}$ $(i=1-l)$ are known scalars, then we have $\mathbf{I}=(u_1\mathbf{u}_1+\cdots+u_l\mathbf{u}_l)$ due to the definition above. However, we will also adopt $\mathbf{I}$ to represent the set $\{u_i\mathbf{u}_i\}$, i.e. $\mathbf{I}=\{u_i\mathbf{u}_i\}$ $(i=1-l)$. The vectors $\mathbf{J}$ and $\mathbf{L}_i$ $(i=1,2)$ to be introduced also have similar properties.

For the convenience of narrating, we introduce the following concept.

Definition. *Adjacent Angular Variable.*

The relative position of the angular variables $\{\tilde{\theta}_i\}$ of (2.1) along the kinematic loop can be described by Fig. 2.3. If $\lambda=1$ as shown in Fig. 2.3(b), then $\tilde{\theta}_1$ has no *adjacent angular variable*. If $\lambda=2$ as shown in Fig. 2.3(a), $\tilde{\theta}_1$ and $\tilde{\theta}_2$ are *adjacent angular variables* of each other. If $\lambda\geq3$, every angular variable of the loop has two *adjacent angular variables*. In Fig. 2.3, $\tilde{\theta}_2$ and $\tilde{\theta}_\lambda$ are adjacent angular variables of $\tilde{\theta}_1$; $\tilde{\theta}_1$ and $\tilde{\theta}_{\lambda-1}$ are adjacent angular variables of $\tilde{\theta}_\lambda$, etc.

Definition. *Output Vector.* (i.e. $\mathbf{J}$)

Let Θ be an angular variable, $\Theta\in\{\tilde{\theta}_i\}$, $(i=1-\lambda,\ 1\leq\lambda\leq6)$, the displacement equation $f(\Theta,\mathbf{\bar{N}})=0$ is to be determined. Then the *output vector* $\mathbf{J}$ is the sum of all those

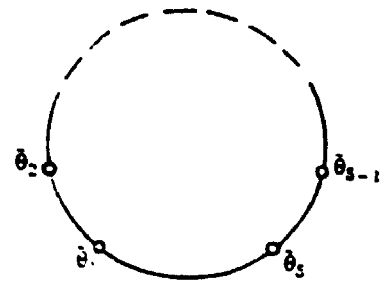


Fig.2.3

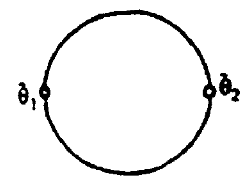


Fig. 2.3(a)

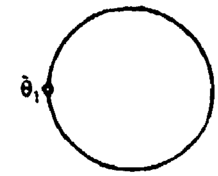


Fig.2.3(b)

unknown constant-magnitude vectors of the kinematic loop between Θ's adjacent angular variables such that (1). $I \cap J = \varnothing$ (i.e. empty set); (2). if the number of this kind of vectors is equal to or greater than two, the relative angle of every two such vectors must be either constant or completely determined by Θ. We call Θ output angle.

In general, it is always feasible and more convenient for the first step to derive at least one of those $f(\tilde{\theta}_\alpha, \varpi)=0$, where $\{\tilde{\theta}_\alpha\} \subset \{\tilde{\theta}_i\}$ $(i=1-\lambda)$ are rotary variables corresponding to *constrained axes*.

After the first I/O equation for any mechanism is obtained, generally, the derivation of the I/O equations corresponding to other pair variables of the mechanism becomes much easier. Therefore, we focus only on the derivation of the first I/O equation.

For certain mechanisms, it is very difficult and even impossible to derive the I/O equations directly; instead, it is feasible to first develop a set of simultaneous equations, namely the ***displacement equations***, which contains the input and output quantities as well as one or more "intermediate" variables. In this case, it is quite clear that the next step should be to eliminate the "intermediate" variables from the displacement equations, in order to obtain the I/O equation. The so-called "intermediate" variables could be angular or translational variables. Here we will pay more attention on angular variables which will be defined as either ***auxiliary angle*** or ***subauxiliary angle***.

Definition. *Auxiliary Angle*.

Given a mechanism, let Θ be an angular variable, i.e. $\Theta \in \{\tilde{\theta}_i\}$ $(i=1-\lambda,\ 1 \le \lambda \le 6)$, the I/O equation $f(\Theta, \varpi)=0$ is to be determined. Suppose it is very difficult or impossible to derive $f(\Theta, \varpi)=0$ directly, and a set of displacement equations is obtained instead:

$$\begin{cases} f_1(\Theta, \varpi, \beta_1, \cdots, \beta_m) = 0 \\ \cdots\cdots \\ f_n(\Theta, \varpi, \beta_1, \cdots, \beta_m) = 0 \end{cases} \tag{2.4}$$

where $\{\beta_k\} \subset \{\bar{\theta}_i\}$, $(k = 1 - m,\ i = 1 - \lambda,\ 1 \le m < \lambda \le 6)$. If $\{\beta_k\}$ $(k = 1 - m)$ can *all* be eliminated from (2.4) *simultaneously* in a single operation and the resultant is the I/O equation $f(\Theta, \varpi) = 0$, then each β_k $(k = 1 - m)$ is defined as an *auxiliary angle*, or *auxiliary* for short.

Proposition 2.1.

In the course of deriving the I/O equations for any RM (i.e. Regular Mechanism) by the algebraic method, we have the following conclusions based on the value of *Small* λ,

(1) if $\lambda < 4$, no auxiliary is needed;

(2) if $\lambda = 4$ and there is no special geometric condition imposed, then, one auxiliary is needed;

(3) if $\lambda > 4$ and there is no special geometric condition imposed, then, two auxiliaries are needed;

If there do exist special geometric condition in the length of the links or the relative direction of the pair axes, the required number of auxiliary in conclusions (2) and (3) could be smaller.

Proposition 2.2.

In the course of deriving the I/O equations for ORM (i.e. Over-constrained Regular Mechanism by algebraic method, according to the value of *Small* λ, we have the following conclusions,

(1) if $\lambda < 5$, no auxiliary is needed;

(2) if $\lambda = 5$ and there is no special geometric condition imposed, then, one auxiliary is needed; otherwise, no auxiliary is needed.

Propositions 2.1 and 2.2 are summarized from the results of many published research papers and author' extensive review of the unanalyzed mechanisms as well as those analyzed mechanisms.

Definition. *Subauxiliary Angle*.

If more angular variables are involved in the course of deriving a set of (simultaneous) displacement equations which contains only the required number of auxiliaries, then each of those extra angular variables involved is defined as an *ubauxiliary angle*, or *subauxiliary* for short.

For any mechanism which needs *auxiliary* in the course of deriving its I/O equation, suppose an output angle has been designated, now one needs to know how to choose an auxiliary $\psi_1 \in \langle \bar{\theta}_i \rangle$. Here is the principle: ψ_1 can be any one of the remaining angular variables except the angular variable corresponding to the triple of *unconstrained rotary axes*. It is always feasible and, in most cases, more convenient for derivation to choose ψ_1 from Θ's two adjacent angular variables.

Definition. *Auxiliary Vector.* (i.e. L_1)

Let ψ_1 be an auxiliary angle, $\psi_1 \in \langle \bar{\theta}_i \rangle$, $(i = 1 - \lambda,\ 4 \le \lambda \le 6)$. The *auxiliary vector* is the sum of all those unknown constant-magnitude vectors of the kinematic loop between ψ_1's adjacent angular variables such that (1). $L_1 \cap (I \cup J) = \varnothing$; (2). if the number of this kind vector is equal to or greater than two, the relative angle of every two such vectors must be either constant or completely determined by ψ_1.

If a second *auxiliary* is needed, it can be chosen from any one of the remaining angular variables. However, it is generally more convenient for derivation to choose ψ_2 from Θ or ψ_1's adjacent angular variables.

The second *Auxiliary Vector* is defined similarly as follow.

Definition. *Auxiliary Vector.* (i.e. L_2)

Let ψ_2 be the second auxiliary angle, $\psi_2 \in \langle \bar{\theta}_i \rangle$, $(i = 1 - \lambda,\ 5 \le \lambda \le 6)$. The second *auxiliary vector* is the sum of all those unknown constant-magnitude vectors of the kinematic loop between ψ_2's adjacent angular variables such that, (1). $L_2 \cap (I \cup J \cup L_1) = \varnothing$; (2). if the number of this kind vector is equal to or greater than two, the relative angle of every two such vectors must be either constant or completely determined by ψ_2.

Definition. *Floating Vector* and *Ground Vector*.

For any given spatial linkage mechanism, after the output angle Θ and (if $\lambda = k \ge 4$) all the required auxiliary angles $\{\psi_i\}$ are chosen, we can imagine that Θ and ψ_i are all "marked" with a paint brush respectively, then three possible situations are as follows:

(1) If $\lambda \geq 3$, and there is no special geometric conditions imposed on the structure of the mechanism, then, generally there *should* be *two* unmarked angular variables left. Suppose θ_μ and θ_ν are the two unmarked angular variables as shown in Fig. 2.4(a), dissembling the loop at the two unmarked pairs (at a and b), we get two separate parts (chains) of the loop as shown in Fig. 2.4(b), where $b'\eta a'$ is the part connecting with the ground and $a\xi b$ is the floating part. Points a and a' (points b and b') are coincident when the two parts of the loop are reassembled together. Now the vector loop equation can be considered as being composed of two parts,

$$\mathbf{R}_g + \mathbf{R}_f = 0 \tag{2.5}$$

$$\begin{cases} \mathbf{R}_g = \mathbf{R}_{b'\eta a'} \\ \mathbf{R}_f = \mathbf{R}_{a\xi b} \end{cases} \tag{2.5a}$$

where $\mathbf{R}_f$ excludes (whereas $\mathbf{R}_g$ includes) the vector components corresponding to translational variables at both ends a and b. The subscripts g and f of (2.5) refer to "ground" and "floating", respectively. We define the *floating vector* $\mathbf{F} = -\mathbf{R}_f$ and call $\mathbf{R}_g$ ($=\mathbf{R}_{b'\eta a'}$) *ground vector*, and then rewrite the loop equation as

$$\mathbf{R}_g = \mathbf{F} \tag{2.6}$$

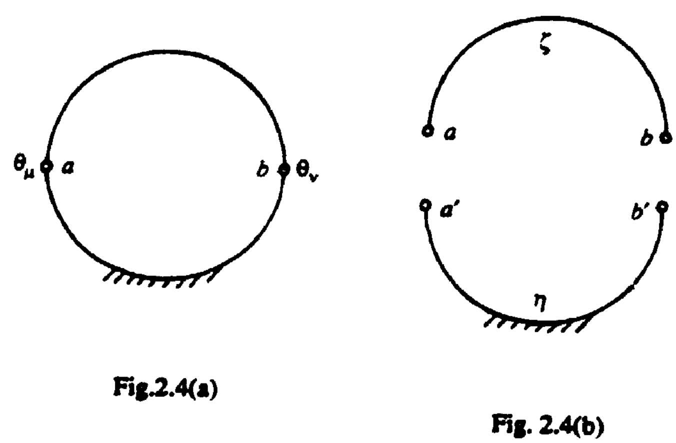


Fig.2.4(a)

Fig. 2.4(b)

(2) If $\lambda \geq 3$, and there is special geometric condition imposed, then, there *should* be *at least* two unmarked angular variables left. we define the *floating vector* $\mathbf{F} = -\mathbf{R}_f$ on the condition that except θ_μ and θ_ν all other unmarked angular variables, if any, are on the floating chain. In this case the loop equation can be written in

the same form as (2.6).

(3) If $\lambda \le 2$, then, there *will* be *none* (for $\lambda \le 1$) or only *one* (for $\lambda = 2$) unmarked angular variable θ_μ left, in this case, there is no *floating vector* exists. However, it is useful to assume that $\mathbf{F} = 0$.

For a given mechanism, the couple of *floating vector* and *ground vector* may not be unique. It is determined by how one chooses the (first) output angle and its related auxiliary angles from $\{\bar{\theta}_i\}$ of (2.1).

Definition. *End-axis Vector.*

Suppose the unmarked angular variable θ_μ is not a basic pair variable corresponding to a triple of *unconstrained rotary axes*; then, the (angular) axis vector $\mathbf{a}_\mu$ corresponding to θ_μ is an *end-axis vector* (of the ground vector or floating vector). The same applies to θ_ν, (there exists a second unmarked angular variable, if $\lambda \ge 3$ and there is no *unconstrained rotary axis* contained in the mechanism).

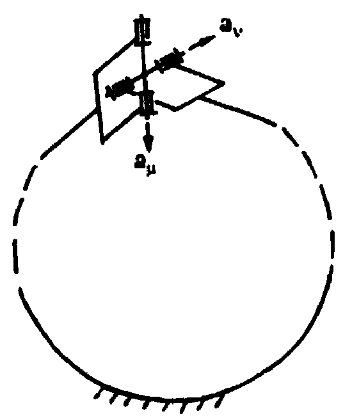


Fig. 2.5

Let $\mathbf{F} = F\mathbf{f}$, where $\mathbf{f}$ is a unit vector. It is necessary to point out that sometimes $\mathbf{f}$ exists and there are two end-axis vectors associated with $\mathbf{f}$, but $F = 0$, thus $\mathbf{F} = 0$. This can be illustrated by Fig. 2.5, where $\mathbf{f} = (\mathbf{a}_\mu \times \mathbf{a}_\nu)/|\mathbf{a}_\mu \times \mathbf{a}_\nu|$, $F = 0$ and $\{\mathbf{a}_\mu, \mathbf{a}_\nu\}$ are the two end-axis vectors of $\mathbf{f}$.

2.4. Four basic operations

For virtually all practical kinematic problems of spatial mechanisms, the displacement analysis can be accomplished by just one or two of the four basic operations as follows.

For any mechanism, the vector loop equation (2.6) can be written as

$$\sum x_i \mathbf{a}_{ii} + \{ \cdots \} = \mathbf{F} \tag{2.7}$$

Three cases:

Case (I). There exist two end-axis vectors:

(a) if $\sum \tilde{N}_T \neq 0$, use operations (1) and (4);

(b) if $\sum \tilde{N}_T = 0$, use (1), (2) and (3).

Case (II). There exists only one end-axis vector:

(a) if $\mathbf{F} \neq 0$, use (2) and (3);

(b) if $\mathbf{F} = 0$, trivial, [use (4)].

Case (III). There exists no end-axis vector:

(a) if $\sum \tilde{N}_T \neq 0$, use (4);

(b) if $\sum \tilde{N}_T = 0$, trivial.

Four basic operations:

(1) $(\mathbf{a}_\mu \cdot \mathbf{a}_\nu) \equiv (\mathbf{a}_\mu \cdot \mathbf{a}_\nu)$, where $\mathbf{a}_\mu$ and $\mathbf{a}_\nu$ are the two *end-axis vectors* and the two sides of the identity are calculated from the floating and the ground part of the loop, respectively.

(2) Square both sides of (2.7);

(3) $\mathbf{a}_\mu \cdot (2.7)$, where $\mathbf{a}_\mu$ is an *end-axis vector*.

(4) In order to eliminate one or two or more elements of $\{x_i \mathbf{a}_i\}$ from (2.7), use an appropriate vector to dot product both sides of (2.7);

2.5. Fundamentals of vector mathematics

(1). Let $\mathbf{a}_1, \mathbf{a}_2$ and $\mathbf{q}$ be three unit vectors, and $\mathbf{q} \cdot \mathbf{a}_1 = 0$, $\mathbf{q} \cdot \mathbf{a}_2 = 0$ and $\alpha_{21} = (2\pi - \alpha_{12})$ as shown in Fig. 2.6, then we have

$$\begin{cases} \mathbf{a}_1 \times \mathbf{a}_2 = \mathbf{q} \sin\alpha_{12} \\ \mathbf{q} = \mathbf{a}_1 \times \mathbf{a}_2 \csc\alpha_{12} \quad (\text{if } \mathbf{a}_1 \cdot \mathbf{a}_2 \neq \pm 1) \end{cases} \tag{2.8}$$

$$\begin{cases} \mathbf{a}_2 = \cos\alpha_{12} \mathbf{a}_1 + \sin\alpha_{12} \mathbf{q} \times \mathbf{a}_1 \\ \mathbf{a}_1 = \cos\alpha_{12} \mathbf{a}_2 - \sin\alpha_{12} \mathbf{q} \times \mathbf{a}_2 \end{cases} \tag{2.9}$$

(2). Let $\mathbf{e}_1, \mathbf{e}_2$ and $\mathbf{e}_3$ be a positive triple of mutually perpendicular unit vectors; $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ are four arbitrary vectors, then

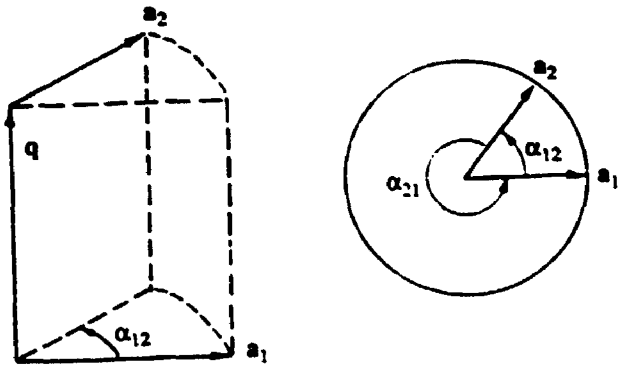


Fig. 2.6

$$\begin{cases} \mathbf{a} = \sum(\mathbf{a}\cdot\mathbf{e}_i)\mathbf{e}_i \\ \mathbf{a}^2 = \sum(\mathbf{a}\cdot\mathbf{e}_i)^2 \\ \mathbf{a}\cdot\mathbf{b} = \sum(\mathbf{a}\cdot\mathbf{e}_i)(\mathbf{b}\cdot\mathbf{e}_i) \\ \mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos(\hat{\mathbf{ab}}) \end{cases} \tag{2.10}$$

$$\begin{cases} \mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\,\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\,\mathbf{c} \\ (\mathbf{a}\times\mathbf{b})\times\mathbf{c} = (\mathbf{a}\cdot\mathbf{c})\,\mathbf{b} - (\mathbf{b}\cdot\mathbf{c})\,\mathbf{a} \end{cases} \tag{2.11}$$

$$\begin{cases} (\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c}) \\ (\mathbf{a}\cdot\mathbf{b})(\mathbf{c}\cdot\mathbf{d}) = (\mathbf{a}\times\mathbf{c})\cdot(\mathbf{b}\times\mathbf{d}) + (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c}) \\ (\mathbf{a}\cdot\mathbf{b})(\mathbf{c}\cdot\mathbf{d}) = (\mathbf{a}\times\mathbf{d})\cdot(\mathbf{b}\times\mathbf{c}) + (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) \end{cases} \tag{2.12}$$

$$(\mathbf{a}\cdot\mathbf{e}_1)^2 + (\mathbf{a}\cdot\mathbf{e}_2)^2 = (\mathbf{a}\times\mathbf{e}_3)^2 \tag{2.13}$$

(3). The quantity of $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ changes sign if the cyclic order of their vectors is changed. Otherwise, its value is unaffected by interchange of vectors or by the exchange of dot and cross product.

(4). Any three vectors of dimension two in two-dimension space are linear correlation; Any four vectors of dimension three in three-dimension space are linear correlation.

The four items above are important mathematical prerequisites.

The formula in Eq.(2.13) might be a new one, for the author has never seen such an identity elsewhere before. Following is the proof.

Using the first equation of Eq.(2.12), the right-hand side of Eq.(2.13) can be transformed as follow,

$$(\mathbf{a}\times\mathbf{e}_3)^2 = (\mathbf{a}\times\mathbf{e}_3)\cdot(\mathbf{a}\times\mathbf{e}_3) = \mathbf{a}^2 - (\mathbf{a}\cdot\mathbf{e}_3)^2 \tag{2.14}$$

Considering Eq.(2.14) and the second equation of Eq.(2.10), the validity of Eq.(2.13) is clear.

2.6. Four vector equations and their solutions

From sections 2.6.1 to 2.6.4, four typical vector equations are introduced and analyzed in detail. These equations are evolved from mechanisms, and they are very important because the vast majority of the *practically* existing kinematic problems of spatial linkage mechanisms can be transformed into the solution of one of the four equations.

2.6.1. The first vector equation

$$\sum_{i=1}^{k} x_i \mathbf{a}_i = \mathbf{M} \tag{2.15}$$

Given: $\mathbf{M}$ and unit vector set $\{\mathbf{a}_i\}$, $(i=1\text{-}k,\ 1\le k\le 6)$, where $\mathbf{a}_{i_0}\cdot\mathbf{a}_{j_0}\ne\pm 1$, $(i_0\ne j_0)$, $\mathbf{a}_{i_0}\in\{\mathbf{a}_i\}$ and $\mathbf{a}_{j_0}\in\{\mathbf{a}_i\}$; Unknown: $\{x_i\}$, $(i=1\text{-}k,\ 1\le k\le 6)$.

Solution:

(1). If $k=1$, from the scalar product of $\mathbf{a}_1$ with both sides of (2.15) we get

$$x_1 = (\mathbf{a}_1\cdot\mathbf{M}) \tag{2.16}$$

(2). If $k\ge 2$ and any three-element subset of $\{\mathbf{a}_i\}$ $(i=1\text{-}k)$ is linear correlation, from the scalar product of $\mathbf{a}_j$ $(j=1,2)$ with both sides of (6.1) we obtain

$$\sum_{i=1}^{2} x_i (\mathbf{a}_j\cdot\mathbf{a}_i) = (\mathbf{a}_j\cdot\mathbf{N}) \qquad (j=1,2) \tag{2.17}$$

$$\mathbf{N} = \begin{cases} \mathbf{M} & (\text{if } k=2); \\ \mathbf{M}-(x_3\mathbf{a}_3+\cdots+x_k\mathbf{a}_k) & (\text{if } k\ge 3). \end{cases} \tag{2.17a}$$

Solving (2.17) yields

$$\begin{cases} x_1 = [(\mathbf{a}_1\cdot\mathbf{N}) - (\mathbf{a}_1\cdot\mathbf{a}_2)(\mathbf{a}_2\cdot\mathbf{N})]\,/\,[1-(\mathbf{a}_1\cdot\mathbf{a}_2)^2] \\ x_2 = [(\mathbf{a}_2\cdot\mathbf{N}) - (\mathbf{a}_1\cdot\mathbf{a}_2)(\mathbf{a}_1\cdot\mathbf{N})]\,/\,[1-(\mathbf{a}_1\cdot\mathbf{a}_2)^2] \end{cases} \tag{2.18}$$

(3). If $k\ge 3$ and $\{\mathbf{a}_i\}$ $(i=1\text{-}k)$ has at least one three-element subset which is not linear correlation. Without loss of generality, we suppose that the three-element subset is $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$, then we have $(\mathbf{a}_1\times\mathbf{a}_2\cdot\mathbf{a}_3)\ne 0$.

The scalar product of $\mathbf{a}_j$ $(j=1\text{-}3)$ with both sides of (2.15) yields

$$\sum_{i=1}^{3} x_i (\mathbf{a}_j\cdot\mathbf{a}_i) = (\mathbf{a}_j\cdot\mathbf{W}) \qquad (j=1,2,3) \tag{2.19}$$

$$\mathbf{W} = \begin{cases} \mathbf{M} & (\text{if } k = 3); \\ \mathbf{M} - (x_4\mathbf{a}_4 + \cdots + x_k\mathbf{a}_k) & (\text{if } k \geq 4). \end{cases} \tag{2.19a}$$

Solving (2.19) yields

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & (\mathbf{a}_1 \cdot \mathbf{a}_2) & (\mathbf{a}_1 \cdot \mathbf{a}_3) \\ (\mathbf{a}_1 \cdot \mathbf{a}_2) & 1 & (\mathbf{a}_2 \cdot \mathbf{a}_3) \\ (\mathbf{a}_1 \cdot \mathbf{a}_3) & (\mathbf{a}_2 \cdot \mathbf{a}_3) & 1 \end{bmatrix}^{-1} \begin{bmatrix} (\mathbf{a}_1 \cdot \mathbf{W}) \\ (\mathbf{a}_2 \cdot \mathbf{W}) \\ (\mathbf{a}_3 \cdot \mathbf{W}) \end{bmatrix} \tag{2.20}$$

The physical meaning relating to a mechanism for $k \geq 3$ in (2.17a) is that the mechanism has at least $(k-2)$ *T-IDoF-2*, (Translational Idle Degree of Freedom of the second kind), whereas for $k \geq 4$ in (2.19a) is that the mechanism has at least $(k-3)$ *T-IDoF-2*.

2.6.2. The second vector equation

$$\begin{cases} \mathbf{U} \cdot \mathbf{q} = V & (1) \\ \mathbf{q} = \mathbf{e}_1 \cos\Theta + \mathbf{e}_2 \sin\Theta & (2) \end{cases} \tag{2.21}$$

Given: $\{\mathbf{U}, V, \mathbf{e}_1, \mathbf{e}_2\}$; Unknown: $\{\Theta, \mathbf{q}\}$.

Solution:

Substituting (2.21-2) into (2.21-1) yields

$$A \cos\Theta + B \sin\Theta = C \tag{2.22}$$

$$\begin{cases} A = \mathbf{U} \cdot \mathbf{e}_1 \\ B = \mathbf{U} \cdot \mathbf{e}_2 \\ C = V \end{cases} \tag{2.22a}$$

Let $y = \tan(\Theta / 2)$, then we have

$$\begin{cases} \cos\Theta = (1 - y^2) / (1 + y^2) \\ \sin\Theta = 2y / (1 + y^2) \end{cases} \tag{2.23}$$

Substituting (2.23) into (2.22) yields

$$(A + C)y^2 - 2B\,y + (C - A) = 0 \tag{2.24}$$

Solving (2.24) yields y, then $\Theta = 2\tan^{-1}y$, i.e.

$$\Theta = 2\tan^{-1}[(B \pm \sqrt{A^2 + B^2 - C^2}) / (A + C)] \tag{2.25}$$

Substituting (2.25) into (2.21-2) yields $\mathbf{q}$.

2.6.3. The third vector equation

$$\begin{cases} (\mathbf{U}\cdot\mathbf{q})^2 + (\mathbf{U}_1\cdot\mathbf{q})(\mathbf{U}_2\cdot\mathbf{q}) + (\mathbf{W}\cdot\mathbf{q}) + V = 0 & (1) \\ \mathbf{q} = \mathbf{e}_1\cos\Theta + \mathbf{e}_2\sin\Theta & (2) \end{cases} \tag{2.26}$$

Given: $\{\mathbf{U}, \mathbf{U}_1, \mathbf{U}_2, \mathbf{W}, V, \mathbf{e}_1, \mathbf{e}_2\}$; Unknown: $\{\Theta, \mathbf{q}\}$.

Solution:

Substituting (2.26-2) into (2.26-1) yields

$$\mu_1 c^2\Theta + \mu_2 s^2\Theta + \mu_3 c\Theta s\Theta + \mu_4 c\Theta + \mu_5 s\Theta + \mu_6 = 0 \tag{2.27}$$

$$\begin{cases} \mu_1 = (\mathbf{U}\cdot\mathbf{e}_1)^2 + (\mathbf{U}_1\cdot\mathbf{e}_1)(\mathbf{U}_2\cdot\mathbf{e}_1) \\ \mu_2 = (\mathbf{U}\cdot\mathbf{e}_2)^2 + (\mathbf{U}_1\cdot\mathbf{e}_2)(\mathbf{U}_2\cdot\mathbf{e}_2) \\ \mu_3 = 2(\mathbf{U}\cdot\mathbf{e}_1)(\mathbf{U}\cdot\mathbf{e}_2) + (\mathbf{U}_1\cdot\mathbf{e}_1)(\mathbf{U}_2\cdot\mathbf{e}_2) \\ \qquad + (\mathbf{U}_2\cdot\mathbf{e}_1)(\mathbf{U}_1\cdot\mathbf{e}_2) \\ \mu_4 = (\mathbf{W}\cdot\mathbf{e}_1) \\ \mu_5 = (\mathbf{W}\cdot\mathbf{e}_2) \\ \mu_6 = V \end{cases} \tag{2.27a}$$

Let $y = \tan(\Theta/2)$, then we have $\cos\Theta = (1-y^2)/(1+y^2)$, $\sin\Theta = 2y/(1+y^2)$, from (2.27) we get

$$\sum_{i=0}^{4} v_i\, y^{4-i} = 0 \tag{2.28}$$

$$\begin{cases} v_0 = \mu_1 \\ v_1 = -2\mu_3 \\ v_i = -2\mu_1 + 4\mu_2 - \mu_4 + \mu_6 \\ v_i = 2\mu_3 + 2\mu_5 \\ v_i = \mu_1 + \mu_4 + \mu_6 \end{cases} \tag{2.28a}$$

Solving (2.28) yields y, then $\Theta = 2\tan^{-1}y$; From (2.26-2) we get $\mathbf{q}$.

2.6.4. The fourth vector equation

$$\begin{cases} \mathbf{U}\cdot\mathbf{q} = V & (1) \\ \mathbf{U}'\cdot\mathbf{q} = V' & (2) \\ \mathbf{q} = \cos\Theta\,\mathbf{e}_1 + \sin\Theta\,\mathbf{e}_2 & (3) \end{cases} \tag{2.29}$$

Where $\mathbf{U}$, $\mathbf{U}'$, V and V' are all linear in $\cos\Theta$ and $\sin\Theta$. Substituting (2.29-3) into (2.29-1) and (2.29-2) yields

$$\begin{cases} A(\Theta)\cos\psi + B(\Theta)\sin\psi = C(\Theta) \\ A'(\Theta)\cos\psi + B'(\Theta)\sin\psi = C'(\Theta) \end{cases} \tag{2.30}$$

$$\begin{cases} A(\Theta) = \mathbf{U}\cdot\mathbf{e}_1 \\ B(\Theta) = \mathbf{U}\cdot\mathbf{e}_2 \\ C(\Theta) = V \end{cases} \tag{2.30a}$$

$$\begin{cases} A'(\Theta) = \mathbf{U}' \cdot \mathbf{e}_1 \\ B'(\Theta) = \mathbf{U}' \cdot \mathbf{e}_2 \\ C'(\Theta) = V' \end{cases} \tag{2.30b}$$

where the expressions for A, B, C, A', B' and C' all have the same form as follow,

$$T(\Theta) = t_1 \cos\Theta + t_2 \sin\Theta + t_3 \tag{2.31}$$

Given: $\{a_i, b_i, c_i, a_i', b_i', c_i'\}$, $(i = 1, 2, 3)$; Unknown: $\{\psi, \Theta\}$. The major requirement is to eliminate ψ from (2.30) and solve Θ.

Solution:

From (2.30) we get

$$\begin{cases} \cos\psi = -Q_2 / Q_3 \\ \sin\psi = Q_1 / Q_3 \end{cases} \tag{2.32}$$

$$\begin{cases} Q_1 = (A\,C' - C\,A') \\ Q_2 = (B\,C' - C\,B') \\ Q_3 = (A\,B' - B\,A') \end{cases} \tag{2.32a}$$

From $\cos^2\Theta + \sin^2\Theta = 1$ and (2.32) we get

$$Q_1^2 + Q_2^2 = Q_3^2 \tag{2.33}$$

Let $y = \tan(\Theta/2)$, then we have

$$\begin{cases} \cos\Theta = (1-y^2)/(1+y^2) \\ \sin\Theta = 2y/(1+y^2) \end{cases} \tag{2.34}$$

Substituting (2.34) into (2.31), then substituting (2.31) into (2.32a), and then substituting (2.32a) into (2.33) we get

$$\sum_{i=0}^{8} \delta_i \, y^{8-i} = 0 \tag{2.35}$$

$$\begin{cases} \delta_0 = -k_9 \\ \delta_1 = 2k_4 - 2k_8 \\ \delta_2 = 4k_3 - 4k_7 - 2k_9 \\ \delta_3 = 8k_2 + 6k_4 - 8k_6 - 2k_8 \\ \delta_4 = 16k_1 + 8k_3 \end{cases} \tag{2.35a}$$

$$\begin{cases} \delta_5 = 8k_2 + 6k_4 + 8k_6 + 2k_8 \\ \delta_6 = 4k_3 + 4k_7 + 2k_9 \\ \delta_7 = 2k_4 + 2k_8 \\ \delta_8 = k_5 + k_9 \end{cases} \tag{2.35a}$$

$$\begin{cases} k_1 = (\xi_1^2 - \xi_2^2) + (\eta_1^2 - \eta_2^2) - (\zeta_1^2 - \zeta_2^2) \\ k_2 = 2(\xi_1\xi_4 - \xi_2\xi_3) + 2(\eta_1\eta_4 - \eta_2\eta_3) - 2(\zeta_1\zeta_4 - \zeta_2\zeta_3) \\ k_3 = (\xi_2^2 - \xi_3^2 + \xi_4^2 + 2\xi_1\xi_5) \\ \quad + (\eta_2^2 - \eta_3^2 + \eta_4^2 + 2\eta_1\eta_5) - (\zeta_2^2 - \zeta_3^2 + \zeta_4^2 + 2\zeta_1\zeta_5) \\ k_4 = 2(\xi_2\xi_3 + \xi_4\xi_5) + 2(\eta_2\eta_3 + \eta_4\eta_5) - 2(\zeta_2\zeta_3 + \zeta_4\zeta_5) \end{cases} \tag{2.35b}$$

$$\begin{cases} k_5 = (\xi_2^2 + \xi_3^2 + \xi_5^2) + (\eta_2^2 + \eta_3^2 + \eta_5^2) - (\zeta_2^2 + \zeta_3^2 + \zeta_5^2) \\ k_6 = 2(\xi_1\xi_2 + \eta_1\eta_2 - \zeta_1\zeta_2) \\ k_7 = 2(\xi_1\xi_3 + \xi_2\xi_4) + 2(\eta_1\eta_3 + \eta_2\eta_4) - 2(\zeta_1\zeta_3 + \zeta_2\zeta_4) \\ k_8 = 2(\xi_2\xi_5 + \xi_3\xi_4) + 2(\eta_2\eta_5 + \eta_3\eta_4) - 2(\zeta_2\zeta_5 + \zeta_3\zeta_4) \\ k_9 = 2(\xi_3\xi_5 + \eta_3\eta_5 + \zeta_3\zeta_5) \end{cases} \tag{2.35b}$$

$$\begin{cases} \xi_1 = (a_2c_2' - c_2a_2') - (a_1c_1' - c_1a_1') \\ \xi_2 = (a_1c_2' + a_2c_1') - (c_1a_2' + c_2a_1') \\ \xi_3 = (a_1c_3' + a_3c_1') - (c_1a_3' + c_3a_1') \\ \xi_4 = (a_2c_3' + a_3c_2') - (c_1a_3' + c_3a_1') \\ \xi_5 = (a_3c_3' - c_3a_3') + (a_1c_1' - c_1a_1') \end{cases} \tag{2.35c}$$

$$\begin{cases} \eta_1 = (b_2c_2' - c_2b_2') - (b_1c_1' - c_1b_1') \\ \eta_2 = (b_1c_2' + b_2c_1') - (c_1b_2' + c_2b_1') \\ \eta_3 = (b_1c_3' + b_3c_1') - (c_1b_3' + c_3b_1') \\ \eta_4 = (b_2c_3' + b_3c_2') - (c_1b_3' + c_3b_1') \\ \eta_5 = (b_3c_3' - c_3b_3') + (b_1c_1' - c_1b_1') \end{cases} \tag{2.35d}$$

$$\begin{cases} \zeta_1 = (a_2b_2' - b_2a_2') - (a_1b_1' - b_1a_1') \\ \zeta_2 = (a_1b_2' + a_2b_1') - (b_1a_2' + b_2a_1') \\ \zeta_3 = (a_1b_3' + a_3b_1') - (b_1a_3' + b_3a_1') \\ \zeta_4 = (a_2b_3' + a_3b_2') - (b_1a_3' + b_3a_1') \\ \zeta_5 = (a_3b_3' - b_3a_3') + (a_1b_1' - b_1a_1') \end{cases} \tag{2.35e}$$

Solving (2.35) yields y, then $\theta = 2\tan^{-1}y$. ψ can be determined from (2.32).

Special Case :

For some mechanisms ($\lambda = 5$, $\Lambda = 4$, the definition for *Big* Λ will be introduced in Chapter 14), the expressions for $\{Q_i\}$, $(i = 1,2,3)$, can be simplified and transformed into expressions linear in $\cos\theta$ and $\sin\theta$ as follow,

$$Q_i = X_i\cos\theta + Y_i\sin\theta + Z_i \qquad (i = 1,2,3) \tag{2.36}$$

Substituting (2.36) into (2.33) yields

$$\mu_1 c^2\theta + \mu_2 s^2\theta + \mu_3 c\theta\, s\theta + \mu_4 c\theta + \mu_5 s\theta + \mu_6 = 0 \tag{2.37}$$

$$\begin{cases} \mu_1 = X_1^2 + X_2^2 - X_3^2 \\ \mu_2 = Y_1^2 + Y_2^2 - Y_3^2 \\ \mu_3 = 2X_1Y_1 + 2X_2Y_2 - 2X_3Y_3 \\ \mu_4 = 2X_1Z_1 + 2X_2Z_2 - 2X_3Z_3 \\ \mu_5 = 2Y_1Z_1 + 2Y_2Z_2 - 2Y_3Z_3 \\ \mu_6 = Z_1^2 + Z_2^2 - Z_3^2 \end{cases} \tag{2.37a}$$

The solution of (2.37) is similar to that of (2.27). From (2.37) we get θ, then from (2.32) we get ψ.

From the results of the above derivations, we can see that the four equations are all turned into polynomials afterwards. The orders of these polynomials are one, two, four and eight, respectively.

The motions of the *general* forms of those mechanisms whose $\Lambda=5$ are governed by 16th order polynomials. Although the *vector algebraic method* is applicable to the analysis of the mechanisms of $\Lambda=5$ with general geometries using the eliminating technique developed by Lee and Liang ([47,49] 1987-1988), we do not discuss the derivation of these 16th order polynomials here for the following reasons, (1) it is a tedious task; (2) the mechanisms governed by 16th order polynomials can hardly have any practical usefulness; and (3) the number of these mechanisms is small. Of the 191 distinct *core-loops* (to be introduced in Chapt.14) of all RMs and ORMs there are only 3 distinct *core-loops* whose $\Lambda=5$ (i.e. *4R-C*, *5R-P* and *6R*). Interested readers are referred to references [47-49](Lee and Liang, 1987-88), [59,60](Raghavan and Roth, 1990) and [42,43](Kohli and Osvatic, 1992).

It is worth noting that the orders of the polynomials of some of the RMs whose $\Lambda=4$ and $\Lambda=5$ may reduce from 8 to 6 and from 16 to 12, respectively, due to certain special geometric conditions of the mechanisms ([120.121] Mavroidis and Roth, 1992).

2.7. Conclusion

The *vector algebraic method* is easy to learn and easy to use. Its analysis procedure is direct, succinct and standardized. In comparison with any other analytical method, the proposed method has shown advantages on its efficiency, uniformity and simplicity. Chapters 4 to 13 will be devoted to the demonstration of these advantages.

CHAPTER 3. VECTOR TETRAHEDRON EQUATIONS

3.1. Introduction

A large number of displacement analysis problems in spatial mechanisms and serial robots can be formulated by the following *vector tetrahedron equation* (i.e. VTE):

$$\mathbf{r} + \mathbf{s} + \mathbf{t} + \mathbf{C} = 0 \tag{3.1}$$

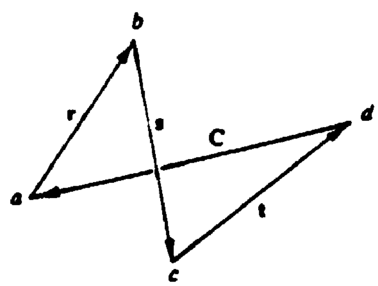


Fig. 3.1(a)

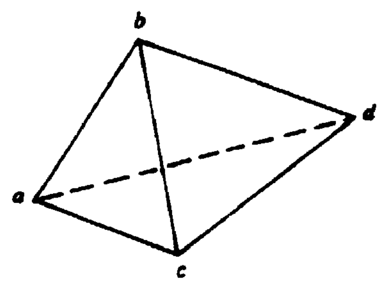


Fig. 3.1(b)

where $\mathbf{r}, \mathbf{s}$ and $\mathbf{t}$ are unknown vectors, and $\mathbf{C}$ is a known vector. Let us draw an arbitrary spatial closed-polyline of four edges (line-segments) *a-b-c-d-a* as shown in Fig. 3.1(a) and connect ac and bd, then a tetrahedron is obtained as shown in Fig. 3.1(b), this is how equation (3.1) gained its name. M.A. Chace([8,9] 1964) was the first who systematically analyzed the vector tetrahedron equation. In his approach, vectors $\mathbf{r}, \mathbf{s}$ and $\mathbf{t}$ are expressed in spherical coordinates, measured from known right-hand Cartesian reference frames $\hat{\lambda}_r, \hat{\mu}_r, \hat{\nu}_r$; $\hat{\lambda}_s, \hat{\mu}_s, \hat{\nu}_s$; and $\hat{\lambda}_t, \hat{\mu}_t, \hat{\nu}_t$. For instance,

$$\mathbf{r} = r\,\{\sin\phi_r\,[\cos\theta_r \hat{\lambda}_r + \sin\theta_r \hat{\mu}_r] + (\cos\phi_r)\hat{\nu}_r\} \tag{3.2}$$

In a given problem any three of the nine quantities r, θ_r, ϕ_r; s, θ_s, ϕ_s; and t, θ_t, ϕ_t may be unknown. Thus nine distinct problems may exist.

However, instead of decomposing each vector into three components, we found that the vector tetrahedron equation problem can be formulated as a set of simultaneous equations, which may include in addition to the vector tetrahedron equation certain constraint equations. The advantages of the new formulation are twofold: (1) it better represents actual problems in the kinematic analysis of mechanisms and serial robots, *directly* expressed by a vector loop equation and a set of structural constraint equations; (2) the new formulation can make the analysis intuitively more direct and computationally easier.

In the following a new systematic analysis procedure for solving vector tetrahedron equations is presented.

3.2. The conventional and the new formulations

Any four vectors of dimension three can be expressed in terms of three non-coplanar vectors. Let us assume that there are three arbitrarily given non-coplanar unit vectors $\{e_1, e_2, e_3\}$, then any vector $Q = pq$ (where $p = |Q|$ and $q = Q/|Q|$) can be written as,

$$Q = xe_1 + ye_2 + ze_3 \tag{3.3}$$

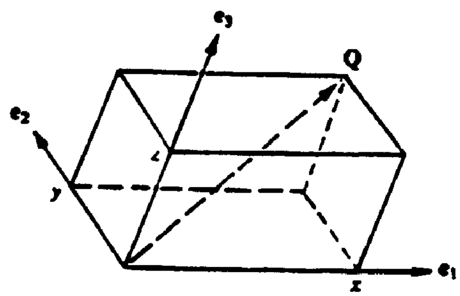


Fig. 3.2

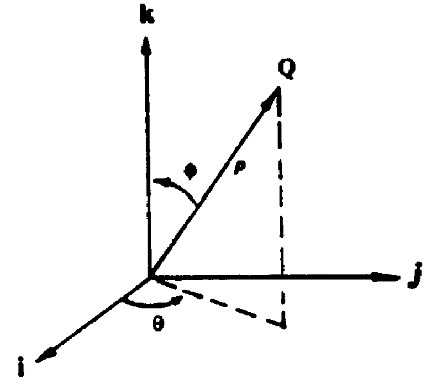


Fig.3.3

Vector Q is shown in Fig. 3.2, where $\{x, y, z\}$ are the lengths of the three edges of the (inclined) parallelepiped. From (3.3) it is clear that in the given reference frame $\{e_1, e_2, e_3\}$, vector Q can be defined in terms of *three* scalars:

$$Q: \quad \{x, y, z\} \tag{3.4}$$

If $\{i, j, k\}$ represent a Cartesian reference frame and two angles $\{\theta, \phi\}$ are introduced as shown in Fig. 3.3, then Q can also be expressed as,

$$Q = p\{\sin\phi[\cos\theta\, i + \sin\theta\, j] + (\cos\phi)\, k\} \tag{3.5}$$

From (3.4) it follows that in a Cartesian reference frame any vector Q can be defined in terms of another set of *three* scalars:

$$Q: \quad \{p, \theta, \phi\} \tag{3.6}$$

Comparing (3.3) with (3.5), we can see that the spherical coordinate system complicates the vector expression and operations; moreover, the physical meaning of the spherical coordinate system is not as direct and intuitively obvious as the expression given in (3.3).

In the new formulation, instead of using the spherical coordinate system, the formulation of (3.3) will be used.

Formulations of the VTE problems.

Chace's approach in formulating the VTE problems is summarized in a table of [4], which is displayed here as Table 3.1 for comparison.

In the proposed new approach, the VTE problems is formulated as the solution of a set of simultaneous equations which contains the VTE and certain constraint equations as shown in Table 3.2.

Comparing Table 3.1 with 3.2, it is clear that in Chace's approach the unknowns are lengths and angles, whereas in the new approach the unknowns are lengths and unit vectors.

An important vector identity and its proof.

The following vector identity will be repeatedly used in this chapter,

$$\mathbf{a}\cdot\mathbf{b}\times\mathbf{c} = \pm\sqrt{\mathbf{a}^2\mathbf{b}^2\mathbf{c}^2 - (\mathbf{b}\cdot\mathbf{c})^2\mathbf{a}^2 - (\mathbf{c}\cdot\mathbf{a})^2\mathbf{b}^2 - (\mathbf{a}\cdot\mathbf{b})^2\mathbf{c}^2 + 2(\mathbf{a}\cdot\mathbf{b})(\mathbf{b}\cdot\mathbf{c})(\mathbf{c}\cdot\mathbf{a})} \tag{3.7}$$

where $\mathbf{a}$, $\mathbf{b}$ or $\mathbf{c}$ can be any vector (i.e. not necessarily unit vectors).

Proof (The first method):

From

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}. \tag{3.8}$$

we can obtain the Jacobian identity:

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) + \mathbf{b}\times(\mathbf{c}\times\mathbf{a}) + \mathbf{c}\times(\mathbf{a}\times\mathbf{b}) = 0 \tag{3.9}$$

The scalar product of $(\mathbf{b}\times\mathbf{c})$ and (3.9) yields

$$(\mathbf{b}\times\mathbf{c})^2\mathbf{a} - [(\mathbf{b}\times\mathbf{c})\cdot\mathbf{a}](\mathbf{b}\times\mathbf{c}) + [(\mathbf{b}\times\mathbf{c})(\mathbf{c}\times\mathbf{a})]\mathbf{b} + [(\mathbf{b}\times\mathbf{c})(\mathbf{a}\times\mathbf{b})]\mathbf{c} = 0 \tag{3.10}$$

and the scalar product of vector $\mathbf{a}$ and (3.10) gives (3.7).

Proof (The second method):

Suppose $\mathbf{b}\times\mathbf{c}\neq 0$, then $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{b}\times\mathbf{c}$ are three non-coplanar vectors, hence vector $\mathbf{a}$ can be expressed as,

$$\mathbf{a} = x\,\mathbf{b} + y\,\mathbf{c} + z\,\mathbf{b}\times\mathbf{c} \tag{3.11}$$

Table 3.1. Chace's modelings of VTE problems.

VTE: $\mathbf{r}+\mathbf{s}+\mathbf{t}+\mathbf{C}=0$					Degree of
Case Number	Unknown	Known			Polynomial Solution
		Vector	Unit Vector	Scalar	
1	r, θ_r, ϕ_r	C			1
2a	$r, \theta_r; s$	C	$\hat{s}, \hat{\omega}_r$	ϕ_r	2
2b	$r, \theta_r; \theta_s$	C	$\hat{\omega}_r, \hat{\omega}_s$	$\phi_r; s, \phi_s$	4
2c	$\theta_r, \phi_r; s$	C	$\hat{s}$	r	2
2d	$\theta_r, \phi_r; \theta_s$	C	$\hat{\omega}_s$	$r; s, \phi_s$	2
3a	$r; s; t$	C	$\hat{r}, \hat{s}, \hat{t}$		1
3b	$r; s; \theta_t$	C	$\hat{r}, \hat{s}, \hat{\omega}_t$	t, ϕ_t	2
3c	$r; \theta_s; \theta_t$	C	$\hat{r}, \hat{\omega}_s, \hat{\omega}_t$	$s, \phi_s; t, \phi_t$	4
3d	$\theta_r; \theta_s; \theta_t$	C	$\hat{\omega}_r, \hat{\omega}_s, \hat{\omega}_t$	$r, \phi_r; s, \phi_s; t, \phi_t$	8

Remarks: (1). Unit vectors $\hat{\omega}_r$, $\hat{\omega}_s$, $\hat{\omega}_t$ are the known directions from which the known angles ϕ_r, ϕ_s and ϕ_t are measured. (2). Whenever any of the vectors $\mathbf{r}$, $\mathbf{s}$ and $\mathbf{t}$ are completely known they are summed into the single constant $\mathbf{C}$.

Table 3.2. New modelings of VTE problems.

VTE: $p_1\mathbf{q}_1+p_2\mathbf{q}_2+p_3\mathbf{q}_3=\mathbf{I}$			Degree of
Case Number	Unknown	Constraints	Polynomial Solution
1	$p_1\mathbf{q}_1$	none	1
2	$p_1\mathbf{q}_1$, p_2	$\mathbf{q}_1 \cdot \mathbf{u}=m$	2
3	$p_1\mathbf{q}_1$, $\mathbf{q}_2$	$\mathbf{q}_1 \cdot \mathbf{u}=m$, $\mathbf{q}_2 \cdot \mathbf{v}=n$	4
4	p_1, $\mathbf{q}_2$	none	2
5	$\mathbf{q}_1$, $\mathbf{q}_2$	$\mathbf{q}_1 \cdot \mathbf{u}=m$	2
6	p_1, p_2, p_3	none	1
7	p_1, p_2, $\mathbf{q}_3$	$\mathbf{q}_3 \cdot \mathbf{u}=m$	2
8	p_1, $\mathbf{q}_2$, $\mathbf{q}_3$	$\mathbf{q}_2 \cdot \mathbf{u}=m$, $\mathbf{q}_3 \cdot \mathbf{v}=n$	4
9	$\mathbf{q}_1$, $\mathbf{q}_2$, $\mathbf{q}_3$	$\mathbf{q}_1 \cdot \mathbf{u}=m$, $\mathbf{q}_2 \cdot \mathbf{v}=n$ $\mathbf{q}_3 \cdot \mathbf{w}=l$	8

Remarks: All parameters (vectors or scalars) in each of the VTE problems are known, except for those indicated as unknowns. Whenever any of the vectors $(p_1\mathbf{q}_1)$, $(p_2\mathbf{q}_2)$ or $(p_3\mathbf{q}_3)$ are known at the beginning, they are summed into the single constant $\mathbf{I}$. $(\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{q}_j)$ are unit vectors.

The scalar product of $\{[\mathbf{c}\times(\mathbf{b}\times\mathbf{c})], [(\mathbf{b}\times\mathbf{c})\times\mathbf{b}], (\mathbf{b}\times\mathbf{c})\}$ with (3.11) yields

$$\begin{cases} x = [(\mathbf{a}\cdot\mathbf{b})c^2 - (\mathbf{c}\cdot\mathbf{a})(\mathbf{b}\cdot\mathbf{c})] / [b^2c^2 - (\mathbf{b}\cdot\mathbf{c})^2] \\ y = [(\mathbf{c}\cdot\mathbf{a})\,b^2 - (\mathbf{a}\cdot\mathbf{b})(\mathbf{b}\cdot\mathbf{c})] / [b^2c^2 - (\mathbf{b}\cdot\mathbf{c})^2] \\ z = [\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})] / [b^2c^2 - (\mathbf{b}\cdot\mathbf{c})^2] \end{cases} \tag{3.11a}$$

The scalar product of $\mathbf{a}$ and (3.10) yields

$$a^2 = x\,(\mathbf{a}\cdot\mathbf{b}) + y\,(\mathbf{c}\cdot\mathbf{a}) + z\,[\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})] \tag{3.12}$$

Substituting (3.11) into (3.12) yields (3.7).

If $\mathbf{b}\times\mathbf{c} = 0$, then (3.7) becomes $0 \equiv 0$.

3.3. Solutions of the VTEs

3.3.1. If $\{x, \mathbf{q}_1\}$ are the unknowns and $\{\mathbf{I}\}$ is known, the problem reduces to:

$$x\,\mathbf{q}_1 = \mathbf{I} \tag{3.13}$$

Solution:

This case is trivial. From (3.13)

$$\begin{cases} x = |\mathbf{I}| \\ \mathbf{q}_1 = \mathbf{I} / |\mathbf{I}| \end{cases} \tag{3.14}$$

3.3.2. If $\{x, y, \mathbf{q}_1\}$ are the unknowns and $\{\mathbf{q}_2, \mathbf{I}, \mathbf{u}, m\}$ are known, the problem becomes:

$$\begin{cases} x\mathbf{q}_1 + y\mathbf{q}_2 = \mathbf{I} & (1) \\ \mathbf{q}_1\cdot\mathbf{u} = m & (2) \end{cases} \tag{3.15}$$

Solution:

The scalar product of $\mathbf{u}$ and (3.15-1) yields

$$m\,x + (\mathbf{u}\cdot\mathbf{q}_2)\,y = (\mathbf{u}\cdot\mathbf{I}) \tag{3.16}$$

Rearranging (3.15-1) and squaring both sides yields

$$x^2 = y^2 - 2(\mathbf{I}\cdot\mathbf{q}_2)\,y + I^2 \tag{3.17}$$

From (3.16) and (3.17) we obtain

$$k_0 y^2 + k_1 y + k_2 = 0 \tag{3.18}$$

$$\begin{cases} k_0 = (\mathbf{u}\cdot\mathbf{q}_2)^2 - m^2 \\ k_1 = 2\,[(\mathbf{I}\cdot\mathbf{q}_2)\,m^2 - (\mathbf{u}\cdot\mathbf{I})(\mathbf{u}\cdot\mathbf{q}_2)] \\ k_2 = (\mathbf{u}\cdot\mathbf{I}) - m^2 I^2 \end{cases} \tag{3.18a}$$

From (3.18) we have

$$y = (-k_1 \pm \sqrt{k_1^2 - 4k_0k_2})/(2k_0) \tag{3.19}$$

Substituting (3.19) into (3.16) yields x, then from (3.15) we obtain

$$\mathbf{q}_1 = [(\mathbf{l} - y\mathbf{q}_2)/x] \tag{3.20}$$

Special case: if $m = 0$,

From (3.16) we get

$$y = (\mathbf{u}\cdot\mathbf{l})/(\mathbf{u}\cdot\mathbf{q}_2) \tag{3.21}$$

From (3.15-1) we get $x\mathbf{q}_1 = (\mathbf{l} - y\mathbf{q}_2)$, and squaring both sides of the equation yields

$$\begin{cases} x = |\mathbf{l} - y\mathbf{q}_2| = \sqrt{\mathbf{l}^2 - 2y(\mathbf{l}\cdot\mathbf{q}_2) + y^2} \\ \mathbf{q}_1 = (\mathbf{l} - y\mathbf{q}_2)/x \end{cases} \tag{3.22}$$

It is clear that if $m \neq 0$, equation (3.15) has two distinct solutions; If $m = 0$, equation (3.15) has only one solution.

3.3.3. If $\{x, \mathbf{q}_1, r\}$ are the unknowns and $\{p_2, \mathbf{l}, m, n, \mathbf{u}, \mathbf{v}\}$ are known, the problem becomes:

$$\begin{cases} x\mathbf{q}_1 + p_2\mathbf{q}_2 = \mathbf{l} & (1) \\ \mathbf{q}_1\cdot\mathbf{u} = m & (2) \\ \mathbf{q}_2\cdot\mathbf{v} = r & (3) \end{cases} \tag{3.23}$$

Solution:

The scalar product of $\{\mathbf{u}, \mathbf{v}, \mathbf{u}\times\mathbf{v}\}$ and (3.23-1) yields

$$x\,m + p_2[\mathbf{u}\cdot\mathbf{q}_2] = (\mathbf{u}\cdot\mathbf{l}) \tag{3.24}$$

$$x\,[\mathbf{v}\cdot\mathbf{q}_1] = (\mathbf{v}\cdot\mathbf{l}) - p_2 n \tag{3.25}$$

$$x\,\mathbf{q}_1\cdot\mathbf{u}\times\mathbf{v} + p_2\mathbf{q}_2\cdot\mathbf{u}\times\mathbf{v} = (\mathbf{l}\cdot\mathbf{u}\times\mathbf{v}) \tag{3.26}$$

Equation (3.7) and taking into account of (3.23) yields

$$\begin{cases} \mathbf{q}_1\cdot\mathbf{u}\times\mathbf{v} = \pm\sqrt{1 - m^2 - (\mathbf{u}\cdot\mathbf{v})^2 - [\mathbf{v}\cdot\mathbf{q}_1]^2 - 2m(\mathbf{u}\cdot\mathbf{v})[\mathbf{v}\cdot\mathbf{q}_1]} \\ \mathbf{q}_2\cdot\mathbf{u}\times\mathbf{v} = \pm\sqrt{1 - n^2 - (\mathbf{u}\cdot\mathbf{v})^2 - [\mathbf{v}\cdot\mathbf{q}_2]^2 - 2n(\mathbf{u}\cdot\mathbf{v})[\mathbf{v}\cdot\mathbf{q}_2]} \end{cases} \tag{3.26a}$$

From (3.25) and (3.24) we obtain

$$\begin{cases} [\mathbf{v}\cdot\mathbf{q}_1] = (\mathbf{v}\cdot\mathbf{l} - p_2 n)/x \\ [\mathbf{u}\cdot\mathbf{q}_2] = (m\,x - \mathbf{u}\cdot\mathbf{l})/p_2 \end{cases} \tag{3.27}$$

Substituting (3.27) into (3.26a) yields

$$\begin{cases} \mathbf{q}_1\cdot\mathbf{u}\times\mathbf{v} = \pm\,(\sqrt{a_1x^2+b_1x+c_1})\,/\,x \\ \mathbf{q}_2\cdot\mathbf{u}\times\mathbf{v} = \pm\,(\sqrt{a_2x^2+b_2x+c_2})\,/\,p_2 \end{cases} \tag{3.28}$$

where

$$\begin{cases} a_1 = 1 - m^2 - (\mathbf{u}\cdot\mathbf{v})^2 \\ b_1 = -\,2m(\mathbf{u}\cdot\mathbf{v})(\mathbf{v}\cdot\mathbf{I} - p_2 n) \\ c_1 = -\,(\mathbf{v}\cdot\mathbf{I} - p_2 n)^2 \end{cases} \tag{3.28a}$$

$$\begin{cases} a_2 = -\,m^2 \\ b_2 = 2m\,[(\mathbf{u}\cdot\mathbf{I}) - p_2 n(\mathbf{u}\cdot\mathbf{v})] \\ c_2 = p_2^2\,[1 - n^2 - (\mathbf{u}\cdot\mathbf{v})^2] \\ \quad -(\mathbf{u}\cdot\mathbf{I})[(\mathbf{u}\cdot\mathbf{I}) - 2p_2 n(\mathbf{u}\cdot\mathbf{v})] \end{cases} \tag{3.28b}$$

Substitutin? (3.28) into (3.26) yields

$$\pm\sqrt{a_1x^2+b_1x+c_1} \pm \sqrt{a_2x^2+b_2x+c_2} = (\mathbf{I}\cdot\mathbf{u}\times\mathbf{v}) \tag{3.29}$$

Squaring both sides of (3.29) yields

$$\begin{aligned} &\pm 2\sqrt{(a_1x^2+b_1x+c_1)(a_2x^2+b_2x+c_2)} \\ &= -(a_1+a_2)x^2 - (b_1+b_2)x - (c_1+c_2) + (\mathbf{I}\cdot\mathbf{u}\times\mathbf{v}) \end{aligned} \tag{3.30}$$

Squaring both sides of (3.29) yields

$$\sum_{i=0}^{4} k_i\, x^{4-i} = 0 \tag{3.31}$$

where

$$\begin{cases} k_0 = (a_1 - a_2)^2 \\ k_1 = 2(a_1 - a_2)(b_1 - b_2) \\ k_2 = 2(a_1 - a_2)(c_1 - c_2) \\ \quad -2(a_1+a_2)(\mathbf{I}\cdot\mathbf{u}\times\mathbf{v}) + (b_1-b_2)^2 \\ k_3 = 2(b_1 - b_2)(c_1 - c_2) - 2(b_1+b_2)(\mathbf{I}\cdot\mathbf{u}\times\mathbf{v}) \\ k_4 = [(c_1 - c_2) - (\mathbf{I}\cdot\mathbf{u}\times\mathbf{v})]^2 \end{cases} \tag{3.31a}$$

Solving equation (3.31) the value of x can be determined. Now let

$$\mathbf{q}_1 = \xi\mathbf{u} + \eta\mathbf{v} + \zeta\mathbf{u}\times\mathbf{v} \tag{3.32}$$

The scalar product of $\{\mathbf{v}\times(\mathbf{u}\times\mathbf{v}),\ (\mathbf{u}\times\mathbf{v})\times\mathbf{u},\ \mathbf{I}\}$ and (3.32) yields

$$\begin{cases} \xi = \{m - (\mathbf{u}\cdot\mathbf{v})(\mathbf{v}\cdot\mathbf{q}_1)\} / \{1 - (\mathbf{u}\cdot\mathbf{v})^2\} \\ \eta = \{(\mathbf{v}\cdot\mathbf{q}_1) - (\mathbf{u}\cdot\mathbf{v})m\} / \{1 - (\mathbf{u}\cdot\mathbf{v})^2\} \\ \zeta = \{[\mathbf{I}\cdot\mathbf{q}_1] - \mathbf{I}\cdot(\xi\mathbf{u} + \eta\mathbf{v})\} / (\mathbf{I}\cdot\mathbf{u}\times\mathbf{v}) \end{cases} \tag{3.32a}$$

From (3.23-1) we obtain $(p_2\mathbf{q}_2)^2 = (\mathbf{I} - x\mathbf{q}_1)^2$, which yields

$$[\mathbf{I}\cdot\mathbf{q}_1] = (p_2^2 - x^2 - \mathbf{I}^2)/(2x) \tag{3.33}$$

Substituting (3.23-1) and (3.33) into (3.32a), and then substituting (3.32a) into (3.32) yields $\mathbf{q}_1$; From (3.23-1) we can obtain

$$\mathbf{q}_2 = (\mathbf{I} - x\mathbf{q}_1)/p_2 \tag{3.34}$$

3.3.4. If $\{x, \mathbf{q}_2\}$ are the unknowns and $\{p_2, \mathbf{q}_1, \mathbf{I}\}$ are known, the problem becomes

$$x\,\mathbf{q}_1 + p_2\mathbf{q}_2 = \mathbf{I} \tag{3.35}$$

Solution:

Rearranging (3.35) yields

$$p_2\mathbf{q}_2 = \mathbf{I} - x\,\mathbf{q}_1 \tag{3.36}$$

Squaring both sides of (3.36) yields

$$x^2 - 2(\mathbf{I}\cdot\mathbf{q}_1)x + (\mathbf{I}^2 - p_2^2) = 0 \tag{3.37}$$

From (3.37) we obtain

$$x = (\mathbf{I}\cdot\mathbf{q}_1) \pm \sqrt{(\mathbf{I}\cdot\mathbf{q}_1)^2 - (\mathbf{I}^2 - p_2^2)} \tag{3.38}$$

From (3.36) we get

$$\mathbf{q}_2 = (\mathbf{I} - x\mathbf{q}_1)/p_2 \tag{3.39}$$

3.3.5. If $\{\mathbf{q}_1, \mathbf{q}_2\}$ are the unknowns and $\{p_1, p_2, \mathbf{I}, \mathbf{u}, m\}$ are known, then

$$\begin{cases} p_1\mathbf{q}_1 + p_2\mathbf{q}_2 = \mathbf{I} & (1) \\ \mathbf{q}_1\cdot\mathbf{u} = m & (2) \end{cases} \tag{3.40}$$

Solution:

Let

$$\mathbf{q}_1 = \xi\mathbf{u} + \eta\mathbf{I} + \zeta\,\mathbf{u}\times\mathbf{I} \tag{3.41}$$

The scalar product of $\{\mathbf{I}\times(\mathbf{u}\times\mathbf{v}), (\mathbf{u}\times\mathbf{I})\times\mathbf{u}, \mathbf{u}\times\mathbf{I}\}$ and (3.41) yields

$$\begin{cases} \xi = [m\,\mathbf{I}^2 - (\mathbf{u}\cdot\mathbf{I})(\mathbf{q}_1\cdot\mathbf{I})] \,/\, [\mathbf{I}^2 - (\mathbf{u}\cdot\mathbf{I})^2] \\ \eta = [(\mathbf{q}_1\cdot\mathbf{I}) - (\mathbf{u}\cdot\mathbf{I})\,m] \,/\, [\mathbf{I}^2 - (\mathbf{u}\cdot\mathbf{I})^2] \\ \zeta = (\mathbf{u}\times\mathbf{I}\cdot\mathbf{q}_1) \,/\, [\mathbf{I}^2 - (\mathbf{u}\cdot\mathbf{I})^2] \end{cases} \tag{3.41a}$$

Using (3.7) we obtain

$$(\mathbf{u}\times\mathbf{I}\cdot\mathbf{q}_1) = \pm\sqrt{\mathbf{I}^2 - (\mathbf{u}\cdot\mathbf{I})^2 - m^2 - (\mathbf{q}_1\cdot\mathbf{I})^2 - 2m(\mathbf{u}\cdot\mathbf{I})(\mathbf{q}_1\cdot\mathbf{I})} \tag{3.42}$$

From (3.40-1) we obtain $(\mathbf{I} - p_1\mathbf{q}_1)^2 = (p_2\mathbf{q}_2)^2$, which yields

$$(\mathbf{q}_1\cdot\mathbf{I}) = (\mathbf{I}^2 + p_1^2 - p_2^2) \,/\, (2p_1) \tag{3.43}$$

Substituting {(3.42), (3.43)} into (3.41a), and then $\mathbf{q}_1$ can be obtained from (3.41); From (3.40-1) we can get

$$\mathbf{q}_2 = (\mathbf{I} - p_1\mathbf{q}_1) \,/\, p_2 \tag{3.44}$$

3.3.6. If $\{x, y, z\}$ are the unknowns and $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{I}\}$ are known.

$$x\mathbf{q}_1 + y\mathbf{q}_2 + z\mathbf{q}_3 = \mathbf{I} \tag{3.45}$$

Solution:

The scalar product of $\{\mathbf{q}_2\times\mathbf{q}_3,\ \mathbf{q}_3\times\mathbf{q}_1,\ \mathbf{q}_1\times\mathbf{q}_2\}$ and (3.45) yields

$$\begin{cases} x = (\mathbf{q}_2\times\mathbf{q}_3\cdot\mathbf{I}) \,/\, (\mathbf{q}_1\times\mathbf{q}_2\cdot\mathbf{q}_3) \\ y = (\mathbf{q}_3\times\mathbf{q}_1\cdot\mathbf{I}) \,/\, (\mathbf{q}_1\times\mathbf{q}_2\cdot\mathbf{q}_3) \\ z = (\mathbf{q}_1\times\mathbf{q}_2\cdot\mathbf{I}) \,/\, (\mathbf{q}_1\times\mathbf{q}_2\cdot\mathbf{q}_3) \end{cases} \tag{3.46}$$

3.3.7. If $\{x, y, \mathbf{q}_3\}$ are the unknowns and $\{\mathbf{q}_1, \mathbf{q}_2, p_3, \mathbf{I}, \mathbf{u}, m\}$ are known, then

$$\begin{cases} x\,\mathbf{q}_1 + y\,\mathbf{q}_2 + p_3\mathbf{q}_3 = \mathbf{I} & (1) \\ \mathbf{q}_3\cdot\mathbf{u} = m & (2) \end{cases} \tag{3.47}$$

Solution:

The scalar product of u and (3.47-1) yields

$$x(\mathbf{u}\cdot\mathbf{q}_1) + y(\mathbf{u}\cdot\mathbf{q}_2) = (\mathbf{u}\cdot\mathbf{I}) - p_3 m \tag{3.48}$$

Rearranging (3.47-1) we get

$$p_3\mathbf{q}_3 = (\mathbf{I} - x\mathbf{q}_1 - y\mathbf{q}_2) \tag{3.49}$$

Squaring both sides of (3.49) yields

$$x^2 + y^2 + 2(\mathbf{q}_1\cdot\mathbf{q}_2)xy - 2(\mathbf{I}\cdot\mathbf{q}_1)x - 2(\mathbf{I}\cdot\mathbf{q}_2)y + (\mathbf{I}^2 - p_3^2) = 0 \tag{3.50}$$

If $(\mathbf{u}\cdot\mathbf{q}_2)\neq 0$, from (3.48) we obtain

$$y = a\,x + b \tag{3.51}$$

$$\begin{cases} a = -(\mathbf{u}\cdot\mathbf{q}_1) / (\mathbf{u}\cdot\mathbf{q}_2) \\ b = [(\mathbf{u}\cdot\mathbf{l}) - p_3 m] / (\mathbf{u}\cdot\mathbf{q}_2) \end{cases} \tag{3.51a}$$

Substituting (3.51) into (3.50) yields

$$k_0 x^2 + k_1 x + k_2 = 0 \tag{3.52}$$

where

$$\begin{cases} k_0 = 1 + a^2 + 2a(\mathbf{q}_1\cdot\mathbf{q}_2) \\ k_1 = 2[a\,b + a(\mathbf{l}\cdot\mathbf{q}_2) + b(\mathbf{q}_1\cdot\mathbf{q}_2) - (\mathbf{l}\cdot\mathbf{q}_1)] \\ k_3 = b^2 - 2b(\mathbf{l}\cdot\mathbf{q}_2) + (\mathbf{l}^2 - p_3^2) \end{cases} \tag{3.52a}$$

From (3.52) we get x; from (3.51) we have y; and from (3.49) obtain $\mathbf{q}_3$.

If $(\mathbf{u}\cdot\mathbf{q}_2)=0$, and $(\mathbf{u}\cdot\mathbf{q}_1)\neq 0$, then from (3.48) we get x; from (3.50) we get y; and from (3.49) we obtain $\mathbf{q}_3$.

If $(\mathbf{u}\cdot\mathbf{q}_2)=0$, and $(\mathbf{u}\cdot\mathbf{q}_1)=0$, then we can derive neither x nor y from (3.48). We need to adopt another approach. The scalar product of $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{l}\}$ and (3.47-1) yields

$$\begin{cases} x + y(\mathbf{q}_1\cdot\mathbf{q}_2) + \{p_3[\mathbf{q}_1\cdot\mathbf{q}_3] - (\mathbf{l}\cdot\mathbf{q}_1)\} = 0 \\ x(\mathbf{q}_1\cdot\mathbf{q}_2) + y + \{p_3[\mathbf{q}_2\cdot\mathbf{q}_3] - (\mathbf{l}\cdot\mathbf{q}_2)\} = 0 \\ x(\mathbf{l}\cdot\mathbf{q}_1) + y(\mathbf{l}\cdot\mathbf{q}_2) + \{p_3[\mathbf{l}\cdot\mathbf{q}_3] - \mathbf{l}^2\} = 0 \end{cases} \tag{3.53}$$

Because the vector $(x, y, 1)\neq 0$, (3.53) yields

$$\begin{vmatrix} 1 & (\mathbf{q}_1\cdot\mathbf{q}_2) & \{p_3[\mathbf{q}_1\cdot\mathbf{q}_3] - (\mathbf{l}\cdot\mathbf{q}_1)\} \\ (\mathbf{q}_1\cdot\mathbf{q}_2) & 1 & \{p_3[\mathbf{q}_2\cdot\mathbf{q}_3] - (\mathbf{l}\cdot\mathbf{q}_2)\} \\ (\mathbf{l}\cdot\mathbf{q}_1) & (\mathbf{l}\cdot\mathbf{q}_2) & \{p_3[\mathbf{l}\cdot\mathbf{q}_3] - \mathbf{l}^2\} \end{vmatrix} = 0 \tag{3.54}$$

Expanding (3.54) yields

$$\mathbf{U}\cdot\mathbf{q}_3 = V \tag{3.55}$$

$$\begin{cases} \mathbf{U} = p_3\{(\mathbf{q}_1\times\mathbf{q}_2)^2\,\mathbf{l} + (\mathbf{q}_1\times\mathbf{q}_2)\cdot(\mathbf{q}_2\times\mathbf{l})\,\mathbf{q}_1 + (\mathbf{q}_1\times\mathbf{q}_2)\cdot(\mathbf{l}\times\mathbf{q}_1)\,\mathbf{q}_2\} \\ V = p_3^{-1}\,\mathbf{U}\cdot\mathbf{l} \end{cases} \tag{3.55a}$$

$\mathbf{q}_3$ is the only unknown (vector) in equation (3.55) which can be used to determine one scalar unknown. On account of the fact that $\mathbf{q}_3$ is a unit vector and constrained by the scalar equation (3.47-2), it is clear that $\mathbf{q}_3$ has only one scalar unknown yet to be determined, namely, $\mathbf{q}_3$ can be expressed by one unknown scalar variable and the known vectors. Let

$$\mathbf{q}_3 = \xi \mathbf{u} + \eta \mathbf{q}_1 + \zeta \mathbf{u}\times\mathbf{q}_1 \tag{3.56}$$

The scalar product of $\mathbf{u}$ and (3.56) yields $\xi = m$.

From (3.56) we can derive

$$(\eta \mathbf{q}_1 + \zeta \mathbf{u}\times\mathbf{q}_1)^2 = (\mathbf{q}_3 - m\,\mathbf{u})^2$$

$$\text{i.e.} \qquad \eta^2 + \zeta^2 = (\sqrt{1-m^2})^2 \tag{3.57}$$

From (3.57) we obtain

$$\begin{cases} \eta = \sqrt{1-m^2}\cos\theta \\ \zeta = \sqrt{1-m^2}\sin\theta \end{cases} \tag{3.57a}$$

Substituting (3.57a) and $\xi = m$ into (3.56) yields

$$\mathbf{q}_3 = m\,\mathbf{u} + \sqrt{1-m^2}\cos\theta\,\mathbf{q}_1 + \sqrt{1-m^2}\sin\theta\,(\mathbf{u}\times\mathbf{q}_1) \tag{3.58}$$

Substituting (3.58) into (3.55) yields

$$A\cos\theta + B\sin\theta = C \tag{3.59}$$

$$\begin{cases} A = \sqrt{1-m^2}(\mathbf{U}\cdot\mathbf{q}_1) \\ B = \sqrt{1-m^2}(\mathbf{U}\cdot\mathbf{u}\times\mathbf{q}_1) \\ C = -(\mathbf{U}\cdot\mathbf{u}) + V \end{cases} \tag{3.59a}$$

Let $\Omega = \tan(\theta/2)$, then we have

$$\begin{cases} \cos\theta = (1-\Omega^2)/(1+\Omega^2) \\ \sin\theta = 2\Omega/(1+\Omega^2) \end{cases} \tag{3.60}$$

Substituting (3.60) into (3.59) yields

$$(A+C)\Omega^2 - 2B\,\Omega + (C-A) = 0 \tag{3.61}$$

From (3.61) we get Ω, then $\theta = 2\tan^{-1}\Omega$, i.e.

$$\theta = 2\tan^{-1}[(B \pm \sqrt{A^2+B^2-C^2})/(A+C)] \tag{3.62}$$

Substituting (3.62) into (3.58) yields $\mathbf{q}_3$. Now from (3.47-1) we obtain

$$x\mathbf{q}_1 + y\mathbf{q}_2 = (\mathbf{I} - p_3\mathbf{q}_3) \tag{3.63}$$

The scalar product of $\{\mathbf{q}_2\times\mathbf{I},\ \mathbf{q}_1\times\mathbf{I}\}$ and (3.63) yields

$$\begin{cases} x = (\mathbf{I}-p_3\mathbf{q}_3)\cdot(\mathbf{q}_2\times\mathbf{I}) / [\mathbf{q}_1\cdot(\mathbf{q}_2\times\mathbf{I})] \\ y = (\mathbf{I}-p_3\mathbf{q}_3)\cdot(\mathbf{I}\times\mathbf{q}_1) / [\mathbf{q}_1\cdot(\mathbf{q}_2\times\mathbf{I})] \end{cases} \tag{3.64}$$

3.3.8. If $\{x, \mathbf{q}_2, \mathbf{q}_3\}$ are the unknowns and $\{\mathbf{q}_1, p_2, p_3, \mathbf{I}, \mathbf{u}, m, \mathbf{v}, n\}$ are known, then

$$\begin{cases} x\mathbf{q}_1 + p_2\mathbf{q}_2 + p_3\mathbf{q}_3 = \mathbf{I} & (1) \\ \mathbf{q}_2 \cdot \mathbf{u} = m & (2) \\ \mathbf{q}_3 \cdot \mathbf{v} = n & (3) \end{cases} \tag{3.65}$$

Solution:

The scalar product of $\{\mathbf{u}, \mathbf{v}, \mathbf{u}\times\mathbf{v}\}$ and (3.65-1) yields

$$x(\mathbf{u}\cdot\mathbf{q}_1) + p_3[\mathbf{u}\cdot\mathbf{q}_3] = (\mathbf{u}\cdot\mathbf{I}) - p_2 m \tag{3.66}$$

$$x(\mathbf{v}\cdot\mathbf{q}_1) + p_2[\mathbf{v}\cdot\mathbf{q}_2] = (\mathbf{v}\cdot\mathbf{I}) - p_3 n \tag{3.67}$$

$$x(\mathbf{u}\times\mathbf{v}\cdot\mathbf{q}_1) + p_2[\mathbf{u}\times\mathbf{v}\cdot\mathbf{q}_2] + p_3[\mathbf{u}\times\mathbf{v}\cdot\mathbf{q}_3] = (\mathbf{u}\times\mathbf{v}\cdot\mathbf{I}) \tag{3.68}$$

Using (3.7) we can obtain

$$\begin{cases} [\mathbf{u}\times\mathbf{v}\cdot\mathbf{q}_2] = \pm\sqrt{1 - m^2 - (\mathbf{u}\cdot\mathbf{v})^2 - [\mathbf{v}\cdot\mathbf{q}_2]^2 - 2m(\mathbf{u}\cdot\mathbf{v})[\mathbf{v}\cdot\mathbf{q}_2]} \\ [\mathbf{u}\times\mathbf{v}\cdot\mathbf{q}_3] = \pm\sqrt{1 - n^2 - (\mathbf{u}\cdot\mathbf{v})^2 - [\mathbf{u}\cdot\mathbf{q}_3]^2 - 2n(\mathbf{u}\cdot\mathbf{v})[\mathbf{u}\cdot\mathbf{q}_3]} \end{cases} \tag{3.68a}$$

From (3.66) and (3.67) we have

$$\begin{cases} [\mathbf{v}\cdot\mathbf{q}_2] = a_1 x + b_1 & (1) \\ [\mathbf{u}\cdot\mathbf{q}_3] = a_2 x + b_2 & (2) \end{cases} \tag{3.69}$$

where

$$\begin{cases} a_1 = -(\mathbf{v}\cdot\mathbf{q}_1) / p_2 \\ a_2 = -(\mathbf{u}\cdot\mathbf{q}_1) / p_3 \\ b_1 = [(\mathbf{v}\cdot\mathbf{I}) - p_3 n] / p_2 \\ b_2 = [(\mathbf{u}\cdot\mathbf{I}) - p_2 m] / p_3 \end{cases} \tag{3.69a}$$

Substituting (3.69) into (3.68a) yields

$$\begin{cases} [\upsilon\times\mathbf{v}\cdot\mathbf{q}_2] = \pm\sqrt{c_1 - d_1 x - a_1^2 x^2} \\ [\mathbf{u}\times\mathbf{v}\cdot\mathbf{q}_3] = \pm\sqrt{c_2 - d_2 x - a_2^2 x^2} \end{cases} \tag{3.70}$$

where

$$\begin{cases} c_1 = 1 - m^2 - b_1^2 - (\mathbf{u}\cdot\mathbf{v})^2 - 2m\, b_1(\mathbf{u}\cdot\mathbf{v}) \\ c_2 = 1 - n^2 - b_2^2 - (\mathbf{u}\cdot\mathbf{v})^2 - 2n\, b_2(\mathbf{u}\cdot\mathbf{v}) \\ d_1 = 2a_1[b_1 + m(\mathbf{u}\cdot\mathbf{v})] \\ d_2 = 2a_2[b_2 + n(\mathbf{u}\cdot\mathbf{v})] \end{cases} \tag{3.70a}$$

Substituting (3.70) into (3.68) yields

$$\pm p_2\sqrt{c_1-d_1x-a_1^2x^2} \pm p_3\sqrt{c_2-d_2x-a_2^2x^2} = (\mathbf{u}\times\mathbf{v}\cdot\mathbf{I}) - x(\mathbf{u}\times\mathbf{v}\cdot\mathbf{q}_1) \tag{3.71}$$

From (3.71) we can obtain

$$\sum_{i=0}^{4} k_i x^{4-i} = 0 \tag{3.72}$$

where

$$\begin{cases} k_0 = (a_1a_2)^2 - \beta_1^2 \\ k_1 = (d_1a_2^2 + d_2a_1^2) + 2\beta_1\beta_2 \\ k_2 = (d_1d_2 - c_1a_2^2 - c_2a_1^2) - (\beta_2^2 + 2\beta_1\beta_3) \\ k_3 = -(c_1d_2 + c_2d_1) + 2\beta_2\beta_3 \\ k_4 = c_1c_2 - \beta_3^2 \end{cases} \tag{3.72a}$$

and

$$\begin{cases} \beta_1 = [(\mathbf{u}\times\mathbf{v}\cdot\mathbf{q}_1)^2 + p_2^2a_1^2 + p_3^2a_2^2]\,/\,(2p_2p_3) \\ \beta_2 = [p_2^2d_1 + p_3^2d_2 - 2(\mathbf{u}\times\mathbf{v}\cdot\mathbf{I})(\mathbf{u}\times\mathbf{v}\cdot\mathbf{q}_1)]/(2p_2p_3) \\ \beta_3 = [(\mathbf{u}\times\mathbf{v}\cdot\mathbf{I})^2 - p_2^2c_1 - p_3^2c_2]\,/\,(2p_2p_3) \end{cases} \tag{3.72b}$$

Solving (3.72) we can determine x. Now let

$$\mathbf{q}_2 = \xi\mathbf{u} + \eta\mathbf{v} + \zeta\mathbf{u}\times\mathbf{v} \tag{3.73}$$

The scalar product of $\{\mathbf{v}\times(\mathbf{u}\times\mathbf{v}),\ (\mathbf{u}\times\mathbf{v})\times\mathbf{u},\ (\mathbf{I}-x\mathbf{q}_1)\}$ and (3.73) yields

$$\begin{cases} \xi = [m - (\mathbf{u}\cdot\mathbf{v})(\mathbf{v}\cdot\mathbf{q}_2)]\,/\,[1-(\mathbf{u}\cdot\mathbf{v})^2] \\ \eta = [(\mathbf{v}\cdot\mathbf{q}_2) - (\mathbf{u}\cdot\mathbf{v})m]\,/\,[1-(\mathbf{u}\cdot\mathbf{v})^2] \\ \zeta = [(\mathbf{I}-x\mathbf{q}_1)\cdot\mathbf{q}_2 - (\mathbf{I}-x\mathbf{q}_1)\cdot(\xi\mathbf{u}+\eta\mathbf{v})]/[1-(\mathbf{u}\cdot\mathbf{v})^2] \end{cases} \tag{3.73a}$$

From (3.65-1) we have

$$[(\mathbf{I}-x\mathbf{q}_1) - p_2\mathbf{q}_2]^2 = (p_3\mathbf{q}_3)^2$$

i.e.

$$(\mathbf{I}-x\mathbf{q}_1)\cdot\mathbf{q}_2 = [(\mathbf{I}-x\mathbf{q}_1)^2 + p_2^2 - p_3^2]/(2p_2) \tag{3.74}$$

Substituting {(3.69-1), (3.74)} into (3.73a), and then substituting (3.7ↄa) into (3.73) we get $\mathbf{q}_2$; From (3.65-1) we obtain

$$\mathbf{q}_3 = (\mathbf{I} - x\mathbf{q}_1 - p_2\mathbf{q}_2)\,/\,p_3 \tag{3.75}$$

3.3.9. If $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3\}$ are the unknowns and $\{p_1, p_2, p_3, \mathbf{I}, \mathbf{u}, \mathbf{v}, \omega, m, n, l\}$ are known, then

$$\begin{cases} p_1\mathbf{q}_1 + p_2\mathbf{q}_2 + p_3\mathbf{q}_3 = \mathbf{I} & (1) \\ \mathbf{q}_1\cdot\mathbf{u} = m & (2) \\ \mathbf{q}_2\cdot\mathbf{v} = n & (3) \\ \mathbf{q}_3\cdot\mathbf{w} = l & (4) \end{cases} \tag{3.76}$$

Solution:

The scalar product of $\mathbf{v}$ and (3.76-1) yields

$$(p_1\mathbf{v})\cdot\mathbf{q}_1 = -(p_3\mathbf{v})\cdot\mathbf{q}_3 - (\mathbf{I}\cdot\mathbf{v} - p_2 n) \tag{3.77}$$

From (3.76-1) we have

$$(p_2\mathbf{q}_2)^2 = (\mathbf{I} - p_1\mathbf{q}_1 - p_3\mathbf{q}_3)^2$$

i.e.

$$[2p_1(p_3\mathbf{q}_3 - \mathbf{I})]\cdot\mathbf{q}_1 = (2p_3\mathbf{I})\cdot\mathbf{q}_3 + (p_2^2 - p_1^2 - p_3^2 - \mathbf{I}^2) \tag{3.78}$$

Let

$$\mathbf{q}_1 = x\,\mathbf{u} + y\,\mathbf{u}\times\mathbf{I} + z\,\mathbf{u}\times(\mathbf{u}\times\mathbf{I}) \tag{3.79}$$

The scalar product of $\mathbf{u}$ and (3.79) yields

$$x = m \tag{3.80}$$

Squaring both sides of (3.79) yields

$$y^2 + z^2 = r^2 \tag{3.81}$$

$$r = (\sqrt{1-m^2}) / \sqrt{\mathbf{I}^2 - (\mathbf{u}\cdot\mathbf{I})^2} \tag{3.81a}$$

It is clear from (3.81) that we can write

$$\begin{cases} y = r\cos\psi \\ z = r\sin\psi \end{cases} \tag{3.82}$$

Substituting (3.80) and (3.82) into (3.79) yields

$$\mathbf{q}_1 = m\,\mathbf{u} + r\cos\psi\,\mathbf{u}\times\mathbf{I} + r\sin\psi\,\mathbf{u}\times(\mathbf{u}\times\mathbf{I}) \tag{3.83}$$

Similarly,

$$\mathbf{q}_3 = l\,\mathbf{w} + \rho\cos\theta\,\mathbf{w}\times\mathbf{I} + \rho\sin\theta\,\mathbf{w}\times(\mathbf{w}\times\mathbf{I}) \tag{3.84}$$

$$\rho = (\sqrt{1-l^2}) / \sqrt{\mathbf{I}^2 - (\mathbf{w}\cdot\mathbf{I})^2} \tag{3.84a}$$

Substituting {(3.93), (3.84)} into {(3.77), (3.78)} yields

$$\begin{cases} A\cos\psi + B\sin\psi = C \\ A'\cos\psi + B'\sin\psi = C' \end{cases} \tag{3.85}$$

where the expressions for {A, B, C, A', B', C'} all have the same form:

$$T(\theta) = t_1\cos\theta + t_2\sin\theta + t_3 \tag{3.85a}$$

$$\begin{cases} a_1 = 0 \\ a_2 = 0 \\ a_3 = p_1 r\, \mathbf{v}\cdot(\mathbf{u}\times\mathbf{I}) \end{cases} \tag{3.85b}$$

$$\begin{cases} b_1 = 0 \\ b_2 = 0 \\ b_3 = p_1 r\, \mathbf{v}\cdot[\mathbf{u}\times(\mathbf{u}\times\mathbf{I})] \end{cases} \tag{3.85c}$$

$$\begin{cases} c_1 = -p_3\rho\, \mathbf{v}\cdot(\mathbf{w}\times\mathbf{I}) \\ c_2 = -p_3\rho\, \mathbf{v}\cdot[\mathbf{w}\times(\mathbf{w}\times\mathbf{I})] \\ c_3 = -p_3 l(\mathbf{v}\cdot\mathbf{w}) - p_1 m(\mathbf{v}\cdot\mathbf{u}) - (\mathbf{v}\cdot\mathbf{I} - p_2 n) \end{cases} \tag{3.85d}$$

$$\begin{cases} a_1' = 2p_1p_3 r\rho\, (\mathbf{u}\times\mathbf{I})\cdot(\mathbf{w}\times\mathbf{I}) \\ a_2' = 2p_1p_3 r\rho\, (\mathbf{u}\times\mathbf{I})\cdot[\mathbf{w}\times(\mathbf{w}\times\mathbf{I})] \\ a_3' = 2p_1p_3\; l\, (\mathbf{u}\times\mathbf{I})\cdot\mathbf{w} \end{cases} \tag{3.85e}$$

$$\begin{cases} b_1' = 2p_1p_3 r\rho\, [\mathbf{u}\times(\mathbf{u}\times\mathbf{I})]\cdot(\mathbf{w}\times\mathbf{I}) \\ b_2' = 2p_1p_3 r\rho\, [\mathbf{u}\times(\mathbf{u}\times\mathbf{I})]\cdot[\mathbf{w}\times(\mathbf{w}\times\mathbf{I})] \\ b_3' = 2p_1p_3 rl\, [\mathbf{u}\times(\mathbf{u}\times\mathbf{I})]\cdot\mathbf{w} + 2p_1 r(\mathbf{u}\times\mathbf{I})^2 \end{cases} \tag{3.85f}$$

$$\begin{cases} c_1' = 2p_3\rho\, (\mathbf{I} - p_1 m\,\mathbf{u})\cdot(\mathbf{w}\times\mathbf{I}) \\ c_2' = 2p_3\rho\, (\mathbf{I} - p_1 m\,\mathbf{u})\cdot[\mathbf{w}\times(\mathbf{w}\times\mathbf{I})] \\ c_3' = 2p_3 l\, (\mathbf{I} - p_1 m\,\mathbf{u})\cdot\mathbf{w} + 2p_1 m\, (\mathbf{I}\cdot\mathbf{u}) \\ \qquad + (p_2^2 - p_1^2 - p_3^3 - \mathbf{I}^2) \end{cases} \tag{3.85g}$$

From (3.85) we have

$$\begin{cases} \cos\psi = -Q_2 / Q_3 \\ \sin\psi = Q_1 / Q_3 \end{cases} \tag{3.86}$$

$$\begin{cases} Q_1 = (AC' - CA') \\ Q_2 = (BC' - CB') \\ Q_3 = (AB' - BA') \end{cases} \tag{3.86a}$$

From $\cos^2\psi + \sin^2\psi = 1$ and (3.86a) we obtain

$$Q_1^2 + Q_2^2 = Q_3^2 \tag{3.87}$$

Let $y = \tan(\theta/2)$, then we have

$$\begin{cases} \cos\theta = (1-y^2)/(1+y^2) \\ \sin\theta = 2y/(1+y^2) \end{cases} \tag{3.88}$$

Substituting (3.88) into (3.85a), then further substituting the expressions for A, B, C, A', B' and C' into (3.86a), and finally substituting (3.86a) into (3.87) we obtain

$$\sum_{i=0}^{8} \delta_i y^{8-i} = 0 \tag{3.89}$$

The coefficients $\{\delta_i\}$ $(i=1\text{-}8)$ of (3.89) are the same as shown from Eq.(2.35a) to (2.35e). Solving (3.89) we get y, then $\theta = 2\tan^{-1}y$. ψ can be determined from (3.86). From (3.83) and (3.84) we can get q_1 and q_3, then q_2 can be determined from (3.76-1).

3.4. Applications

3.4.1. Analysis of a closed-loop robot structure *RRSS*

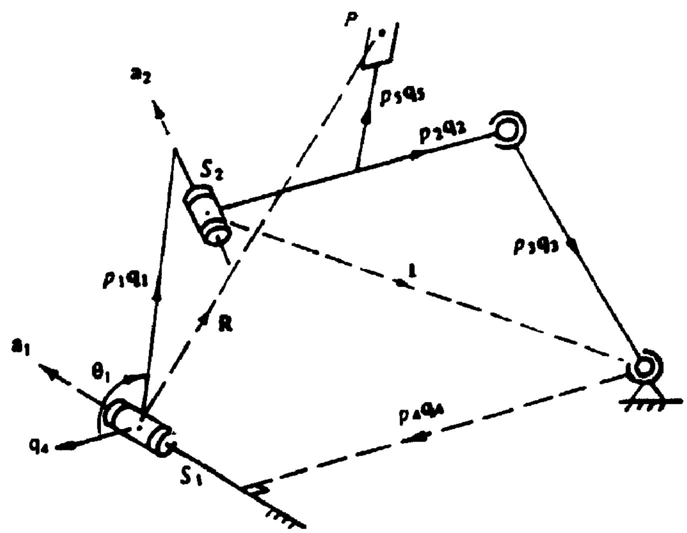


Fig. 3.4

The diagram of the RRSS robot is shown in Fig. 3.4. Angle θ_1 is the input angle controlled by an actu⁻ ᵓr. When θ_1 varies, point P (the center of the robot hand) moves along a spatial curve. It is required to find the function of the trajectory of point P, i.e. $\mathbf{R} = \mathbf{R}(\theta_1)$.

Once the input angle θ_1 is given, vectors $p_1\mathbf{q}_1$ and $S_2\mathbf{a}_2$ are also determined. In order to find $\mathbf{R}(\theta_1)$, i.e.

$$\mathbf{R}(\theta_1) = p_1\mathbf{q}_1 - S_2\mathbf{a}_2 + p_{21}\mathbf{q}_2 + p_5\mathbf{q}_5 \tag{3.90}$$

it is necessary to determine $\mathbf{q}_2$ first. As for vector $\mathbf{q}_3$, its direction relative to $\mathbf{q}_2$ and $\mathbf{a}_2$ are constant, hence it is also determined after $\mathbf{q}_2$ is found.

Now the problem can be modeled as follows. The unknowns are $\{\mathbf{q}_2, \mathbf{q}_3\}$; and $\{p_2=(p_{21}+p_{22}),\ p_3, \mathbf{a}_2, \mathbf{I}\}$ are known.

$$\begin{cases} p_2\mathbf{q}_2 + p_3\mathbf{q}_3 = \mathbf{I} & (1) \\ \mathbf{q}_2 \cdot \mathbf{a}_2 = 0 & (2) \end{cases} \tag{3.91}$$

It is clear that (3.91) is the fifth case of VTE when $m=0$ as shown in (3.40). Here $\mathbf{I}$ is a function of θ_1:

$$\mathbf{I} = S_2\mathbf{a}_2 - p_1\mathbf{q}_1 - S_1\mathbf{a}_1 - p_4\mathbf{q}_4 \tag{3.92}$$

$$\begin{cases} \mathbf{q}_1 = c\theta_1\mathbf{q}_4 + s\theta_1\mathbf{a}_1\times\mathbf{q}_4 \\ \mathbf{a}_2 = c\alpha_{12}\mathbf{a}_1 + s\alpha_{12}\mathbf{q}_1\times\mathbf{a}_1 \\ \quad = c\alpha_{12}\mathbf{a}_1 + s\theta_1 s\alpha_{12}\mathbf{q}_4 - c\theta_1 s\alpha_{12}\mathbf{a}_1\times\mathbf{q}_4 \end{cases} \tag{3.92a}$$

Substituting (3.92a) into (3.92) yields

$$\mathbf{I} = \mathbf{I}(\theta_1) = k_1\mathbf{a}_1 + k_2\mathbf{q}_4 + k_3\mathbf{a}_1\times\mathbf{q}_4 \tag{3.93}$$

where

$$\begin{cases} k_1 = (S_2 c\alpha_{12} - S_1) \\ k_2 = (S_2 s\alpha_{12} s\theta_1 - p_1 c\theta_1 - p_4) \\ k_3 = -(S_2 s\alpha_{12} c\theta_1 - p_1 s\theta_1) \end{cases} \tag{3.93a}$$

Vector $\mathbf{q}_3$ can be expressed in terms of $\mathbf{a}_2$ and $\mathbf{q}_2$:

$$\mathbf{q}_3 = \rho_1\mathbf{a}_2 + \rho_2\mathbf{q}_2 + \rho_3\mathbf{a}_2\times\mathbf{q}_2 \tag{3.94}$$

The scalar product of $\{\mathbf{a}_2, \mathbf{q}_2, \mathbf{a}_2\times\mathbf{q}_2\}$ and (3.94) yields

$$\begin{cases} \rho_1 = (\mathbf{a}_2\cdot\mathbf{q}_3) \\ \rho_2 = (\mathbf{q}_2\cdot\mathbf{q}_3) \\ \rho_3 = (\mathbf{a}_2\times\mathbf{q}_2\cdot\mathbf{q}_3) \end{cases} \tag{3.94a}$$

Using the results of §3.3.5 we obtain

$$\mathbf{q}_2 = \beta_1\mathbf{a}_2 + \beta_2\mathbf{I} + \beta_3\mathbf{a}_2\times\mathbf{I} \tag{3.95}$$

where

$$\begin{cases} \beta_1 = -(\mathbf{a}_2\cdot\mathbf{I})(\mathbf{q}_2\cdot\mathbf{I}) \,/\, [\mathbf{I}^2 - (\mathbf{a}_2\cdot\mathbf{I})^2] \\ \beta_2 = (\mathbf{q}_2\cdot\mathbf{I}) \,/\, [\mathbf{I}^2 - (\mathbf{a}_2\cdot\mathbf{I})^2] \\ \beta_3 = (\mathbf{a}_2\times\mathbf{I}\cdot\mathbf{q}_2) \,/\, [\mathbf{I}^2 - (\mathbf{a}_2\cdot\mathbf{I})^2] \end{cases} \tag{3.95a}$$

$$\begin{cases} (\mathbf{a}_2 \times \mathbf{I} \cdot \mathbf{q}_2) = \pm\sqrt{\mathbf{I}^2 - (\mathbf{a}_2 \cdot \mathbf{I})^2 - (\mathbf{q}_2 \cdot \mathbf{I})^2} \\ (\mathbf{q}_2 \cdot \mathbf{I}) = (\mathbf{I}^2 + p_2^2 - p_3^2)\,/\,(2p_2) \end{cases} \tag{3.95b}$$

and

$$\begin{cases} (\mathbf{a}_2 \cdot \mathbf{I}) = S_2 - S_1 c\alpha_{12} - p_4 s\theta_1 s\alpha_{12} \\ \mathbf{I}^2 = S_2^2 + p_1^2 + S_1^2 + p_4^2 - 2S_1S_2c\alpha_{12} \\ \quad -2p_4S_2s\theta_1 s\alpha_{12} - 2p_1p_4c\theta_1 \end{cases} \tag{3.95c}$$

Substituting (3.93) and (3.94) into (3.95) yields

$$\mathbf{q}_2 = m_1\mathbf{a}_1 + m_2\mathbf{q}_4 + m_3\mathbf{a}_1 \times \mathbf{q}_4 \tag{3.96}$$

$$\begin{cases} m_1 = \beta_1 c\alpha_{12} + \beta_2 k_1 + \beta_3(k_3 s\theta_1 + k_2 c\theta_1)s\alpha_{12} \\ m_2 = \beta_1 s\theta_1 s\alpha_{12} + \beta_2 k_2 - \beta_3(k_1 c\theta_1 s\alpha_{12} + k_3 c\alpha_{12}) \\ m_3 = -\beta_1 c\theta_1 s\alpha_{12} + \beta_2 k_3 + \beta_3(k_2 c\alpha_{12} - k_1 s\theta_1 s\alpha_{12}) \end{cases} \tag{3.96a}$$

Substituting (3.92a) and (3.96) into (3.94) yields

$$\mathbf{q}_5 = n_1\mathbf{a}_1 + n_2\mathbf{q}_4 + n_3\mathbf{a}_1 \times \mathbf{q}_4 \tag{3.97}$$

$$\begin{cases} n_1 = \rho_1 c\alpha_{12} + \rho_2 m_1 + \rho_3(m_3 s\theta_1 + m_2 c\theta_1)s\alpha_{12} \\ n_2 = \rho_1 s\theta_1 s\alpha_{12} + \rho_2 m_2 - \rho_3(m_1 c\theta_1 s\alpha_{12} + m_3 c\alpha_{12}) \\ n_3 = -\rho_1 c\theta_1 s\alpha_{12} + \rho_2 m_3 + \rho_3(m_2 c\alpha_{12} - m_1 s\theta_1 s\alpha_{12}) \end{cases} \tag{3.97a}$$

Substituting (3.92a), (3.96) and (3.97) into (3.90) yields

$$\mathbf{R} = r_1\mathbf{a}_1 + r_2\mathbf{q}_4 + r_3\mathbf{a}_1 \times \mathbf{q}_4 \tag{3.98}$$

$$\begin{cases} r_1 = -S_2 c\alpha_{12} + p_{21} m_1 + p_5 n_1 \\ r_2 = p_1 c\theta_1 - S_2 s\theta_1 s\alpha_{12} + p_{21} m_2 + p_5 n_2 \\ r_3 = p_1 s\theta_1 + S_2 c\theta_1 s\alpha_{12} + p_{21} m_3 + p_5 n_3 \end{cases} \tag{3.98a}$$

3.4.2. Analysis of an R_0-RSC mechanism

The diagram of the R_0-RSC mechanism is shown in Fig. 3.5. Angle θ_1 is the input angle controlled by an actuator. When θ_1 varies, an output of translation and rotation is obtained at the cylindrical pair. It is required to find the displacement relationship, i.e. output in terms of the input.

The problem can be modeled as follows. The unknowns are $\{x, \mathbf{q}_2, \mathbf{q}_3\}$; and $\{p_2, p_3, \mathbf{a}_3, \mathbf{I}, \mathbf{a}_2\}$ are known.

$$\begin{cases} p_2\mathbf{q}_2 + p_3\mathbf{q}_3 + x\,\mathbf{a}_3 = \mathbf{I} \\ \mathbf{q}_2 \cdot \mathbf{a}_2 = 0 \\ \mathbf{q}_3 \cdot \mathbf{a}_3 = 0 \end{cases} \tag{3.99}$$

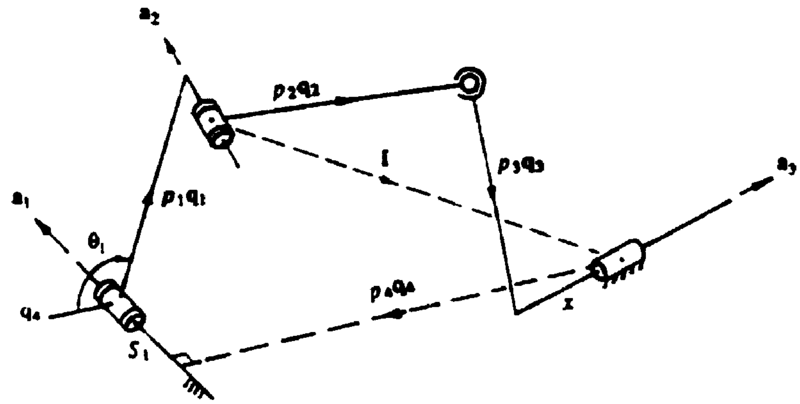


Fig. 3.5

where

$$\mathbf{I} = S_2\mathbf{a}_2 - p_1\mathbf{q}_1 - S_1\mathbf{a}_1 - p_4\mathbf{q}_4 \tag{3.100}$$

and

$$\begin{cases} \mathbf{a}_2 = \cos\alpha_{12}\,\mathbf{a}_1 + \sin\alpha_{12}\,\mathbf{q}_1\times\mathbf{a}_1 \\ \mathbf{q}_1 = \cos\theta_1\,\mathbf{q}_4 + \sin\theta_1\,\mathbf{a}_1\times\mathbf{q}_4 \end{cases} \tag{3.100a}$$

Clearly (3.99) is just a special version of the eighth case of VTE when $m = n = 0$, as shown in (3.65).

3.4.3. Inverse kinem. 'ic analysis of an RRPRRR robot

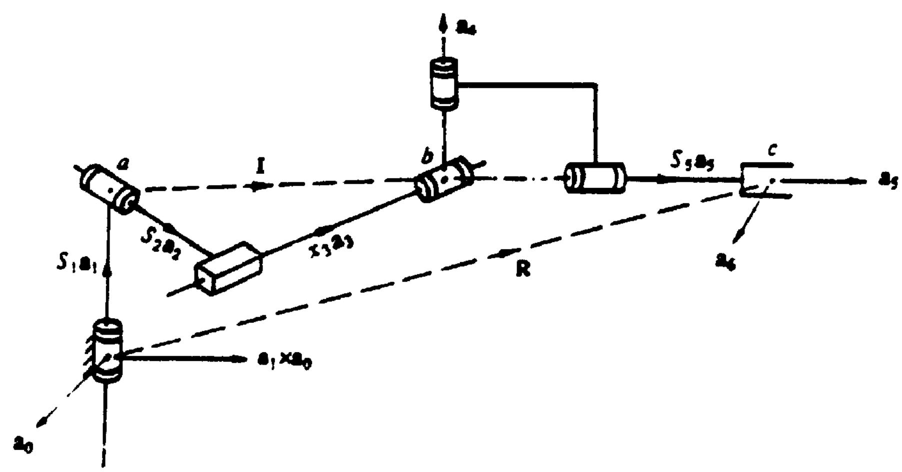


Fig. 3.6

The diagram of the RRPRRR robot is shown in Fig. 3.6. Given (a) the location of the robot and (b) the position and orientation of the end-effector, it is required to determine the configuration of the robot arm.

The problem can be formulated as follows. The unknowns are $\{\mathbf{a}_2, x_3, \mathbf{a}_3, \mathbf{a}_4\}$; and $\{S_2, \mathbf{I}, S_1\mathbf{a}_1, S_5\mathbf{a}_5, \mathbf{a}_6, \mathbf{R}\}$ are known.

$$\begin{cases} S_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{I} & (1) \\ \mathbf{a}_i \cdot \mathbf{a}_{i+1} = 0 \quad (i = 1, 2, 3, 4) & (2) \end{cases} \tag{3.101}$$

where

$$\mathbf{I} = \mathbf{R} - S_1\mathbf{a}_1 - S_5\mathbf{a}_5 \tag{3.102}$$

$\mathbf{I}$ is measured from a to b and $(S_5\mathbf{a}_5)$ from b to c, as shown in Fig. 3.6.

One vector equation is equivalent to three scalar equations, so (3.101) is equivalent to a total of 7 scalar equations, which is just the necessary number of equations to solve for the one unknown scalar x_3 and the three unknown unit vectors $\{\mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$, for each unknown unit vector can be defined in terms of two unknown scalars. Because $\mathbf{a}_2 \cdot \mathbf{a}_3 = 0$, the three vectors of (3.101-1) constitute a right-angled triangle.

Equation (3.101) has no exact corresponding equivalence in the nine cases of VTE. However, the vector tetrahedron equation (3.101-1) is obviously the same as (3.23-1); the difference of the two sets of equations (3.23) and (3.101) are their constraint equations.

Squaring both sides of (3.101-1) yields

$$x_3 = \sqrt{I^2 - S_2^2} \tag{3.103}$$

The minus sign in front of the square root is omitted, for it is obviously unreasonable according to the structure of the robot as shown in Fig. 3.6.

Vectors $\mathbf{I}$ and $\mathbf{a}_1$ are known vectors and $\mathbf{I} \times \mathbf{a}_1 \neq 0$. Let

$$\mathbf{a}_2 = \xi \mathbf{I} + \eta \mathbf{a}_1 + \zeta \mathbf{I} \times \mathbf{a}_1 \tag{3.104}$$

From the scalar product of $\{[\mathbf{a}_1 \times (\mathbf{I} \times \mathbf{a}_1)], [(\mathbf{I} \times \mathbf{a}_1) \times \mathbf{I}], (\mathbf{I} \times \mathbf{a}_1)\}$ with both sides of (3.104), and considering that $(\mathbf{I} \cdot \mathbf{a}_2) = S_2$, we obtain

$$\begin{cases} \xi = S_2 / [I^2 - (\mathbf{I} \cdot \mathbf{a}_1)^2] \\ \eta = -(\mathbf{I} \cdot \mathbf{a}_1) S_2 / [I^2 - (\mathbf{I} \cdot \mathbf{a}_1)^2] \\ \zeta = [\mathbf{a}_2 \cdot (\mathbf{I} \times \mathbf{a}_1)] / [I^2 - (\mathbf{I} \cdot \mathbf{a}_1)^2] \end{cases} \tag{3.104a}$$

$[\mathbf{a}_2 \cdot (\mathbf{I} \times \mathbf{a}_1)]$ of (3.104a) is unknown. Using (3.7) we get

$$[\mathbf{a}_2 \cdot (\mathbf{I} \times \mathbf{a}_1)] = \sqrt{I^2 - S_2^2 - (\mathbf{I} \cdot \mathbf{a}_1)^2} \tag{3.104b}$$

Substituting (3.104b) into (3.104a) yields $\{\xi, \eta, \zeta\}$, and from (3.104) we obtain $\mathbf{a}_2$.

From (3.101-1) we have

$$\mathbf{a}_3 = (\mathbf{I} - S_2\mathbf{a}_2)/x_3 \qquad (3.105)$$

From (3.101-2) or directly from the diagram of the robot we can see that

$$\mathbf{a}_4 = \pm\, \mathbf{a}_3 \times \mathbf{a}_5 \qquad (3.106)$$

If $\mathbf{a}_3$ and $\mathbf{a}_5$ are coaxial, i.e. $|\mathbf{a}_3 \cdot \mathbf{a}_5| = 1$, then $\mathbf{a}_4$ can be any direction perpendicular to $\mathbf{a}_3$, namely

$$\mathbf{a}_4 = (\mathbf{a}_3 \times \mathbf{e}) / |\mathbf{a}_3 \times \mathbf{e}| \qquad (3.107)$$

where $\mathbf{e}$ can be any vector.

3.4.4. Inverse kinematic analysis of a 6R robot

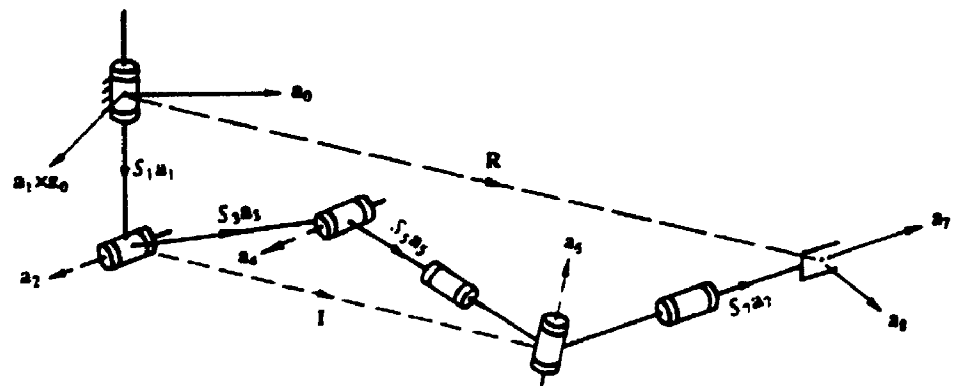


Fig. 3.7

The diagram of the 6R robot is shown in Fig. 3.7. Given (a) the robot location and (b) the position and orientation of the end-effector, it is required to determine the configuration of the robot arm.

The problem can be formulated as follows: the unknowns are $\{\mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_5, \mathbf{a}_6\}$; and $\{S_3, S_5, \mathbf{I}, \mathbf{a}_1, \mathbf{a}_7\}$ are known.

$$\begin{cases} S_3\mathbf{a}_3 + S_5\mathbf{a}_5 = \mathbf{I} & (1) \\ \mathbf{a}_1 \cdot \mathbf{a}_2 = 0 & (2) \\ \mathbf{a}_2 \cdot \mathbf{a}_3 = 0 & (3) \\ \mathbf{a}_2 \cdot \mathbf{a}_5 = 0 & (4) \\ \mathbf{a}_5 \cdot \mathbf{a}_6 = 0 & (5) \\ \mathbf{a}_6 \cdot \mathbf{a}_7 = 0 & (6) \end{cases} \qquad (3.108)$$

The relationship of (3.108) and (3.40) is similar to the relationship of (3.97) and (3.23).

Let $\mathbf{a}_0$ be any fixed unit vector perpendicular to $\mathbf{a}_1$, then $\mathbf{a}_2$ can be expressed as,

$$\mathbf{a}_2 = \cos(\widehat{\mathbf{a}_0,\mathbf{a}_2})\,\mathbf{a}_0 + \sin(\widehat{\mathbf{a}_0,\mathbf{a}_2})\,\mathbf{a}_1\times\mathbf{a}_0 \tag{3.109}$$

where $(\widehat{\mathbf{a}_0,\mathbf{a}_2})$ is the right hand rotation angle from $\mathbf{a}_0$ to $\mathbf{a}_2$ about $\mathbf{a}_1$. Taking the scalar product of $\mathbf{a}_2$ and (3.108-1) yields

$$\mathbf{I}\cdot\mathbf{a}_2 = 0 \tag{3.110}$$

Substituting (3.109) into (3.110) yields

$$(\widehat{\mathbf{a}_0,\mathbf{a}_2}) = \tan^{-1}[-(\mathbf{I}\cdot\mathbf{a}_0)/(\mathbf{I}\cdot\mathbf{a}_1\times\mathbf{a}_0)] \tag{3.111}$$

Substituting (3.111) into (3.109) yields $\mathbf{a}_2$.

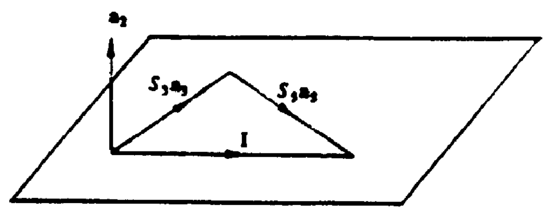


Fig.3.7(a)

Geometrically, the relationship of $\mathbf{a}_2$ to elements of (3.108-1) can be shown in Fig. 3.7(a), where vectors $\mathbf{a}_3$, $\mathbf{a}_5$ and $\mathbf{I}$ are coplanar and $\mathbf{a}_2$ is perpendicular to the plane. Let

$$\mathbf{a}_3 = x\,\mathbf{I} + y\,\mathbf{a}_2\times\mathbf{I} \tag{3.112}$$

From the scalar product of $\mathbf{I}$ and (3.112), and squaring both sides of (3.112) yields

$$\begin{cases} x = (\mathbf{a}_3\cdot\mathbf{I})/\mathbf{I}^2 \\ y = \pm\sqrt{[\mathbf{I}^2-(\mathbf{a}_3\cdot\mathbf{I})^2]/\{[\mathbf{I}^2-(\mathbf{a}_2\cdot\mathbf{I})^2]\mathbf{I}^2\}} \end{cases} \tag{3.112a}$$

$(\mathbf{a}_3\cdot\mathbf{I})$ of (3.112a) is unknown, however, it can be determined easily. From (3.108-1) we have

$$(\mathbf{I} - S_3\mathbf{a}_3)^2 = (S_5\mathbf{a}_5)^2$$

i.e.

$$(\mathbf{a}_3\cdot\mathbf{I}) = (\mathbf{I}^2 + S_3^2 - S_5^2)/(2S_3) \tag{3.113}$$

Substituting (3.113) into (3.112a), and further substituting (3.112a) into (3.112) yields $\mathbf{a}_3$. From (3.108-1) we have

$$\mathbf{a}_5 = (\mathbf{I} - S_3\mathbf{a}_3)/S_5 \tag{3.114}$$

From (3.108-5) and (3.108-6), or from the figure of the robot we can see that $\mathbf{a}_6=(\mathbf{a}_5\times\mathbf{a}_7)/|\mathbf{a}_5\times\mathbf{a}_7|$, namely

$$\mathbf{a}_6 = (\mathbf{a}_5\times\mathbf{a}_7)/[1-(\mathbf{a}_5\cdot\mathbf{a}_7)^2] \tag{3.115}$$

3.5. Conclusion

In this chapter, a series of systematic analysis steps were developed for solving the VTEs problems.

The advantages of the new modelings, as compared to previous modelings ([9] Chace), have been stated at the beginning of this chapter. From the above analysis, one additional conclusion can be drawn: to solve the VTE problems, what matters is the *direct* relationship between the vectors *within* the system, and each vector within the system can be *directly* related to the others. Thus the complication of carefully choosing proper (right-angled) Cartesian reference frames, in order to define the vectors using spherical coordinates, is avoided. The actual directions of all the vectors of a system relative to an external fixed reference frame is unimportant.

CHAPTER 4. PLANAR MECHANISMS

4.1. Introduction

Some planar mechanisms are analyzed using vector algebraic method. As compared to other methods, the approach in this chapter is more flexible and simpler.

4.2. Vector triangle equations

A large number of displacement analysis problems in planar mechanisms can be formulated by the following *vector triangle equation*

$$p_1\mathbf{q}_1 + p_2\mathbf{q}_2 = \mathbf{I} \tag{4.1}$$

A planar vector equation is equivalent to two scalar equations. In a given problem, any two of the four quantities $\{p_1, \mathbf{q}_1, p_2, \mathbf{q}_2\}$ may be unknown. Thus four distinct problems may exist. Namely

Case 1. $\{p_1, \mathbf{q}_1\}$ are unknown;

Case 2. $\{p_1, p_2\}$ are unknown;

Case 3. $\{p_1, \mathbf{q}_2\}$ are unknown;

Case 4. $\{\mathbf{q}_1, \mathbf{q}_2\}$ are unknown.

4.2.1. The solution for case 1

If $\{x, \mathbf{q}_1\}$ are unknown and $\{\mathbf{I}\}$ is known

$$x\,\mathbf{q}_1 = \mathbf{I} \tag{4.2}$$

Solution:

This case is trivial. From (4.2) we can directly obtain

$$\begin{cases} x = |\mathbf{I}| \\ \mathbf{q}_1 = \mathbf{I} / |\mathbf{I}| \end{cases} \tag{4.3}$$

4.2.2. The solution for case 2.

If $\{x, y\}$ are unknown and $\{\mathbf{q}_1, \mathbf{q}_2, \mathbf{I}\}$ are known.

$$x\,\mathbf{q}_1 + y\,\mathbf{q}_2 = \mathbf{I} \tag{4.4}$$

Solution (The first approach):

The scalar product of $\{q_1, q_2\}$ with both sides of (4.4) yields

$$\begin{cases} x + (q_1 \cdot q_2) y = (q_1 \cdot I) \\ (q_1 \cdot q_2) x + y = (q_2 \cdot I) \end{cases} \tag{4.5}$$

Solving (4.5) yields

$$\begin{cases} x = [(q_1 \cdot I) - (q_1 \cdot q_2)(q_2 \cdot I)] / [1 - (q_1 \cdot q_2)^2] \\ y = [(q_2 \cdot I) - (q_1 \cdot q_2)(q_1 \cdot I)] / [1 - (q_1 \cdot q_2)^2] \end{cases} \tag{4.6}$$

Solution (The second approach):

Suppose n is a unit vector that is perpendicular to the plane of the vector triangle. Dot product $n \times q_2$ with both sides of (4.4) yields

$$x = (n \times q_2 \cdot I) / (n \times q_2 \cdot q_1) \tag{4.7}$$

Dot product $n \times q_1$ with both sides of (4.4) yields

$$y = (n \times q_1 \cdot I) / (n \times q_1 \cdot q_2) \tag{4.8}$$

4.2.3. The solution for case 3

If $\{x, q_2\}$ are unknown and $\{q_1, p_2, I\}$ are known.

$$x\, q_1 + p_2 q_2 = I \tag{4.9}$$

Solution:

Rearranging (4.9),

$$x\, q_1 - I = -p_2 q_2 \tag{4.10}$$

Squaring both sides of (4.10) yields

$$x^2 - 2(q_1 \cdot I) x + (I^2 - p_2^2) = 0 \tag{4.11}$$

Solving (4.11) yields

$$x = (q_1 \cdot I) \pm \sqrt{(q_1 \cdot I)^2 - (I^2 - p_2^2)} \tag{4.12}$$

Substituting (4.12) into (4.10) yields

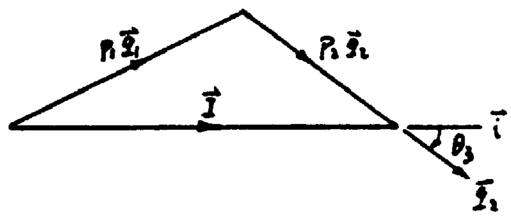
$$q_2 = (I - x\, q_1) / p_2 \tag{4.13}$$

4.2.4. The solution for case 4

If $\{\mathbf{q}_1, \mathbf{q}_2\}$ are unknown and $\{p_1, p_2, \mathbf{l}\}$ are known.

$$p_1\mathbf{q}_1 + p_2\mathbf{q}_2 = \mathbf{l} \tag{4.14}$$

Solution (The first approach):



Denote $\mathbf{l}=l\,\mathbf{i}$. Let θ_3 be the angle between $\mathbf{l}$ and $\mathbf{q}_2$ as shown in Fig. 4.1, then the vector $\mathbf{q}_2$ can be expressed as

$$\mathbf{q}_2 = c\theta_3\mathbf{i} + s\theta_3\mathbf{n}\times\mathbf{i} \tag{4.15}$$

Rearranging (4.14),

$$\mathbf{l} - p_2\mathbf{q}_2 = p_1\mathbf{q}_1 \tag{4.16}$$

Squaring both sides of (4.16) yields

$$(\mathbf{l}\cdot\mathbf{q}_2) = (\mathbf{l}^2 + p_2^2 - p_1^2)/(2p_2) \tag{4.17}$$

Substituting (4.15) into (4.17) yields

$$A\cos\theta_3 + B\sin\theta_3 = C \tag{4.18}$$

$$\begin{cases} A = l \\ B = 0 \\ C = (\mathbf{l}^2 + p_2^2 - p_1^2) \end{cases} \tag{4.18a}$$

From (4.18) we obtain

$$\theta_3 = \cos^{-1}(C/A) = \cos^{-1}[(l^2 + p_2^2 - p_1^2)/l] \tag{4.19}$$

Substituting (4.19) into (4.15) yields $\mathbf{q}_2$. Then from (4.14) we obtain

$$\mathbf{q}_1 = (\mathbf{l} - p_2\mathbf{q}_2)/p_1 \tag{4.20}$$

Solution (The second approach):

Let

$$\mathbf{q}_1 = x_1\mathbf{i} + x_2\mathbf{n}\times\mathbf{i} \tag{4.21}$$

Dot product $\mathbf{i}$ with both sides of (4.21) yields

$$x_1 = (\mathbf{q}_1 \cdot \mathbf{i}) \tag{4.22}$$

Squaring both sides of (4.21) yields

$$x_2 = \pm\sqrt{1 - x_1^2} \tag{4.23}$$

Rearranging (4.14)

$$l\mathbf{i} - p_1\mathbf{q}_1 = p_2\mathbf{q}_2 \tag{4.24}$$

Squaring both sides of (4.24) yields

$$(\mathbf{q}_1 \cdot \mathbf{i}) = (p_2^2 - p_1^2 - l^2)/(2 l p_1) \tag{4.25}$$

Substituting (4.25) into (4.22) yields x_1, then from (4.23) we get x_2, thus $\mathbf{q}_1$ is determined from (4.21). Finally, $\mathbf{q}_2$ is obtained from (4.14)

$$\mathbf{q}_2 = (\mathbf{l} - p_1\mathbf{q}_1)/p_2 \tag{4.26}$$

4.3.1. Analysis of the P_0–PP mechanism

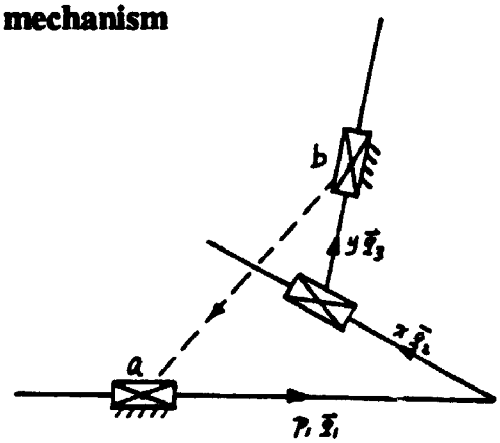


Fig. 4.2

The diagram of the P_0–PP mechanism is shown in Fig. 4.2, where the input and the output are translational motions at a and b, respectively. We can formulate the problem as follow: the unknowns are $\{x, y\}$; and $\{\mathbf{q}_2, \mathbf{q}_3, \mathbf{I}\}$ are known.

$$x\mathbf{q}_2 + y\,\mathbf{q}_3 = \mathbf{I} \tag{4.27}$$

$$\mathbf{I} = -(\mathbf{R} + p_1\mathbf{q}_1) \tag{4.27a}$$

Solution:

This is the second of the *vector triangle equation*. Using $\mathbf{q}_2$ and $\mathbf{q}_3$ to dot product both sides of (1) yields

$$\begin{cases} x + (\mathbf{q}_2 \cdot \mathbf{q}_3)y = (\mathbf{q}_2 \cdot \mathbf{I}) \\ (\mathbf{q}_2 \cdot \mathbf{q}_3)x + y = (\mathbf{q}_3 \cdot \mathbf{I}) \end{cases} \tag{4.28}$$

Solving (4.28) yields

$$\begin{cases} x = [(\mathbf{q}_2 \cdot \mathbf{I}) - (\mathbf{q}_2 \cdot \mathbf{q}_3)(\mathbf{q}_3 \cdot \mathbf{I})] / [1 - (\mathbf{q}_2 \cdot \mathbf{q}_3)^2] \\ y = [(\mathbf{q}_3 \cdot \mathbf{I}) - (\mathbf{q}_2 \cdot \mathbf{q}_3)(\mathbf{q}_2 \cdot \mathbf{I})] / [1 - (\mathbf{q}_2 \cdot \mathbf{q}_3)^2] \end{cases} \tag{4.29}$$

4.3.2. Analysis of the R_0-$(RP)P$ mechanism

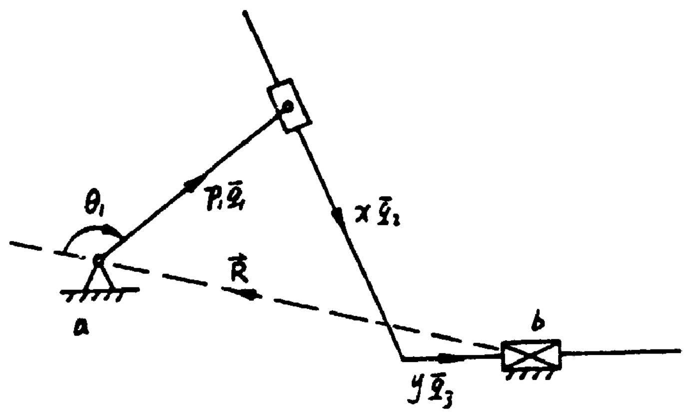


Fig.4.3

The diagram of the R_0-$(RP)P$ mechanism is shown in Fig. 4.3, where the input and the output are rotational and translational motions at a and b, respectively. We can formulate the problem as follow: the unknowns are $\{x, y\}$; and $\{\theta_1, p_1\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{R}\}$ are known.

$$x\,\mathbf{q}_2 + y\,\mathbf{q}_3 = \mathbf{I} \tag{4.30}$$

$$\begin{cases} \mathbf{I} = -(p_1\mathbf{q}_1 + \mathbf{R}) \\ \mathbf{q}_1 = \cos\theta_1\,\mathbf{r} + \sin\theta_1\,\mathbf{n}\times\mathbf{r} \end{cases} \tag{4.30a}$$

Obviously this problem is identical to the one in section 4.3.1.

4.3.3. Analysis of the cam mechanism R_0-$(RP)P$

The diagram of the cam (mechanism) R_0-$(RP)P$ is shown in Fig. 4.4, where the input and the output are rotational and translational motions at a and b, respectively. We can formulate the problem as follow: the unknowns are $\{x, y\}$; and $\{\theta_1, p_1\mathbf{q}_1, p_2\mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4, \mathbf{R}\}$ are known.

$$x\,\mathbf{q}_3 + y\,\mathbf{q}_4 = \mathbf{I} \tag{4.31}$$

$$\begin{cases} \mathbf{I} = -(p_1\mathbf{q}_1 + p_2\mathbf{q}_2 + \mathbf{R}) \\ \mathbf{q}_1 = \cos\theta_1\,\mathbf{r} - \sin\theta_1\,\mathbf{n}\times\mathbf{r} \end{cases} \tag{4.31a}$$

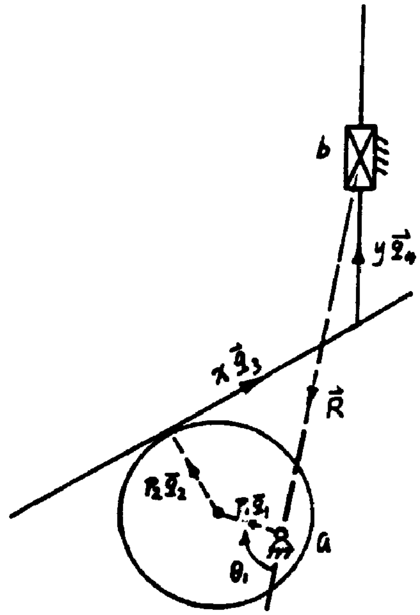


Fig. 4.4

Obviously this problem is also identical to the one in section 4.3.1.

4.4 Analysis of the P_0-$(RP)R$ mechanism

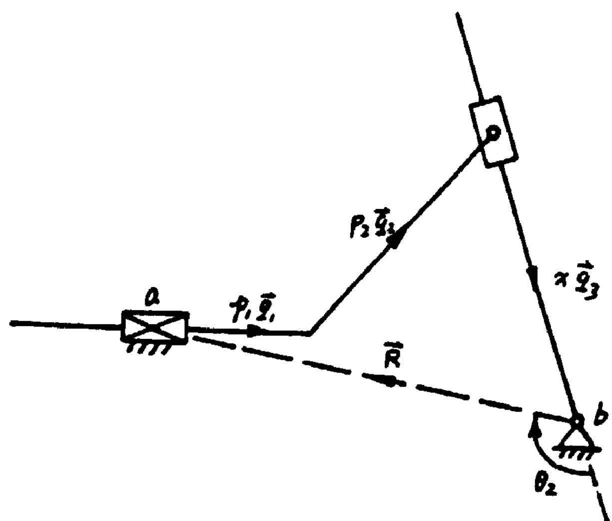


Fig. 4.5

The diagram of the P_0-$(RP)R$ mechanism is shown in Fig. 4.5, where the input and the output are translational and rotational motions at a and b, respectively. We can formulate the problem as follow: the unknowns are $\{x, \theta_2\}$; and $\{p_1\mathbf{q}_1, p_2\mathbf{q}_2, \mathbf{R}\}$ are known.

$$x\,\mathbf{q}_3 = \mathbf{l} \tag{4.32}$$

$$\begin{cases} \mathbf{l} = -p_1\mathbf{q}_1 - (p_2\mathbf{q}_2 + \mathbf{R}) \\ \mathbf{q}_3 = c\theta_2\,\mathbf{r} + s\theta_2\,\mathbf{n}\times\mathbf{r} \end{cases} \tag{4.32a}$$

Solution:

This is the first case of the *vector triangle equation*. Squaring both sides of (4.32) yields

$$x = \pm\sqrt{\mathbf{I}^2} = \pm\sqrt{p_1^2+p_2^2+R^2+2p_1p_2(\mathbf{q}_1\cdot\mathbf{q}_2)+2p_1R(\mathbf{q}_1\cdot\mathbf{r})+2p_2R(\mathbf{q}_2\cdot\mathbf{r})} \tag{4.33}$$

From (4.33) we obtain

$$\mathbf{q}_3 = \mathbf{I}/x \tag{4.34}$$

Substituting (4.34) into the second equation of (4.32a) yields

$$c\theta_2\mathbf{r} + s\theta_2\mathbf{n}\times\mathbf{r} = \mathbf{I}/x \tag{4.35}$$

The scalar product of $\{\mathbf{r}, \mathbf{n}\times\mathbf{r}\}$ with both sides of (4.35) yields

$$\begin{cases} \cos\theta_2 = (\mathbf{r}\cdot\mathbf{I})/x \\ \sin\theta_2 = (\mathbf{n}\times\mathbf{r}\cdot\mathbf{I})/x \end{cases} \tag{4.36}$$

4.5 Analysis of the R_0-$R(RP)$ mechanism

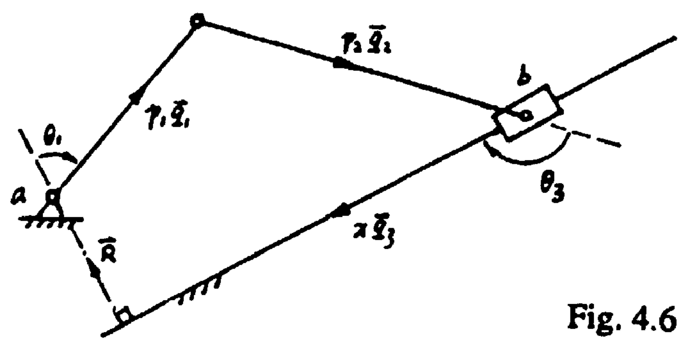


Fig. 4.6

The diagram of the R_0-$R(RP)$ mechanism is shown in Fig. 4.6, where the input is θ_1. Let $\theta_3=\Theta_3$ be the output angle, the first rotary variable to be determined. Then the vector loop equation of the mechanism can be written as

$$x\,\mathbf{q}_3+\mathbf{I}+\mathbf{J} = 0 \tag{4.37}$$

$$\begin{cases} \mathbf{J} = p_2\mathbf{q}_2 & (1) \\ \mathbf{I} = p_1\mathbf{q}_1+\mathbf{R} & (2) \end{cases} \tag{4.37a}$$

$$\begin{cases} \mathbf{q}_3 = c\Theta_3\mathbf{q}_3-s\Theta_3\mathbf{n}\ \ \mathbf{q}_3 & (1) \\ \mathbf{q}_1 = c\theta_1\mathbf{r}-s\theta_1\mathbf{n}\times\mathbf{r} & (2) \end{cases} \tag{4.37b}$$

Solution (The first approach):

This is the third case of the *vector triangle equation*. The scalar product of $\mathbf{r}$ with both sides of (4.37) yields

$$\mathbf{r}\cdot\mathbf{J} = -(\mathbf{r}\cdot\mathbf{I}) \tag{4.38}$$

Substituting (4.37a-1) into (4.38) yields

$$\mathbf{U}\cdot\mathbf{q}_2 = V \tag{4.39}$$

$$\begin{cases} \mathbf{U} = p_2\mathbf{r} \\ V = -(\mathbf{r}\cdot\mathbf{I}) \end{cases} \tag{4.39a}$$

Substituting (4.37b-1) into (4.39) yields

$$A\cos\Theta_3 + B\sin\Theta_3 = C \tag{4.40}$$

$$\begin{cases} A = \mathbf{U}\cdot\mathbf{q}_3 = 0 \\ B = \mathbf{U}\cdot\mathbf{q}_3\times\mathbf{n} = -p_2 \\ C = V = -(p_1 c\theta_1 + R) \end{cases} \tag{4.40a}$$

From (4.40) we obtain

$$\Theta_3 = \sin^{-1}(C/B) = \sin^{-1}[(p_1 c\theta_1 + R)/p_2] \tag{4.41}$$

The scalar product of $\mathbf{q}_3$ with both sides of (4.37) yields

$$x = -\mathbf{q}_3\cdot(\mathbf{I}+\mathbf{J}) \tag{4.42}$$

Solution (The second approach):

Rearranging (4.37) yields

$$x\,\mathbf{q}_3 + \mathbf{I} = -p_2\mathbf{q}_2 \tag{4.43}$$

Squaring both sides of (4.43) yields

$$x^2 + 2(\mathbf{q}_3\cdot\mathbf{I})x + (\mathbf{I}^2 - p_2^2) = 0 \tag{4.44}$$

Solving (4.44) yields

$$x = (\mathbf{q}_3\cdot\mathbf{I}) \pm \sqrt{(\mathbf{q}_3\cdot\mathbf{I})^2 - (\mathbf{I}^2 - p_2^2)} \tag{4.45}$$

From (4.37) we obtain

$$\mathbf{q}_2 = -(x\,\mathbf{q}_3 + \mathbf{I})/p_2 \tag{4.46}$$

4.6. Analysis of the R_0-RRR mechanism

The diagram of the R_0-RRR mechanism is shown in Fig. 4.7, where the input is θ_1. Let $\theta_4 = \Theta_4$ be the output angle, then the vector loop equation of the mechanism can be written as

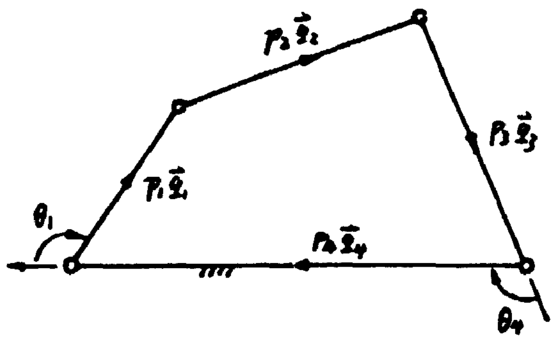


Fig. 4.7

$$\mathbf{J}+\mathbf{I}=\mathbf{F} \tag{4.47}$$

$$\begin{cases}\mathbf{J}=p_3\mathbf{q}_3 & (1)\\ \mathbf{I}=p_1\mathbf{q}_1+p_4\mathbf{q}_4 & (2)\\ \mathbf{F}=-p_2\mathbf{q}_2\end{cases} \tag{4.47a}$$

$$\begin{cases}\mathbf{q}_3=\cos\Theta_4\mathbf{q}_4-\sin\Theta_4\mathbf{n}\times\mathbf{q}_4 & (1)\\ \mathbf{q}_1=\cos\theta_1\mathbf{q}_4+\sin\theta_1\mathbf{n}\times\mathbf{q}_4 & (2)\end{cases} \tag{4.47b}$$

Solution:

This is the fourth case of the *vector triangle equation*. Squaring both sides of (4.47) yields

$$2\mathbf{I}\cdot\mathbf{J}=(\mathbf{F}^2-\mathbf{I}^2-\mathbf{J}^2) \tag{4.48}$$

Substituting (4.47a-1) into (4.48) yields

$$\mathbf{U}\cdot\mathbf{q}_3=V \tag{4.49}$$

$$\begin{cases}\mathbf{U}=2p_3\mathbf{I}\\ V=\mathbf{F}^2-\mathbf{I}^2-p_3^2\end{cases} \tag{4.49a}$$

Substituting (4.47b-1) into (4.49) yields

$$A\cos\Theta_4+B\sin\Theta_4=C \tag{4.50}$$

$$\begin{cases}A=\mathbf{U}\cdot\mathbf{q}_4\\ B=\mathbf{U}\cdot\mathbf{q}_4\times\mathbf{n}\\ C=V\end{cases} \tag{4.50a}$$

Let $y=\tan(\Theta_4/2)$, then we have

$$\begin{cases}\cos\Theta_4=(1-y^2)/(1+y^2)\\ \sin\Theta_4=2y/(1+y^2)\end{cases} \tag{4.51}$$

Substituting (4.51) into (4.50) yields

$$(A+C)y^2-2By+(C-A)=0 \tag{4.52}$$

Solving (4.52) we get y; then $\Theta_4 = 2\tan^{-1}y$, i.e.

$$\Theta_4 = 2\tan^{-1}[(B \pm \sqrt{A^2+B^2-C^2})/(A+C)] \tag{4.53}$$

4.7. Analysis of a multi-loop planar mechanism

The diagram of the multi-loop planar mechanism is shown in Fig. 4.8, where the input is θ_1. Unknowns are $\{x, y, q_2, q_4, q_5, q_6\}$.

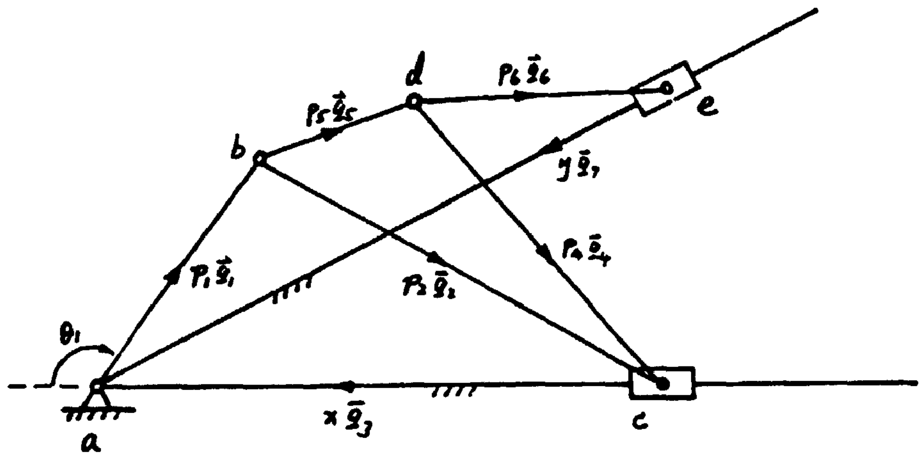


Fig. 4.8

Solution:

Let us first consider the loop $a-b-c-a$. The vector loop equation is

$$x\,q_3 + p_2 q_2 + p_1 q_1 = 0 \tag{4.54}$$

It is clear that this is the third case of the *vector triangle equation*. Rearranging (4.54) yields

$$x\,q_3 + p_1 q_1 = -p_2 q_2 \tag{4.55}$$

Squaring both sides of (4.55) yields

$$x^2 + 2p_1(q_1 \cdot q_3)x + (p_1^2 - p_2^2) = 0 \tag{4.56}$$

Solving (4.56) yields

$$x = -p_1(q_1 \cdot q_3) \pm \sqrt{p_1^2(q_1 \cdot q_3)^2 + p_2^2 - p_1^2} \tag{4.57}$$

Substituting (4.57) into (4.54) yields

$$q_2 = -(x\,q_3 + p_1 q_1)/p_2 \tag{4.58}$$

Since the links bc, cd and bd constitute a rigid body, the vectors q_4 and q_5 can be obtained directly,

$$\begin{cases} \mathbf{q}_4 = \cos\theta_{24}\mathbf{q}_2 + \sin\theta_{24}\mathbf{n}\times\mathbf{q}_2 \\ \mathbf{q}_5 = \cos\theta_{25}\mathbf{q}_2 + \sin\theta_{25}\mathbf{n}\times\mathbf{q}_2 \end{cases} \tag{4.59}$$

Now let us consider the loop $-b-d-e-a$. The vector loop equation can be written as

$$y\,\mathbf{q}_7 + p_6\mathbf{q}_6 + \mathbf{I} = 0 \tag{4.60}$$

$$\mathbf{I} = p_1\mathbf{q}_1 + p_5\mathbf{q}_5 \tag{4.60a}$$

Obviously this is also the third case of the *vector triangle equation*. Rearranging the terms of (4.60) yields

$$y\,\mathbf{q}_7 + \mathbf{I} = -p_6\mathbf{q}_6 \tag{4.61}$$

Squaring both sides of (4.61) yields

$$y^2 + 2(\mathbf{I}\cdot\mathbf{q}_7)y + (\mathbf{I}^2 - p_6^2) = 0 \tag{4.62}$$

Solving (4.62) yields

$$y = (\mathbf{I}\cdot\mathbf{q}_7) \pm \sqrt{(\mathbf{I}\cdot\mathbf{q}_7)^2 - (\mathbf{I}^2 - p_6^2)} \tag{4.63}$$

Substituting (4.63) into (4.60) yields

$$\mathbf{q}_6 = -(\mathbf{I} - y\,\mathbf{q}_7)/p_6 \tag{4.64}$$

4.8. Conclusion

There are only a few books specifically addressing the kinematics of spatial mechanisms. However, there are many books dealing with the kinematics of planar mechanisms. Based on the analysis in this chapter, there is evidence that the majority of the kinematics problems of planar mechanisms are simply covered by *only* four cases of the *vector triangle equation*.

Comparing the mechanisms in Figures 4.4, 4.5 and 4.6, the physical differences of the three planar mechanisms are quite obvious. However, the equations governing the motions of the mechanisms, Eqs.(4.27), (4.30) and (4.31), respectively, are the same. Consequently the analyses are also the same. Being able to expose the very essence of the kinematic features of mechanisms is one of the advantages of the *vector algebraic method*.

CHAPTER 5. CERTAIN SPATIAL MECHANISMS CONTAINING HIGHER PAIRS

5.1. Introduction

Kinematic pairs can be classified as lower or higher pairs. Those having surface contact between the two elements of a pair are lower pairs, whereas those with line or point contacts are higher pairs ([61] Reuleaux, 1876). Physically, a "lower pair can be more heavily loaded and is considerably more wear-resistant", whereas the advantage of higher pairs is that "their line or point contact can result in lower frictional losses" ([33] Hain, 1961).

The degree of freedom (DoF) of a lower pair is $1 \le DoF \le 3$, whereas that of a higher pair is $3 \le DoF \le 5$. Thus, from the kinematic point of view, higher pairs are generally more complex than lower pairs. This is probably one of the reasons that the overwhelming majority of the literature on spatial mechanisms has been devoted to the study of mechanisms employing lower pairs.

In this chapter certain spatial mechanisms containing higher pairs, such as S_G (Sphere-groove), B_B (Bar-bar) and S_P (Sphere-plane), are kinematically analyzed using the *vector algebraic method*.

Other pertinent research dealing with spatial mechanisms containing higher pairs include the (Direction Cosine) Matrix Method, developed by Denavit and Hartenberg [10], and the Differential Constraint Method proposed by Sandor and Kohli [64]. Using the Matrix Method, Denavit [11] solved the displacement problem of the R_0-$B_B R$ mechanism by replacing the higher pair (B_B) by three revolute and two prismatic pairs using (2x2) dual matrices. Beggs [4](pp.136-141) analyzed the same mechanism using (4x4) matrices and Zhang [106] examined the R_0-$B_B R$, R_0-$S_P R$ and R_0-$S_G R$ mechanisms by using (3x3) matrices. Using the differential constraint conditions for the higher pairs, Sandor and Kohli [64] studied the kinematic problems of the R_T-$S_G R$ and R_0-$S_P R$ mechanisms.

In order to utilize the homogeneous matrix transformation process using the matrix method to find the input/output equations for spatial mechanisms containing higher pairs, one must employ at least three Cartesian coordinate systems to specify the relative position and orientation of elements of the higher pair. As a result, the matrix analysis procedure becomes complex.

The approach of utilizing the differential constraint conditions of the (higher) pairs to analyze mechanisms is novel [64], but its applicability is limited. For the vast majority of spatial mechanisms, this method may not work.

The new approach presented in this chapter is free of the limitations stated above. It uses standardized vector expressions and operations. There is no need to specify the relative orientation of elements of higher pairs for the purpose of the analysis. Consequently, the new approach is more direct and much simpler than existing methods.

5.2. Standard analysis procedure

Using the vector algebraic method, the kinematic analysis problem can be formulated as follows: (1). Draw a simplified vector loop diagram based on the original diagram of the mechanism; (2). Find the input and the output vectors, then specify the vector loop equation; (3). Specify the direction equations, based on the vector loop diagram of the mechanism. At this point the problem is reduced to finding the solution of vector equations.

Before we carry out the detailed analysis for any specific mechanism, we will develop the canonical descriptions and solutions for two vector equations.

(1). The first vector equation:

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{I} \tag{5.1}$$

where $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{I}\}$ are given and $\{x_1, x_2, x_3\}$ are unknown.

Solution:

Take the scalar product of Eq.(5.1) in turn with $\{\mathbf{a}_2\times\mathbf{a}_3, \mathbf{a}_3\times\mathbf{a}_1, \mathbf{a}_1\times\mathbf{a}_2\}$ to yield

$$\begin{cases} x_1 = (\mathbf{a}_2\times\mathbf{a}_3\cdot\mathbf{I})/(\mathbf{a}_1\cdot\mathbf{a}_2\times\mathbf{a}_3) \\ x_2 = (\mathbf{a}_3\times\mathbf{a}_1\cdot\mathbf{I})/(\mathbf{a}_1\cdot\mathbf{a}_2\times\mathbf{a}_3) \\ x_3 = (\mathbf{a}_1\times\mathbf{a}_2\cdot\mathbf{I})/(\mathbf{a}_1\cdot\mathbf{a}_2\times\mathbf{a}_3) \end{cases} \tag{5.2}$$

(2). The second vector equation:

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \mathbf{K} = 0 \tag{5.3}$$

$$\begin{cases} \mathbf{K} = \mathbf{I} + \mathbf{J} & (1) \\ \mathbf{J} = \mathbf{J}(\mathbf{q}) & (2) \end{cases} \tag{5.3a}$$

$$\begin{cases} \mathbf{a}_2 = \mathbf{a}_2(\mathbf{q}) & (1) \\ \mathbf{q} = \cos\Theta\,\mathbf{e}_1 + \sin\Theta\,\mathbf{e}_2 & (2) \end{cases} \tag{5.3b}$$

where $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{I}, \mathbf{a}_1\}$ are given and $\{x_1, x_2, \Theta\}$ are unknown.

Solution:

(i). *Displacement analysis*.

The scalar product of Eq.(5.3) with $\mathbf{a}_1 \times \mathbf{a}_2$ yields

$$\mathbf{a}_1 \times \mathbf{a}_2 \cdot \mathbf{K} = 0 \tag{5.4}$$

Substituting Eqs.(5.3a) and (5.3b) into (5.4) gives

$$\mathbf{U} \cdot \mathbf{q} = V \tag{5.5}$$

$$\begin{cases} \mathbf{U} = (\textit{a known vector}) \\ V = (\textit{a known scalar}) \end{cases} \tag{5.5a}$$

Here the step from Eq.(5.4) to Eq.(5.5) may not seem clear, but for a real mechanism this step is feasible and can be easily accomplished.

Substituting Eq.(5.3b-2) into (5.5) yields

$$A\cos\Theta + B\sin\Theta = C \tag{5.6}$$

$$\begin{cases} A = \mathbf{U} \cdot \mathbf{e}_1 \\ B = \mathbf{U} \cdot \mathbf{e}_2 \\ C = V \end{cases} \tag{5.6a}$$

Let $y = \tan(\Theta/2)$, we have

$$\begin{cases} \cos\Theta = (1-y^2)/(1+y^2) \\ \sin\Theta = 2y/(1+y^2) \end{cases} \tag{5.7}$$

Substituting Eq.(5.7) into (5.6) yields

$$(A+C)y^2 - 2By + (C-A) = 0 \tag{5.8}$$

Solving Eq.(5.8) for y and substituting into $\Theta = 2\tan^{-1}y$ yields

$$\Theta = 2\tan^{-1}[(B \pm \sqrt{A^2+B^2-C^2})/(A+C)] \tag{5.9}$$

At this point, $\{x_1, x_2\}$ are the only unknowns in Eq.(5.3). In order to obtain the solution we can use the scalar product of two known vectors with Eq.(5.3). Here, for instance, we use the scalar products of $\mathbf{a}_1$ and $\mathbf{a}_2$ and obtain the solution as follows:

$$\begin{cases} x_1 = [(\mathbf{a}_1 \cdot \mathbf{a}_2)(\mathbf{K} \cdot \mathbf{a}_2) - (\mathbf{K} \cdot \mathbf{a}_1)] \,/\, [1 - (\mathbf{a}_1 \cdot \mathbf{a}_2)^2] \\ x_2 = [(\mathbf{a}_1 \cdot \mathbf{a}_2)(\mathbf{K} \cdot \mathbf{a}_1) - (\mathbf{K} \cdot \mathbf{a}_2)] \,/\, [1 - (\mathbf{a}_1 \cdot \mathbf{a}_2)^2] \end{cases} \tag{5.10}$$

(ii). *Velocity analysis*. Unknowns: $\{\dot{x}_1, \dot{x}_2, \dot{\Theta}\}$.

From the time derivative of Eq.(5.6)

$$\dot{\Theta} = (\dot{A}\cos\Theta + \dot{B}\sin\Theta - \dot{C})/(A\sin\Theta - B\cos\Theta) \tag{5.11}$$

$$\begin{cases} \dot{A} = \dot{\mathbf{U}} \cdot \mathbf{e}_1 \\ \dot{B} = \dot{\mathbf{U}} \cdot \mathbf{e}_2 \\ \dot{C} = \dot{V} \end{cases} \tag{5.11a}$$

From the time derivative of Eq.(5.3)

$$\dot{x}_1 \mathbf{a}_1 + \dot{x}_2 \mathbf{a}_2 + \mathbf{M} = 0 \tag{5.12}$$

where $\mathbf{M}$ is a known vector:

$$\mathbf{M} = x_1 \dot{\mathbf{a}}_1 + x_2 \dot{\mathbf{a}}_2 + \dot{\mathbf{K}} \tag{5.12a}$$

It is clear that Eq.(5.12) has the same form as Eq.(5.3). Thus, $\{\dot{x}_1, \dot{x}_2\}$ can be obtained by simply replacing $\mathbf{K}$ of Eq.(5.10) with $\mathbf{M}$ of Eq.(5.12a):

$$\begin{cases} \dot{x}_1 = [(\mathbf{a}_1 \cdot \mathbf{a}_2)(\mathbf{M}\cdot\mathbf{a}_2) - (\mathbf{M}\cdot\mathbf{a}_1)] \,/\, [1-(\mathbf{a}_1\cdot\mathbf{a}_2)^2] \\ \dot{x}_2 = [(\mathbf{a}_1 \cdot \mathbf{a}_2)(\mathbf{M}\cdot\mathbf{a}_1) - (\mathbf{M}\cdot\mathbf{a}_2)] \,/\, [1-(\mathbf{a}_1\cdot\mathbf{a}_2)^2] \end{cases} \tag{5.13}$$

Obviously, $\{\dot{x}_1, \dot{x}_2\}$ can also be obtained by differentiating Eq.(5.10).

(iii). *Acceleration analysis.* Unknowns: $\{\ddot{x}_1, \ddot{x}_2, \dot{\Theta}\}$.

From the second derivative of Eq.(5.6)

$$\ddot{\Theta} = \frac{(\ddot{A} + 2\dot{B}\dot{\Theta} - A\dot{\Theta}^2)\cos\Theta + (\ddot{B} - 2\dot{A}\dot{\Theta} - B\dot{\Theta}^2)\sin\Theta - \ddot{C}}{A\sin\Theta - B\cos\Theta} \tag{5.14}$$

where

$$\begin{cases} \ddot{A} = \ddot{\mathbf{U}}\cdot\mathbf{e}_1 \\ \ddot{B} = \ddot{\mathbf{U}}\cdot\mathbf{e}_2 \\ \ddot{C} = \ddot{V} \end{cases} \tag{5.14a}$$

Similarly, the second derivatives of Eqs.(5.3) and (5.10) yield $\{\ddot{x}_1, \ddot{x}_2\}$.

For a real mechanism vector $\mathbf{I}$ of Eq.(5.3a) usually represents an *input vector*; it is the sum of all known vectors at the very beginning. Vector $\mathbf{J}$ of Eq.(5.3a) is the *output vector*; it is the sum of all those constant-magnitude vectors of the kinematic loop of the mechanism that can be expressed as a function of the *output angle* Θ.

5.3. The R_0-S_pC mechanism

The R_0-S_pC mechanism is shown in Fig. 5.1(a), where ab, db and $\mathbf{a}_1$ are perpendicular to each other and bc and $\mathbf{a}_1$ are co-planar. The input of the mechanism is the rotation of ab around axis $\mathbf{a}_1$. The output is a translation and a rotation at h, where the rotation is an idle degree of freedom. To analyze the mechanism by using the vector algebraic method, the first step is to construct a vector loop diagram as shown in Fig. 5.1(b). The loop is a-b-e-f-g-h-a, where point f is the contact point of the sphere and the plane and g is the center of the sphere. It is clear from Fig. 5.1(b) that if the input angle, θ_1, is given, the only unknowns in the kinematic loop are $\{x_2, x_3, x_4\}$. For simplicity, we denote $c\alpha = \cos\alpha$ and $s\alpha = \sin\alpha$.

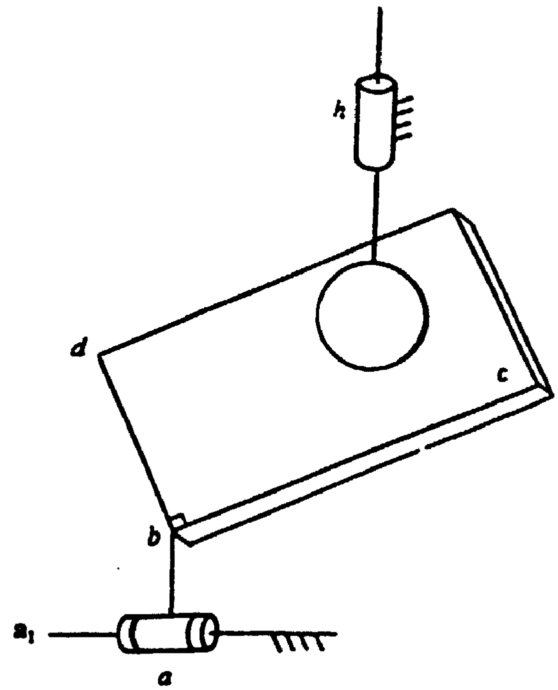


Fig. 5.1(a). The R_0–S_pC mechanism.

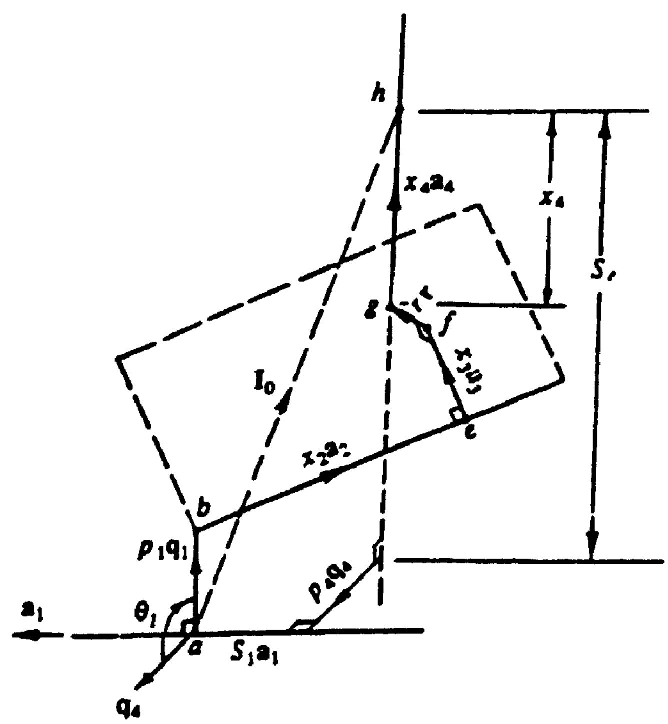


Fig. 5.1(b). The vector diagram of the R_0–S_pC mechanism.

(i). *Displacement analysis*.

Given: $\{\mathbf{a}_1, \mathbf{a}_4, \mathbf{I}\}$; Unknowns: $\{x_2, x_3, x_4\}$.

The loop equation is

$$x_2\mathbf{a}_2+x_3\mathbf{a}_3+x_4\mathbf{a}_4-\mathbf{I}=0 \tag{5.15}$$

$$\begin{cases} \mathbf{I}=-(p_1\mathbf{q}_1+r\mathbf{r})+\mathbf{I}_0 & (1) \\ \mathbf{I}_0=S_4\mathbf{a}_4-p_4\mathbf{q}_4-S_1\mathbf{a}_1 & (2) \end{cases} \tag{5.15a}$$

The direction equations are

$$\begin{cases} \mathbf{a}_3=\mathbf{a}_1\times\mathbf{q}_1 & (1) \\ \mathbf{a}_2=(c\alpha_{12}\mathbf{a}_1+s\alpha_{12}\mathbf{a}_3\times\mathbf{a}_1) & \\ \quad =c\alpha_{12}\mathbf{a}_1-s\alpha_{12}\mathbf{q}_1 & (2) \\ \mathbf{r}=(\mathbf{a}_2\times\mathbf{a}_3)=-c\alpha_{12}\mathbf{q}_1-s\alpha_{12}\mathbf{a}_1 & (3) \\ \mathbf{q}_1=c\theta_1\mathbf{q}_4+s\theta_1\mathbf{a}_1\times\mathbf{q}_4 & (4) \\ \mathbf{a}_1=c\alpha_{41}\mathbf{a}_4+s\alpha_{41}\mathbf{q}_4\times\mathbf{a}_4 & (5) \end{cases} \tag{5.15b}$$

The direction equations of Eq.(5.15b) specify the relationship (i.e. relative direction) of every unit vector in the kinematic loop relative to its neighboring unit vectors. They can be obtained directly from the simplified vector loop diagram, using Eqs. (2.8) and (2.9). Here in Eq.(5.15a) the input vector $\mathbf{I}$, which is divided into two parts $(p_1\mathbf{q}_1+r\mathbf{r})$, is a function of the input angle θ_1 and $\mathbf{I}_0$ is a constant vector.

The vector loop equation, Eq.(5.15), has the same form as Eq.(5.1). Therefore, according to Eq.(5.2)

$$\begin{cases} x_2=(\mathbf{a}_3\times\mathbf{a}_4\cdot\mathbf{I})/(\mathbf{a}_2\times\mathbf{a}_3\cdot\mathbf{a}_4) & (1) \\ x_3=(\mathbf{a}_4\times\mathbf{a}_2\cdot\mathbf{I})/(\mathbf{a}_2\times\mathbf{a}_3\cdot\mathbf{a}_4) & (2) \\ x_4=(\mathbf{a}_2\times\mathbf{a}_3\cdot\mathbf{I})/(\mathbf{a}_2\times\mathbf{a}_3\cdot\mathbf{a}_4) & (3) \end{cases} \tag{5.16}$$

Eq.(5.16) is based on the condition that $(\mathbf{a}_2\times\mathbf{a}_3\cdot\mathbf{a}_4)\neq 0$. If $(\mathbf{a}_2\times\mathbf{a}_3\cdot\mathbf{a}_4)=0$, then $\{x_2, x_3, x_4\}$ have infinite solutions. At this point, the displacement analysis is complete. If we require numerical solutions for a specific mechanism, the vector expressions of Eq.(5.16) need to be transformed into scalar expressions. For this analysis, it is best to separately scalarize the numerators and the denominators of Eq.(5.16) as follows. (As an example, the detailed derivation of Eq.(5.17) is given in Appendix 5.A.)

The numerator of Eq.(5.16-1) is

$$(\mathbf{a}_3\times\mathbf{a}_4\cdot\mathbf{I})=(\mathbf{U}_1\cdot\mathbf{q}_1+V_1)=a_1c\theta_1+a_2s\theta_1+a_3 \tag{5.17}$$

where

$$\begin{cases} \mathbf{U}_1=(S_1-r\,s\alpha_{12})\mathbf{a}_4-p_4c\alpha_{41}\mathbf{q}_4 \\ V_1=(r\,c\alpha_{12}-p_1)c\alpha_{41} \end{cases} \tag{5.17a}$$

$$\begin{cases} a_1=(\mathbf{U}_1\cdot\mathbf{q}_4)=-p_4c\alpha_{41} \\ a_2=(\mathbf{U}_1\cdot\mathbf{a}_1\times\mathbf{q}_4)=(S_1-r\,s\alpha_{12})s\alpha_{41} \\ a_3=V_1=(r\,c\alpha_{12}-p_1)c\alpha_{41} \end{cases} \tag{5.17b}$$

The numerator of Eq.(5.16-2) is

$$(\mathbf{a}_4\times\mathbf{a}_2\cdot\mathbf{I}) = (\mathbf{U}_2\cdot\mathbf{q}_1+V_2) = b_1c\theta_1+b_2s\theta_1+b_3 \tag{5.18}$$

where

$$\begin{cases} \mathbf{U}_2=(r-p_1c\alpha_{12}-S_1s\alpha_{12})s\alpha_{41}\mathbf{q}_4+p_4s\alpha_{12}\mathbf{q}_4\times\mathbf{a}_4 \\ V_2=-p_4s\alpha_{41}c\alpha_{12} \end{cases} \tag{5.18a}$$

$$\begin{cases} b_1=(\mathbf{U}_2\cdot\mathbf{q}_4)=(r-p_1c\alpha_{12}-S_1s\alpha_{12})s\alpha_{41} \\ b_2=(\mathbf{U}_2\cdot\mathbf{a}_1\times\mathbf{q}_4)=-p_4c\alpha_{41}s\alpha_{12} \\ b_3=V_2=-p_4s\alpha_{41}c\alpha_{12} \end{cases} \tag{5.18b}$$

The numerator of Eq.(5.16-3) is

$$(\mathbf{a}_2\times\mathbf{a}_3\cdot\mathbf{I}) = (\mathbf{U}_3\cdot\mathbf{q}_1+V_3) = c_1c\theta_1+c_2s\theta_1+c_3 \tag{5.19}$$

where

$$\begin{cases} \mathbf{U}_3=c\alpha_{12}(p_4\mathbf{q}_4-S_4\mathbf{a}_4) \\ V_3=(S_1-S_4c\alpha_{41})s\alpha_{12}+(p_1c\alpha_{12}-r) \end{cases} \tag{5.19a}$$

$$\begin{cases} c_1=(\mathbf{U}_3\cdot\mathbf{q}_4)=p_4c\alpha_{12} \\ c_2=(\mathbf{U}_3\cdot\mathbf{a}_1\times\mathbf{q}_4)=-S_4s\alpha_{41}c\alpha_{12} \\ c_3=V_3=(S_1-S_4c\alpha_{41})s\alpha_{12}+(p_1c\alpha_{12}-r) \end{cases} \tag{5.19b}$$

Finally, the denominator is

$$(\mathbf{a}_2\times\mathbf{a}_3\cdot\mathbf{a}_4)=d_2s\theta_1+d_3 \tag{5.20}$$

where

$$\begin{cases} d_2=-s\alpha_{41}c\alpha_{12} \\ d_3=-c\alpha_{41}s\alpha_{12} \end{cases} \tag{5.20a}$$

$\mathbf{U}_i$ and V_i $(i=1,2,3)$ of Eqs. (5.17), (5.18) and (5.19) are respectively constant vectors and constant scalars, whereas $\mathbf{q}_1$ is a vector function of the input angle θ_1, as shown in Fig. 5.1(b) and Eq.(5.15b-4).

Substituting Eqs. (5.17), (5.18), (5.19) and (5.20) into Eq.(5.16) yields

$$\begin{cases} x_2=(a_1c\theta_1+a_2s\theta_1+a_3)/(d_2s\theta_1+d_3) \\ x_3=(b_1c\theta_1+b_2s\theta_1+b_3)/(d_2s\theta_1+d_3) \\ x_4=(c_1c\theta_1+c_2s\theta_1+c_3)/(d_2s\theta_1+d_3) \end{cases} \tag{5.21}$$

(ii). *Velocity analysis*. Unknowns: $\{\dot{x}_2, \dot{x}_3, \dot{x}_4\}$.

Differentiating Eq.(5.21) yields

$$\begin{cases} \dot{x}_2 = \dot{\theta}_1[(a_2d_3 - a_3d_2)c\theta_1 - a_1d_3s\theta_1 - a_1d_2]/(d_2s\theta_1 + d_3)^2 \\ \dot{x}_3 = \dot{\theta}_1[(b_2d_3 - b_3d_2)c\theta_1 - b_1d_3s\theta_1 - b_1d_2]/(d_2s\theta_1 + d_3)^2 \\ \dot{x}_4 = \dot{\theta}_1[(c_2d_3 - c_3d_2)c\theta_1 - c_1d_3s\theta_1 - c_1d_2]/(d_2s\theta_1 + d_3)^2 \end{cases} \tag{5.22}$$

(iii). *Acceleration analysis*. Unknowns: $\{\ddot{x}_2, \ddot{x}_3, \ddot{x}_4\}$.

$\{\ddot{x}_2, \ddot{x}_3, \ddot{x}_4\}$ can be obtained by differentiating Eq.(5.22).

5.4. The R_G-S_GC mechanism

The R_G-S_GC mechanism is shown in Fig. 5.2(a). The simplified vector loop diagram of this mechanism is given in Fig. 5.2(b). The input pair is at point a. The input angle is θ_1 and the output angle is θ_2, i.e. $\theta_2 = \theta_2$.

(i). *Displacement analysis*.

Given: $\{\mathbf{a}_1, \mathbf{a}_4, \mathbf{I}\}$; Unknowns: $\{x_3, x_4, \theta_2\}$.

The loop equation is

$$x_2\mathbf{a}_2 - x_4\mathbf{a}_4 + \mathbf{K} = 0 \tag{5.23}$$

$$\begin{cases} \mathbf{K} = \mathbf{I} + \mathbf{J} & (1) \\ \mathbf{J} = S_3\mathbf{a}_3 + p_2\mathbf{q}_2 & (2) \\ \mathbf{I} = p_1\mathbf{q}_1 + (S_1\mathbf{a}_1 + p_3\mathbf{q}_3) & (3) \end{cases} \tag{5.23a}$$

The direction equations are

$$\begin{cases} \mathbf{a}_3 = c\alpha_{34}\mathbf{a}_4 - s\alpha_{34}\mathbf{q}_2\mathrm{x}\mathbf{a}_4 & (1) \\ \mathbf{q}_2 = \cos\theta_2\mathbf{q}_3 - \sin\theta_2\mathbf{a}_4\mathrm{x}\mathbf{q}_3 & (2) \\ \mathbf{a}_2 = c\alpha_{12}\mathbf{a}_1 + s\alpha_{12}\mathbf{q}_1\mathrm{x}\mathbf{a}_1 & (3) \\ \mathbf{q}_1 = c\theta_1\mathbf{q}_3 + s\theta_1\mathbf{a}_1\mathrm{x}\mathbf{q}_3 & (4) \end{cases} \tag{5.23b}$$

The scalar product of $\mathbf{a}_2\mathrm{x}\mathbf{a}_4$ with Eq.(5.23) and considering Eq.(5.23a-1) yields

$$\mathbf{a}_2\mathrm{x}\mathbf{a}_4\cdot\mathbf{J} = -\mathbf{a}_2\mathrm{x}\mathbf{a}_4\cdot\mathbf{I} \tag{5.24}$$

Substituting Eq.(5.23a-2) into Eq.(5.24) and considering Eq.(5.23b-1) results in

$$\mathbf{U}\cdot\mathbf{q}_2 = V \tag{5.25}$$

$$\begin{cases} \mathbf{U} = p_2\mathbf{a}_2\mathrm{x}\mathbf{a}_4 - S_3s\alpha_{34}\mathbf{a}_2 \\ V = -\mathbf{I}\cdot\mathbf{a}_2\mathrm{x}\mathbf{a}_4 \end{cases} \tag{5.25a}$$

$\mathbf{U}$ and V of Eq.(5.25) are respectively a vector function and a scalar function of the input angle θ_1. $\mathbf{q}_2$ is a function of the output angle θ_2.

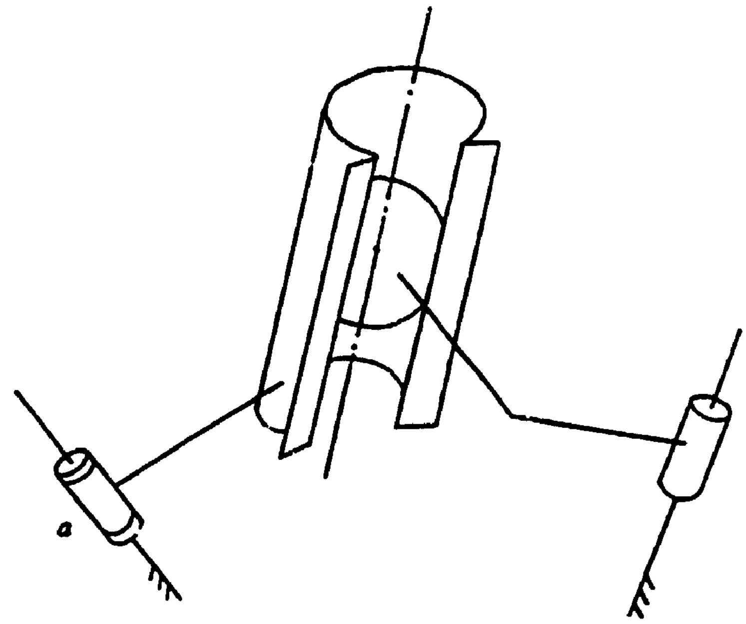


Fig. 5.2(a). The R_0–$S_G C$ mechanism.

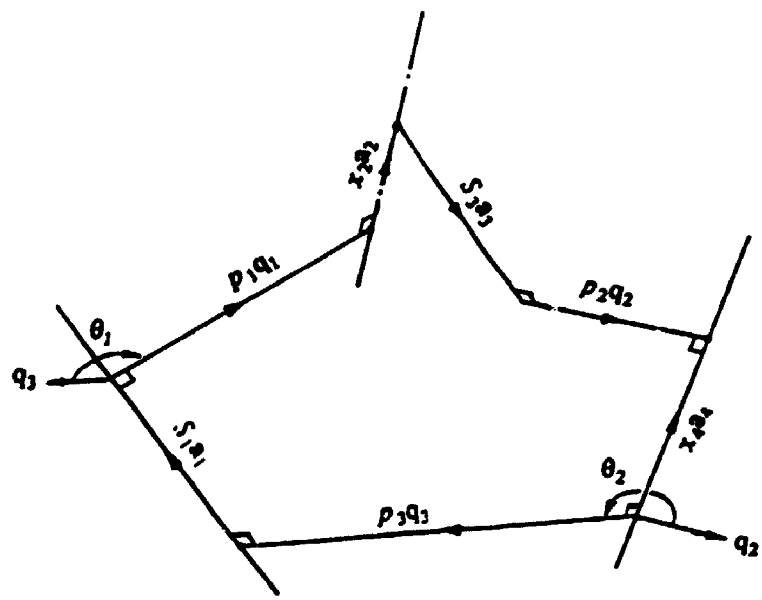


Fig. 5.2(b). The vector diagram of the R_0–$S_G C$ mechanism.

Substituting Eq.(5.23b-2) into (5.25) yields

$$A\cos\Theta_2 + B\sin\Theta_2 = C \tag{5.26}$$

$$\begin{cases} A = \mathbf{U}\cdot\mathbf{q}_3 \\ B = \mathbf{U}\cdot\mathbf{q}_3\times\mathbf{a}_4 \\ C = V \end{cases} \tag{5.26a}$$

From Eq.(5.26) we have

$$\Theta_2 = 2\tan^{-1}[(B \pm \sqrt{A^2+B^2-C^2})/(A+C)] \tag{5.27}$$

Now $\{x_2, x_4\}$ are the only unknowns in Eq.(5.23). From the scalar product of $\{\mathbf{q}_3, \mathbf{q}_1\}$ with Eq.(5.23)

$$\begin{cases} x_2 = -\mathbf{q}_3\cdot(\mathbf{K})/(\mathbf{q}_3\cdot\mathbf{a}_2) \\ x_4 = \ \mathbf{q}_1\cdot(\mathbf{K})/(\mathbf{q}_1\cdot\mathbf{a}_4) \end{cases} \tag{5.28}$$

(ii). *Velocity analysis*. Unknowns: $\{\dot{x}_2, \dot{x}_4, \dot{\Theta}\}$.

Differentiating Eq.(5.26) yields

$$\dot{\Theta}_2 = (\dot{A}\cos\Theta_2 + \dot{B}\sin\Theta_2 - \dot{C})/(A\sin\Theta_2 - B\cos\Theta_2) \tag{5.29}$$

$$\begin{cases} \dot{A} = \dot{\mathbf{U}}\cdot\mathbf{e}_1 \\ \dot{B} = \dot{\mathbf{U}}\cdot\mathbf{e}_2 \\ \dot{C} = \dot{V} \end{cases} \tag{5.29a}$$

The derivative of Eq.(5.23) is

$$\dot{x}_4\mathbf{a}_4 - \dot{x}_2\mathbf{a}_2 = (\dot{\mathbf{K}} + x_2\dot{\mathbf{a}}_2) \tag{5.30}$$

The scalar product of $\{\mathbf{q}_3, \mathbf{q}_1\}$ with Eq.(5.30) yields

$$\begin{cases} \dot{x}_2 = -\mathbf{q}_3\cdot(\dot{\mathbf{K}} + x_2\dot{\mathbf{a}}_2)/(\mathbf{q}_3\cdot\mathbf{a}_2) \\ \dot{x}_4 = \ \mathbf{q}_1\cdot(\dot{\mathbf{K}} + x_2\dot{\mathbf{a}}_2)/(\mathbf{q}_1\cdot\mathbf{a}_4) \end{cases} \tag{5.31}$$

(iii). *Acceleration analysis*. Unknowns: $\{\ddot{x}_2, \ddot{x}_4, \ddot{\Theta}\}$.

The second derivative of Eq.(5.26) yields $\ddot{\Theta}_2$, which is a similar expression to Eq.(5.14) and

$$\begin{cases} \ddot{A} = \ddot{\mathbf{U}}\cdot\mathbf{q}_3 \\ \ddot{B} = \ddot{\mathbf{U}}\cdot\mathbf{q}_3\times\mathbf{a}_4 \\ \ddot{C} = \ddot{V} \end{cases} \tag{5.32a}$$

The second derivative of Eq.(5.23) yields

$$\begin{cases} \ddot{x}_2 = -\mathbf{q}_3\cdot(\ddot{\mathbf{K}} + \dot{x}_2\dot{\mathbf{a}}_2 + 2x_2\ddot{\mathbf{a}}_2)/(\mathbf{q}_3\cdot\mathbf{a}_2) \\ \ddot{x}_4 = \ \mathbf{q}_1\cdot(\ddot{\mathbf{K}} + \dot{x}_2\dot{\mathbf{a}}_2 + 2x_2\ddot{\mathbf{a}}_2)/(\mathbf{q}_1\cdot\mathbf{a}_4) \end{cases} \tag{5.33}$$

The kinematic analysis is now complete. If numerical analysis is required for a mechanism, then Eqs. (5.26a), (5.28), (5.29a), (5.31), (5.32a) and (5.33) should be transformed into scalar expressions. For instance, the scalerized form of Eq.(5.26a) is as follows:

$$\begin{cases} A = a_1\cos\theta_1 + a_2\sin\theta_1 + a_3 \\ B = b_1\cos\theta_1 + b_2\sin\theta_1 + b_3 \\ C = c_1\cos\theta_1 + c_2\sin\theta_1 + c_3 \end{cases} \tag{5.34}$$

where $\{a_i, b_i, c_i\}$ $(i = 1 - 3)$ are constants:

$$\begin{cases} a_1 = -p_2 c\alpha_{41} s\alpha_{12} \\ a_2 = S_3 s\alpha_{34} s\alpha_{12} \\ a_3 = -p_2 s\alpha_{41} c\alpha_{12} \end{cases} \tag{5.34a}$$

$$\begin{cases} b_1 = S_3 c\alpha_{41} s\alpha_{12} s\alpha_{34} \\ b_2 = p_2 s\alpha_{12} \\ b_3 = -S_3 s\alpha_{41} c\alpha_{12} s\alpha_{34} \end{cases} \tag{5.34b}$$

$$\begin{cases} c_1 = (p_1 s\alpha_{41} c\alpha_{12} - p_3 c\alpha_{41} s\alpha_{12}) \\ c_2 = -S_1 s\alpha_{41} s\alpha_{12} \\ c_3 = (p_1 c\alpha_{41} s\alpha_{12} - p_3 s\alpha_{41} c\alpha_{12}) \end{cases} \tag{5.34c}$$

Eq.(5.28) can be expressed as,

$$\begin{cases} x_2 = -(S_3 s\alpha_{34} s\theta_2 + p_2 c\theta_2 + p_1 c\theta_1 + p_3)/(s\theta_1 s\alpha_{12}) \\ x_4 = (S_3\rho_1 + p_2\rho_2 + p_1 + p_3 c\theta_1)/(s\alpha_{41} s\theta_1) \end{cases} \tag{5.35}$$

$$\begin{cases} \rho_1 = (c\theta_1 c\theta_2 - s\theta_1 s\theta_2 c\alpha_{41}) \\ \rho_2 = (c\theta_1 s\theta_2 + s\theta_1 c\theta_2 c\alpha_{41}) s\alpha_{34} + (s\theta_1 s\alpha_{41}) \end{cases} \tag{5.35a}$$

The scalarized form of Eq.(5.29a) can be obtained from the derivative of Eq.(5.34),

$$\begin{cases} \dot{A} = (-a_1\sin\theta_1 + a_2\cos\theta_1)\dot{\theta}_1 \\ \dot{B} = (-b_1\sin\theta_1 + b_2\cos\theta_1)\dot{\theta}_1 \\ \dot{C} = (-c_1\sin\theta_1 + c_2\cos\theta_1)\dot{\theta}_1 \end{cases} \tag{5.36}$$

The scalarized form of Eqs. (5.31), (5.32a) and (5.33) can be similarly obtained, although they are not displayed here.

5.5. The R_0-S_pR mechanism

The $R_0 - S_pR$ mechanism is shown in Fig. 5.3(a) and the corresponding vector diagram is shown in Fig. 5.3(b). The input pair is located at point a. $\{\mathbf{a}_2, \mathbf{a}_3, \mathbf{r}\}$ are three mutually perpendicular unit vectors. Point b is the center of the sphere. Point c is the contact point of the sphere and the plane.

(i). *Displacement analysis*.

Given: $\{\mathbf{a}_1, \mathbf{a}_4, \mathbf{l}\}$; Unknowns: $\{x_2, x_3, \Theta_2\}$.

The loop equation is

$$x_2\mathbf{a}_2 + x_3\mathbf{a}_3 + \mathbf{K} = 0 \tag{5.37}$$

$$\begin{cases} \mathbf{K} = \mathbf{I} + \mathbf{J} & (1) \\ \mathbf{J} = r\mathbf{r} & (2) \\ \mathbf{I} = p_1\mathbf{q}_1 - \mathbf{I}_0 & (3) \\ \mathbf{I}_0 = S_1\mathbf{a}_1 - p_3\mathbf{q}_3 + S_4\mathbf{a}_4 & (4) \end{cases} \tag{5.37a}$$

The direction equations are

$$\begin{cases} \mathbf{a}_2 = \cos\Theta_2\mathbf{q}_3 - \sin\Theta_2\mathbf{a}_4 \times \mathbf{q}_3 & (1) \\ \mathbf{a}_3 = c\alpha_{34}\mathbf{a}_4 - s\alpha_{34}\mathbf{a}_2 \times \mathbf{a}_4 & (2) \\ \mathbf{r} = (\mathbf{a}_2 \times \mathbf{a}_3) = c\alpha_{34}\mathbf{a}_2 \times \mathbf{a}_4 + s\alpha_{34}\mathbf{a}_4 & (3) \\ \mathbf{q}_1 = \cos\theta_1\mathbf{q}_3 + \sin\theta_1\mathbf{a}_1 \times \mathbf{q}_3 & (4) \end{cases} \tag{5.37b}$$

The scalai product of $\mathbf{r}$ with Eq.(5.37) yields

$$\mathbf{r} \cdot \mathbf{K} = 0 \tag{5.38}$$

From Eq.(5.38) we have

$$\begin{cases} \mathbf{r} \cdot \mathbf{K} = \mathbf{r} \cdot (\mathbf{I} + \mathbf{J}) \\ \quad = \mathbf{r} \cdot (\mathbf{I} + r\mathbf{r}) \\ \quad = \mathbf{r} \cdot \mathbf{I} + r \\ \quad = (c\alpha_{34}\mathbf{a}_2 \times \mathbf{a}_4 + s\alpha_{34}\mathbf{a}_4) \cdot \mathbf{I} + r \\ \quad = -(c\alpha_{34}\mathbf{I} \times \mathbf{a}_4) \cdot \mathbf{a}_2 + [s\alpha_{34}(\mathbf{I} \cdot \mathbf{a}_4) + r] = 0 \end{cases} \tag{5.39}$$

and from Eq.(5.39)

$$\mathbf{U} \cdot \mathbf{a}_2 = V \tag{5.40}$$

$$\begin{cases} \mathbf{U} = c\alpha_{34}\mathbf{I} \times \mathbf{a}_4 \\ V = s\alpha_{34}(\mathbf{I} \cdot \mathbf{a}_4) + r \end{cases} \tag{5.40a}$$

Substituting Eq.(5.37b-1) into Eq.(5.40) yields

$$A\cos\Theta_2 + B\sin\Theta_2 = C \tag{5.41}$$

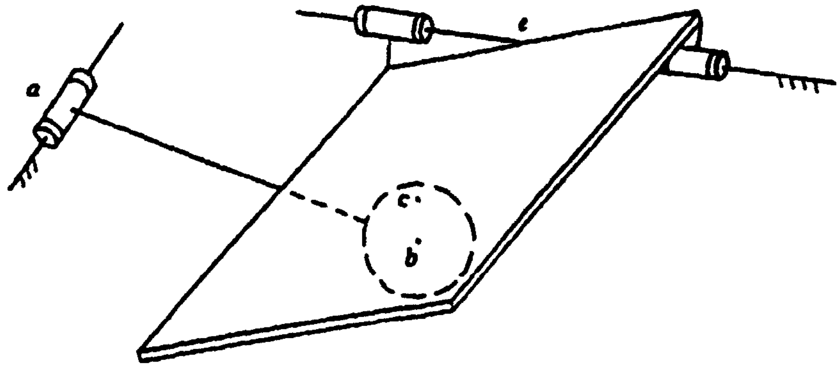


Fig. 5.3(a). The R_0–S_pR mechanism.

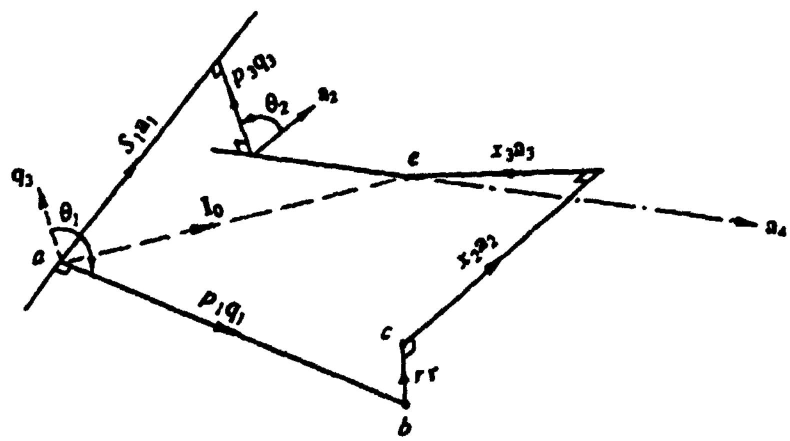


Fig. 5.3(b). The vector diagram of the R_0–S_pR mechanism.

$$\begin{cases} A = (\mathbf{U}\cdot\mathbf{q}_3) \\ B = (\mathbf{U}\cdot\mathbf{q}_3\times\mathbf{a}_4) \\ C = V \end{cases} \tag{5.41a}$$

From Eq.(5.41)

$$\Theta_2 = 2\tan^{-1}[(B \pm \sqrt{A^2+B^2-C^2})/(A+C)] \tag{5.42}$$

The scalar product of Eq.(5.37) with $\mathbf{q}_2$ and $\mathbf{a}_3$ yields

$$\begin{cases} x_2 = -(\mathbf{K}\cdot\mathbf{a}_2) \\ x_3 = -(\mathbf{K}\cdot\mathbf{a}_3) \end{cases} \tag{5.43}$$

The displacement analysis is now complete. If we need to calculate numerical values of these variables, we can expand Eqs. (5.41a) and (5.43) as follows:

$$\begin{cases} A = a_2 s\theta_1 + a_3 \\ B = b_1 c\theta_1 + b_3 \\ C = c_2 s\theta_1 + c_3 \end{cases} \tag{5.41b}$$

$$\begin{cases} a_2 = p_1 c\alpha_{34} c\alpha_{41} \\ a_3 = S_1 c\alpha_{34} s\alpha_{41} \\ b_1 = p_1 c\alpha_{34} \\ b_3 = p_3 c\alpha_{34} \\ c_2 = p_1 c\alpha_{34} s\alpha_{41} \\ c_3 = S_1 c\alpha_{34} s\alpha_{41} + r - S_4 \end{cases} \tag{5.41c}$$

$$\begin{cases} x_2 = -(p_1 c\theta_1 + p_3)c\Theta_2 + (p_1 c\alpha_{41} s\theta_1 + S_1 s\alpha_{41})s\Theta_2 \\ x_3 = s\alpha_{34}[(p_1 c\alpha_{41} s\theta_1 + S_1 s\alpha_{41})c\Theta_2 \\ \quad + (p_1 c\theta_1 + p_3)s\Theta_2] - c\alpha_{34}(p_1 s\alpha_{41} s\theta_1 - S_1 c\alpha_{41} - S_4) \end{cases} \tag{5.43a}$$

(ii). *Velocity analysis*.

Unknowns: $\{\dot{x}_2, \dot{x}_3, \dot{\Theta}_2\}$.

From the derivative of Eq.(5.41) we have

$$\dot{\Theta}_2 = (\dot{A}\cos\Theta_2 + \dot{B}\sin\Theta_2 - \dot{C})/(A\sin\Theta_2 - B\cos\Theta_2) \tag{5.44}$$

$$\begin{cases} \dot{A} = (\dot{\theta}_1 c\theta_1)a_2 \\ \dot{B} = (\dot{\theta}_1 s\theta_1)(-b_1) \\ \dot{C} = (\dot{\theta}_1 c\theta_1)c_2 \end{cases} \tag{5.44a}$$

Differentiating Eq.(5.43a) yields

$$\begin{cases} \dot{x}_2 = p_1\dot{\theta}_1[s\theta_1 c\Theta_2 + c\alpha_{41}c\theta_1 s\Theta_2] \\ \quad +\dot{\Theta}_2[(p_1 c\theta_1 + p_3)s\Theta_2 + (p_1 c\alpha_{41}s\theta_1 + S_1 s\alpha_{41})c\Theta_2] \\ \dot{x}_3 = p_1\dot{\theta}_1[s\alpha_{34}s\theta_1 s\Theta_2 - s\alpha_{34}c\alpha_{41}c\theta_1 c\Theta_2 - c\alpha_{34}s\alpha_{41}c\theta_1] \\ \quad +\dot{\Theta}_2 s\alpha_{34}[p_1 c\alpha_{41}s\theta_1 + S_1 s\alpha_{41})s\Theta_2 - (p_1 c\theta_1 + p_3)c\Theta_2] \end{cases} \tag{5.45}$$

(iii). *Acceleration analysis*.

Unknowns: $\{\ddot{x}_2, \ddot{x}_3, \ddot{\Theta}_2\}$

The second derivative of Eq.(5.41) yields $\ddot{\Theta}_2$, where $\{\ddot{A}, \ddot{B}, \ddot{C}\}$ can be obtained from the derivative of Eq.(5.44a).

$$\begin{cases} \ddot{A} = (\ddot{\theta}_1 c\theta_1 - \dot{\theta}_1^2 s\theta_1)a_2 \\ \ddot{B} = (\ddot{\theta}_1 s\theta_1 + \dot{\theta}_1^2 c\theta_1)(-b_1) \\ \ddot{C} = (\ddot{\theta}_1 c\theta_1 - \dot{\theta}_1^2 s\theta_1)c_2 \end{cases} \tag{5.46a}$$

The derivative of Eq.(5.45) yields $\{\ddot{x}_2, \ddot{x}_3\}$.

5.6. The R_0-B_8R mechanism

The R_0-B_8R mechanism is shown in Fig. 5.4(a) and Fig.5.4(b). The input pair is at point a. The radii of the two round bars of pair B_8 are r_1 and r_2, respectively. Denote $r = (r_1 + r_2)$.

(i). *Displacement analysis*.

Given: $\{\mathbf{a}_1, \mathbf{a}_4, \mathbf{I}\}$; Unknowns: $\{x_2, x_3, \Theta_2\}$.

The loop equation is

$$x_2\mathbf{a}_2 + x_3\mathbf{a}_3 + \mathbf{K} = 0 \tag{5.47}$$

$$\begin{cases} \mathbf{K} = \mathbf{J} + \mathbf{I} & (1) \\ \mathbf{J} = r\mathbf{r} + p_2\mathbf{q}_2 & (2) \\ \mathbf{I} = p_1\mathbf{q}_1 + \mathbf{I}_0 & (3) \\ \mathbf{I}_0 = S_4\mathbf{a}_4 + p_3\mathbf{q}_3 + S_1\mathbf{a}_1 & (4) \end{cases} \tag{5.47a}$$

The direction equations are

$$\begin{cases} \mathbf{r} = (\mathbf{a}_2 \times \mathbf{a}_3)/|\mathbf{a}_2 \times \mathbf{a}_3| & (1) \\ \mathbf{a}_3 = c\alpha_{34}\mathbf{a}_4 - s\alpha_{34}\mathbf{q}_2 \times \mathbf{a}_4 & (2) \\ \mathbf{q}_2 = \cos\Theta_2\mathbf{q}_3 - \sin\Theta_2\mathbf{a}_4 \times \mathbf{q}_3 & (3) \\ \mathbf{a}_2 = c\alpha_{12}\mathbf{a}_1 + s\alpha_{12}\mathbf{q}_1 \times \mathbf{a}_1 & (4) \\ \mathbf{q}_1 = c\theta_1\mathbf{q}_3 + s\theta_1\mathbf{a}_1 \times \mathbf{q}_3 & (5) \end{cases} \tag{5.47b}$$

The scalar product of $\mathbf{a}_2 \times \mathbf{a}_3$ with Eq.(5.47) yields

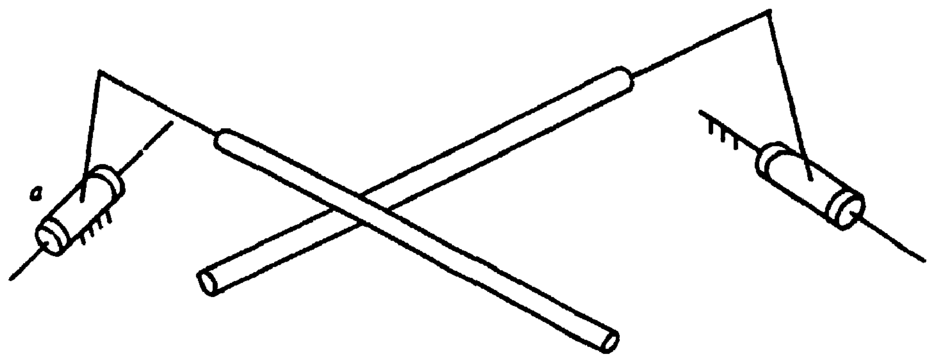


Fig. 5.4(a). The R_0–$B_B R$ mechanism.

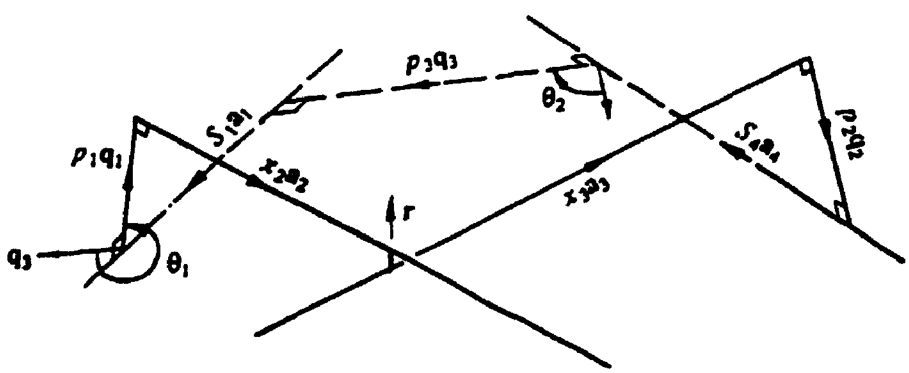


Fig. 5.4(b). The vector diagram of the R_0–$B_B R$ mechanism.

$$\mathbf{a}_2\times\mathbf{a}_3\cdot\mathbf{K} = 0 \tag{5.48}$$

Approximate solution. ($r = 0$)

If r is very small as compared to the geometrical size of the mechanism, then we can let $r = 0$.

Substituting Eqs. (5.47a-1) and (5.47b-2) into Eq.(5.48) yields

$$\mathbf{U}\cdot\mathbf{q}_3 = V \tag{5.49}$$

$$\begin{cases} \mathbf{U} = s\alpha_{34}[(\mathbf{I}\times\mathbf{a}_2)\times\mathbf{a}_4] + p_2 c\alpha_{34}\mathbf{a}_2\times\mathbf{a}_4 \\ V = c\alpha_{34}[(\mathbf{I}\times\mathbf{a}_2)\cdot\mathbf{a}_4] - p_2 s\alpha_{34}(\mathbf{a}_2\cdot\mathbf{a}_4) \end{cases} \tag{5.49a}$$

Substituting Eq.(5.47b-3) into Eq.(5.49) yields

$$A\cos\Theta_2 + B\sin\Theta_2 = C \tag{5.50}$$

$$\begin{cases} A = \mathbf{U}\cdot\mathbf{q}_3 \\ B = \mathbf{U}\cdot\mathbf{q}_3\times\mathbf{a}_4 \\ C = V \end{cases} \tag{5.50a}$$

From Eq.(5.50) we have

$$\Theta_2 = 2\tan^{-1}[(B \pm \sqrt{A^2+B^2-C^2})/(A+C)] \tag{5.51}$$

From the scalar product of $\{\mathbf{a}_2, \mathbf{a}_3\}$ with Eq.(5.47) we can obtain $\{x_2, x_3\}$. The velocity and acceleration analysis procedures are similar to (ii) and (iii) of sections 5.2 and 5.3.

Accurate solution. ($r \neq 0$)

From Eq.(5.48) we get

$$\mathbf{a}_2\times\mathbf{a}_3\cdot\left(r\frac{\mathbf{a}_2\times\mathbf{a}_3}{|\mathbf{a}_2\times\mathbf{a}_3|} + p_2\mathbf{q}_2 + \mathbf{I}\right) = 0$$

i.e.

$$[(p_2\mathbf{a}_2)\cdot\mathbf{a}_3\times\mathbf{q}_2 + (\mathbf{I}\times\mathbf{a}_2)\cdot\mathbf{a}_3] = -r\,|\mathbf{a}_2\times\mathbf{a}_3| \tag{5.52}$$

Squaring Eq.(5.52) yields

$$[(p_2\mathbf{a}_2)\cdot\mathbf{a}_3\times\mathbf{q}_2 + (\mathbf{I}\times\mathbf{a}_2)\cdot\mathbf{a}_3]^2 = r^2[1-(\mathbf{a}_2\cdot\mathbf{a}_3)^2] \tag{5.53}$$

Let

$$[(p_2\mathbf{a}_2)\cdot\mathbf{a}_3\times\mathbf{q}_2 + (\mathbf{I}\times\mathbf{a}_2)\cdot\mathbf{a}_3] = \mathbf{U}_1\cdot\mathbf{q}_2 + V_1 \tag{5.54}$$

$$\begin{cases} \mathbf{U}_1 = p_2 c\alpha_{34}\mathbf{a}_2\times\mathbf{a}_4 - s\alpha_{34}\mathbf{a}_4\times(\mathbf{I}\times\mathbf{a}_2) \\ V_1 = -p_2 s\alpha_{34}\mathbf{a}_2\cdot\mathbf{a}_4 + c\alpha_{34}(\mathbf{a}_4\cdot\mathbf{I}\times\mathbf{a}_2) \end{cases} \tag{5.54a}$$

Let

$$(\mathbf{a}_2 \cdot \mathbf{a}_3) = \mathbf{U}_2 \cdot \mathbf{q}_2 + V_2 \tag{5.55}$$

$$\begin{cases} \mathbf{U}_2 = s\alpha_{34}\mathbf{a}_2 \times \mathbf{a}_4 \\ V_2 = c\alpha_{34}(\mathbf{a}_2 \cdot \mathbf{a}_4) \end{cases} \tag{5.55a}$$

Substituting Eqs. (5.54) and (5.55) into Eq.(5.53) yields

$$(\mathbf{U}_1 \cdot \mathbf{q}_2)^2 + r^2(\mathbf{U}_2 \cdot \mathbf{q}_2)^2 + \mathbf{W} \cdot \mathbf{q}_2 + V_3 = 0 \tag{5.56}$$

$$\begin{cases} \mathbf{W} = 2V_1\mathbf{U}_1 + 2r^2V_2\mathbf{U}_2 \\ V_3 = V_1^2 + (V_2^2 - 1)r^2 \end{cases} \tag{5.56a}$$

Vector $\mathbf{q}_2$ is the only unknown in (5.56) and it is a function of the output angle Θ_2. Substituting Eq.(5.47b-3) into Eq.(5.56) yields

$$\mu_1 c^2\Theta_2 + \mu_2 s^2\Theta_2 + \mu_3 c\Theta_2 s\Theta_2 + \mu_4 c\Theta_2 + \mu_5 s\Theta_2 + \mu_6 = 0 \tag{5.57}$$

$$\begin{cases} \mu_1 = (\mathbf{U}_1 \cdot \mathbf{q}_3)^2 + r^2(\mathbf{U}_2 \cdot \mathbf{q}_3)^2 \\ \mu_2 = (\mathbf{U}_1 \cdot \mathbf{q}_3 \times \mathbf{a}_4)^2 + r^2(\mathbf{U}_2 \cdot \mathbf{q}_3 \times \mathbf{a}_4)^2 \\ \mu_3 = 2(\mathbf{U}_1 \cdot \mathbf{q}_3)(\mathbf{U}_1 \cdot \mathbf{q}_3 \times \mathbf{a}_4) + 2r^2(\mathbf{U}_2 \cdot \mathbf{q}_3)(\mathbf{U}_2 \cdot \mathbf{q}_3 \times \mathbf{a}_4) \\ \mu_4 = (\mathbf{W} \cdot \mathbf{q}_3) \\ \mu_5 = (\mathbf{W} \cdot \mathbf{q}_3 \times \mathbf{a}_4) \\ \mu_6 = V_3 \end{cases} \tag{5.57a}$$

Let $y = \tan(\Theta_2/2)$, then, from Eq.(5.57) we get

$$\sum_{i=0}^{4} \nu_i y^{4-i} = 0 \tag{5.58}$$

$$\begin{cases} \nu_0 = \mu_1 \\ \nu_1 = -2\mu_3 \\ \nu_i = -2\mu_1 + 4\mu_2 - \mu_4 + \mu_6 \\ \nu_i = 2\mu_3 + 2\mu_5 \\ \nu_i = \mu_1 + \mu_4 + \mu_6 \end{cases} \tag{5.58a}$$

From Eq.(5.58) we obtain y, then $\Theta_2 = 2\tan^{-1}y$. At this point, $\{x_2, x_3\}$ are the only unknowns in Eq.(5.47), and they can be obtained by considering the scalar product of $\mathbf{a}_2$ and $\mathbf{a}_3$ with both sides of Eq.(5.47).

5.7. Conclusion

From sections 5.4, 5.5 and 5.6 we can see that, using the vector algebraic method, the analysis procedures for the $R_0\text{–}S_OC$, $R_0\text{–}S_PR$ and $R_0\text{–}B_BR$ mechanisms are just a matter of routine and are almost identical. Only the accurate solution procedure for R_0-$B_s R$ is somewhat different, but the basic idea is still the same.

From Fig. 5.1(b) and section 5.3 we can see that before we proceed to the detailed analysis, there is no need to specify the relative direction of vectors $\mathbf{r}$ and $\mathbf{a}_4$. The same applies to vectors $\{\mathbf{a}_2, \mathbf{a}_3\}$ of Fig.2(b), vectors $\{\mathbf{q}_1, \mathbf{r}\}$ of Fig. 5.3(b) and vectors $\{\mathbf{a}_2, \mathbf{a}_3\}$ of Fig. 5.4(b). The determination of the relative direction of these vector couples is a "by-product", namely, after we determine other variables, the relative directions of these vector couples become known. However, if the Matrix Method is used, at least three Cartesian coordinate systems are required, specifying the relative directions of each of the vector couples of the higher pair, in order to perform the matrix transformation.

The diagram of the R_0-$B_s R$ mechanism shown in Fig. 5.5 comes from Begg's book [4](Page 137), in which the kinematic analysis of the mechanism was carried out by the matrix method, which required 9 pages of algebraic manipulation. But here we required less than two and a half pages. Comparing Fig. 5.4(b) with Fig. 5.5, it is clear that "a picture says one thousand words!"

The mechanisms analyzed in this chapter are all three-link mechanisms. Since the DoF of higher pairs is $3 \le DoF \le 5$, the number of links, N_l, of a mechanism containing higher pairs can only be $2 \le N_l \le 4$. From the diagrams of the mechanisms discussed in this chapter, it is not difficult to perceive that these mechanisms are all very compact structurally, due to their small number of links. In fact, compact structure is an attribute of spatial mechanisms containing higher pairs.

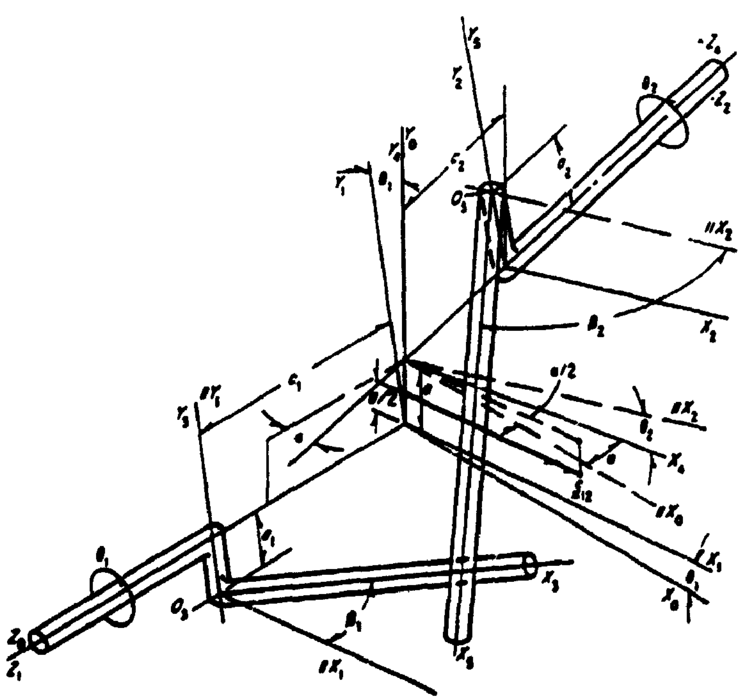


Fig. 5.5

Appendix 5.A. The Detailed Derivation of Eq.(5.17).

Since

$$(\mathbf{a}_3 \times \mathbf{a}_4 \cdot \mathbf{I}) = (\mathbf{a}_1 \times \mathbf{q}_1) \times \mathbf{a}_4 \cdot \mathbf{I} = [(\mathbf{a}_4 \cdot \mathbf{a}_1)\mathbf{q}_1 - (\mathbf{a}_4 \cdot \mathbf{q}_1)\mathbf{a}_1] \cdot [-p_1 \mathbf{q}_1 - r\mathbf{r} + \mathbf{I}_0]$$

$$= [c\alpha_{41}\mathbf{q}_1 - (\mathbf{a}_4 \cdot \mathbf{q}_1)\mathbf{a}_1] \cdot [-p_1 \mathbf{q}_1 + r\, c\alpha_{12}\mathbf{q}_1 + r\, s\alpha_{12}\mathbf{a}_1 + S_4\mathbf{a}_4 - p_4\mathbf{q}_4 - S_1\mathbf{a}_1]$$

$$= [c\alpha_{41}\mathbf{q}_1 - (\mathbf{a}_4 \cdot \mathbf{q}_1)\mathbf{a}_1] \cdot [(r\, c\alpha_{12} - p_1)\mathbf{q}_1 + (r\, s\alpha_{12} - S_1)\mathbf{a}_1 + S_4\mathbf{a}_4 - p_4\mathbf{q}_4]$$

$$= [(S_1 - r\, s\alpha_{12})\mathbf{a}_4 - p_4 c\alpha_{41}\mathbf{q}_4] \cdot \mathbf{q}_1 + (r\, c\alpha_{12} - p_1)c\alpha_{41}$$

Hence

$$(\mathbf{a}_3 \times \mathbf{a}_4 \cdot \mathbf{I}) = \mathbf{U}_1 \cdot \mathbf{q}_1 + V_1 \tag{A.1}$$

$$\begin{cases} \mathbf{U}_1 = (S_1 - r\, s\alpha_{12})\mathbf{a}_4 - p_4 c\alpha_{41}\mathbf{q}_4 \\ V_1 = (r\, c\alpha_{12} - p_1)c\alpha_{41} \end{cases}$$

Substituting Eq.(5.15b-4) into Eq.(A.1) yields

$$(\mathbf{a}_3 \times \mathbf{a}_4 \cdot \mathbf{I}) = a_1 c\theta_1 + a_2 s\theta_1 + a_3 \tag{A.2}$$

where a_1, a_2 and a_3 are given in Eq.(5.17b).

CHAPTER 6. THE X_0–R–C–S MECHANISM

6.1. Introduction

In this chapter the R_0-R-C-S mechanisms are kinematically analyzed using the *vector algebraic method*. The I/O equations are obtained as fourth order polynomials.

6.2. Analysis of the R_0-CSR mechanism

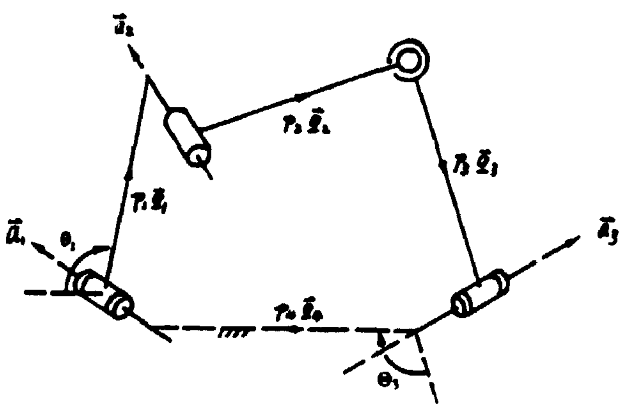


Fig. 6.1

The diagram of the R_0-CSR mechanism is shown in Fig. 6.1. The input angle is θ_1; the output angle is θ_3, i.e. $\theta_3=\Theta_3$. The vector loop equation and the direction equations can be written as,

$$-x\,\mathbf{a}_2 + \mathbf{K} = \mathbf{F} \tag{6.1}$$

$$\begin{cases} \mathbf{K} = \mathbf{J} + \mathbf{I} & (1) \\ \mathbf{J} = p_3\mathbf{q}_3 & (2) \\ \mathbf{I} = p_1\mathbf{q}_1 + (S_1\mathbf{a}_1 + p_4\mathbf{q}_4 - S_3\mathbf{a}_3) & (3) \\ \mathbf{F} = -p_2\mathbf{q}_2 & (4) \end{cases} \tag{6.1a}$$

$$\begin{cases} \mathbf{q}_3 = \cos\Theta_3\,\mathbf{q}_4 - \sin\Theta_3\,\mathbf{a}_3\times\mathbf{q}_4 & (1) \\ \mathbf{q}_2 = \cos\theta_2\,\mathbf{q}_1 + \sin\theta_2\,\mathbf{a}_2\times\mathbf{q}_1 & (2) \\ \mathbf{a}_2 = \cos\alpha_{12}\,\mathbf{a}_1 + \sin\alpha_{12}\,\mathbf{q}_1\times\mathbf{a}_1 & (3) \\ \mathbf{q}_1 = \cos\theta_1\,\mathbf{q}_4 + \sin\theta_1\,\mathbf{a}_1\times\mathbf{q}_4 & (4) \\ \mathbf{a}_3\times\mathbf{a}_1 = \mathbf{q}_4\sin\alpha_{31} & (5) \end{cases} \tag{6.1b}$$

Squaring both sides of (6.1) yields

$$x^2 - 2[\mathbf{a}_2\cdot\mathbf{K}]\,x + [\mathbf{K}]^2 = \mathbf{F}^2 \tag{6.2}$$

The scalar product of $\mathbf{a}_2$ with both sides of (6.1) yields

$$x = [\mathbf{a}_2 \cdot \mathbf{K}] \tag{6.3}$$

Substituting (6.3) into (6.2) yields

$$-[\mathbf{a}_2 \cdot \mathbf{K}]^2 + [\mathbf{K}]^2 = \mathbf{F}^2 \tag{6.4}$$

Substituting (6.1a-1), (6.1a-2) and (6.1a-4) into (6.4) yields

$$[\mathbf{U} \cdot \mathbf{q}_3]^2 + [\mathbf{W} \cdot \mathbf{q}_3] + V = 0 \tag{6.5}$$

$$\begin{cases} \mathbf{U} = p_3 \mathbf{a}_3 \\ \mathbf{W} = 2 p_3 \{(\mathbf{I} \cdot \mathbf{a}_2)\mathbf{a}_2 - \mathbf{I}\} \\ V = (\mathbf{I} \cdot \mathbf{a}_2)^2 - \mathbf{I}^2 + p_2^2 - p_3^2 \end{cases} \tag{6.5a}$$

Substituting (6.1b-1) into (6.5) yields

$$\mu_1 c^2\Theta_3 + \mu_2 s^2\Theta_3 + \mu_3 c\theta_3 s\Theta_3 + \mu_4 c\Theta_3 + \mu_5 s\Theta_3 + \mu_6 = 0 \tag{6.6}$$

$$\begin{cases} \mu_1 = (\mathbf{U} \cdot \mathbf{q}_4)^2 \\ \mu_2 = (\mathbf{U} \cdot \mathbf{a}_3 \times \mathbf{q}_4)^2 \\ \mu_3 = -2(\mathbf{U} \cdot \mathbf{q}_4)(\qquad \mathbf{q}_4) \\ \mu_4 = (\mathbf{W} \cdot \mathbf{q}_4) \\ \mu_5 = -(\mathbf{W} \cdot \mathbf{a}_3 \times \mathbf{q}_4) \\ \mu_6 = V \end{cases} \tag{6.6a}$$

Let $y = \tan(\Theta_3/2)$, then we have

$$\begin{cases} \cos\Theta_3 = (1-y^2)/(1+y^2) \\ \sin\Theta_3 = 2y/(1+y^2) \end{cases} \tag{6.7}$$

Substituting (6.7) into (6.6) yields

$$\sum_{i=0}^{4} v_i \, y^{4-i} = 0 \tag{6.8}$$

$$\begin{cases} v_0 = \mu_1 \\ v_1 = -2\mu_3 \\ v_2 = -2\mu_1 + 4\mu_2 - \mu_4 + \mu_6 \\ v_3 = 2\mu_3 + 2\mu_5 \\ v_4 = \mu_1 + \mu_4 + \mu_6 \end{cases} \tag{6.8a}$$

Solving (6.8) we get y; then $\Theta_3 = 2\tan^{-1} y$.

6.3. Analysis of the R_σ-SCR mechanism

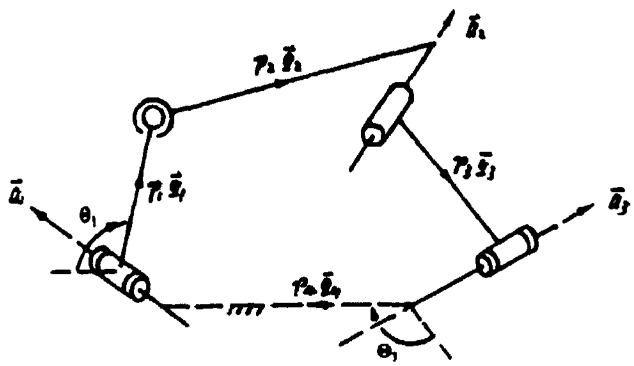


Fig. 6.2

The diagram of the R_σ-SCR mechanism is shown in Fig. 6.2. The input angle is θ_1; the output angle is θ_3, i.e. $\theta_3 = \Theta_3$. The vector loop equation and the direction equations can be written as,

$$-x\,\mathbf{a}_2 + \mathbf{K} = \mathbf{F} \tag{6.9}$$

$$\begin{cases} \mathbf{K} = \mathbf{J} + \mathbf{I} & (1) \\ \mathbf{J} = p_3\mathbf{q}_3 & (2) \\ \mathbf{I} = p_1\mathbf{q}_1 + (S_1\mathbf{a}_1 + p_4\mathbf{q}_4 - S_3\mathbf{a}_3) & (3) \\ \mathbf{F} = -p_2\mathbf{q}_2 & (4) \end{cases} \tag{6.9a}$$

$$\begin{cases} \mathbf{q}_2 = \cos\theta_2\,\mathbf{q}_3 - \sin\theta_2\,\mathbf{a}_2\times\mathbf{q}_3 & (1) \\ \mathbf{a}_2 = \cos\alpha_{23}\,\mathbf{a}_3 - \sin\alpha_{23}\,\mathbf{q}_3\times\mathbf{a}_3 & (2) \\ \mathbf{q}_3 = \cos\Theta_3\,\mathbf{q}_4 - \sin\Theta_3\,\mathbf{a}_3\times\mathbf{q}_4 & (3) \\ \mathbf{q}_1 = \cos\theta_1\,\mathbf{q}_4 + \sin\theta_1\,\mathbf{a}_1\times\mathbf{q}_4 & (4) \\ \mathbf{a}_3\times\mathbf{a}_1 = \mathbf{q}_4\sin\alpha_{31} & (5) \end{cases} \tag{6.9b}$$

Squaring both sides of (6.9) yields

$$x^2 - 2[\mathbf{a}_2\cdot\mathbf{K}]x + [\mathbf{K}]^2 = \mathbf{F}^2 \tag{6.10}$$

The scalar product of $\mathbf{a}_2$ with both sides of (6.9) yields

$$x = [\mathbf{a}_2\cdot\mathbf{K}] \tag{6.11}$$

Substituting (6.11) into (6.10) yields

$$-[\mathbf{a}_2\cdot\mathbf{K}]^2 + [\mathbf{K}]^2 = \mathbf{F}^2 \tag{6.12}$$

Substituting (6.9a-1), (6.9a-2), (6.9a-4) and (6.9b-2) into (6.12) yields

$$[\mathbf{U}\cdot\mathbf{q}_3]^2 + [\mathbf{W}\cdot\mathbf{q}_3] + V = 0 \tag{6.13}$$

$$\begin{cases} \mathbf{U} = s\alpha_{23}\,\mathbf{I}\times\mathbf{a}_3 \\ \mathbf{W} = 2\{c\alpha_{23}\,s\alpha_{23}(\mathbf{I}\cdot\mathbf{a}_3)\,\mathbf{I}\times\mathbf{a}_3 - p_3\mathbf{I}\} \\ V = c^2\alpha_{23}(\mathbf{I}\cdot\mathbf{a}_3)^2 - \mathbf{I}^2 + p_2^2 - p_3^2 \end{cases} \tag{6.13a}$$

The remaining derivation is identical to the one given in section 6.2.

6.4. Analysis of the R_0–SRC mechanism

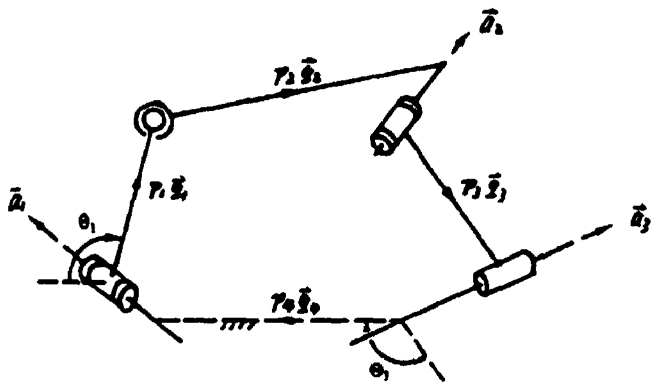


Fig. 6.3

The diagram of the R_0–SRC mechanism is shown in Fig. 6.3. The input angle is θ_1; the output angle is θ_3, i.e. $\theta_3 = \Theta_3$. The vector loop equation and the direction equations can be written as,

$$-x\,\mathbf{a}_3 + \mathbf{K} = \mathbf{F} \tag{6.14}$$

$$\begin{cases} \mathbf{K} = \mathbf{J} + \mathbf{I} & (1) \\ \mathbf{J} = p_3\mathbf{q}_3 - S_2\mathbf{a}_2 & (2) \\ \mathbf{I} = p_1\mathbf{q}_1 + (S_1\mathbf{a}_1 + p_4\mathbf{q}_4) & (3) \\ \mathbf{F} = -p_2\mathbf{q}_2 & (4) \end{cases} \tag{6.14a}$$

$$\begin{cases} \mathbf{q}_2 = \cos\theta_2\,\mathbf{q}_3 - \sin\theta_2\,\mathbf{a}_2\times\mathbf{q}_3 & (1) \\ \mathbf{a}_2 = \cos\alpha_{23}\,\mathbf{a}_3 - \sin\alpha_{23}\,\mathbf{q}_3\times\mathbf{a}_3 & (2) \\ \mathbf{q}_3 = \cos\Theta_3\,\mathbf{q}_4 - \sin\Theta_3\,\mathbf{a}_3\times\mathbf{q}_4 & (3) \\ \mathbf{q}_1 = \cos\theta_1\,\mathbf{q}_4 + \sin\theta_1\,\mathbf{a}_1\times\mathbf{q}_4 & (4) \\ \mathbf{a}_3\times\mathbf{a}_1 = \mathbf{q}_4\sin\alpha_{31} & (5) \end{cases} \tag{6.14b}$$

Squaring both sides of (6.14) yields

$$x^2 - 2(\mathbf{a}_3\cdot\mathbf{K})x + [\mathbf{K}]^2 = \mathbf{F}^2 \tag{6.15}$$

The scalar product of $\mathbf{a}_2$ with both sides of (6.14) yields

$$x\,c\alpha_{23} = [\mathbf{a}_2\cdot\mathbf{K}] \tag{6.16}$$

Substituting (6.16) into (6.15) yields

$$[\mathbf{a}_2\cdot\mathbf{K}]^2 - 2c\alpha_{23}(\mathbf{a}_3\cdot\mathbf{K})[\mathbf{a}_2\cdot\mathbf{K}] + c^2\alpha_{23}[\mathbf{K}]^2 - c^2\alpha_{23}\mathbf{F}^2 = 0 \tag{6.17}$$

Substituting (6.14a-1), (6.14a-2) and (6.14b-2) into (6.17) yields

$$[\mathbf{U}\cdot\mathbf{q}_3]^2 + [\mathbf{W}\cdot\mathbf{q}_3] + V = 0 \tag{6.18}$$

$$\begin{cases} \mathbf{U} = s\alpha_{23}\mathbf{I}\times\mathbf{a}_3 \\ \mathbf{W} = 2(p_3c^2\alpha_{23}\mathbf{I}^2 - S_2s\alpha_{23}\mathbf{I}\times\mathbf{a}_3) \\ V = S_2^2 + c^2\alpha_{23}(\mathbf{I}^2 - (\mathbf{I}\cdot\mathbf{a}_3)^2 + (p_3 - \mathbf{F}^2 - S_2^2)) \end{cases} \tag{6.18a}$$

6.5. Analysis of the R_0–RSC mechanism

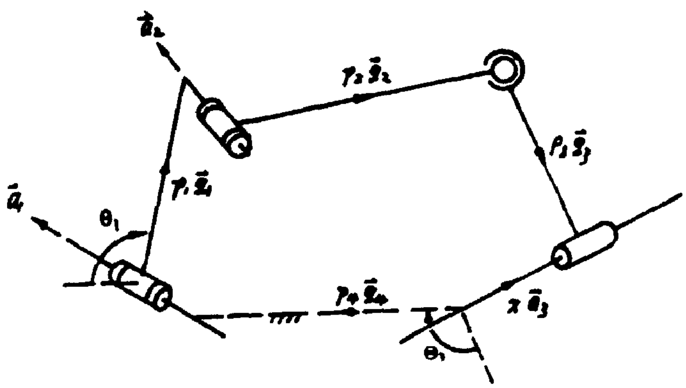


Fig. 6.4

The diagram of the R_0–RSC mechanism is shown in Fig. 6.4. The input angle is θ_1; the output angle is θ_3, i.e. $\theta_3 = \Theta_3$. The vector loop equation and the direction equations can be written as,

$$-x\,\mathbf{a}_3 + \mathbf{K} = \mathbf{F} \tag{6.19}$$

$$\begin{cases} \mathbf{K} = \mathbf{J} + \mathbf{I} & (1) \\ J = p_3\mathbf{q}_3 & (2) \\ \mathbf{I} = p_1\mathbf{q}_1 - S_2\mathbf{a}_2 + (S_1\mathbf{a}_1 + p_4\mathbf{q}_4) & (3) \\ \mathbf{F} = -p_2\mathbf{q}_2 & (4) \end{cases} \tag{6.19a}$$

$$\begin{cases} \mathbf{q}_3 = \cos\Theta_3\,\mathbf{q}_4 - \sin\Theta_3\,\mathbf{a}_3\times\mathbf{q}_4 & (1) \\ \mathbf{q}_2 = \cos\theta_2\,\mathbf{q}_1 + \sin\theta_2\,\mathbf{a}_2\times\mathbf{q}_1 & (2) \\ \mathbf{q}_1 = \cos\theta_1\,\mathbf{q}_4 + \sin\theta_1\,\mathbf{a}_1\times\mathbf{q}_4 & (3) \\ \mathbf{a}_2 = \cos\alpha_{21}\,\mathbf{a}_1 + \sin\alpha_{21}\,\mathbf{q}_1\times\mathbf{a}_1 & (4) \\ \mathbf{a}_3\times\mathbf{a}_1 = \mathbf{q}_4\sin\alpha_{31} & (5) \end{cases} \tag{6.19b}$$

Squaring both sides of (6.19) yields

$$x^2 - 2(\mathbf{a}_3 \cdot \mathbf{K})x + [\mathbf{K}]^2 = \mathbf{F}^2 \tag{6.20}$$

The scalar product of $\mathbf{a}_2$ with both sides of (6.19) yields

$$x(\mathbf{a}_2 \cdot \mathbf{a}_3) = [\mathbf{a}_2 \cdot \mathbf{K}] \tag{6.21}$$

Substituting (6.21) into (6.20) yields

$$[\mathbf{a}_2 \cdot \mathbf{K}]^2 - 2(\mathbf{a}_2 \cdot \mathbf{a}_3)(\mathbf{a}_3 \cdot \mathbf{K})[\mathbf{a}_2 \cdot \mathbf{K}] + (\mathbf{a}_2 \cdot \mathbf{a}_3)^2 [\mathbf{K}]^2 - (\mathbf{a}_2 \cdot \mathbf{a}_3)^2 \mathbf{F}^2 = 0 \tag{6.22}$$

Substituting (6.19a-1) and (6.19a-2) into (6.22) yields

$$[\mathbf{U} \cdot \mathbf{q}_3]^2 + [\mathbf{W} \cdot \mathbf{q}_3] + V = 0 \tag{6.23}$$

$$\begin{cases} \mathbf{U} = p_3 \mathbf{a}_2 \\ \mathbf{W} = 2 p_3 \{(\mathbf{I} \times \mathbf{a}_3) \cdot (\mathbf{a}_2 \times \mathbf{a}_3) \mathbf{a}_2 + (\mathbf{a}_2 \cdot \mathbf{a}_3)^2 \mathbf{I}\} \\ V = (\mathbf{I} \cdot \mathbf{a}_2) - 2(\mathbf{a}_2 \cdot \mathbf{a}_3)(\mathbf{I} \cdot \mathbf{a}_3)(\mathbf{I} \cdot \mathbf{a}_2) \\ \qquad + (\mathbf{a}_2 \cdot \mathbf{a}_3)^2 (\mathbf{I}^2 + p_3^2 - \mathbf{F}^2) \end{cases} \tag{6.23a}$$

6.6. Analysis of the R_G–RCS mechanism

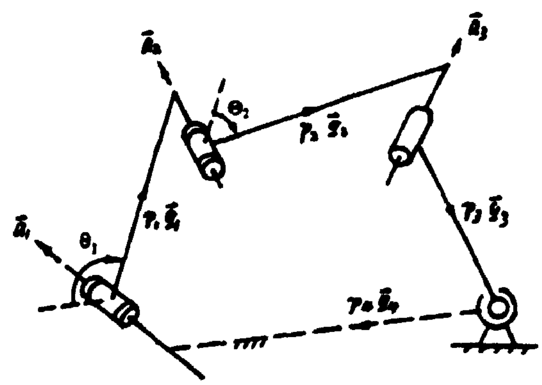


Fig. 6.5

The diagram of the R_G–RCS mechanism is shown in Fig. 6.5. The input angle is θ_1; the output angle is θ_2, i.e. $\theta_2 = \Theta_2$. The vector loop equation and the direction equations can be written as,

$$-x\,\mathbf{a}_3 + \mathbf{K} = \mathbf{F} \tag{6.24}$$

$$\begin{cases} \mathbf{K} = \mathbf{J} + \mathbf{I} & (1) \\ \mathbf{J} = p_2 \mathbf{q}_2 & (2) \\ \mathbf{I} = p_1 \mathbf{q}_1 - S_2 \mathbf{a}_2 + (S_1 \mathbf{a}_1 + p_4 \mathbf{q}_4) & (3) \\ \mathbf{F} = -p_3 \mathbf{q}_3 & (4) \end{cases} \tag{6.24a}$$

$$\begin{cases} \mathbf{q}_3 = \cos\theta_3\,\mathbf{q}_2 + \sin\theta_3\,\mathbf{a}_3\times\mathbf{q}_2 & (1) \\ \mathbf{a}_3 = c\,\alpha_{23}\,\mathbf{a}_2 + s\,\alpha_{23}\,\mathbf{q}_2\times\mathbf{a}_2 & (2) \\ \mathbf{q}_2 = \cos\theta_2\,\mathbf{q}_1 + \sin\theta_2\,\mathbf{a}_2\times\mathbf{q}_1 & (3) \\ \mathbf{a}_2 = \cos\alpha_{21}\,\mathbf{a}_1 + \sin\alpha_{21}\,\mathbf{q}_1\times\mathbf{a}_1 & (4) \\ \mathbf{q}_1 = \cos\theta_1\,\mathbf{q}_4 + \sin\theta_1\,\mathbf{a}_1\times\mathbf{q}_4 & (5) \end{cases} \tag{6.24b}$$

Squaring both sides of (6.24) yields

$$x^2 - 2[\mathbf{a}_3\cdot\mathbf{K}]x + [\mathbf{K}]^2 = \mathbf{F}^2 \tag{6.25}$$

The scalar product of $\mathbf{a}_2$ with both sides of (6.24) yields

$$x = [\mathbf{a}_3\cdot\mathbf{K}] \tag{6.26}$$

Substituting (6.26) into (6.25) yields

$$-[\mathbf{K}\cdot\mathbf{a}_3]^2 + [\mathbf{K}]^2 = \mathbf{F}^2 \tag{6.27}$$

Substituting (6.24a-1), (6.24a-2) and (6.24b-2) into (6.27) yields

$$[\mathbf{U}\cdot\mathbf{q}_2]^2 + [\mathbf{W}\cdot\mathbf{q}_2] + V = 0 \tag{6.28}$$

$$\begin{cases} \mathbf{U} = s\,\alpha_{23}\,\mathbf{a}_2\times\mathbf{I} \\ \mathbf{W} = 2c\,\alpha_{23}\,s\,\alpha_{23}(\mathbf{I}\cdot\mathbf{a}_2)\,\mathbf{a}_2\times\mathbf{I} - 2p_2\,\mathbf{I} \\ V = c^2\alpha_{23}(\mathbf{a}_2\cdot\mathbf{I})^2 - (\mathbf{I}^2 + p_2^2 - p_3^2) \end{cases} \tag{6.28a}$$

6.7. Analysis of the R_0-RCS mechanism

The diagram of the R_0-RCS mechanism is shown in Fig. 6.6. The input angle is θ_1; the output angle is θ_3, i.e. $\theta_3 = \Theta_3$. The vector loop equation and the direction equations can be written as,

$$\mathbf{I} = \mathbf{F} \tag{6.29}$$

$$\begin{cases} \mathbf{I} = (p_1\mathbf{q}_1 - S_2\mathbf{a}_2) + (S_1\mathbf{a}_1 + p_4\mathbf{q}_4) & (1) \\ \mathbf{F} = -p_2\mathbf{q}_2 + x\,\mathbf{a}_3 - p_3\mathbf{q}_3 & (2) \end{cases} \tag{6.29a}$$

$$\begin{cases} \mathbf{q}_3 = \cos\theta_3\,\mathbf{q}_2 + \sin\theta_3\,\mathbf{a}_3\times\mathbf{q}_2 & (1) \\ \mathbf{a}_3 = c\,\alpha_{23}\,\mathbf{a}_2 + s\,\alpha_{23}\,\mathbf{q}_2\times\mathbf{a}_2 & (2) \\ \mathbf{q}_2 = \cos\theta_2\,\mathbf{q}_1 + \sin\theta_2\,\mathbf{a}_2\times\mathbf{q}_1 & (3) \\ \mathbf{a}_2 = \cos\alpha_{21}\,\mathbf{a}_1 + \sin\alpha_{21}\,\mathbf{q}_1\times\mathbf{a}_1 & (4) \\ \mathbf{q}_1 = \cos\theta_1\,\mathbf{q}_4 + \sin\theta_1\,\mathbf{a}_1\times\mathbf{q}_4 & (5) \end{cases} \tag{6.29b}$$

Squaring both sides of (6.29) yields

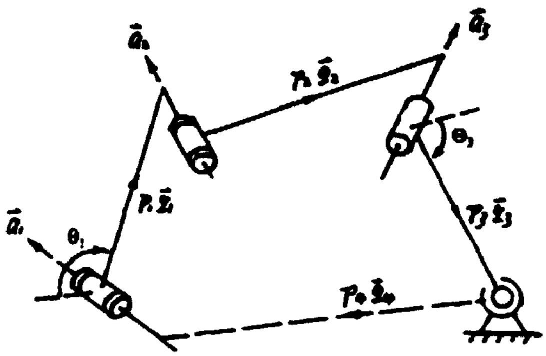


Fig. 6.6

$$I^2 = p_2^2 + p_3^2 + x^2 + 2p_2p_3\cos\Theta_3 \tag{6.30}$$

The scalar product of a_2 with both sides of (6.29) yields

$$(a_2\cdot I) = x\,c\alpha_{23} - p_3 s\alpha_{23}\sin\Theta_3 \tag{6.31}$$

From (6.30) and (6.31) we obtain

$$\begin{cases} \cos\Theta_3 = [(I^2 - p_2^2 - p_3^2) - x^2]\,/\,(2p_2p_3) \\ \sin\Theta_3 = [x\,c\alpha_{23} - (a_2\cdot I)]\,/\,(p_3 s\alpha_{23}) \end{cases} \tag{6.32}$$

From $\cos^2\Theta_3 + \sin^2\Theta_3 = 1$ and (6.32) we get

$$x^4 + k_2x^2 + k_3x + k_4 = 0 \tag{6.33}$$

$$\begin{cases} k_2 = (2p_2\cot\alpha_{23})^2 - 2(I^2 - p_2^2 - p_3^2) \\ k_3 = -8p_2^2\cot\alpha_{23}\csc\alpha_{23}(a_2\cdot I) \\ k_4 = [2p_2\csc\alpha_{23}(a_2\cdot I)]^2 + (I^2 - p_2^2 - p_3^2)^2 - (2p_2p_3)^2 \end{cases} \tag{6.33a}$$

From (6.33) we obtain x; From (6.32) we obtain Θ_3; Substituting (6.29b-1) and (6.29b-2) into (6.29a-2), then substituting (6.29a-2) into (6.29) yields

$$\rho_1 q_2 + \rho_2 q_2\times a_2 = (I - \rho_3 a_2) \tag{6.34}$$

$$\begin{cases} \rho_1 = -(p_2 + p_3 c\Theta_3) \\ \rho_2 = (s\alpha_{23}x + p_3 c\alpha_{23} s\Theta_3) \\ \rho_3 = (c\alpha_{23}x - p_3 s\alpha_{23} s\Theta_3) \end{cases} \tag{6.34a}$$

The scalar product of $\{I,\ (I\times a_2)\}$ with both sides of (6.34) yields

$$\begin{cases} (\rho_1 I + \rho_2 a_2\times I)\cdot q_2 = (I^2 - \rho_3 a_2\cdot I) \\ [\rho_1 I\times a_2 + \rho_2 a_2\times(I\times a_2)]\cdot q_2 = 0 \end{cases} \tag{6.35}$$

Substituting (6.29b-3) into (6.35) yields

$$\begin{cases} A\cos\theta_2 + B\sin\theta_2 = C \\ A'\cos\theta_2 + B'\sin\theta_2 = 0 \end{cases} \tag{6.36}$$

$$\begin{cases} A = (\rho_1 \mathbf{l} + \rho_2 \mathbf{a}_2 \times \mathbf{l}) \cdot \mathbf{q}_1 \\ B = (\rho_1 \mathbf{l} + \rho_2 \mathbf{a}_2 \times \mathbf{l}) \cdot (\mathbf{a}_2 \times \mathbf{q}_1) \\ C = (\mathbf{l}^2 - \rho_3 \mathbf{a}_2 \cdot \mathbf{l}) \\ A' = [\rho_1 \mathbf{l} \times \mathbf{a}_2 + \rho_2 \mathbf{a}_2 \times (\mathbf{l} \times \mathbf{a}_2)] \cdot \mathbf{q}_1 \\ B' = [\rho_1 \mathbf{l} \times \mathbf{a}_2 + \rho_2 \mathbf{a}_2 \times (\mathbf{l} \times \mathbf{a}_2)] \cdot (\mathbf{a}_2 \times \mathbf{q}_1) \end{cases} \tag{6.36a}$$

From (6.36) we obtain θ_2 :

$$\begin{cases} \cos\theta_2 = CB' \,/\, (AB' - BA') \\ \sin\theta_2 = CA' \,/\, (A'B - B'A) \end{cases} \tag{6.37}$$

6.8. Conclusion

Comparing the analyses procedures in Sections 6.2, 6.3, 6.4, 6.5 and 6.6, the uniformity of the vector algebraic method is apparant. In Section 6.7 a slightly different approach is adopted, for showing the flexibility of using the method.

CHAPTER 7. THE R_0-3R-E MECHANISM

7.1. Introduction

The tracta coupling mechanism is a useful engineering device applied as a constant-velocity universal joint for nonparallel, intersecting shafts ([75] Wallace and Freudenstein, 1970). There exist several different versions of tracta coupling mechanisms. One of them is composed of four R pairs and one E pair with E pair sitting at the middle, R_0-$RERR$.

The displacement analysis of the generalized tracta coupling R_0-$RERR$ was first performed by Wallace and Freudenstein ([75] 1968, [76] 1970). They successfully obtained a fourth-order polynomial displacement equation using the *geometric configuration method*, by first disassembling the linkage into two configurations and then reassembled it under appropriate geometric constraints.

The variant mechanisms of the generalized tracta coupling are R_0-$ERRR$, R_0-$RRER$ and R_0-$RRRE$. Mechanism R_0-$ERRR$ has been analyzed by Duffy and Keen ([22] 1972) using the *spherical trigonometry method* by simulating the E pair (as shown in Fig. 7.1) with PRP and PPR joint arrangements (Fig. 7.2), and the mechanism was considered a special case of the spatial seven-link R_0-$PPRRRR$ and R_0-$PRPRRR$ mechanisms.

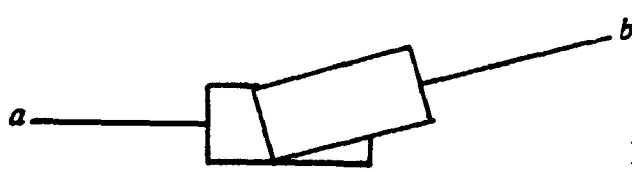


Fig. 7.1 Plane Pair (E)

In 1980, Duffy ([28] pp.129-131, pp.369-384) introduced three more forms of simulations of E pair, i.e. RPR, RRP and R-R-R combinations, and analyzed R_0-$RERR$ mechanism by simulating the E pair with R-R-R joint arrangement, i.e. three-parallel-revolute-pair combination.

The geometric configuration method can solve the symmetric R_0-$RERR$ mechanism but it does not work for unsymmetric cases: R_0-$ERRR$, R_0-$RRER$ and R_0-$RRRE$. The spherical trigonometry method, by simulating the E pair with PRP, PPR joint arrangement, solved the R_0-$ERRR$ mechanism, however, it becomes awkward for the symmetric case R_0-$RERR$, for "the formation of the required second equation has

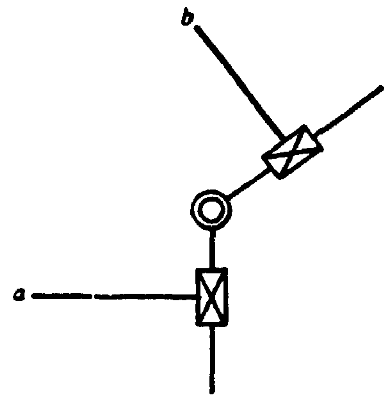


Fig. 7.2(a)

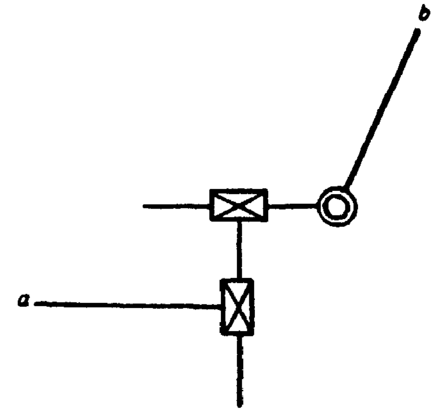


Fig. 7.2(b)

proved to be a most difficult problem," (Duffy and Keen, [22] p213, 1972).

The geometric configuration method is not a general method. In the vast domain of spatial mechanisms, its applicability is quite limited. It can only be used to solve a small number of simple mechanisms or the mechanisms with special geometric conditions, such as symmetry, etc.

Spherical trigonometry method is a general method, but its way of tackling those mechanisms with an E pair appears rather tortuous. The E pair is simple as shown in Fig. 7.1. Its simulations are complex, as shown in Fig. 7.2. As a matter of fact, kinematically, the simulation models of E pair are not entirely equivalent to E pair. Equivalence holds *only* within certain motion ranges.

Now the following questions naturally arise:

(1). Is it possible to develop a technique which is equally applicable for analyzing *all* the four mechanisms R_0–*ERRR*, R_0–*RERR*, R_0–*RRER* and R_0–*RRRE*? Moreover, can we make it not only *equally applicable* but also *methodologically simple*?

(2). Using the geometric configuration method, the "algebra involved in carrying out this procedure is formidable, but feasible"([76] Wallace and Freudenstein, p.719, 1970). Is it possible to make the algebraic operation not only feasible, but also *simple*?

(3). What if we perform the analysis directly using the E pair, instead of using the complex simulated models?

(4). Is it possible to develop not only standardized analysis steps but also standardized algebraic expressions, so as to permit the analysis of different mechanisms using the *same routine* —— not only methodologically but also the appearance of the algebraic operations and expressions?

In this chapter, the generalized tracta coupling and its variant mechanisms are all kinematically analyzed by ***vector algebraic method***. Of the four mechanisms, R_0-*RRRE* and R_0-*RRER* have not been analyzed before. It is demonstrated that the new method successfully answers the questions posed above. The new approach presented in this chapter is much simpler than previous approaches.

Following is a brief summary of the analysis steps of the new approach:

(a). Write down the vector loop equation and constraint equations, based on the structure of the mechanism;

(b). Derive the first equation relating the input, output and auxiliary angles using a vector, which is perpendicular to the plane of the E pair, to dot product both sides of the vector loop equation;

(c). Derive the second equation relating the input, output and auxiliary angle using the property $\mathbf{a}_i \cdot \mathbf{a}_j = \mathbf{a}_i \cdot \mathbf{a}_j$, where one side of the identity is calculated from one part of the mechanism loop, another side of the identity is calculated from the other part of the loop.

(d). Eliminate the auxiliary angle from the two equations derived in steps (b) and (c) to obtain the desired input-output displacement equation.

7.2. Analysis of the Generalized Tracta Coupling R_0-*RERR*

(1). Configuration analysis.

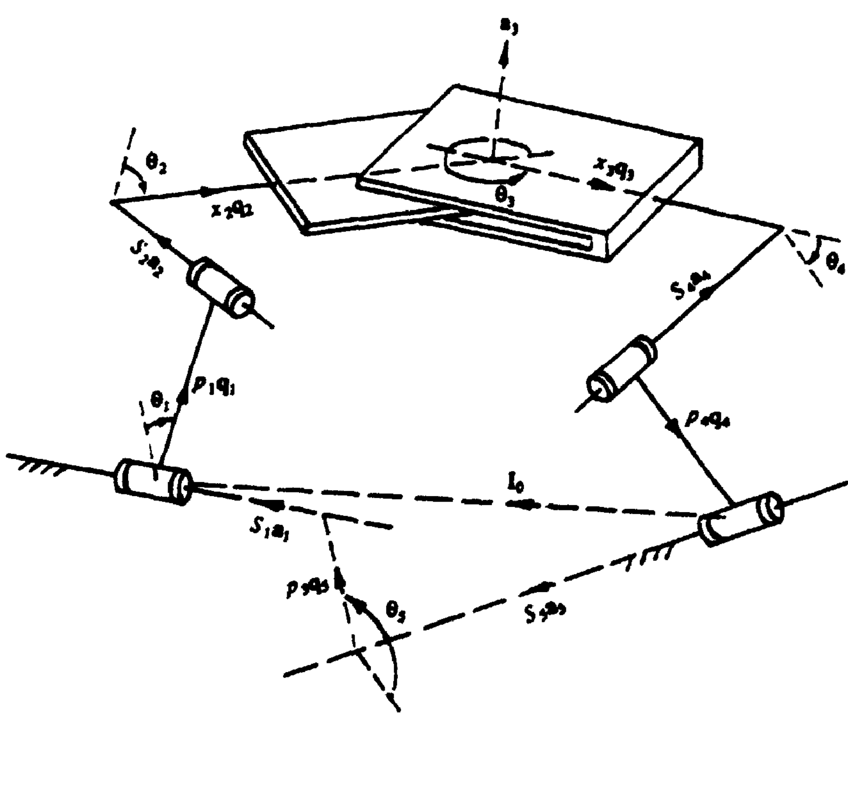


Fig. 7.3

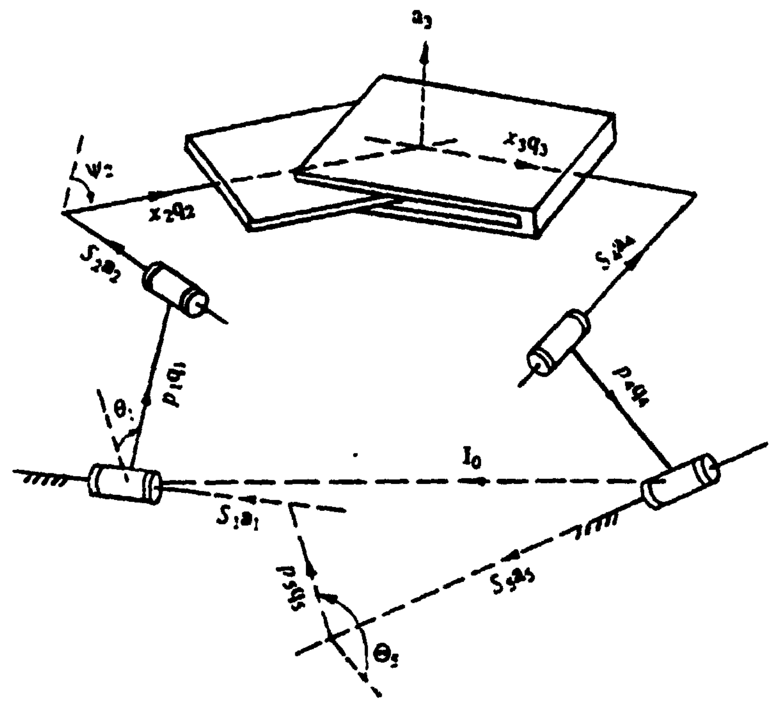


Fig. 7.3(a)

Mechanism R_0-$RERR$ is shown in Fig. 7.3. When the input angle θ_1 is given, unknowns to be determined are $\{x_2, x_3, \theta_2, \theta_3, \theta_4, \theta_5\}$. Let θ_5 be the output angle, θ_2 be the auxiliary angle, i.e. $\theta_5 = \Theta_5$, $\theta_2 = \psi_2$, as shown in Fig. 7.3(a). The vector loop equation can be written as:

$$x_2\mathbf{q}_2 + x_3\mathbf{q}_3 + \mathbf{K} = 0 \tag{7.1}$$

$$\begin{cases} \mathbf{K} = \mathbf{J} + \mathbf{I} & (1) \\ \mathbf{J} = p_4\mathbf{q}_4 - S_4\mathbf{a}_4 & (2) \\ \mathbf{I} = (p_1\mathbf{q}_1 + S_2\mathbf{a}_2) + \mathbf{I}_0 & (3) \\ \mathbf{I}_0 = S_5\mathbf{a}_5 + p_5\mathbf{q}_5 + S_1\mathbf{a}_1 & (4) \end{cases} \tag{7.1a}$$

$$\begin{cases} \mathbf{q}_3 = \mathbf{a}_3 \times \mathbf{a}_4 \csc\alpha_{34} & (1) \\ \mathbf{a}_4 = c\alpha_{45}\mathbf{a}_5 - s\alpha_{45}\mathbf{q}_4 \times \mathbf{a}_5 & (2) \\ \mathbf{q}_4 = \cos\Theta_5\mathbf{q}_5 - \sin\Theta_5\mathbf{a}_5 \times \mathbf{q}_5 & (3) \end{cases} \tag{7.1b}$$

$$\begin{cases} \mathbf{a}_3 = c\alpha_{23}\mathbf{a}_2 + s\alpha_{23}\mathbf{q}_2 \times \mathbf{a}_2 & (4) \\ \mathbf{q}_2 = \cos\psi_2\mathbf{q}_1 + \sin\psi_2\mathbf{a}_2 \times \mathbf{q}_1 & (5) \\ \mathbf{a}_2 = c\alpha_{12}\mathbf{a}_1 + s\alpha_{12}\mathbf{q}_1 \times \mathbf{a}_1 & (6) \\ \mathbf{q}_1 = c\theta_1\mathbf{q}_5 + s\theta_1\mathbf{a}_1 \times \mathbf{q}_5 & (7) \end{cases} \tag{7.1b}$$

where $\mathbf{I}$ is the sum of those vectors in the kinematic loop which are known at the very beginning. $\mathbf{I}$ is called *input vector*; $\mathbf{J}$ is the sum of those unknown constant-magnitude vectors of the kinematic loop which can be directly expressed in terms of the output angle, Θ_5 as in this case. $\mathbf{J}$ is called *output vector*. The more strict definitions of $\mathbf{I}$, $\mathbf{J}$ are given in chapter 2.

(i). *Derivation of first equation relating* θ_1, Θ_5 *and* ψ_2.

The scalar product of $\mathbf{a}_3$ with both sides of (7.1) yields

$$\mathbf{K}\cdot\mathbf{a}_3 = 0 \tag{7.2}$$

Substituting (7.1b-4) into (7.2) yields

$$\mathbf{U}\cdot\mathbf{q}_2 = V \tag{7.3}$$

$$\begin{cases} \mathbf{U} = s\alpha_{23}(\mathbf{a}_2\times\mathbf{K}) \\ V = -c\alpha_{23}(\mathbf{a}_2\cdot\mathbf{K}) \end{cases} \tag{7.3a}$$

Substituting (7.1b-5) into (7.3) yields

$$A\cos\psi_2 + B\sin\psi_2 = C \tag{7.4}$$

$$\begin{cases} A = (\mathbf{U}\cdot\mathbf{q}_1) \\ B = (\mathbf{U}\cdot\mathbf{a}_2\times\mathbf{q}_1) \\ C = V \end{cases} \tag{7.4a}$$

(ii). *Derivation of a second equation relating* θ_1, Θ_5 *and* ψ_2.

According to the structure of the mechanism, we can write:

$$\mathbf{a}_3(\theta_1,\psi_2)\cdot\mathbf{a}_4(\Theta_5) = c\alpha_{34} \tag{7.5}$$

Substituting (7.1b-4) into (7.5) yields

$$\mathbf{U}'\cdot\mathbf{q}_2 = V' \tag{7.6}$$

$$\begin{cases} \mathbf{U}' = s\alpha_{23}(\mathbf{a}_2\times\mathbf{a}_4) \\ V' = -c\alpha_{23}(\mathbf{a}_2\cdot\mathbf{a}_4) + c\alpha_{34} \end{cases} \tag{7.6a}$$

Substituting (7.1b-5) into (7.6) yields

$$A'\cos\psi_2 + B'\sin\psi_2 = C' \tag{7.7}$$

$$\begin{cases} A' = (\mathbf{U}'\cdot\mathbf{q}_1) \\ B' = (\mathbf{U}'\cdot\mathbf{a}_2\times\mathbf{q}_1) \\ C' = V' \end{cases} \tag{7.7a}$$

(iii). *Derivation of the input-output displacement equations.*

From (7.4) and (7.7) we get

$$\begin{cases} \cos\psi_2 = -Q_2/Q_3 \\ \sin\psi_2 = Q_1/Q_3 \end{cases} \tag{7.8}$$

$$\begin{cases} Q_1 = (AC' - A'C) \\ Q_2 = (BC' - B'C) \\ Q_3 = (AB' - A'B) \end{cases} \tag{7.8a}$$

From $\cos^2\psi + \sin^2\psi = 1$ and (7.8) we obtain

$$Q_1^2 + Q_2^2 = Q_3^2 \tag{7.9}$$

$$\begin{cases} Q_1 = (\mathbf{U}\cdot\mathbf{q}_1)V' - (\mathbf{U}'\cdot\mathbf{q}_1)V \\ \quad = s\alpha_{23}c\alpha_{23}\{[(\mathbf{q}_1\times\mathbf{a}_2)\times\mathbf{a}_2]\cdot(\mathbf{a}_4\times\mathbf{K})\} + s\alpha_{23}c\alpha_{34}(\mathbf{q}_1\times\mathbf{a}_2)\cdot\mathbf{K} \\ \quad = -(s\alpha_{23}c\alpha_{23}\mathbf{q}_1)\cdot(\mathbf{a}_4\times\mathbf{K}) + (s\alpha_{23}c\alpha_{34}\mathbf{q}_1)\cdot\mathbf{K} \\ Q_2 = s\alpha_{23}c\alpha_{23}(\mathbf{q}_1\times\mathbf{a}_2)\cdot(\mathbf{a}_4\times\mathbf{K}) + (s\alpha_{23}c\alpha_{34}\mathbf{q}_1)\cdot\mathbf{K} \\ Q_3 = -s^2\alpha_{23}\mathbf{a}_2\cdot(\mathbf{a}_4\times\mathbf{K}) \end{cases} \tag{7.9a}$$

Because $(\mathbf{a}_4\times\mathbf{K})$ and $\mathbf{K}$ are linear in $\sin\Theta_5$ and $\cos\Theta_5$, we can see from (7.9a) that Q_1, Q_2 and Q_3 are also linear in $\sin\Theta_5$ and $\cos\Theta_5$. Hence, we can obtain a fourth-order polynomial displacement equation from (7.9).

Substituting (7.1a) into (7.9a) yields

$$\begin{cases} Q_1 = (\mathbf{W}_1\cdot\mathbf{q}_4 + Z_1) = (\mathbf{W}_1\cdot\mathbf{q}_5)\cos\Theta_5 + (\mathbf{W}_1\cdot\mathbf{q}_5\times\mathbf{a}_5)\sin\Theta_5 + Z_1 \\ Q_2 = (\mathbf{W}_2\cdot\mathbf{q}_4 + Z_2) = (\mathbf{W}_2\cdot\mathbf{q}_5)\cos\Theta_5 + (\mathbf{W}_2\cdot\mathbf{q}_5\times\mathbf{a}_5)\sin\Theta_5 + Z_2 \\ Q_3 = (\mathbf{W}_3\cdot\mathbf{q}_4 + Z_3) = (\mathbf{W}_3\cdot\mathbf{q}_5)\cos\Theta_5 + (\mathbf{W}_3\cdot\mathbf{q}_5\times\mathbf{a}_5)\sin\Theta_5 + Z_3 \end{cases} \tag{7.10}$$

$$\begin{cases} \mathbf{W}_1 = s\alpha_{23}c\alpha_{23}\{s\alpha_{45}(\mathbf{a}_5\cdot\mathbf{q}_1)\mathbf{I} + s\alpha_{45}(\mathbf{a}_5\cdot\mathbf{I})\mathbf{q}_1 + p_1c\alpha_{45}\mathbf{a}_5\times\mathbf{q}_1\} \\ \qquad + s\alpha_{23}c\alpha_{34}\{p_4\mathbf{q}_1\times\mathbf{a}_2 + S_4s\alpha_{45}\mathbf{q}_5\times(\mathbf{q}_1\times\mathbf{a}_2)\} \\ \mathbf{W}_2 = s\alpha_{23}c\alpha_{23}\{p_4c\alpha_{45}(\mathbf{q}_1\times\mathbf{a}_2)\times\mathbf{a}_5 - s\alpha_{45}(\mathbf{q}_1\times\mathbf{a}_2\cdot\mathbf{a}_5)\mathbf{I} \\ \qquad - s\alpha_{45}(\mathbf{I}\cdot\mathbf{a}_5)\mathbf{q}_1\times\mathbf{a}_2\} + s\alpha_{23}c\alpha_{34}\{p_4\mathbf{q}_1 + S_4s\alpha_{45}\mathbf{a}_5\times\mathbf{q}_1\} \\ \mathbf{W}_3 = s^2\alpha_{23}\{p_4c\alpha_{45}\mathbf{a}_5\times\mathbf{a}_2 + s\alpha_{45}(\mathbf{a}_5\cdot\mathbf{a}_2)\mathbf{I} + s\alpha_{45}(\mathbf{I}\cdot\mathbf{a}_5)\mathbf{a}_2\} \end{cases} \tag{7.10a}$$

$$\begin{cases} Z_1 = s\alpha_{23}c\alpha_{23}\{p_4s\alpha_{45}(\mathbf{a}_5\cdot\mathbf{q}_1) - c\alpha_{45}(\mathbf{a}_5\cdot\mathbf{I}\times\mathbf{q}_1)\} \\ \qquad + s\alpha_{23}c\alpha_{34}\{(\mathbf{I}\cdot\mathbf{q}_1\times\mathbf{a}_2) - S_4c\alpha_{45}(\mathbf{a}_5\cdot\mathbf{q}_1\times\mathbf{a}_2)\} \\ Z_2 = s\alpha_{23}c\alpha_{23}\{c\alpha_{45}(\mathbf{q}_1\times\mathbf{a}_2)\cdot(\mathbf{a}_5\times\mathbf{I}) - p_4s\alpha_{45}(\mathbf{q}_1\times\mathbf{a}_2\cdot\mathbf{a}_5)\} \\ \qquad + s\alpha_{23}c\alpha_{34}\{(\mathbf{I}\cdot\mathbf{q}_1) - S_4c\alpha_{45}(\mathbf{a}_5\cdot\mathbf{q}_1)\} \\ Z_3 = s^2\alpha_{23}\{p_4s\alpha_{45}(\mathbf{a}_5\cdot\mathbf{a}_2) + c\alpha_{45}(\mathbf{a}_5\cdot\mathbf{I}\times\mathbf{a}_2)\} \end{cases} \tag{7.10a}$$

Substituting (7.10) into (7.9) yields

$$\mu_1\cos^2\Theta_5 + \mu_2\sin^2\Theta_5 + \mu_3\sin\Theta_5\cos\Theta_5 + \mu_4\cos\Theta_5 + \mu_5\sin\Theta_5 + \mu_6 = 0 \tag{7.11}$$

$$\begin{cases} \mu_1 = (W_1 \cdot q_5)^2 + (W_2 \cdot q_5)^2 - (W_3 \cdot q_5)^2 \\ \mu_2 = (W_1 \cdot q_5 \times a_5)^2 + (W_2 \cdot q_5 \times a_5)^2 - (W_3 \cdot q_5 \times a_5)^2 \\ \mu_3 = 2(W_1 \cdot q_5)(W_1 \cdot q_5 \times a_5) + 2(W_2 \cdot q_5)(W_2 \cdot q_5 \times q_5) - 2(W_3 \cdot q_5)(W_3 \cdot q_5 \times q_5) \\ \mu_4 = 2(W_1 \cdot q_5)Z_1 + 2(W_2 \cdot q_5)Z_2 - 2(W_3 \cdot q_5)Z_3 \\ \mu_5 = 2(W_1 \cdot q_5 \times a_5)Z_1 + 2(W_2 \cdot q_5 \times a_5)Z_2 - 2(W_3 \cdot q_5 \times a_5)Z_3 \\ \mu_6 = Z_1^2 + Z_2^2 - Z_3^2 \end{cases} \quad (7.11a)$$

Let $y = \tan(\Theta_5/2)$, then $\cos\Theta_5 = (1-y^2)/(1+y^2)$, $\sin\Theta = 2y/(1+y^2)$. From (7.11) we get

$$\sum_{i=0}^{4} v_i y^{4-i} = 0 \quad (7.12)$$

$$\begin{cases} v_0 = \mu_1 \\ v_1 = -2\mu_3 \\ v_2 = -2\mu_1 + 4\mu_2 - \mu_4 + \mu_6 \\ v_3 = 2\mu_3 + 2\mu_5 \\ v_4 = \mu_1 + \mu_4 + \mu_6 \end{cases} \quad (7.12a)$$

Solving (7.12) yields y, then $\theta_5 = 2\tan^{-1}y$; From (7.8) we get ψ_2, namely θ_2; To this stage, $\{x_2, x_3\}$ are the only unknowns in (7.1). The scalar products of a_4 and a_2 with both sides of (7.1) yield

$$\begin{cases} x_2 = -(K \cdot a_4)/(q_2 \cdot a_4) \\ x_3 = -(K \cdot a_2)/(q_3 \cdot a_2) \end{cases} \quad (7.13)$$

From the structure of the mechanism, we can write:

$$\cos\theta_3 q_2 + \sin\theta_3 a_3 \times q_2 = q_3 \quad (7.14)$$

$$\cos\theta_4 q_3 + \sin\theta_4 a_4 \times q_3 = q_4 \quad (7.15)$$

The scalar product of q_2 with both sides of (7.14) and the scalar product of $a_3 \times q_2$ with both sides of (7.14) yield θ_3:

$$\begin{cases} \cos\theta_3 = q_2 \cdot q_3 \\ \sin\theta_3 = a_3 \times q_2 \cdot q_3 \end{cases} \quad (7.16)$$

The scalar product of q_3 with both sides of (7.15) and the scalar product of $a_4 \times q_3$ with bo'h sides of (7.15) yield θ_4:

$$\begin{cases} \cos\theta_4 = q_3 \cdot q_4 \\ \sin\theta_4 = a_4 \times q_3 \cdot q_4 \end{cases} \quad (7.17)$$

To this stage, theoretical displacement analysis for the generalized tracta coupling is complete. If we need to calculate numerical values for a specific generalized tracta coupling, we can easily expand (7.10a) by substituting the related identities of (7.1a) and (7.1b). For instance, $\mathbf{w}_1$ and z_1 can be expressed as,

$$\mathbf{W}_1 = w_{11}\mathbf{q}_5 + w_{12}\mathbf{a}_5 + w_{13}\mathbf{q}_5 \times \mathbf{a}_5 \tag{7.18}$$

$$\begin{cases} w_{11} = s\alpha_{23}c\alpha_{23}[\rho_1 s\alpha_{45}s\alpha_{51}s\theta_1 + \rho_2 s\alpha_{45}c\theta_1 - p_1 c\alpha_{45}c\alpha_{51}s\theta_1] \\ \quad + s\alpha_{23}c\alpha_{34}[p_4(\beta_2 c\alpha_{51} - \beta_1 s\alpha_{51})] \\ w_{12} = s\alpha_{23}c\alpha_{23}[2\rho_2 s\alpha_{45}s\alpha_{51}s\theta_1] \\ \quad + s\alpha_{23}c\alpha_{34}[p_4 s\alpha_{51} + S_4 s\alpha_{45}c\alpha_{51}]s\theta_1 s\alpha_{12} \\ w_{13} = s\alpha_{23}c\alpha_{23}[s\alpha_{45}s\alpha_{51}s\theta_1 - \rho_2 s\alpha_{45}c\alpha_{51}s\theta_1 - p_1 c\alpha_{45}c\theta_1] \\ \quad + s\alpha_{23}c\alpha_{34}[-p_4 c\alpha_{51} + S_4 s\alpha_{45}s\alpha_{51}] \end{cases} \tag{7.18a}$$

$$\begin{cases} Z_1 = s\alpha_{23}c\alpha_{23}[p_4 s\alpha_{45}s\alpha_{51}s\theta_1 - c\alpha_{45}(\rho_1 c\alpha_{51}s\theta_1 + \rho_3 c\theta_1)] \\ \quad + s\alpha_{23}c\alpha_{34}[\rho_1(\beta_2 c\alpha_{51} - \beta_1 s\alpha_{51})] \\ \quad + s\alpha_{23}c\alpha_{34}[s\theta_1 s\alpha_{12}(\rho_2 s\alpha_{51} - \rho_3 c\alpha_{51} - S_4 c\alpha_{45}s\alpha_{51})] \end{cases} \tag{7.19}$$

where ρ_i and β_i are given as follow:

$$\mathbf{I} = \rho_1\mathbf{q}_5 + \rho_2\mathbf{a}_5 + \rho_3\mathbf{q}_5 \times \mathbf{a}_5 \tag{7.20}$$

$$\begin{cases} \rho_1 = S_2 s\theta_1 s\alpha_{12} + p_1 c\theta_1 + p_5 \\ \rho_2 = S_2\beta_1 + p_1 s\alpha_{51}s\theta_1 + S_1 c\alpha_{51} + S_5 \\ \rho_3 = S_2\beta_2 - p_1 c\alpha_{51}s\theta_1 + S_1 s\alpha_{51} \end{cases} \tag{7.20a}$$

$$\begin{cases} \beta_1 = (c\alpha_{51}c\alpha_{12} - s\alpha_{51}s\alpha_{12}c\theta_1) \\ \beta_2 = (s\alpha_{51}c\alpha_{12} + c\alpha_{51}s\alpha_{12}c\theta_1) \end{cases} \tag{7.20b}$$

$\mathbf{w}_2$, $\mathbf{w}_3$, z_2 and z_3 can be similarly expanded. It is clear that

$$\begin{cases} (\mathbf{W}_i \cdot \mathbf{q}_5) = w_{i1} \\ (\mathbf{W}_i \cdot \mathbf{q}_5 \times \mathbf{a}_5) = w_{i3} \end{cases} \quad (i = 1, 2, 3) \tag{7.21}$$

Substitute (7.21) and z_i into (7.11a), $\{\mu_k\}$ $(k = 1 - 6)$ can be easily solved.

Velocity and acceleration analysis.

From $\frac{d}{dt}$(7.11) we get

$$\dot{\Theta}_5 = \frac{\dot{\mu}_1\cos^2\Theta_5 + \dot{\mu}_2\sin^2\Theta_5 + \dot{\mu}_3\sin\Theta_5\cos\Theta_5 + \dot{\mu}_4\cos\Theta_5 + \dot{\mu}_5\sin\Theta_5 + \dot{\mu}_6}{(\mu_1 - \mu_2)\sin(2\Theta_5) - \mu_3\cos(2\Theta_5) + \mu_4\sin\Theta_5 - \mu_5\cos\Theta_5} \tag{7.22}$$

$\{\dot{\mu}_i\}$ $(i = 1 - 6)$ of (7.22) can be calculated from (7.11a) and (7.10a).

From $\frac{d^2}{dt^2}$(7.11) we obtain $\ddot{\theta}_5$.

Other velocity variables $\{\dot{x}_2, \dot{x}_3, \dot{\theta}_2, \dot{\theta}_3, \dot{\theta}_4\}$ and the corresponding acceleration variables can also be easil, determined. Here we only write out the expression of $\dot{\theta}_5$, for it is the most important one among all the velocity and acceleration variables.

7.3. Displacement analysis of the R_0-ERRR mechanism

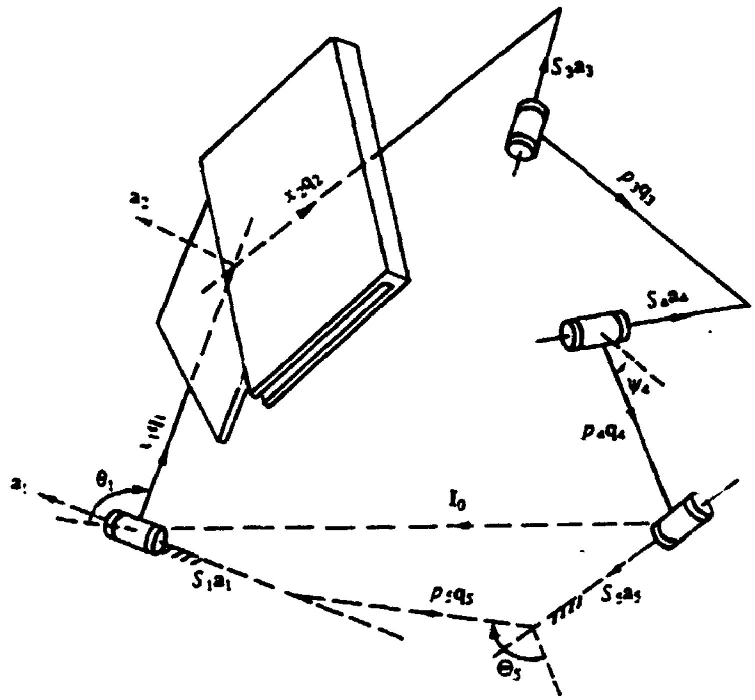


Fig. 7.4

Mechanism R_0-ERRR is shown in Fig. 7.4. When the input angle θ_1 is given, unknowns to be determined are $\{x_1, x_2, \theta_2, \theta_3, \theta_4, \theta_5\}$. Let θ_5 be the output angle, θ_4 be the auxiliary angle, i.e. $\theta_5 = \Theta_5$, $\theta_4 = \psi_4$. The vector loop equation can be written as:

$$x_1 \mathbf{q}_1 + x_2 \mathbf{q}_2 + \mathbf{K} + \mathbf{L} = 0 \tag{7.23}$$

$$\begin{cases} \mathbf{L} = p_3 \mathbf{q}_3 - S_3 \mathbf{a}_3 & (1) \\ \mathbf{K} = \mathbf{J} + \mathbf{I} & (2) \\ \mathbf{J} = p_4 \mathbf{q}_4 - S_4 \mathbf{a}_4 & (3) \\ \mathbf{I} = S_5 \mathbf{a}_5 + p_5 \mathbf{q}_5 + S_1 \mathbf{a}_1 & (4) \end{cases} \tag{7.23a}$$

$$\begin{cases} \mathbf{q}_2 = \mathbf{a}_2 \times \mathbf{a}_3 \csc\alpha_{23} & (1) \\ \mathbf{a}_3 = c\alpha_{34} \mathbf{a}_4 - s\alpha_{34} \mathbf{q}_3 \times \mathbf{a}_4 & (2) \\ \mathbf{q}_3 = \cos\psi_4 \mathbf{q}_4 - \sin\psi_4 \mathbf{a}_4 \times \mathbf{q}_4 & (3) \\ \mathbf{a}_4 = c\alpha_{45} \mathbf{a}_5 - s\alpha_{45} \mathbf{q}_4 \times \mathbf{a}_5 & (4) \\ \mathbf{q}_4 = \cos\Theta_5 \mathbf{q}_5 - \sin\Theta_5 \mathbf{a}_5 \times \mathbf{q}_5 & (5) \end{cases} \tag{7.23b}$$

$$\begin{cases} \mathbf{a}_2 = c\alpha_{12} \mathbf{a}_1 + s\alpha_{12} \mathbf{q}_1 \times \mathbf{a}_1 & (6) \\ \mathbf{q}_1 = c\theta_1 \mathbf{q}_5 + s\theta_1 \mathbf{a}_1 \times \mathbf{q}_5 & (7) \\ \mathbf{q}_5 = \mathbf{a}_5 \times \mathbf{a}_1 \csc\alpha_{51} & (8) \end{cases} \tag{7.23b}$$

where $\mathbf{L}$ is defined as the sum of those unknown constant-magnitude vectors of the kinematic loop which can be expressed in terms of the auxiliary angle, θ_4 as in this case. $\mathbf{L}$ is called *auxiliary vector*. The strict definition of $\mathbf{L}$ is given in chapter 2.

(i). *Derivation of first equation relating θ_1, Θ_5 and ψ_4.*

The scalar product of $\mathbf{a}_1$ with both sides of (7.23) yields

$$\mathbf{a}_2 \cdot \mathbf{L} = -\mathbf{a}_2 \cdot \mathbf{K} \tag{7.24}$$

Substituting (7.23a-1) into (7.24) yields

$$\mathbf{U} \cdot \mathbf{q}_3 = V \tag{7.25}$$

$$\begin{cases} \mathbf{U} = p_3 \mathbf{a}_2 + S_3 s\alpha_{34} (\mathbf{a}_4 \times \mathbf{a}_2) \\ V = -(\mathbf{a}_2 \cdot \mathbf{K}) + S_3 c\alpha_{34} (\mathbf{a}_2 \cdot \mathbf{a}_4) \end{cases} \tag{7.25a}$$

Substituting (7.23b-3) into (7.25) yields

$$A \cos\psi_4 + B \sin\psi_4 = C \tag{7.26}$$

$$\begin{cases} A = (\mathbf{U} \cdot \mathbf{q}_4) \\ B = (\mathbf{U} \cdot \mathbf{q}_4 \times \mathbf{a}_4) \\ C = V \end{cases} \tag{7.26a}$$

(ii). *Derivation of a second equation relating θ_1, Θ_5 and ψ_4.*

From the structure of the mechanism, we can write:

$$\mathbf{a}_2(\theta_1) \cdot \mathbf{a}_3(\Theta_5, \psi_4) = c\alpha_{23} \tag{7.27}$$

Substituting (7.23b-2) into (7.27) yields

$$\mathbf{U}' \cdot \mathbf{q}_3 = V' \tag{7.28}$$

$$\begin{cases} \mathbf{U}' = s\alpha_{34} (\mathbf{a}_2 \times \mathbf{a}_4) \\ V' = -c\alpha_{34} (\mathbf{a}_2 \cdot \mathbf{a}_4) + c\alpha_{23} \end{cases} \tag{7.28a}$$

Substituting (7.23b-3) into (7.28) yields

$$A' \cos\psi_4 + B' \sin\psi_4 = C' \tag{7.29}$$

$$\begin{cases} A' = (\mathbf{U}' \cdot \mathbf{q}_4) \\ B' = (\mathbf{U}' \cdot \mathbf{q}_4 \times \mathbf{a}_4) \\ C' = V' \end{cases} \tag{7.29a}$$

(iii). *Derivation of the input-output displacement equation relating Θ_5.*

From (7.26) and (7.29) we can get

$$Q_1^2 + Q_2^2 = Q_3^2 \tag{7.30}$$

$$\begin{cases} Q_1 = (AC' - A'C) \\ Q_2 = (BC' - B'C) \\ Q_3 = (AB' - A'B) \end{cases} \tag{7.30a}$$

$$\begin{cases} Q_1 = -p_3 c\alpha_{34}(\mathbf{a}_2\cdot\mathbf{q}_4)(\mathbf{a}_2\cdot\mathbf{a}_4) - s\alpha_{34}(\mathbf{a}_2\cdot\mathbf{K})(\mathbf{a}_2\cdot\mathbf{q}_4\times\mathbf{a}_4) \\ \qquad + p_3 c\alpha_{23}(\mathbf{a}_2\cdot\mathbf{q}_4) + S_3 c\alpha_{23} s\alpha_{34}(\mathbf{a}_2\cdot\mathbf{q}_4\times\mathbf{a}_4) \\ Q_2 = -p_3 c\alpha_{34}(\mathbf{a}_2\cdot\mathbf{q}_4\times\mathbf{a}_4)(\mathbf{a}_2\cdot\mathbf{a}_4) + s\alpha_{34}(\mathbf{a}_2\cdot\mathbf{K})(\mathbf{a}_2\cdot\mathbf{q}_4) \\ \qquad + p_3 c\alpha_{23}(\mathbf{a}_2\cdot\mathbf{q}_4\times\mathbf{a}_4) - S_3 c\alpha_{23} s\alpha_{34}(\mathbf{a}_2\cdot\mathbf{q}_4) \\ Q_3 = p_3 s\alpha_{34}\{(\mathbf{a}_2\cdot\mathbf{a}_4)^2 + (\mathbf{a}_2\cdot\mathbf{q}_4\times\mathbf{a}_4)^2\} = p_3 s\alpha_{34}\{1 - (\mathbf{a}_2\cdot\mathbf{a}_4)^2\} \end{cases} \tag{7.30b}$$

Substituting (7.30b) into (7.30) we get

$$p_3^2\{(\mathbf{a}_2\cdot\mathbf{a}_4) - c\alpha_{23}c\alpha_{34}\}^2 + s^2\alpha_{34}\{(\mathbf{a}_2\cdot\mathbf{K}) - S_3 c\alpha_{23}\}^2 - p_3 s^2\alpha_{23}s^2\alpha_{34} = 0 \tag{7.31}$$

$$\begin{cases} \{(\mathbf{a}_2\cdot\mathbf{a}_4) - c\alpha_{23}c\alpha_{34}\} = \mathbf{W}_1\cdot\mathbf{q}_4 + Z_1 \\ \qquad = (\mathbf{W}_1\cdot\mathbf{q}_5)\cos\Theta_5 + (\mathbf{W}_1\cdot\mathbf{q}_5\times\mathbf{a}_5)\sin\Theta_5 + Z_1 \\ \{(\mathbf{a}_2\cdot\mathbf{K}) - S_3 c\alpha_{23}\} = \mathbf{W}_2\cdot\mathbf{q}_4 + Z_2 \\ \qquad = (\mathbf{W}_2\cdot\mathbf{q}_5)\cos\Theta_5 + (\mathbf{W}_2\cdot\mathbf{q}_5\times\mathbf{a}_5)\sin\Theta_5 + Z_2 \end{cases} \tag{7.31a}$$

$$\begin{cases} \mathbf{W}_1 = s\alpha_{45}(\mathbf{a}_2\times\mathbf{a}_5) \\ \mathbf{W}_2 = p_4\mathbf{a}_2 + S_4 s\alpha_{45}(\mathbf{a}_5\times\mathbf{a}_2) \\ Z_1 = c\alpha_{45}(\mathbf{a}_2\cdot\mathbf{a}_5) - c\alpha_{23}c\alpha_{34} \\ Z_2 = (\mathbf{I}\cdot\mathbf{a}_2) - S_4 c\alpha_{45}(\mathbf{a}_5\cdot\mathbf{a}_2) - S_3 c\alpha_{23} \end{cases} \tag{7.31b}$$

Substituting (7.31a) into (7.31) yields

$$\mu_1\cos^2\Theta_5 + \mu_2\sin^2\Theta_5 + \mu_3\sin\Theta_5\cos\Theta_5 + \mu_4\cos\Theta_5 + \mu_5\sin\Theta_5 + \mu_6 = 0 \tag{7.32}$$

$$\begin{cases} \mu_1 = p_3^2(\mathbf{W}_1\cdot\mathbf{q}_5)^2 + s^2\alpha_{34}(\mathbf{W}_2\cdot\mathbf{q}_5)^2 \\ \mu_2 = p_3^2(\mathbf{W}_1\cdot\mathbf{q}_5\times\mathbf{a}_5)^2 + s^2\alpha_{34}(\mathbf{W}_2\cdot\mathbf{q}_5\times\mathbf{a}_5)^2 \\ \mu_3 = 2p_3^2(\mathbf{W}_1\cdot\mathbf{q}_5)(\mathbf{W}_1\cdot\mathbf{q}_5\times\mathbf{a}_5) + 2s^2\alpha_{34}(\mathbf{W}_2\cdot\mathbf{q}_5)(\mathbf{W}_2\cdot\mathbf{q}_5\times\mathbf{a}_5) \\ \mu_4 = 2p_3^2(\mathbf{W}_1\cdot\mathbf{q}_5)Z_1 + 2s^2\alpha_{34}(\mathbf{W}_2\cdot\mathbf{q}_5)Z_2 \\ \mu_5 = 2p_3^2(\mathbf{W}_1\cdot\mathbf{q}_5\times\mathbf{a}_5)Z_1 + 2s^2\alpha_{34}(\mathbf{W}_2\cdot\mathbf{q}_5\times\mathbf{a}_5)Z_2 \\ \mu_6 = p_3^2 Z_1^2 + s^2\alpha_{34}Z_2^2 - p_3^2 s^2\alpha_{23}s^2\alpha_{34} \end{cases} \tag{7.32a}$$

Let $y = \tan(\Theta_5/2)$, then from (7.32) we have

$$\sum_{i=0}^{4} v_i y^{4-i} = 0 \tag{7.33}$$

$$\begin{cases} v_0 = \mu_1 \\ v_1 = -2\mu_3 \\ v_2 = -2\mu_1 + 4\mu_2 - \mu_4 + \mu_6 \\ v_3 = 2\mu_3 + 2\mu_5 \\ v_4 = \mu_1 + \mu_4 + \mu_6 \end{cases} \tag{7.33a}$$

Solving (7.33) we get y, then $\Theta_5 = 2\tan^{-1}y$; Other displacement variables can also be easily determined and the procedure is similar to that of section 7.2.

7.4. Displacement analysis of R_0-$RRER$ mechanism

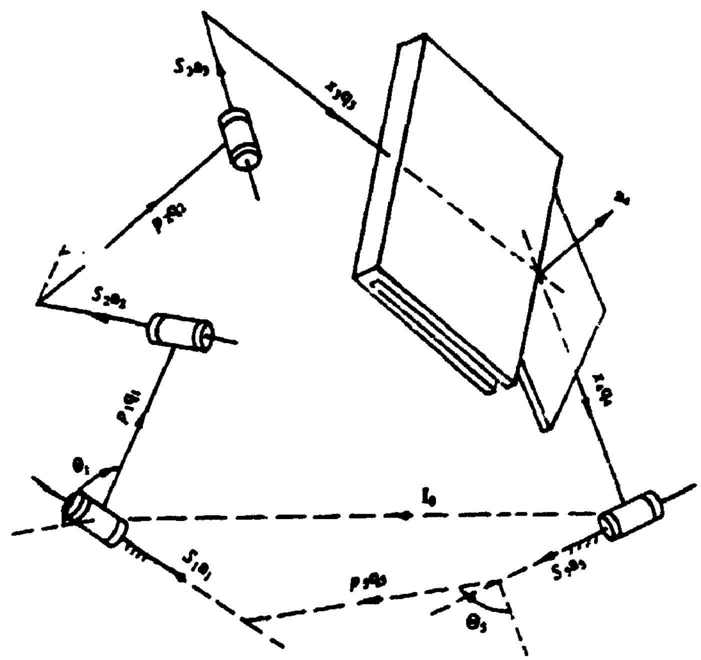


Fig. 7.5

Mechanism R_0-$RRER$ is shown in Fig. 7.5. When the input angle θ_1 is given, unknowns to be determined are $\{x_3, x_4, \theta_2, \theta_3, \theta_4, \theta_5\}$. Let θ_5 be the output angle, θ_2 be the auxiliary angle, i.e. $\theta_5 = \Theta_5$, $\theta_2 = \psi_2$. The vector loop equation can be written as:

$$x_3\mathbf{q}_3 + x_4\mathbf{q}_4 + \mathbf{L} + \mathbf{I} = 0 \tag{7.34}$$

$$\begin{cases} \mathbf{L} = p_2\mathbf{q}_2 + S_3\mathbf{a}_3 & (1) \\ \mathbf{I} = (p_1\mathbf{q}_1 + S_2\mathbf{a}_2) - \mathbf{I}_0 & (2) \\ \mathbf{I}_0 = S_5\mathbf{a}_5 + p_5\mathbf{q}_5 + S_1\mathbf{a}_1 & (3) \end{cases} \tag{7.34a}$$

$$\begin{cases} \mathbf{q}_3 = \mathbf{a}_3 \times \mathbf{a}_4 \csc\alpha_{34} & (1) \\ \mathbf{a}_3 = c\alpha_{23}\,\mathbf{a}_2 + s\alpha_{23}\,\mathbf{q}_2 \times \mathbf{a}_2 & (2) \\ \mathbf{q}_2 = \cos\psi_2\,\mathbf{q}_1 + \sin\psi_2\,\mathbf{a}_2 \times \mathbf{q}_1 & (3) \\ \mathbf{a}_2 = c\alpha_{12}\,\mathbf{a}_1 + s\alpha_{12}\,\mathbf{q}_1 \times \mathbf{a}_1 & (4) \\ \mathbf{q}_1 = c\theta_1\,\mathbf{q}_5 + s\theta_1\,\mathbf{a}_1 \times \mathbf{q}_5 & (5) \end{cases} \tag{7.34b}$$

$$\begin{cases} \mathbf{a}_4 = c\alpha_{45}\,\mathbf{a}_5 - s\alpha_{45}\,\mathbf{q}_4 \times \mathbf{a}_5 & (6) \\ \mathbf{q}_4 = \cos\Theta_5\,\mathbf{q}_5 - \sin\Theta_5\,\mathbf{a}_5 \times \mathbf{q}_5 & (7) \\ \mathbf{q}_5 = \mathbf{a}_5 \times \mathbf{a}_1 \csc\alpha_{51} & (8) \end{cases} \tag{7.34b}$$

(i). *Derivation of first equation relating* θ_1, Θ_5 *and* ψ_2.

The scalar product of $\mathbf{a}_4$ with both sides of (7.34) yields

$$\mathbf{a}_4 \cdot \mathbf{L} = -\,\mathbf{a}_4 \cdot \mathbf{I} \tag{7.35}$$

Substituting (7.34a-1) into (7.35) yields

$$\mathbf{U} \cdot \mathbf{q}_2 = V \tag{7.36}$$

$$\begin{cases} \mathbf{U} = p_2\,\mathbf{a}_4 + S_3\,s\alpha_{23}\,(\mathbf{a}_2 \times \mathbf{a}_4) \\ V = -\,(\mathbf{I} \cdot \mathbf{a}_4) - S_3\,c\alpha_{23}\,(\mathbf{a}_2 \cdot \mathbf{a}_4) \end{cases} \tag{7.36a}$$

Substituting (7.34b-3) into (7.36) yields

$$A\cos\psi_2 + B\sin\psi_2 = C \tag{7.37}$$

$$\begin{cases} A = (\mathbf{U} \cdot \mathbf{q}_1) \\ B = (\mathbf{U} \cdot \mathbf{a}_2 \times \mathbf{q}_1) \\ C = V \end{cases} \tag{7.37a}$$

(ii). *Derivation of a second equation relating* θ_1, Θ_5 *and* ψ_2.

From the structure of the mechanism, we can write:

$$\mathbf{a}_3(\psi_2, \theta_1) \cdot \mathbf{a}_4(\Theta_5) = c\alpha_{34} \tag{7.38}$$

Substituting (7.34b-2) into (7.38) yields

$$\mathbf{U}' \cdot \mathbf{q}_2 = V' \tag{7.39}$$

$$\begin{cases} \mathbf{U}' = s\alpha_{23}\,(\mathbf{a}_2 \times \mathbf{a}_4) \\ V' = -\,c\alpha_{23}\,(\mathbf{a}_2 \cdot \mathbf{a}_4) + c\alpha_{34} \end{cases} \tag{7.39a}$$

Substituting (7.34b-3) into (7.39) yields

$$A'\cos\psi_2 + B'\sin\psi_2 = C' \tag{7.40}$$

$$\begin{cases} A' = (\mathbf{U}'\cdot\mathbf{q}_1) \\ B' = (\mathbf{U}'\cdot\mathbf{a}_2\times\mathbf{q}_1) \\ C' = V' \end{cases} \tag{7.40a}$$

(iii). ***Derivation of the input-output displacement equation relating*** Θ_5.

From (7.37) and (7.40) we get

$$Q_1^2 + Q_2^2 = Q_3^2 \tag{7.41}$$

$$\begin{cases} Q_1 = (AC' - A'C) \\ Q_2 = (BC' - B'C) \\ Q_3 = (AB' - A'B) \end{cases} \tag{7.41a}$$

$$\begin{cases} Q_1 = -p_2 c\alpha_{23}(\mathbf{q}_1\cdot\mathbf{a}_4)(\mathbf{a}_2\cdot\mathbf{a}_4) + s\alpha_{23}(\mathbf{q}_1\times\mathbf{a}_2\cdot\mathbf{a}_4)(\mathbf{I}\cdot\mathbf{a}_4) \\ \qquad + p_2 c\alpha_{34}(\mathbf{q}_1\cdot\mathbf{a}_4) + S_3 s\alpha_{23} c\alpha_{34}(\mathbf{q}_1\times\mathbf{a}_2\cdot\mathbf{a}_4) \\ Q_2 = p_2 c\alpha_{23}(\mathbf{q}_1\times\mathbf{a}_2\cdot\mathbf{a}_4)(\mathbf{a}_2\cdot\mathbf{a}_4) + s\alpha_{23}(\mathbf{q}_1\cdot\mathbf{a}_4)(\mathbf{I}\cdot\mathbf{a}_4) \\ \qquad - p_2 c\alpha_{34}(\mathbf{q}_1\times\mathbf{a}_2\cdot\mathbf{a}_4) + S_3 s\alpha_{23} c\alpha_{34}(\mathbf{q}_1\cdot\mathbf{a}_4) \\ Q_3 = p_2 s\alpha_{23}\{(\mathbf{q}_1\cdot\mathbf{a}_4)^2 + (\mathbf{q}_1\times\mathbf{a}_2\cdot\mathbf{a}_4)^2\} = p_2 s\alpha_{23}\{1 - (\mathbf{a}_2\cdot\mathbf{a}_4)^2\} \end{cases} \tag{7.41b}$$

Substituting (7.41b) into (7.41) yields

$$p_2^2\{(\mathbf{a}_2\cdot\mathbf{a}_4) - c\alpha_{23}c\alpha_{34}\}^2 + s^2\alpha_{23}\{(\mathbf{I}\cdot\mathbf{a}_4) + S_3 c\alpha_{34}\}^2 - p_2{}^2 s^2\alpha_{23} s^2\alpha_{34} = 0 \tag{7.42}$$

$$\begin{cases} p_2\{(\mathbf{a}_2\cdot\mathbf{a}_4) - c\alpha_{23}c\alpha_{34}\} = \mathbf{W}_1\cdot\mathbf{q}_4 + Z_1 \\ \qquad = (\mathbf{W}_1\cdot\mathbf{q}_5)\cos\Theta_5 + (\mathbf{W}_1\cdot\mathbf{q}_5\times\mathbf{a}_5)\sin\Theta_5 + Z_1 \\ s\alpha_{23}\{(\mathbf{a}_2\cdot\mathbf{K}) + S_3 c\alpha_{34}\} = \mathbf{W}_2\cdot\mathbf{q}_4 + Z_2 \\ \qquad = (\mathbf{W}_2\cdot\mathbf{q}_5)\cos\Theta_5 + (\mathbf{W}_2\cdot\mathbf{q}_5\times\mathbf{a}_5)\sin\Theta_5 + Z_2 \end{cases} \tag{7.42a}$$

$$\begin{cases} \mathbf{W}_1 = p_2 s\alpha_{45}(\mathbf{a}_2\times\mathbf{a}_5) \\ \mathbf{W}_2 = s\alpha_{23} s\alpha_{45}(\mathbf{I}\times\mathbf{a}_5) \\ Z_1 = p_2[c\alpha_{45}(\mathbf{a}_5\cdot\mathbf{a}_2) - c\alpha_{23}c\alpha_{34}] \\ Z_2 = s\alpha_{23}[c\alpha_{45}(\mathbf{a}_5\cdot\mathbf{I}) + S_3 c\alpha_{34}] \end{cases} \tag{7.42b}$$

Substituting (7.42a) into (7.42) yields

$$\mu_1\cos^2\Theta_5 + \mu_2\sin^2\Theta_5 + \mu_3\sin\Theta_5\cos\Theta_5 + \mu_4\cos\Theta_5 + \mu_5\sin\Theta_5 + \mu_6 = 0 \tag{7.43}$$

$$\begin{cases} \mu_1 = (\mathbf{W}_1\cdot\mathbf{q}_5)^2 + (\mathbf{W}_2\cdot\mathbf{q}_5)^2 \\ \mu_2 = (\mathbf{W}_1\cdot\mathbf{q}_5\times\mathbf{a}_5)^2 + (\mathbf{W}_2\cdot\mathbf{q}_5\times\mathbf{a}_5)^2 \\ \mu_3 = 2(\mathbf{W}_1\cdot\mathbf{q}_5)(\mathbf{W}_1\cdot\mathbf{q}_5\times\mathbf{a}_5) + 2(\mathbf{W}_2\cdot\mathbf{q}_5)(\mathbf{W}_2\cdot\mathbf{q}_5\times\mathbf{a}_5) \\ \mu_4 = 2(\mathbf{W}_1\cdot\mathbf{q}_5)Z_1 + 2(\mathbf{W}_2\cdot\mathbf{q}_5)Z_2 \\ \mu_5 = 2(\mathbf{W}_1\cdot\mathbf{q}_5\times\mathbf{a}_5)Z_1 + 2(\mathbf{W}_2\cdot\mathbf{q}_5\times\mathbf{a}_5)Z_2 \\ \mu_6 = Z_1^2 + Z_2^2 - p_2^2 s^2\alpha_{23} s^2\alpha_{34} \end{cases} \tag{7.43a}$$

Let $y = \tan(\Theta_5/2)$, then from (7.43) we obtain $\sum_{i=0}^{4} v_i y^{4-i} = 0$, where $\{v_i\}$ $(i = 0 - 4)$

is the same as that of (7.12a) or (7.33a).

7.5. Displacement analysis of R_0-$RRRE$ mechanism

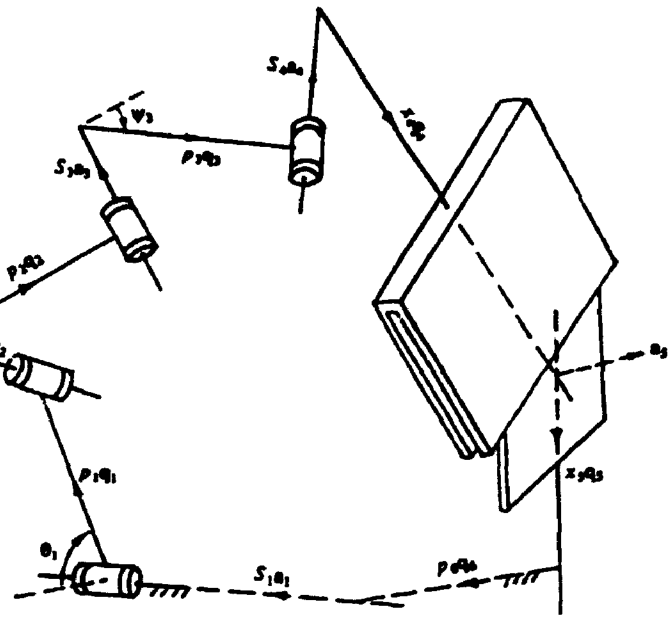


Fig. 7.6

Mechanism R_0-$RRRE$ is shown in Fig. 7.6. This mechanism has never been analyzed before. When the input angle θ_1 is given, unknowns to be determined are $\{x_4, x_5, \theta_2, \theta_3, \theta_4, \theta_5\}$. Let θ_2 be the output angle, θ_3 be the auxiliary angle, i.e. $\theta_2 = \Theta_2$, $\theta_3 = \psi_3$. The vector loop equation can be written as:

$$x_4 \mathbf{q}_4 + x_5 \mathbf{q}_5 + \mathbf{L} + \mathbf{K} = 0 \tag{7.44}$$

$$\begin{cases} \mathbf{L} = p_3 \mathbf{q}_3 + S_4 \mathbf{a}_4 & (1) \\ \mathbf{K} = \mathbf{J} + \mathbf{I} & (2) \\ \mathbf{J} = p_2 \mathbf{q}_2 + S_3 \mathbf{a}_3 & (3) \\ \mathbf{I} = (p_1 \mathbf{q}_1 + S_2 \mathbf{a}_2) + \mathbf{I}_0 & (4) \\ \mathbf{I}_0 = p_6 \mathbf{q}_6 + S_1 \mathbf{a}_1 & (5) \end{cases} \tag{7.44a}$$

$$\begin{cases} \mathbf{a}_4 = c\alpha_{34} \mathbf{a}_3 + s\alpha_{34} \mathbf{q}_3 \times \mathbf{a}_3 & (1) \\ \mathbf{q}_3 = \cos\psi_3 \mathbf{q}_2 + \sin\psi_3 \mathbf{a}_3 \times \mathbf{q}_2 & (2) \\ \mathbf{a}_3 = c\alpha_{23} \mathbf{a}_2 + s\alpha_{23} \mathbf{q}_2 \times \mathbf{a}_2 & (3) \\ \mathbf{q}_2 = \cos\Theta_2 \mathbf{q}_1 + \sin\Theta_2 \mathbf{a}_2 \times \mathbf{q}_1 & (4) \\ \mathbf{a}_2 = c\alpha_{12} \mathbf{a}_1 + s\alpha_{12} \mathbf{q}_1 \times \mathbf{q}_6 & (5) \\ \mathbf{q}_1 = c\theta_1 \mathbf{q}_6 + s\theta_1 \mathbf{a}_1 \times \mathbf{q}_6 & (6) \end{cases} \tag{7.44b}$$

$$\begin{cases} \mathbf{q}_5 = \mathbf{a}_5 \times \mathbf{q}_6 \csc(\mathbf{a}_5 \hat{,} \mathbf{q}_6) & (7) \\ \mathbf{q}_6 = \mathbf{q}_5 \times \mathbf{a}_1 \csc(\mathbf{q}_5 \hat{,} \mathbf{a}_1) & (8) \end{cases} \tag{7.44b}$$

(i). *Derivation of first equation relating* θ_1, Θ_2 *and* ψ_3.

The scalar product of $\mathbf{a}_5$ with both sides of (7.44) yields

$$\mathbf{a}_5 \cdot \mathbf{L} = -\mathbf{a}_5 \cdot \mathbf{K} \tag{7.45}$$

Substituting (7.44a-1) into (7.45) yields

$$\mathbf{U} \cdot \mathbf{q}_3 = V \tag{7.46}$$

$$\begin{cases} \mathbf{U} = p_3 \mathbf{a}_5 + S_4 s\alpha_{34} (\mathbf{a}_3 \times \mathbf{a}_5) \\ V = -(\mathbf{a}_5 \cdot \mathbf{K}) - S_4 c\alpha_{34} (\mathbf{a}_5 \cdot \mathbf{a}_3) \end{cases} \tag{7.46a}$$

Substituting (7.44b-2) into (7.46) yields

$$A \cos\psi_3 + B \sin\psi_3 = C \tag{7.47}$$

$$\begin{cases} A = (\mathbf{U} \cdot \mathbf{q}_2) \\ B = (\mathbf{U} \cdot \mathbf{a}_3 \times \mathbf{q}_2) \\ C = V \end{cases} \tag{7.47a}$$

(ii). *Derivation of a second equation relating* θ_1, Θ_2 *and* ψ_3.

From the structure of the mechanism, we can write:

$$\mathbf{a}_4(\psi_3, \Theta_2, \theta_1) \cdot \mathbf{a}_5 = c\alpha_{45} \tag{7.48}$$

Substituting (7.44b-1) into (7.48) yields

$$\mathbf{U}' \cdot \mathbf{q}_3 = V' \tag{7.49}$$

$$\begin{cases} \mathbf{U}' = s\alpha_{34} (\mathbf{a}_3 \times \mathbf{a}_5) \\ V' = -c\alpha_{34} (\mathbf{a}_5 \cdot \mathbf{a}_3) + c\alpha_{45} \end{cases} \tag{7.49a}$$

Substituting (7.44b-2) into (7.49) yields

$$A' \cos\psi_2 + B' \sin\psi_2 = C' \tag{7.50}$$

$$\begin{cases} A' = (\mathbf{U}' \cdot \mathbf{q}_2) \\ B' = (\mathbf{U}' \cdot \mathbf{a}_3 \times \mathbf{q}_2) \\ C' = V' \end{cases} \tag{7.50a}$$

(iii). *Derivation of the input-output displacement equation relating* Θ_2.

From (7.47) and (7.50) we get

$$Q_1^2 + Q_2^2 = Q_3^2 \tag{7.51}$$

$$\begin{cases} Q_1 = (AC' - A'C) \\ Q_2 = (BC' - B'C) \\ Q_3 = (AB' - A'B) \end{cases} \tag{7.51a}$$

$$\begin{cases} Q_1 = -p_3 c\alpha_{34}(\mathbf{a}_5 \cdot \mathbf{q}_2)(\mathbf{a}_5 \cdot \mathbf{a}_3) + s\alpha_{34}(\mathbf{a}_5 \cdot \mathbf{q}_2 \times \mathbf{q}_2 \times \mathbf{a}_3)(\mathbf{a}_5 \cdot \mathbf{K}) \\ \quad + p_3 c\alpha_{45}(\mathbf{a}_5 \cdot \mathbf{q}_2) + S_4 s\alpha_{34} c\alpha_{45}(\mathbf{a}_5 \cdot \mathbf{q}_2 \times \mathbf{a}_3) \\ Q_2 = p_3 c\alpha_{34}(\mathbf{a}_5 \cdot \mathbf{q}_2 \times \mathbf{a}_3)(\mathbf{a}_5 \cdot \mathbf{a}_3) + s\alpha_{34}(\mathbf{a}_5 \cdot \mathbf{q}_2)(\mathbf{a}_5 \cdot \mathbf{K}) \\ \quad - p_3 c\alpha_{45}(\mathbf{a}_5 \cdot \mathbf{q}_2 \times \mathbf{a}_3) + S_4 s\alpha_{34} c\alpha_{45}(\mathbf{a}_5 \cdot \mathbf{q}_2) \\ Q_3 = p_3 s\alpha_{34}\{(\mathbf{a}_5 \cdot \mathbf{q}_2)^2 + (\mathbf{a}_5 \cdot \mathbf{q}_2 \times \mathbf{a}_3)^2\} = p_3 s\alpha_{34}\{1 - (\mathbf{a}_5 \cdot \mathbf{a}_3)^2\} \end{cases} \tag{7.51b}$$

Substituting (7.51b) into (7.51) yields

$$p_3^2\{(\mathbf{a}_5 \cdot \mathbf{a}_3) - c\alpha_{34} c\alpha_{45}\}^2 + s^2\alpha_{34}\{(\mathbf{a}_5 \cdot \mathbf{K}) + S_4 c\alpha_{45}\}^2 - p_3^2 s^2\alpha_{34} s^2\alpha_{45} = 0 \tag{7.52}$$

$$\begin{cases} p_3\{(\mathbf{a}_5 \cdot \mathbf{a}_3) - c\alpha_{34} c\alpha_{45}\} = \mathbf{W}_1 \cdot \mathbf{q}_2 + Z_1 \\ \qquad = (\mathbf{W}_1 \cdot \mathbf{q}_1)\cos\Theta_2 + (\mathbf{W}_1 \cdot \mathbf{a}_2 \times \mathbf{q}_1)\sin\Theta_2 + Z_1 \\ s\alpha_{34}\{(\mathbf{a}_5 \cdot \mathbf{K}) + S_4 c\alpha_{45}\} = \mathbf{W}_2 \cdot \mathbf{q}_2 + Z_2 \\ \qquad = (\mathbf{W}_2 \cdot \mathbf{q}_1)\cos\Theta_2 + (\mathbf{W}_2 \cdot \mathbf{a}_2 \times \mathbf{q}_1)\sin\Theta_2 + Z_2 \end{cases} \tag{7.52a}$$

$$\begin{cases} \mathbf{W}_1 = p_3 s\alpha_{23}(\mathbf{a}_2 \times \mathbf{a}_5) \\ \mathbf{W}_2 = s\alpha_{34}[p_2\mathbf{a}_5 + S_3 s\alpha_{23}(\mathbf{a}_2 \times \mathbf{a}_5)] \\ Z_1 = p_3[c\alpha_{23}(\mathbf{a}_5 \cdot \mathbf{a}_2) - c\alpha_{34} c\alpha_{45}] \\ Z_2 = s\alpha_{34}[S_3 c\alpha_{23}(\mathbf{a}_5 \cdot \mathbf{a}_2) + (\mathbf{a}_5 \cdot \mathbf{I}) + S_4 c\alpha_{45}] \end{cases} \tag{7.52b}$$

Substituting (7.52a) into (7.52) yields

$$\mu_1\cos^2\Theta_5 + \mu_2\sin^2\Theta_5 + \mu_3\sin\Theta_5\cos\Theta_5 + \mu_4\cos\Theta_5 + \mu_5\sin\Theta_5 + \mu_6 = 0 \tag{7.53}$$

$$\begin{cases} \mu_1 = (\mathbf{W}_1 \cdot \mathbf{q}_1)^2 + (\mathbf{W}_2 \cdot \mathbf{q}_1)^2 \\ \mu_2 = (\mathbf{W}_1 \cdot \mathbf{a}_2 \times \mathbf{q}_1)^2 + (\mathbf{W}_2 \cdot \mathbf{a}_2 \times \mathbf{q}_1)^2 \\ \mu_3 = 2(\mathbf{W}_1 \cdot \mathbf{q}_1)(\mathbf{W}_1 \cdot \mathbf{a}_2 \times \mathbf{q}_1) + 2(\mathbf{W}_2 \cdot \mathbf{q}_1)(\mathbf{W}_2 \cdot \mathbf{a}_2 \times \mathbf{q}_1) \\ \mu_4 = 2(\mathbf{W}_1 \cdot \mathbf{q}_1)Z_1 + 2(\mathbf{W}_2 \cdot \mathbf{q}_1)Z_2 \\ \mu_5 = 2(\mathbf{W}_1 \cdot \mathbf{a}_2 \times \mathbf{q}_1)Z_1 + 2(\mathbf{W}_2 \cdot \mathbf{a}_2 \times \mathbf{q}_1)Z_2 \\ \mu_6 = Z_1^2 + Z_2^2 - p_3^2 s^2\alpha_{34} s^2\alpha_{45} \end{cases} \tag{7.53a}$$

Let $y = \tan(\Theta_2/2)$, then from (7.53) we obtain $\sum_{i=0}^{4} v_i y^{4-i} = 0$, where $\{v_i\}$ $(i = 0 \sim 4)$ is the same as that of (7.12a).

7.6. Conclusion

From the above analysis, it is clear that the generalized tracta coupling and its variant mechanisms can all be analyzed directly; there is no need and also no advantage to complicate the E pair by simulating it in various models for the purpose of analysis.

Using the simulated models of the E pair, even for the unsymmetric case R_0-*ERRR*, the spherical trigonometry method required more than three pages of analysis ([22] Duffy and Keen, pp.215-218, 1972) to derive the second equation relating the input, output and auxiliary angles. Using the new approach of this chapter, the same derivation requires only a few lines as shown from equation (7.27) to (7.29). The analysis for the symmetric case R_0-*RERR* in the new approach also shares the same level of simplicity, as shown from equation (7.5) to (7.7).

The new vector algebraic approach presented in this chapter is free of the limitations encountered by previous approaches. For any of the generalized tracta coupling and its variant mechanisms, the analysis procedure is systematic and direct, resulting in simple and compact expressions.

CHAPTER 8. THE R_0–3R–S MECHANISM

8.1. Introduction

The Generalized Clemens Coupling R_0-$RSRR$ and its variant mechanisms R_0-$SRRR$, R_0-$RRSR$ and R_0-$RRRS$ are very useful 5-link spatial mechanisms. They have already had much practical application in engineering, especially in agricultural and textile machines. The kinematic features of these mechanisms have been analyzed by some researchers using various methods which include geometric method, Torfason and Sharma ([73] 1973) (generated surfaces method), Wallace and Freudenstein ([77] 1975) (geometric-configuration method); matrix method, Xie, Zheng and Ou ([79] 1980) (tensor rotaional transformation method), Zhang ([106] 1980), Youm and Huang ([86] 1987) (direction cosine matrix method); spherical trigonometry method, Duffy ([28] pp.384-410, 1980); decompositions method, Alizade, Duffy and Azizov ([2] 1987); and so on.

In this chapter the generalized Clemens Coupling R_0-$RSRR$ and its variant mechanisms R_0-$SRRR$, R_0-$RRSR$ and R_0-$RRRS$ are kinematically analyzed by using the *vector algebraic method*. The closed-form input-output displacement equations of those mechanisms are obtained as fourth order polynomials. The distinctiveness of the new approach presented in this chapter lies in two aspects: the analysis procedure for any one of the four mechanisms is identical; and the two unified steps and the pure vector operation make the kinematic analysis procedure very compact and simple.

Following is a brief summary of the analysis steps of the new approach:

(a) Write down the vector loop equation and the direction equations, based on the structure of the mechanism;

(b) Derive the first equation relating the input, output and auxiliary angle by squaring both sides of the vector loop equation;

(c) Derive the second equation relating the input, output and auxiliary angle using the *end axis vector* to scalar product both sides of the vector loop equation;

(d) Eliminate the auxiliary angle from the two equations derived in step (b) and (c) to obtain the desired input-output displacement equation.

8.2. Analysis of the generalized Clemens coupling R_0-$RSRR$

(1). Displacement analysis.

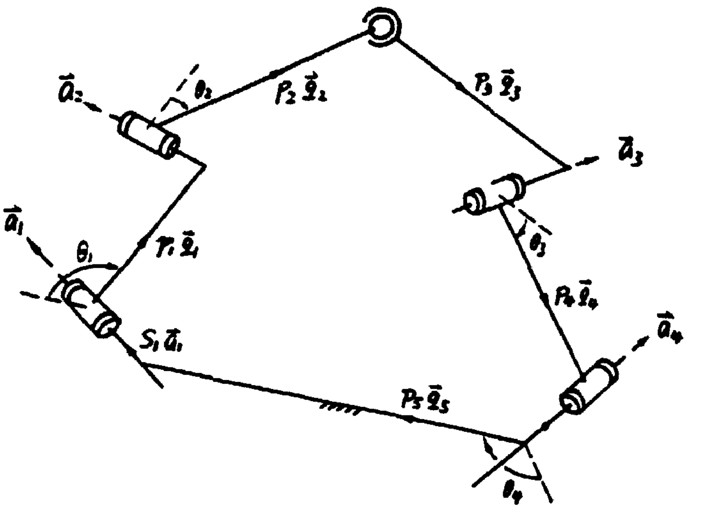


Fig. 8.1

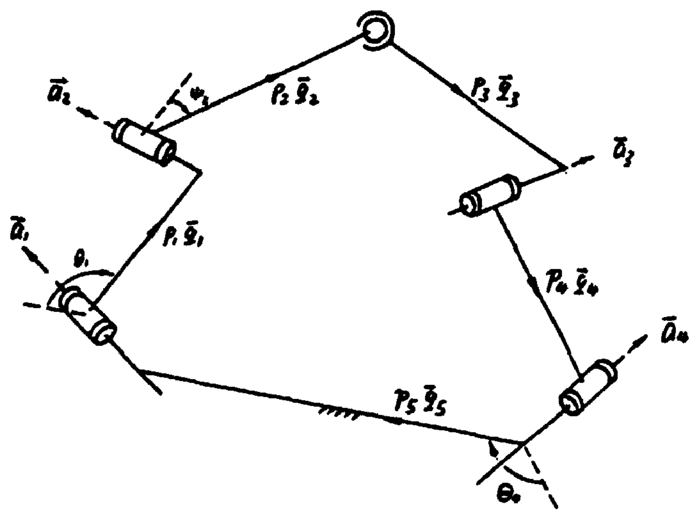


Fig. 8.1(a)

Mechanism R_0-*RSRR* is shown in Fig. 8.1. When the input angle θ_1 is given, unknowns to be determined are $\{\theta_2, \theta_3, \theta_4\}$. Let θ_4 be the output angle, θ_2 be the auxiliary angle, i.e. $\theta_4 = \Theta_4$, $\theta_2 = \psi_2$, as shown in Fig. 8.1(a). The vector loop equation of the mechanism can now be written as:

$$\mathbf{K} + \mathbf{L} = \mathbf{F} \tag{8.1}$$

$$\begin{cases} \mathbf{L} = p_2 \mathbf{q}_2 & (1) \\ \mathbf{K} = \mathbf{J} + \mathbf{I} & (2) \\ \mathbf{J} = p_4 \mathbf{q}_4 - S_3 \mathbf{a}_3 & (3) \\ \mathbf{I} = (S_2 \mathbf{a}_2 + p_1 \mathbf{q}_1) + \mathbf{I}_0 & (4) \\ \mathbf{I}_0 = S_1 \mathbf{a}_1 + p_5 \mathbf{q}_5 - S_4 \mathbf{a}_4 & (5) \\ \mathbf{F} = -p_3 \mathbf{q}_3 & (6) \end{cases} \tag{8.1a}$$

$\mathbf{I}$ is defined as the *input vector* of the mechanism. It is the sum of those vectors in the loop of the mechanism which are given or known at the beginning. $\mathbf{J}$ is the *output vector* of the mechanism. It is the sum of those constant-magnitude vectors in the loop of the mechanism that can be expressed as a function of the output angle (Θ_4 in this case). L is the *auxiliary vector*. It is the sum of those constant-magnitude vectors in the loop of the mechanism that can be expressed as a function of the auxiliary angle (ψ_2 in this case). F is the *floating vector*. Cutting off the loop at the two ends of the floating vector, we get two separate chains, where one part is fixed on the ground and the other part is *floating*. This is how floating vector was named. The strict definition of these vectors are introduced in chapter 2.

The following set of vector equations is called *direction equations*, which specifies the relative direction of any individual (unit) vector with its two adjacent (unit) vectors on the vector loop.

$$\begin{cases} \mathbf{q}_2 = c\psi_2\mathbf{q}_1 + s\psi_2\mathbf{a}_2\times\mathbf{q}_1 & (1) \\ \mathbf{a}_2 = c\alpha_{12}\mathbf{a}_1 + s\alpha_{12}\mathbf{q}_1\times\mathbf{a}_1 & (2) \\ \mathbf{q}_1 = c\theta_1\mathbf{q}_5 + s\theta_1\mathbf{a}_1\times\mathbf{q}_5 & (3) \\ \mathbf{q}_3 = c\theta_3\mathbf{q}_4 - s\theta_3\mathbf{a}_3\times\mathbf{q}_4 & (4) \\ \mathbf{a}_3 = c\alpha_{34}\mathbf{a}_4 - s\alpha_{34}\mathbf{q}_4\times\mathbf{a}_4 & (5) \\ \mathbf{q}_4 = c\Theta_4\mathbf{q}_5 - s\Theta_4\mathbf{a}_4\times\mathbf{q}_5 & (6) \end{cases} \tag{8.1b}$$

(i). *Derivation of the first equation relating* θ_1, Θ_4 *and* ψ_2.

Squaring both sides of (8.1) yields

$$2(\mathbf{K}\cdot\mathbf{L}) = -\mathbf{K}^2 + (\mathbf{F}^2 - \mathbf{L}^2) \tag{8.2}$$

Substituting (8.1a-1) into (8.2) yields

$$\mathbf{U}\cdot\mathbf{q}_2 = V \tag{8.3}$$

$$\begin{cases} \mathbf{U} = 2p_2\mathbf{K} \\ V = -\mathbf{K}^2 + (p_1^2 - p_2^2) \end{cases} \tag{8.3a}$$

Substituting (8.1b-1) into (8.3) yields

$$A\cos\psi_2 + B\sin\psi_2 = C \tag{8.4}$$

$$\begin{cases} A = \mathbf{U}\cdot\mathbf{q}_1 \\ B = \mathbf{U}\cdot\mathbf{a}_2\times\mathbf{q}_1 \\ C = V \end{cases} \tag{8.4a}$$

Since ψ_2 is the auxiliary angle, the variable to be eliminated from simultaneous equations, thus we transform (8.2) in to the standard form (8.3), where only $\mathbf{q}_2$ is a function of ψ_2. Equation (8.3) is a transition from (8.2) to (8.4).

(ii). *Derivation of the second equation relating* θ_1, Θ_4 *and* ψ_2.

We can see from Fig. 8.1 or 8.1(a) that $\mathbf{a}_3$ is an axial vector between L and K.

Dot product of $\mathbf{a}_3$ with both sides of (8.1) yields

$$\mathbf{a}_3 \cdot \mathbf{L} = -\mathbf{a}_3 \cdot \mathbf{K} \tag{8.5}$$

Substituting (8.1a-1) into (8.5) yields

$$\mathbf{U}' \cdot \mathbf{q}_2 = V' \tag{8.6}$$

$$\begin{cases} \mathbf{U}' = p_2 \mathbf{a}_3 \\ V' = -(\mathbf{a}_3 \cdot \mathbf{K}) \end{cases} \tag{8.6a}$$

Substituting (8.1b-1) into (8.6) yields

$$A' \cos\psi_2 + B' \sin\psi_2 = C' \tag{8.7}$$

$$\begin{cases} A' = \mathbf{U}' \cdot \mathbf{q}_1 \\ B' = \mathbf{U}' \cdot \mathbf{a}_2 \times \mathbf{q}_1 \\ C' = V' \end{cases} \tag{8.7a}$$

(iii). *Derivation of the input-output displacement equation relating* Θ_4.

Since $\{A, B, C\}$ and $\{A', B', C'\}$ of (8.4) and (8.7) contain only the input and output angles, therefore, eliminating the *auxiliary angle* ψ_2 from the two equations, we obtain the required input-output displacement equations.

Solving (8.4) and (8.7) yields

$$\begin{cases} \cos\psi_2 = -Q_2 / Q_3 \\ \sin\psi_2 = Q_1 / Q_3 \end{cases} \tag{8.8}$$

$$\begin{cases} Q_1 = (AC' - A'C) \\ Q_2 = (BC' - B'C) \\ Q_3 = (AB' - A'B) \end{cases} \tag{8.8a}$$

From (8.8) and the identity $\cos^2\psi + \sin^2\psi = 1$ we get

$$Q_1^2 + Q_2^2 = Q_3^2 \tag{8.9}$$

Substituting (8.4a) and (8.7a) into (8.9a) yields

$$\begin{cases} Q_1 = (V'\mathbf{U} - V\mathbf{U}') \cdot \mathbf{q}_1 \\ Q_2 = (V'\mathbf{U} - V\mathbf{U}') \cdot \mathbf{a}_2 \times \mathbf{q}_1 \\ Q_3 = (\mathbf{U} \times \mathbf{U}') \cdot \mathbf{a}_2 \end{cases} \tag{8.10}$$

Substituting (8.3a) and (8.6a) into (8.10) we get

$$\begin{cases} Q_1 = -2p_2(\mathbf{a}_3 \cdot \mathbf{K})(\mathbf{q}_1 \cdot \mathbf{K}) - p_2(-\mathbf{K}^2 + p_3^2 - p_2^2)(\mathbf{a}_3 \cdot \mathbf{q}_1) \\ \quad = -2p_2[\mathbf{a}_3 \cdot (\mathbf{I} + \mathbf{J})][\mathbf{q}_1 \cdot (\mathbf{I} + \mathbf{J})] - p_2[-(\mathbf{I} + \mathbf{J})^2 + p_3^2 - p_2^2](\mathbf{a}_3 \cdot \mathbf{q}_1) \\ \quad = 2p_2(\mathbf{q}_1 \times \mathbf{I}) \cdot (\mathbf{a}_3 \times \mathbf{J}) + p_2(\beta_1 \mathbf{q}_1 - 2\beta_2 \mathbf{I}) \cdot \mathbf{a}_3 + (2p_2 p_4 S_3 \mathbf{q}_1) \cdot \mathbf{q}_4 + 2p_2 S_3 \beta_2 \\ Q_3 = 2p_2^2\, \mathbf{a}_2 \cdot (\mathbf{K} \times \mathbf{a}_3) = 2p_2^2\, [(\mathbf{a}_2 \times \mathbf{I}) \cdot \mathbf{a}_3 - \mathbf{a}_2 \cdot (\mathbf{a}_3 \times \mathbf{J})] \end{cases} \tag{8.11}$$

where

$$\begin{cases} \beta_1 = (\mathbf{l}^2 + p_2^2 - p_3^2 + p_4^2 - S_3^2) \\ \quad = S_2^2 + p_1^2 + 2S_2[S_1 c\alpha_{12} + p_5 s\theta_1 s\alpha_{12} - S_4(c\alpha_{41} c\alpha_{12} - s\alpha_{41} s\alpha_{12} c\theta_1)] \\ \qquad + 2p_1(p_1 c\theta_1 - S_4 s\alpha_{41} s\theta_1) + (S_1^2 + p_5^2 + S_4^2 - 2S_1 S_4 c\alpha_{41}) \\ \beta_2 = (\mathbf{q}_1 \cdot \mathbf{l}) = (p_1 + p_5 c\theta_1 - S_4 s\alpha_{41} s\theta_1) \end{cases} \tag{8.11a}$$

Replacing $\mathbf{q}_1$ by $(\mathbf{a}_2 \times \mathbf{q}_1)$ in Q_1 of (8.11), we obtain the expression for Q_2. Since $(\mathbf{a}_3 \times \mathbf{J})$, $\mathbf{a}_3$ and $\mathbf{q}_4$ are linear in $\cos\Theta_4$ and $\sin\Theta_4$, hence we can see from (8.11) that Q_1, Q_2 and Q_3 are also linear in $\cos\Theta_4$ and $\sin\Theta_4$. This implies that we can derive a fourth-order polynomial displacement equation relating Θ_4 from (8.9).

Substituting (8.1a-3) and (8.1b-6) into (8.11) we obtain

$$\begin{cases} Q_1 = (\mathbf{W}_1 \cdot \mathbf{q}_4 + Z_1) \\ Q_2 = (\mathbf{W}_2 \cdot \mathbf{q}_4 + Z_2) \\ Q_3 = (\mathbf{W}_3 \cdot \mathbf{q}_4 + Z_3) \end{cases} \tag{8.12}$$

where

$$\begin{cases} \mathbf{W}_1 = \mathbf{H}(\mathbf{q}_1) \\ \mathbf{W}_2 = \mathbf{H}(\mathbf{a}_2 \times \mathbf{q}_1) \\ \mathbf{W}_3 = 2p_2^2[s\alpha_{34}(\mathbf{a}_2 \times \mathbf{l}) \times \mathbf{a}_4 - p_4 c\alpha_{34} \mathbf{a}_2 \times \mathbf{a}_4] \\ Z_1 = h(\mathbf{q}_1) \\ Z_2 = h(\mathbf{a}_2 \times \mathbf{q}_1) \\ Z_3 = 2p_2^2[c\alpha_{34}(\mathbf{a}_2 \times \mathbf{l}) \cdot \mathbf{a}_4 + p_4 s\alpha_{34} \mathbf{a}_2 \cdot \mathbf{a}_4] \end{cases} \tag{8.12a}$$

$$\begin{cases} \mathbf{H}(\mathbf{x}) = 2p_2 p_4 c\alpha_{34}(\mathbf{x} \times \mathbf{l}) \times \mathbf{a}_4 + p_2 s\alpha_{34}(\beta_1 \mathbf{x} - 2\beta_2 \mathbf{l}) \times \mathbf{a}_4 + 2p_2 p_4 S_3 \tau \\ h(\mathbf{x}) = 2p_2 p_4 s\alpha_{34}(\mathbf{x} \times \mathbf{l}) \cdot \mathbf{a}_4 + p_2 c\alpha_{34}(\beta_1 \mathbf{x} - 2\beta_2 \mathbf{l}) \cdot \mathbf{a}_4 + 2p_2 S_3(\mathbf{l} \cdot \mathbf{x}) \end{cases} \tag{8.12b}$$

Substituting (8.12) into (8.9) yields

$$(\mathbf{W}_1 \cdot \mathbf{q}_4)^2 + (\mathbf{W}_2 \cdot \mathbf{q}_4)^2 - (\mathbf{W}_3 \cdot \mathbf{q}_4)^2 + 2[Z_1(\mathbf{W}_1 \cdot \mathbf{q}_4) + Z_2(\mathbf{W}_2 \cdot \mathbf{q}_4) - Z_3(\mathbf{W}_3 \cdot \mathbf{q}_4)] + (Z_1^2 + Z_2^2 - Z_3^2) = 0 \tag{8.13}$$

Substituting (8.1b-6) into (8.13) yields

$$\mu_1 \cos^2\Theta_4 + \mu_2 \sin^2\Theta_4 + \mu_3 \cos\Theta_4 \sin\Theta_4 + \mu_4 \cos\Theta_4 + \mu_5 \sin\Theta_4 + \mu_6 = 0 \tag{8.14}$$

$$\begin{cases} \mu_1 = (\mathbf{W}_1 \cdot \mathbf{q}_5)^2 + (\mathbf{W}_2 \cdot \mathbf{q}_5)^2 - (\mathbf{W}_3 \cdot \mathbf{q}_5)^2 \\ \mu_2 = (\mathbf{W}_1 \cdot \mathbf{q}_5 \times \mathbf{a}_4)^2 + (\mathbf{W}_2 \cdot \mathbf{q}_5 \times \mathbf{a}_4)^2 - (\mathbf{W}_3 \cdot \mathbf{q}_5 \times \mathbf{a}_4)^2 \\ \mu_3 = 2(\mathbf{W}_1 \cdot \mathbf{q}_5)(\mathbf{W}_1 \cdot \mathbf{q}_5 \times \mathbf{a}_4) + 2(\mathbf{W}_2 \cdot \mathbf{q}_5)(\mathbf{W}_2 \cdot \mathbf{q}_5 \times \mathbf{a}_4) - 2(\mathbf{W}_3 \cdot \mathbf{q}_5)(\mathbf{W}_3 \cdot \mathbf{q}_5 \times \mathbf{a}_4) \\ \mu_4 = 2(\mathbf{W}_1 \cdot \mathbf{q}_5)Z_1 + 2(\mathbf{W}_2 \cdot \mathbf{q}_5)Z_2 - 2(\mathbf{W}_3 \cdot \mathbf{q}_5)Z_3 \\ \mu_5 = 2(\mathbf{W}_1 \cdot \mathbf{q}_5 \times \mathbf{a}_4)Z_1 + 2(\mathbf{W}_2 \cdot \mathbf{q}_5 \times \mathbf{a}_4)Z_2 - 2(\mathbf{W}_3 \cdot \mathbf{q}_5 \times \mathbf{a}_4)Z_3 \\ \mu_6 = Z_1^2 + Z_2^2 - Z_3^2 \end{cases} \tag{8.14a}$$

Let $y = \tan(\Theta_4/2)$, then $\cos\Theta_4 = (1-y^2)/(1+y^2)$, $\sin\Theta_4 = 2y/(1+y^2)$. From (8.14) we get

$$\sum_{i=0}^{4} v_i y^{4-i} = 0 \tag{8.15}$$

$$\begin{cases} v_0 = \mu_1 \\ v_1 = -2\mu_3 \\ v_2 = -2\mu_1 + 4\mu_2 - \mu_4 + \mu_6 \\ v_3 = 2\mu_3 + 2\mu_5 \\ v_4 = \mu_1 + \mu_4 + \mu_6 \end{cases} \tag{8.15a}$$

Solving (8.15) we obtain y, then $\theta_4 = 2\tan^{-1}y$; From (8.8) we get ψ_2; Now the vectors $\{\mathbf{K}, \mathbf{L}\}$ can be determined from (8.1a) and (8.1b). From (8.1) and (8.1a-6) we get

$$\mathbf{q}_3 = -(\mathbf{K}+\mathbf{L})/p_3 = \cos\theta_3 \mathbf{q}_4 - \sin\theta_3 \mathbf{a}_3 \times \mathbf{q}_4 \tag{8.16}$$

Using $\mathbf{q}_4$ and $\mathbf{a}_3 \times \mathbf{a}_4$, respectively, to dot product both sides of (8.16), we obtain θ_3:

$$\begin{cases} \cos\theta_3 = -(\mathbf{K}+\mathbf{L})\cdot \mathbf{q}_4 / p_4 \\ \sin\theta_3 = (\mathbf{K}+\mathbf{L})\cdot \mathbf{a}_3 \times \mathbf{q}_4 / p_4 \end{cases} \tag{8.17}$$

To this stage, theoretical displacement analysis for the Generalized Clemens Coupling is complete. If we want to calculate numerical values for a specific mechanism, we need to scalarize (8.10a) and (8.14a). Since there are quite a few dot products of $\mathbf{W}_i$ $(i=1\text{-}3)$ with $\mathbf{q}_5$ and $\mathbf{q}_5 \times \mathbf{a}_5$ in (8.14a), respectively, it is convenient for calculation to express $\mathbf{W}_i$ as

$$\mathbf{W}_i = w_{i1} \mathbf{q}_5 + w_{i2} \mathbf{q}_5 \times \mathbf{a}_4 + w_{i3} \mathbf{a}_4 \qquad (i=1\text{-}3) \tag{8.18}$$

Hence we have

$$\begin{cases} \mathbf{W}_i \cdot \mathbf{q}_5 = w_{i1} \\ \mathbf{W}_i \cdot \mathbf{q}_5 \times \mathbf{a}_4 = w_{i2} \end{cases} \qquad (i=1\text{-}3) \tag{8.19}$$

As an example, the scalarized z_1 and the coefficients of $\mathbf{w}_1$ are as follows

$$\begin{aligned} Z_1 =& -2p_2p_4 s\alpha_{34}(c\theta_1\rho_2 + c\alpha_{41}s\theta_1\rho_1) + 2p_2S_3\beta_2 \\ &+ p_2 c\alpha_{34}(\beta_1 s\alpha_{41}s\theta_1 - 2\beta_2\rho_3) \end{aligned} \tag{8.20}$$

$$\begin{cases} w_{11} = 2p_2p_4c\alpha_{34}(s\alpha_{41}s\theta_1\rho_1 - c\theta_1\rho_3) + p_2 s\alpha_{34}(\beta_1 c\alpha_{41}s\theta_1 + 2\beta_2\rho_2) + 2p_2p_4S_3c\theta_1 \\ w_{12} = 2p_2p_4c\alpha_{34}(s\alpha_{41}s\theta_1\rho_2 + c\alpha_{41}s\theta_1\rho_3) + p_2 s\alpha_{34}(\beta_1 c\theta_1 - 2\beta_2\rho_1) - 2p_2p_4S_3c\alpha_{41}s\theta_1 \\ w_{13} = 2p_2p_4c\alpha_{34}(s\alpha_{41}s\theta_1\rho_3 - s\alpha_{41}c\theta_1\rho_3) + 2p_2p_4S_3s\alpha_{41}s\theta_1 \end{cases} \tag{8.21}$$

where

$$\begin{cases} \mathbf{I} = \rho_1\mathbf{q}_5 + \rho_2\mathbf{q}_5\times\mathbf{a}_4 + \rho_3\mathbf{a}_4 \\ \rho_1 = S_2 s\theta_1 s\alpha_{12} + p_1 c\theta_1 + p_5 \\ \rho_2 = S_2 s\alpha_{41}c\alpha_{12} - p_1 c\alpha_{41}s\theta_1 + S_1 s\alpha_{41} \\ \rho_3 = S_2(c\alpha_{41}c\alpha_{12} - s\alpha_{41}s\alpha_{12}c\theta_1) + p_1 s\alpha_{41}s\theta_1 + S_1 c\alpha_{41} - S_4 \end{cases} \tag{8.21a}$$

(2). **Velocity and acceleration analysis.**

Differentiating both sides of (8.14) yields

$$\dot{\Theta}_4 = \frac{\dot{\mu}_1\cos^2\Theta_4+\dot{\mu}_2\sin^2\Theta_4+\dot{\mu}_3\cos\Theta_4\sin\Theta_4+\dot{\mu}_4\cos\Theta_4+\dot{\mu}_5\sin\Theta_4+\dot{\mu}_6}{(\mu_1-\mu_2)\sin(2\Theta_4)-\mu_3\cos(2\Theta_4)+\mu_4\sin\Theta_4-\mu_5\cos\Theta_4} \tag{8.22}$$

$\{\dot{\mu}_i\}$ $(i=1-6)$ of (8.22) can be calculated from (8.14a).

From $\frac{d^2}{dt^2}$(8.14) we get $\ddot{\Theta}_4$. Other velocity variables $\{\dot{\theta}_2, \dot{\theta}_3\}$ and the corresponding acceleration variables $\{\ddot{\theta}_2, \ddot{\theta}_3\}$ can also be easily determined. Here we only write out the expression of $\dot{\theta}_3$, for it is the most important one among all the velocity and acceleration variables.

8.3. Displacement analysis of the R_0-SRRR mechanism

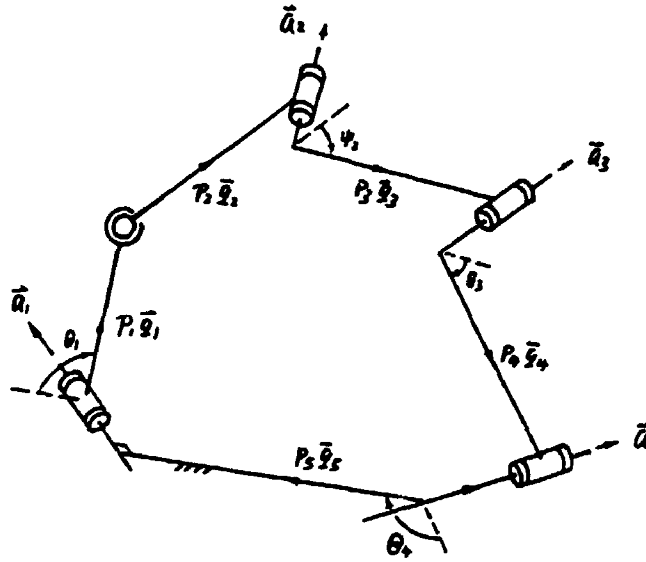


Fig. 8.2

Mechanism R_0-SRRR is shown in Fig. 8.2. When the input angle θ_1 is given, unknowns to be determined are $\{\theta_2, \theta_3, \theta_4\}$. If the first displacement equation we want to derive is $f(\theta_1, \theta_4)=0$, we can let $\theta_4=\Theta_4$. In this case, the auxiliary angle can be chosen between θ_2 and θ_3. If we choose θ_3 as the auxiliary angle, the procedure of deriving $f(\theta_1, \theta_4)=0$ will be quite similar to that of section 8.2. If we choose θ_2 as the auxiliary angle, the procedure of deriving $f(\theta_1, \theta_4)=0$ will then be somewhat different from that of section 8.2. Here we let $\theta_4=\Theta_4$, $\theta_2=\psi_2$, then, the vector loop equation of the mechanism can be written as follow,

$$\mathbf{K}=\mathbf{F} \tag{8.23}$$

$$\begin{cases} \mathbf{K}=\mathbf{I}+\mathbf{J} & (1) \\ \mathbf{J}=p_4\mathbf{q}_4-S_3\mathbf{a}_3 & (2) \\ \mathbf{I}=p_1\mathbf{q}_1+(S_1\mathbf{a}_1+p_5\mathbf{a}_5-S_4\mathbf{a}_4) & (3) \\ \mathbf{F}=-(p_2\mathbf{q}_2-S_2\mathbf{a}_2+p_3\mathbf{q}_3) & (4) \end{cases} \tag{8.23a}$$

$$\begin{cases} \mathbf{q}_1 = c\theta_1 \mathbf{q}_5 + s\theta_1 \mathbf{a}_1 \times \mathbf{q}_5 & (1) \\ \mathbf{q}_2 = c\psi_2 \mathbf{q}_3 - s\psi_2 \mathbf{a}_2 \times \mathbf{q}_3 & (2) \\ \mathbf{a}_2 = c\alpha_{23} \mathbf{a}_3 - s\alpha_{23} \mathbf{q}_3 \times \mathbf{a}_3 & (3) \\ \mathbf{q}_3 = c\theta_3 \mathbf{q}_4 - s\theta_3 \mathbf{a}_3 \times \mathbf{q}_4 & (4) \\ \mathbf{a}_3 = c\alpha_{34} \mathbf{a}_4 - s\alpha_{34} \mathbf{q}_4 \times \mathbf{a}_4 & (5) \\ \mathbf{q}_4 = c\Theta_4 \mathbf{q}_5 - s\Theta_4 \mathbf{a}_4 \times \mathbf{q}_5 & (6) \end{cases} \tag{8.23b}$$

(i). *Derivation of the first equation relating* θ_1, Θ_4 *and* ψ_2.

Squaring both sides of (8.23) yields

$$\mathbf{K}^2 = \mathbf{F}^2 \tag{8.24}$$

Substituting (8.23a-4) into (8.24) yielus

$$\cos\psi_2 = [\mathbf{K}^2 - (p_2^2 + S_2^2 + p_3^2]/(2p_2p_3) \tag{8.25}$$

(ii). *Derivation of the second equation relating* θ_1, Θ_4 *and* ψ_2.

We can see from Fig. 8.2 that $\mathbf{a}_3$ is an axial vector between $\mathbf{L}$ and $\mathbf{K}$. Dot product $\mathbf{a}_3$ with both sides of (8.23) yields

$$\mathbf{K}\cdot\mathbf{a}_3 = \mathbf{F}\cdot\mathbf{a}_3 \tag{8.26}$$

Substituting (8.23a-4) into (8.26) we get

$$\sin\psi_2 = [-(\mathbf{K}\cdot\mathbf{a}_3) + S_2 c\alpha_{23}]/(p_2 s\alpha_{23}) \tag{8.27}$$

(iii). *Derivation of the input-output displacement equation relating* Θ_4.

From (8.25), (8.27) and the identity $\cos^2\psi_2 + \sin^2\psi_2 = 1$ we obtain

$$Q_1^2 + Q_2^2 = Q_3^2 \tag{8.28}$$

$$\begin{cases} Q_1 = s\alpha_{23}\mathbf{K}^2 - s\alpha_{23}(p_2^2 + S_2^2 + p_3^2) \\ Q_2 = -2p_3(\mathbf{K}\cdot\mathbf{a}_3) + 2p_3 S_2 c\alpha_{23} \\ Q_3 = 2p_2 p_3 s\alpha_{23} \end{cases} \tag{8.28a}$$

$$\begin{cases} Q_1 = (\mathbf{W}_1\cdot\mathbf{q}_4) + Z_1 = (\mathbf{W}_1\cdot\mathbf{q}_5)\cos\Theta_4 + (\mathbf{W}_1\cdot\mathbf{q}_5\times\mathbf{a}_4)\sin\Theta_4 + Z_1 \\ Q_2 = (\mathbf{W}_2\cdot\mathbf{q}_4) + Z_2 = (\mathbf{W}_2\cdot\mathbf{q}_5)\cos\Theta_4 + (\mathbf{W}_2\cdot\mathbf{q}_5\times\mathbf{a}_4)\sin\Theta_4 + Z_2 \end{cases} \tag{8.28b}$$

$$\begin{cases} \mathbf{W}_1 = 2s\alpha_{23}(p_4\mathbf{I} - S_3 s\alpha_{34}\mathbf{a}_4\times\mathbf{I}) \\ \mathbf{W}_2 = 2p_3 s\alpha_{34}\mathbf{a}_4\times\mathbf{I} \\ Z_1 = -2S_3 s\alpha_{23} c\alpha_{34}(\mathbf{I}\cdot\mathbf{a}_4) + s\alpha_{23}(\mathbf{I}^2 + \mathbf{J}^2 - p_2^2 - S_2^2 - p_3^2) \\ Z_2 = -2p_3 c\alpha_{34}(\mathbf{I}\cdot\mathbf{a}_4) + 2p_3(S_3 + S_2 c\alpha_{23}) \end{cases} \tag{8.28c}$$

Substituting (8.28b) into (8.28) yields

$$\mu_1\cos^2\Theta_4 + \mu_2\sin^2\Theta_4 + \mu_3\cos\Theta_4\sin\Theta_4 + \mu_4\cos\Theta_4 + \mu_5\sin\Theta_4 + \mu_6 = 0 \tag{8.29}$$

$$
\begin{cases}
\mu_1 = (\mathbf{W}_1 \cdot \mathbf{q}_3)^2 + (\mathbf{W}_2 \cdot \mathbf{q}_3)^2 \\
\mu_2 = (\mathbf{W}_1 \cdot \mathbf{q}_3 \times \mathbf{a}_4)^2 + (\mathbf{W}_2 \cdot \mathbf{q}_3 \times \mathbf{a}_4)^2 \\
\mu_3 = 2(\mathbf{W}_1 \cdot \mathbf{q}_3)(\mathbf{W}_1 \cdot \mathbf{q}_3 \times \mathbf{a}_4) + 2(\mathbf{W}_2 \cdot \mathbf{q}_3)(\mathbf{W}_2 \cdot \mathbf{q}_3 \times \mathbf{a}_4) \\
\mu_4 = 2(\mathbf{W}_1 \cdot \mathbf{q}_3) Z_1 + 2(\mathbf{W}_2 \cdot \mathbf{q}_3) Z_2 \\
\mu_5 = 2(\mathbf{W}_1 \cdot \mathbf{q}_3 \times \mathbf{a}_4) Z_1 + 2(\mathbf{W}_2 \cdot \mathbf{q}_3 \times \mathbf{a}_4) Z_2 \\
\mu_6 = Z_1^2 + Z_2^2 - Q_3^2
\end{cases}
\tag{8.29a}
$$

Let $y = \tan(\theta_4/2)$, then $\cos\theta_4 = (1-y^2)/(1+y^2)$, $\sin\theta_4 = 2y/(1+y^2)$. From (8.29) we obtain

$$
\sum_{i=0}^{4} \nu_i y^{4-i} = 0 \tag{8.30}
$$

$$
\begin{cases}
\nu_0 = \mu_1 \\
\nu_1 = -2\mu_3 \\
\nu_2 = -2\mu_1 + 4\mu_2 - \mu_4 + \mu_6 \\
\nu_3 = 2\mu_3 + 2\mu_5 \\
\nu_4 = \mu_1 + \mu_4 + \mu_6
\end{cases}
\tag{8.30a}
$$

Solving (8.30) we get y, then $\theta_4 = 2\tan^{-1} y$; From (8.25) and (8.27) we obtain θ_2; Using $\mathbf{q}_4$ and $\mathbf{a}_3 \times \mathbf{q}_4$, respectively, to dot product both sides of (8.23) yields

$$
\begin{cases}
(\mathbf{I}+\mathbf{J}) \cdot \mathbf{q}_4 = -(p_2 \mathbf{q}_2 - S_2 \mathbf{a}_2 + p_3 \mathbf{q}_3) \cdot \mathbf{q}_4 \\
(\mathbf{I}+\mathbf{J}) \cdot \mathbf{a}_3 \times \mathbf{q}_4 = -(p_2 \mathbf{q}_2 - S_2 \mathbf{a}_2 + p_3 \mathbf{q}_3) \cdot \mathbf{a}_3 \times \mathbf{q}_4
\end{cases}
\tag{8.31}
$$

i.e.

$$
\begin{cases}
(\mathbf{I} \cdot \mathbf{q}_4) + p_4 = -p_2 [c\theta_2 \cos\theta_3 - c\alpha_{23} s\theta_2 \sin\theta_3] + S_2 s\alpha_{23} \sin\theta_3 - p_3 \cos\theta_3 \\
(\mathbf{I} \cdot \mathbf{a}_3 \times \mathbf{q}_4) = p_2 [c\theta_2 \sin\theta_3 + c\alpha_{23} s\theta_2 \cos\theta_3] + S_2 s\alpha_{23} \cos\theta_3 + p_3 \sin\theta_3
\end{cases}
\tag{8.31a}
$$

i.e.

$$
\begin{cases}
-(p_2 c\theta_2 + p_3)\cos\theta_3 + (p_2 c\alpha_{23} s\theta_2 + S_2 s\alpha_{23})\sin\theta_3 = (\mathbf{I} \cdot \mathbf{q}_4) + p_4 \\
(p_2 c\alpha_{23} s\theta_2 + S_2 s\alpha_{23})\cos\theta_3 + (p_2 c\theta_2 + p_3)\sin\theta_3 = (\mathbf{I} \cdot \mathbf{a}_3 \times \mathbf{q}_4)
\end{cases}
\tag{8.31b}
$$

Solving (8.31b) yields θ_3:

$$
\begin{cases}
\cos\theta_3 = \dfrac{(p_2 c\alpha_{23} s\theta_2 + S_2 s\alpha_{23})(\mathbf{I} \cdot \mathbf{a}_3 \times \mathbf{q}_4) - (p_2 c\theta_2 + p_3)(\mathbf{I} \cdot \mathbf{q}_4 + p_4)}{(p_2 c\theta_2 + p_3)^2 + (p_2 c\alpha_{23} s\theta_2 + S_2 s\alpha_{23})^2} \\[2ex]
\sin\theta_3 = \dfrac{(p_2 c\alpha_{23} s\theta_2 + S_2 s\alpha_{23})(\mathbf{I} \cdot \mathbf{q}_4 + p_4) + (p_2 c\theta_2 + p_3)(\mathbf{I} \cdot \mathbf{a}_3 \times \mathbf{q}_4)}{(p_2 c\theta_2 + p_3)^2 + (p_2 c\alpha_{23} s\theta_2 + S_2 s\alpha_{23})^2}
\end{cases}
\tag{8.32}
$$

To this stage, theoretical displacement analysis for mechanism R_0-$SRRR$ is complete.

8.4. Displacement analysis of the R_0-$RRRS$ mechanism

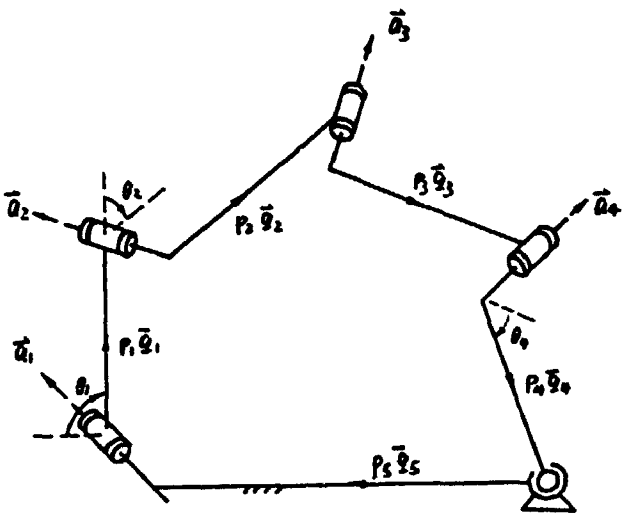


Fig. 8.3

Mechanism R_0-$RRRS$ is shown in Fig. 8.3. As far as we know, this mechanism has never been analyzed before. When the input angle θ_1 is given, unknowns to be determined are $\{\theta_2, \theta_3, \theta_4\}$. Suppose $f(\theta_1, \theta_2)=0$ is the first displacement equation to be derived. Let $\theta_2=\Theta_2$, $\theta_4=\psi_4$. The vector loop equation of the mechanism can be written as

$$\mathbf{K} = \mathbf{F} \tag{8.33}$$

$$\begin{cases} \mathbf{K} = \mathbf{I}+\mathbf{J} & (1) \\ \mathbf{J} = (p_2\mathbf{q}_2 - S_3\mathbf{a}_3) & (2) \\ \mathbf{I} = (-S_2\mathbf{a}_2 + p_1\mathbf{q}_1) + (S_1\mathbf{a}_1 + p_5\mathbf{a}_5) & (3) \\ \mathbf{F} = -p_3\mathbf{q}_3 + S_4\mathbf{a}_4 - p_4\mathbf{q}_4 & (4) \end{cases} \tag{8.33a}$$

$$\begin{cases} \mathbf{q}_4 = c\psi_4\mathbf{q}_3 + s\psi_4\mathbf{a}_4\times\mathbf{q}_3 & (1) \\ \mathbf{a}_4 = c\alpha_{34}\mathbf{a}_3 + s\alpha_{34}\mathbf{q}_3\times\mathbf{a}_3 & (2) \\ \mathbf{q}_3 = c\theta_3\mathbf{q}_2 + s\theta_3\mathbf{a}_3\times\mathbf{q}_2 & (3) \\ \mathbf{a}_3 = c\alpha_{23}\mathbf{a}_2 + s\alpha_{23}\mathbf{q}_2\times\mathbf{a}_2 & (4) \\ \mathbf{q}_2 = c\Theta_2\mathbf{q}_1 + s\Theta_2\mathbf{a}_2\times\mathbf{q}_1 & (5) \\ \mathbf{a}_2 = c\alpha_{12}\mathbf{a}_1 + s\alpha_{12}\mathbf{q}_1\times\mathbf{a}_1 & (6) \\ \mathbf{q}_1 = c\theta_1\mathbf{q}_5 + s\theta_1\mathbf{a}_1\times\mathbf{q}_5 & (7) \end{cases} \tag{8.33b}$$

(i). *Derivation of the first equation relating* θ_1, Θ_2 *and* ψ_4.

Squaring both sides of (8.33) yields

$$\mathbf{K}^2 = \mathbf{F}^2 \tag{8.34}$$

Substituting (8.33a-4) into (8.34) yields

$$\cos\psi_4 = [\mathbf{K}^2 - (p_3^2 + S_4^2 + p_4^2)]/(2p_3p_4) \tag{8.35}$$

(ii). *Derivation of the second equation relating* θ_1, Θ_2 *and* ψ_4.

We can see from Fig. 8.3 that $\mathbf{a}_3$ is an axial vector between $\mathbf{L}$ and $\mathbf{K}$. Dot product $\mathbf{a}_3$ with both sides of (8.33) yields

$$\mathbf{K}\cdot\mathbf{a}_3 = \mathbf{F}\cdot\mathbf{a}_3 \tag{8.36}$$

Substituting (8.33a-4) into (8.36) yields

$$\sin\psi_4 = [-(\mathbf{K}\cdot\mathbf{a}_3)+S_4 c\alpha_{34}]/(p_4 s\alpha_{34}) \tag{8.37}$$

(iii). *Derivation of the input-output displacement equation relating* Θ_2.

From (8.35), (8.37) and the identity $\cos^2\psi_2+\sin^2\psi_2=1$ we obtain

$$Q_1^2+Q_2^2=Q_3^2 \tag{8.38}$$

$$\begin{cases} Q_1 = s\alpha_{34}\mathbf{K}^2 - s\alpha_{34}(p_3^2+S_4^2+p_4^2) \\ Q_2 = -2p_3(\mathbf{K}\cdot\mathbf{a}_3)+2p_3 S_4 c\alpha_{34} \\ Q_3 = 2p_3 p_4 s\alpha_{34} \end{cases} \tag{8.38a}$$

$$\begin{cases} Q_1 = (\mathbf{W}_1\cdot\mathbf{q}_2)+Z_1 = (\mathbf{W}_1\cdot\mathbf{q}_1)\cos\Theta_2+(\mathbf{W}_1\cdot\mathbf{a}_2\times\mathbf{q}_1)\sin\Theta_2+Z_1 \\ Q_2 = (\mathbf{W}_2\cdot\mathbf{q}_2)+Z_2 = (\mathbf{W}_2\cdot\mathbf{q}_1)\cos\Theta_2+(\mathbf{W}_2\cdot\mathbf{a}_2\times\mathbf{q}_1)\sin\Theta_2+Z_2 \end{cases} \tag{8.38b}$$

$$\begin{cases} \mathbf{W}_1 = 2s\alpha_{34}(p_2\mathbf{I}-S_3 s\alpha_{23}\mathbf{a}_2\times\mathbf{I}) \\ \mathbf{W}_2 = -2p_3 s\alpha_{34}\mathbf{a}_2\times\mathbf{I} \\ Z_1 = -2S_3 c\alpha_{23} s\alpha_{34}(\mathbf{I}\cdot\mathbf{a}_2)+s\alpha_{34}(\mathbf{I}^2+\mathbf{J}^2-p_3^2-S_4^2-p_4^2) \\ Z_2 = -2p_3 c\alpha_{23}(\mathbf{I}\cdot\mathbf{a}_2)+2p_3(S_3+S_4 c\alpha_{34}) \end{cases} \tag{8.38c}$$

Substituting (8.38b) into (8.38) yields

$$\mu_1\cos^2\Theta_2+\mu_2\sin^2\Theta_2+\mu_3\cos\Theta_2\sin\Theta_2+\mu_4\cos\Theta_2+\mu_5\sin\Theta_2+\mu_6=0 \tag{8.39}$$

$$\begin{cases} \mu_1 = (\mathbf{W}_1\cdot\mathbf{q}_1)^2+(\mathbf{W}_2\cdot\mathbf{q}_1)^2 \\ \mu_2 = (\mathbf{W}_1\cdot\mathbf{a}_2\times\mathbf{q}_1)^2+(\mathbf{W}_2\cdot\mathbf{a}_2\times\mathbf{q}_1)^2 \\ \mu_3 = 2(\mathbf{W}_1\cdot\mathbf{q}_1)(\mathbf{W}_1\cdot\mathbf{a}_2\times\mathbf{q}_1)+2(\mathbf{W}_2\cdot\mathbf{q}_1)(\mathbf{W}_2\cdot\mathbf{a}_2\times\mathbf{q}_1) \\ \mu_4 = 2(\mathbf{W}_1\cdot\mathbf{q}_1)Z_1+2(\mathbf{W}_2\cdot\mathbf{q}_1)Z_2 \\ \mu_5 = 2(\mathbf{W}_1\cdot\mathbf{a}_2\times\mathbf{q}_1)Z_1+2(\mathbf{W}_2\cdot\mathbf{a}_2\times\mathbf{q}_1)Z_2 \\ \mu_6 = Z_1^2+Z_2^2-Q_3^2 \end{cases} \tag{8.39a}$$

Let $y=\tan(\Theta_4/2)$, then $\cos\Theta_4=(1-y^2)/(1+y^2)$, $\sin\Theta_4=2y/(1+y^2)$. From (8.39) we get

$$\sum_{i=0}^{4} v_i y^{4-i} = 0 \tag{8.40}$$

$$\begin{cases} \nu_0 = \mu_1 \\ \nu_1 = -2\mu_3 \\ \nu_2 = -2\mu_1 + 4\mu_2 - \mu_4 + \mu_6 \\ \nu_3 = 2\mu_3 + 2\mu_5 \\ \nu_4 = \mu_1 + \mu_4 + \mu_6 \end{cases} \tag{8.40a}$$

Solving (8.40) we get y, then $\theta_2 = 2\tan^{-1}y$; From (8.35) and (8.37) we obtain θ_4; Using $\mathbf{q}_2$ and $(\mathbf{a}_3 \times \mathbf{q}_2)$, respectively, to dot product both sides of (8.33) yields θ_3:

$$\begin{cases} \cos\theta_3 = -\dfrac{(p_3 + p_4 c\theta_4)(\mathbf{I}\cdot\mathbf{q}_2 + p_2) + (S_4 s\alpha_{34} + p_4 s\theta_4 c\alpha_{34})(\mathbf{I}\cdot\mathbf{a}_3\times\mathbf{q}_2)}{(p_3 + p_4 c\theta_4)^2 + (S_4 s\alpha_{34} + p_4 s\theta_4 c\alpha_{34})^2} \\ \sin\theta_3 = \dfrac{(S_4 s\alpha_{34} + p_4 s\theta_4 c\alpha_{34})(\mathbf{I}\cdot\mathbf{q}_2 + p_2) - (p_3 + p_4 c\theta_4)(\mathbf{I}\cdot\mathbf{a}_3\times\mathbf{q}_2)}{(p_3 + p_4 c\theta_4)^2 + (S_4 s\alpha_{34} + p_4 s\theta_4 c\alpha_{34})^2} \end{cases} \tag{8.41}$$

To this stage, theoretical displacement analysis for mechanism R_0-$RRRS$ is complete.

8.5. Analysis of the R_0-$RRSR$ mechanism

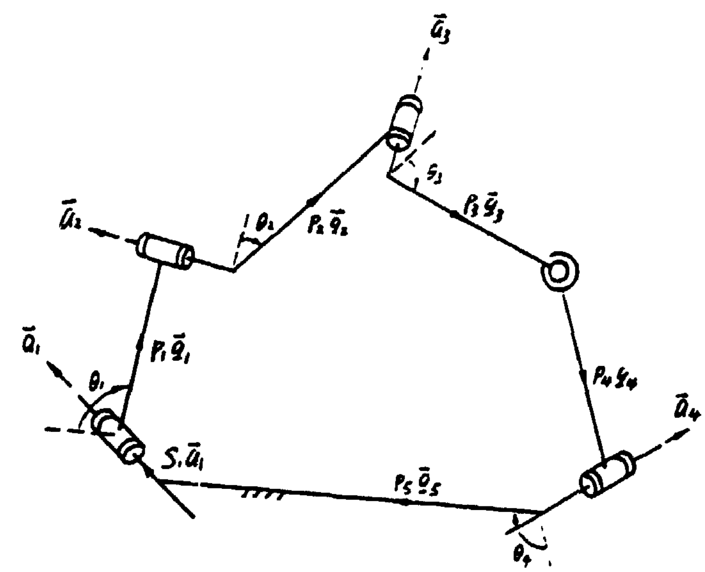


Fig. 8.4

Mechanism R_0-$RRSR$ is shown in Fig. 8.4. The main purpose of this section is to show the flexibility of the vector algebraic method in the analysis of spatial mechanisms, therefore, details are not displayed.

(1). Suppose the first displacement equation we want to derive is $f(\theta_1, \theta_2) = 0$, then we denote $\theta_2 = \Theta_2$, the auxiliary angle can be chosen between θ_3 and θ_4.

Case 1. Let $\theta_2=\Theta_2$, $\theta_3=\psi_3$. The vector loop equation of the mechanisms can be written as follow,

$$\mathbf{K}+\mathbf{L}=\mathbf{F} \tag{8.42}$$

$$\begin{cases} \mathbf{L}= p_3\mathbf{q}_3 \\ \mathbf{K}=\mathbf{I}+\mathbf{J} \\ \mathbf{J}=p_2\mathbf{q}_2-S_3\mathbf{a}_3 \\ \mathbf{I}=(-S_2\mathbf{a}_2+p_1\mathbf{q}_1)+\mathbf{I}_0 \\ \mathbf{I}_0=(S_1\mathbf{a}_1+p_5\mathbf{q}_5-S_4\mathbf{a}_4) \\ \mathbf{F}=-p_4\mathbf{q}_4 \end{cases} \tag{8.42a}$$

Case 2. Let $\theta_2=\Theta_2$, $\theta_4=\psi_4$. The vector loop equation is

$$\mathbf{K}+\mathbf{L}=\mathbf{F} \tag{8.43}$$

$$\begin{cases} \mathbf{L}=p_4\mathbf{q}_4 \\ \mathbf{K}=\mathbf{I}+\mathbf{J} \\ \mathbf{J}=p_2\mathbf{q}_2-S_3\mathbf{a}_3 \\ \mathbf{I}=(-S_2\mathbf{a}_2+p_1\mathbf{q}_1)+\mathbf{I}_0 \\ \mathbf{I}_0=(S_1\mathbf{a}_1+p_5\mathbf{q}_5-S_4\mathbf{a}_4) \\ \mathbf{F}=-p_3\mathbf{q}_3 \end{cases} \tag{8.43a}$$

(2). Suppose the first displacement equation we want to derive is $f(\theta_1, \theta_3)=0$, then let $\theta_3=\Theta_3$, the auxiliary angle can be chosen between θ_2 and θ_4.

Case 1. Let $\theta_3=\Theta_3$, $\theta_2=\psi_2$.

$$\mathbf{I}+\mathbf{L}=\mathbf{F} \tag{8.44}$$

$$\begin{cases} \mathbf{L}=p_2\mathbf{q}_2-S_3\mathbf{a}_3+p_3\mathbf{q}_3 \\ \mathbf{I}=(-S_2\mathbf{a}_2+p_1\mathbf{q}_1)+\mathbf{I}_0 \\ \mathbf{I}_0=(S_1\mathbf{a}_1+p_5\mathbf{q}_5-S_4\mathbf{a}_4) \\ \mathbf{F}=-p_4\mathbf{q}_4 \end{cases} \tag{8.44a}$$

Case 2. Let $\theta_3=\Theta_3$, $\theta_4=\psi_4$.

$$\mathbf{I}+\mathbf{L}=\mathbf{F} \tag{8.45}$$

$$\begin{cases} \mathbf{L}=p_4\mathbf{q}_4 \\ \mathbf{I}=(-S_2\mathbf{a}_2+p_1\mathbf{q}_1)+(S_1\mathbf{a}_1+p_5\mathbf{q}_5-S_4\mathbf{a}_4) \\ \mathbf{F}=-(p_2\mathbf{q}_2-S_3\mathbf{a}_3+p_3\mathbf{q}_3) \end{cases} \tag{8.45a}$$

(3). Suppose the first displacement equation we want to derive is $f(\theta_1, \theta_4)=0$, then let $\theta_4=\Theta_4$, the auxiliary angle can be chosen between θ_2 and θ_3.

Case 1. Let $\theta_4=\Theta_4$, $\theta_2=\psi_2$.

$$\mathbf{K}+\mathbf{L}=\mathbf{F} \tag{8.46}$$

$$\begin{cases} \mathbf{L} = p_2\mathbf{q}_2 - S_3\mathbf{a}_3 \\ \mathbf{K} = \mathbf{I} + \mathbf{J} \\ \mathbf{J} = p_4\mathbf{q}_4 \\ \mathbf{I} = (-S_2\mathbf{a}_2 + p_1\mathbf{q}_1) + \mathbf{I}_0 \\ \mathbf{I}_0 = (S_1\mathbf{a}_1 + p_5\mathbf{q}_5 - S_4\mathbf{a}_4) \\ \mathbf{F} = -p_3\mathbf{q}_3 \end{cases} \tag{8.46a}$$

Case 2. Let $\theta_4 = \Theta_4$, $\theta_3 = \psi_3$.

$$\mathbf{K} = \mathbf{F} \tag{8.47}$$

$$\begin{cases} \mathbf{K} = \mathbf{I} + \mathbf{J} \\ \mathbf{J} = p_4\mathbf{q}_4 \\ \mathbf{I} = (-S_2\mathbf{a}_2 + p_1\mathbf{q}_1) + \mathbf{I}_0 \\ \mathbf{I}_0 = (S_1\mathbf{a}_1 + p_5\mathbf{q}_5 - S_4\mathbf{a}_4) \\ \mathbf{F} = -(p_2\mathbf{q}_2 - S_3\mathbf{a}_3 + p_3\mathbf{q}_3) \end{cases} \tag{8.47a}$$

8.6. Conclusion

Comparing the analyses presented in this chapter with those in Refs. [2, 28, 73, 77, 79, 106], one will find that the vector algebraic approach is simpler than all previous related approaches.

Comparing the analysis procedures of Section 8.2 for the R_0-$RSRR$ mechanism and the previous chapter, what conclusion we can draw? It is clear that the analyses procedures and the algebraic expressions are almost identical, though the mechanisms are different.

CHAPTER 9. THE R_0–2R–2C MECHANISMS

9.1. Introduction

In this chapter the displacement problem of the R_0-2R-2C mechanism is analyzed using the *vector algebraic method*. The I/O (polynomial) displacement equations are derived. Before listing the advantages of the new approach, let us briefly review previous related works.

There are a total of six variants for the R_0-2R-2C mechanism: R_0-CRCR, R_0-RCRC, R_0-CCRR, R_0-RCCR, R_0-RRCC and R_0-CRRC. The R_0-RCCR and R_0-CRCR mechanisms were first analyzed by Dimentberg [12] in 1948, using screw algebra. The I/O displacement equations were obtained as 8th-order polynomials. In 1969 Yang [84] obtained a 4th-order polynomial for the R_0-CRCR mechanism, using (3x3) matrices with dual-number elements. In 1970 Yuan [93,95] studied the R_0-RCCR and R_0-CRCR mechanisms, using the method of line coordinates, and verified corresponding results obtained earlier by Dimentberg and Yang. In 1971 the R_0-RCCR mechanism was studied by Soni and Pamidi[6], also using (3x3) matrices with dual-number elements. Duffy and Habib-Olahi [19-21] investigated the R_0-CRCR, R_0-RCRC and R_0-CRRC mechanisms using spherical trigonometry. Lee and Bagci [50], using (3x3) screw matrices and dual vectors, deduced a 16th-order polynomial equation for the R_0-CRRC mechanism in 1975, where at least eight roots of the polynomial equation are extraneous solutions. Using direction cosines and projections, Lakshminarayan [44,45] derived solutions for the R_0-CRCR, R_0-RCRC and R_0-RRCC mechanisms in 1976. Eight years later, all the variants of mechanism of R_0-2R-2C were, once again, systematically analyzed by Zhang [106], using the direction cosine matrix method.

Recently Raghvan and Roth [59,60], and Kohli and Osvatic [42,43] presented new approaches for analyzing serial manipulators. Though the two papers do not specifically address the problem of the R_0-2R-2C mechanism, they are relevant.

The objective of the displacement analysis of spatial mechanisms is to determine the relative position and orientation of all connected (rigid) bodies of the mechanism in 3-dimensional space. We will show in this chapter that we can use vector expression and vector operations to perform the displacement analysis of spatial mechanisms. As compared to previous works, the new approach is characterized by its standard analysis steps, compact expressions and simplicity.

9.2. Standard analysis procedure

The simplified vector loop diagram of the R_0-$2R$-$2C$ mechanism is shown in Fig. 9.1, where $\{\mathbf{a}_i, \mathbf{q}_i\}$ $(i=1-5)$ are unit vectors of pair axes and links, respectively. $\{\theta_i\}$ $(i=1-5)$ are joint angles. $\{\alpha_i\}$ $(i=1-5)$ are twist angles between the axes of joint i and joint $i+1$, where α_5 is measured between $\mathbf{a}_5$ and $\mathbf{a}_1$. $\{p_i, S_i\}$ $(i=1-5)$ are link lengths and offsets, respectively. θ_1 is the input angle of the mechanism.

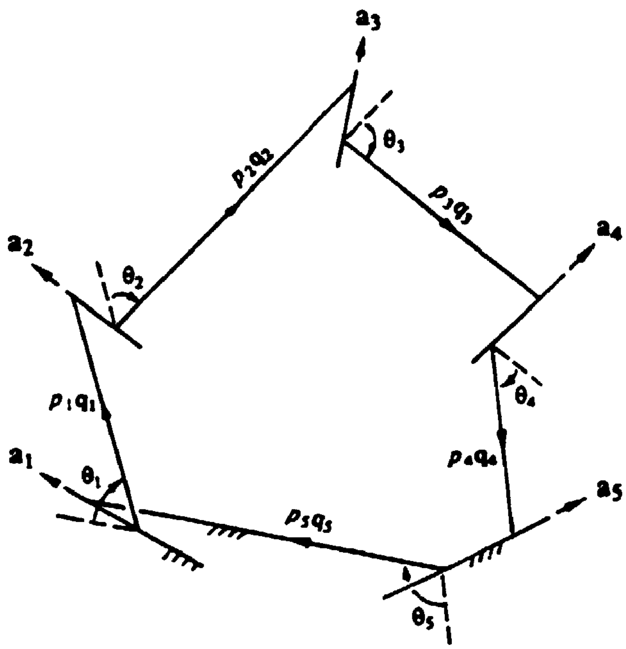


Fig. 9.1

Let $\{\mathbf{a}_k, \mathbf{a}_l\}$ $(2 \le k < l \le 5)$ be axial vectors of the two cylindrical pairs and $\{x_k, x_l\}$ are the corresponding variable offsets. Let $\theta_5 = \Theta_5$ be the "output angle", the first rotary variable to be determined. Let $\theta_2 = \psi_2$ be the "auxiliary angle", the variable to be eliminated from simultaneous equations. The vector loop equation of the mechanism can now be written as,

$$- x_k \mathbf{a}_k - x_l \mathbf{a}_l + \mathbf{K} + \mathbf{L} = \mathbf{F} \tag{9.1}$$

$$\begin{cases} \mathbf{L} = \mathbf{L}(\psi_2) \\ \mathbf{K} = \mathbf{I} + \mathbf{J} \\ \mathbf{J} = \mathbf{J}(\Theta_5) \\ \mathbf{I} = \mathbf{I}(\theta_1) \\ \mathbf{F} = -p_3 \mathbf{q}_3 \end{cases} \tag{9.1a}$$

$\mathbf{I}$ is the *input vector* of the mechanism. It is the sum of those vectors in the loop of the mechanism which are given or known at the beginning. $\mathbf{J}$ is the *output vector* of the mechanism. It is the sum of those constant-magnitude vectors in the loop of the mechanism that can be expressed as a function of the output angle (Θ_5 in this case). $\mathbf{L}$

is the *auxiliary vector*. It is the sum of those constant-magnitude vectors in the loop of the mechanism that can be expressed as a function of the auxiliary angle (ψ_2 in this case). $\mathbf{F}$ is the *floating vector*. Cutting the loop at the two ends of the floating vector, we obtain two separate chains, where one is fixed to the ground and the other is *floating*. For detailed definitions of these vectors see Chapter 2.

For any given k and l, vectors $\mathbf{I}$, $\mathbf{J}$, $\mathbf{K}$, $\mathbf{L}$ and $\mathbf{F}$ can be easily obtained from the vector loop diagram of the mechanism. The following set of vector equations, called *direction equations*, specify the relative direction of any individual (unit) vector with respect to its two adjacent (unit) vectors in the vector loop.

$$\begin{cases} \mathbf{a}_3 = c\alpha_{23}\mathbf{a}_2 + s\alpha_{23}\mathbf{q}_2\times\mathbf{a}_2 & (1) \\ \mathbf{q}_2 = c\psi_2\mathbf{q}_1 + s\psi_2\mathbf{a}_2\times\mathbf{q}_1 & (2) \\ \mathbf{a}_2 = c\alpha_{12}\mathbf{a}_1 + s\alpha_{12}\mathbf{q}_1\times\mathbf{a}_1 & (3) \\ \mathbf{q}_1 = c\theta_1\mathbf{q}_5 + s\theta_1\mathbf{a}_1\times\mathbf{q}_5 & (4) \\ \mathbf{q}_3 = \csc\alpha_{34}\mathbf{a}_3\times\mathbf{a}_4 & (5) \\ \mathbf{a}_4 = c\alpha_{45}\mathbf{a}_5 - s\alpha_{45}\mathbf{q}_4\times\mathbf{a}_5 & (6) \\ \mathbf{q}_4 = c\Theta_5\mathbf{q}_5 - s\Theta_5\mathbf{a}_5\times\mathbf{q}_5 & (7) \end{cases} \tag{9.1b}$$

The set of vector equations in (9.1b) can also be expressed as $\mathbf{a}_i\times\mathbf{a}_{i+1} = \mathbf{q}_i \sin\alpha_{i\,i+1}$ $(i = 1 - 4)$ and $\mathbf{a}_5\times\mathbf{a}_1 = \mathbf{q}_5 \sin\alpha_{51}$. However, representing the direction equations in the form of (9.1b) is more convenient for use.

Step 1. *Derivation of the first equation relating* θ_1, Θ_5 and ψ_2.

Equating $(\mathbf{a}_3\cdot\mathbf{a}_4)$ for the fixed and floating parts of the vector loop, we get

$$\mathbf{a}_3(\psi_2)\cdot\mathbf{a}_4(\Theta_5) = c\alpha_{34} \tag{9.2}$$

From Fig. 9.1 we can see that $\mathbf{a}_4$ is a function of Θ_5, i.e. $\mathbf{a}_4 = \mathbf{a}_4(\Theta_5)$ and $\mathbf{a}_3$ is a function of ψ_2 and θ_1. Since the input angle, θ_1, is given, we have $\mathbf{a}_3 = \mathbf{a}_3(\psi_2)$.

Substituting (9.1b-1) into (9.2) yields

$$\mathbf{U}\cdot\mathbf{q}_2 = V \tag{9.3}$$

$$\begin{cases} \mathbf{U} = \mathbf{U}(\Theta_5) = s\alpha_{23}\mathbf{a}_4\times\mathbf{a}_2 \\ V = V(\Theta_5) = c\alpha_{23}(\mathbf{a}_2\cdot\mathbf{a}_4) - c\alpha_{34} \end{cases} \tag{9.3a}$$

Substituting (9.1b-2) into (9.3) yields

$$A\cos\psi_2 + B\sin\psi_2 = C \tag{9.4}$$

$$\begin{cases} A = \mathbf{U}\cdot\mathbf{q}_1 \\ B = \mathbf{U}\cdot\mathbf{a}_2\times\mathbf{q}_1 \\ C = V \end{cases} \tag{9.4a}$$

Since ψ_2 is the auxiliary angle, the variable to be eliminated from the simultaneous equations, we transform (9.2) into the standard form (9.3), where only $\mathbf{q}_2$ is a function of ψ_2. Eq.(9.3) is a transition from (9.2) to (9.4).

Step 2. *Derivation of the second equation relating* θ_1, Θ_5 and ψ_2.

The scalar product of $\mathbf{a}_k \times \mathbf{a}_l$ with both sides of (9.1) yields

$$\mathbf{a}_k \times \mathbf{a}_l \cdot (\mathbf{K} + \mathbf{L} - \mathbf{F}) = 0 \tag{9.5}$$

Substituting the appropriate equations of (9.1a) and (9.1b) into (9.5) we can obtain

$$\mathbf{U}' \cdot \mathbf{q}_2 = V' \tag{9.6}$$

$$\begin{cases} \mathbf{U}' = \mathbf{U}'(\Theta_5) \\ V' = V'(\Theta_5) \end{cases} \tag{9.6a}$$

Substituting (9.1b-2) into (9.6) yields

$$A' \cos\psi_2 + B' \sin\psi_2 = C' \tag{9.7}$$

$$\begin{cases} A' = \mathbf{U}' \cdot \mathbf{q}_1 \\ B' = \mathbf{U}' \cdot \mathbf{a}_2 \times \mathbf{q}_1 \\ C' = V' \end{cases} \tag{9.7a}$$

Here the step from (9.5) to (9.6) may not seem clear, but for any given k and l this step can be easily accomplished. The expressions for $\mathbf{U}'$ and V' in (2.6a) and (9.7a) are also determined by k and l. Since $\{A, B, C\}$ of (9.4) and $\{A', B', C'\}$ of (9.7) are only functions of the input and output angles, by eliminating ψ_2 from (9.4) and (9.7) we can obtain the input-output displacement equations. The elimination procedure is shown in Section 2.6.4.

After the I/O displacement equation is obtained and the output angle is determined, finding the remaining angular variables and translational offsets becomes easy.

9.3. Analysis of the R_G-$CCRR$ mechanism

In this case, $k=2$ and $l=3$. This mechanism is shown in Fig. 9.2. The loop equation is

$$-x_2\mathbf{a}_2 - x_3\mathbf{a}_3 + \mathbf{K} + \mathbf{L} = \mathbf{F} \tag{9.8}$$

$$\begin{cases} \mathbf{L} = \mathbf{L}(\psi_2) = p_2\mathbf{q}_2 & (1) \\ \mathbf{K} = \mathbf{I} + \mathbf{J} & (2) \\ \mathbf{J} = \mathbf{J}(\Theta_5) = p_4\mathbf{q}_4 - S_4\mathbf{a}_4 & (3) \\ \mathbf{I} = \mathbf{I}(\theta_1) = p_1\mathbf{q}_1 + (p_5\mathbf{q}_5 - S_5\mathbf{a}_5 - S_1\mathbf{a}_1) & (4) \\ \mathbf{F} = -p_3\mathbf{q}_3 & (5) \end{cases} \tag{9.8a}$$

$$\begin{cases} \mathbf{a}_3 = c\alpha_{23}\mathbf{a}_2 + s\alpha_{23}\mathbf{q}_2\times\mathbf{a}_2 & (1) \\ \mathbf{q}_2 = c\psi_2\mathbf{q}_1 + s\psi_2\mathbf{a}_2\times\mathbf{q}_1 & (2) \\ \mathbf{a}_2 = c\alpha_{12}\mathbf{a}_1 + s\alpha_{12}\mathbf{q}_1\times\mathbf{a}_1 & (3) \\ \mathbf{q}_1 = c\theta_1\mathbf{q}_5 + s\theta_1\mathbf{a}_1\times\mathbf{q}_5 & (4) \\ \mathbf{q}_3 = \csc\alpha_{34}\mathbf{a}_3\times\mathbf{a}_4 & (5) \\ \mathbf{a}_4 = c\alpha_{45}\mathbf{a}_5 - s\alpha_{45}\mathbf{q}_4\times\mathbf{a}_5 & (6) \\ \mathbf{q}_4 = c\Theta_5\mathbf{q}_5 - s\Theta_5\mathbf{a}_5\times\mathbf{q}_5 & (7) \end{cases} \qquad (9.8b)$$

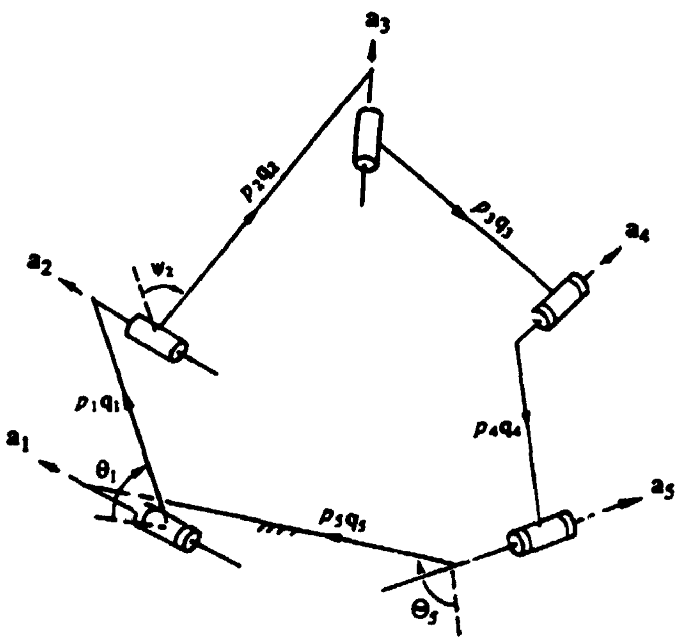


Fig. 9.2

Step 1. *Derivation of the first equation relating* θ_1, Θ_5 and ψ_2.

Equating $(\mathbf{a}_3 \cdot \mathbf{a}_4)$ for the fixed and floating parts of the vector loop, we get

$$\mathbf{a}_3(\psi_2)\cdot\mathbf{a}_4(\Theta_5) = c\alpha_{34} \qquad (9.9)$$

Substituting (9.8b-1) into (9.9) yields

$$\mathbf{U}\cdot\mathbf{q}_2 = V \qquad (9.10)$$

$$\begin{cases} \mathbf{U} = s\alpha_{23}\mathbf{a}_4\times\mathbf{a}_2 \\ V = c\alpha_{23}(\mathbf{a}_2\cdot\mathbf{a}_4) - c\alpha_{34} \end{cases} \qquad (9.10a)$$

Substituting (9.8b-2) into (9.10) yields

$$A\cos\psi_2 + B\sin\psi_2 = C \qquad (9.11)$$

$$\begin{cases} A = \mathbf{U}\cdot\mathbf{q}_1 \\ B = \mathbf{U}\cdot\mathbf{a}_2\times\mathbf{q}_1 \\ C = V \end{cases} \qquad (9.11a)$$

Step 2. *Derivation of the second equation relating* θ_1, Θ_5 and ψ_2.

The dot product of $\mathbf{a}_2 \times \mathbf{a}_3$ with both sides of (9.8) yields

$$\mathbf{a}_2 \times \mathbf{a}_3 \cdot (\mathbf{K} + p_2 \mathbf{q}_2) = \mathbf{a}_2 \times \mathbf{a}_3 \cdot (-p_3 \mathbf{q}_3) \tag{9.12}$$

i.e.

$$(\mathbf{K} \times \mathbf{a}_2) \cdot \mathbf{a}_3 + (s\alpha_{23} \mathbf{q}_2) \cdot (p_2 \mathbf{q}_2) = \mathbf{a}_2 \times \mathbf{a}_3 \cdot (-p_3 \csc\alpha_{34} \mathbf{a}_3 \times \mathbf{a}_4) \tag{9.13}$$

i.e.

$$(\mathbf{K} \times \mathbf{a}_2) \cdot \mathbf{a}_3 + p_2 s\alpha_{23} = p_3 \csc\alpha_{34} \mathbf{a}_2 \cdot \mathbf{a}_4 - p_3 c\alpha_{23} \cot\alpha_{34} \tag{9.14}$$

Substituting (9.8b-1) into (9.14) yields

$$\mathbf{U}' \cdot \mathbf{q}_2 = V' \tag{9.15}$$

$$\begin{cases} \mathbf{U}' = s\alpha_{23} \mathbf{K} \\ V' = p_3 \csc\alpha_{34} (\mathbf{a}_2 \cdot \mathbf{a}_4) - p_3 c\alpha_{23} \cot\alpha_{34} - p_2 s\alpha_{23} \end{cases} \tag{9.15a}$$

Substituting (9.8b-2) into (9.15) yields

$$A' \cos\psi_2 + B' \sin\psi_2 = C' \tag{9.16}$$

$$\begin{cases} A' = \mathbf{U}' \cdot \mathbf{q}_1 \\ B' = \mathbf{U}' \cdot \mathbf{a}_2 \times \mathbf{q}_1 \\ C' = V' \end{cases} \tag{9.16a}$$

Step 3. *Derivation of the I/O displacement equation.*

Expanding (9.11a) yields

$$\begin{cases} A = a_1 \cos\Theta_5 + a_2 \sin\Theta_5 + a_3 \\ B = b_1 \cos\Theta_5 + b_2 \sin\Theta_5 + b_3 \\ C = c_1 \cos\Theta_5 + c_2 \sin\Theta_5 + c_3 \end{cases} \tag{9.17}$$

where

$$\begin{cases} a_1 = \mathbf{D}(\mathbf{q}_1) \cdot \mathbf{q}_5 \\ a_2 = \mathbf{D}(\mathbf{q}_1) \cdot \mathbf{q}_5 \times \mathbf{a}_5 \\ a_3 = \mathbf{d} \cdot \mathbf{q}_1 \end{cases} \tag{9.17a}$$

$$\begin{cases} b_1 = \mathbf{D}(\mathbf{a}_2 \times \mathbf{q}_1) \cdot \mathbf{q}_5 \\ b_2 = \mathbf{D}(\mathbf{a}_2 \times \mathbf{q}_1) \cdot \mathbf{q}_5 \times \mathbf{a}_5 \\ b_3 = \mathbf{d} \cdot \mathbf{a}_2 \times \mathbf{q}_1 \end{cases} \tag{9.17b}$$

$$\begin{cases} c_1 = \mathbf{G} \cdot \mathbf{q}_5 \\ c_2 = \mathbf{G} \cdot \mathbf{q}_5 \times \mathbf{a}_5 \\ c_3 = g \end{cases} \tag{9.17c}$$

$$\begin{cases} \mathbf{D}(\Phi) = s\alpha_{23}s\alpha_{45}(\mathbf{a}_2\times\Phi)\times\mathbf{a}_5 \\ \mathbf{d} = s\alpha_{23}c\alpha_{45}(\mathbf{a}_5\times\mathbf{a}_2) \\ \mathbf{G} = c\alpha_{23}s\alpha_{45}(\mathbf{a}_2\times\mathbf{a}_5) \\ g = c\alpha_{23}c\alpha_{45}(\mathbf{a}_2\cdot\mathbf{a}_5) - c\alpha_{34} \end{cases} \tag{9.17d}$$

Expanding (9.16a) yields

$$\begin{cases} A' = a_1'\cos\Theta_5 + a_2'\sin\Theta_5 + a_3' \\ B' = b_1'\cos\Theta_5 + b_2'\sin\Theta_5 + b_3' \\ C' = c_1'\cos\Theta_5 + c_2'\sin\Theta_5 + c_3' \end{cases} \tag{9.18}$$

where

$$\begin{cases} a_1' = \mathbf{D}'(\mathbf{q}_1)\cdot\mathbf{q}_5 \\ a_2' = \mathbf{D}'(\mathbf{q}_1)\cdot\mathbf{q}_5\times\mathbf{a}_5 \\ a_3' = \mathbf{d}'\cdot\mathbf{q}_1 \end{cases} \tag{9.18a}$$

$$\begin{cases} b_1' = \mathbf{D}'(\mathbf{a}_2\times\mathbf{q}_1)\cdot\mathbf{q}_5 \\ b_2' = \mathbf{D}'(\mathbf{a}_2\times\mathbf{q}_1)\cdot\mathbf{q}_5\times\mathbf{a}_5 \\ b_3' = \mathbf{d}'\cdot\mathbf{a}_2\times\mathbf{q}_1 \end{cases} \tag{9.18b}$$

$$\begin{cases} c_1' = \mathbf{G}'\cdot\mathbf{q}_5 \\ c_2' = \mathbf{G}'\cdot\mathbf{q}_5\times\mathbf{a}_5 \\ c_3' = g' \end{cases} \tag{9.18c}$$

$$\begin{cases} \mathbf{D}'(\Phi) = p_4 s\alpha_{23}\Phi + S_4 s\alpha_{23}s\alpha_{45}(\mathbf{a}_5\times\Phi) \\ \mathbf{d}' = s\alpha_{23}\mathbf{I} - S_4 s\alpha_{23}c\alpha_{45}\mathbf{a}_5 \\ \mathbf{G}' = p_3\csc\alpha_{34}s\alpha_{45}(\mathbf{a}_2\times\mathbf{a}_5) \\ g' = p_3\csc\alpha_{34}c\alpha_{45}(\mathbf{a}_2\cdot\mathbf{a}_5) - p_3 c\alpha_{23}\cot\alpha_{34} - p_2 s\alpha_{23} \end{cases} \tag{9.18d}$$

Solving (9.11) and (9.16) we can obtain the I/O displacement equation $f(\Theta_1, \Theta_5)=0$ and the solution for $\{\Theta_5, \psi_2\}$, as shown from Eq.(2.30) to Eq.(2.35) in Section 2.6.4. Then the vectors $\{\mathbf{K}, \mathbf{L}, \mathbf{F}\}$ can be determined from (9.8a) and (9.8b).

Rearranging (9.8) yields

$$x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = (\mathbf{K} + \mathbf{L} - \mathbf{F}) \tag{9.19}$$

The scalar product of $\mathbf{q}_3$ and $\mathbf{q}_1$ with both sides of (9.19) yields

$$\begin{cases} x_2 = (\mathbf{K}+\mathbf{L}-\mathbf{F})\cdot\mathbf{q}_3 \,/\, (\mathbf{a}_2\cdot\mathbf{q}_3) \\ x_3 = (\mathbf{K}+\mathbf{L}-\mathbf{F})\cdot\mathbf{q}_1 \,/\, (\mathbf{a}_3\cdot\mathbf{q}_1) \end{cases} \tag{9.20}$$

From the structure of the mechanism (Fig. 9.2) we can obtain

$$\mathbf{q}_3 = \cos\theta_3\,\mathbf{q}_2 + \sin\theta_3\,\mathbf{a}_3\times\mathbf{q}_2 \tag{9.21}$$

$$\mathbf{q}_4 = \cos\theta_4\,\mathbf{q}_3 + \sin\theta_4\,\mathbf{a}_4\times\mathbf{q}_3 \tag{9.22}$$

Forming the scalar products of $\mathbf{q}_2$ and $(\mathbf{a}_3\times\mathbf{q}_2)$ with both sides of (9.21) we get θ_3:

$$\begin{cases} \cos\theta_3 = \mathbf{q}_2\cdot\mathbf{q}_3 \\ \sin\theta_3 = (\mathbf{a}_3\times\mathbf{q}_2)\cdot\mathbf{q}_3 \end{cases} \tag{9.23}$$

Forming the scalar products of $\mathbf{q}_3$ and $(\mathbf{a}_4\times\mathbf{q}_3)$ with both sides of (9.21) we get θ_4:

$$\begin{cases} \cos\theta_4 = \mathbf{q}_3\cdot\mathbf{q}_4 \\ \sin\theta_4 = (\mathbf{a}_4\times\mathbf{q}_3)\cdot\mathbf{q}_4 \end{cases} \tag{9.24}$$

The rationale for transforming (9.12) to (9.14) is as follows: (a) since we need an equation in the form $\mathbf{U}\cdot\mathbf{q}_2=V$, we write $\mathbf{a}_2\times\mathbf{a}_3\cdot\mathbf{K}=(\mathbf{K}\times\mathbf{a}_2)\cdot\mathbf{a}_3$, separating $\mathbf{a}_3$, a function of $\mathbf{q}_2$, from $(\mathbf{K}\times\mathbf{a}_2)$, a function of θ_1 and θ_5; (b) we can see from Fig. 9.2 that the relative directions of $\mathbf{a}_2$, $\mathbf{a}_3$ and $\mathbf{q}_2$ are fixed, thus the value of their scalar triple product is constant. It is necessary to be able to perceive right away that $\mathbf{a}_2\times\mathbf{a}_3=\mathbf{q}_2\sin\alpha_{23}$ (or see the first equation of Eq.(2.8)); (c) $\mathbf{q}_3$ is a function of $\{\theta_1, \psi_2, \theta_3\}$ or a function of $\{\theta_5, \theta_4\}$, however, $\mathbf{q}_3$ is also a function of $\{\mathbf{a}_3, \mathbf{a}_4\}$, i.e. if $\mathbf{a}_3$ and $\mathbf{a}_4$ are known, $\mathbf{q}_3$ can also be determined. Hence using $\mathbf{q}_3=\mathbf{a}_3\times\mathbf{a}_4\csc\alpha_{34}$, we can eliminate θ_3 and θ_4, for $\mathbf{a}_3$ is a function of $\{\theta_1, \psi_2\}$ and $\mathbf{a}_4$ a function of θ_5.

The vectorial expressions of (9.17a-c), (9.18a-c), (9.20), (9.23) and (9.24) can be easily transformed into scalar expressions. As an example, the scalarized (9.17a) is

$$\begin{cases} a_1 = s\alpha_{23}s\alpha_{45}(c\alpha_{51}c\alpha_{12}c\theta_1 - s\alpha_{51}s\alpha_{12}) \\ a_2 = -s\alpha_{23}s\alpha_{45}s\alpha_{12}s\theta_1 \\ a_3 = s\alpha_{23}c\alpha_{45}(s\alpha_{51}c\alpha_{12}c\theta_1 - c\alpha_{51}s\alpha_{12}) \end{cases} \tag{9.25}$$

where a_1 is derived as follows,

$$\begin{aligned} a_1 = \mathbf{D}(\mathbf{q}_1)\cdot\mathbf{q}_5 &= [s\alpha_{23}s\alpha_{45}(\mathbf{a}_2\times\mathbf{q}_1)\times\mathbf{a}_5]\cdot\mathbf{q}_5 \\ &= s\alpha_{23}s\alpha_{45}[(c\alpha_{12}\mathbf{a}_1 + s\alpha_{12}\mathbf{q}_1\times\mathbf{a}_1)\times\mathbf{q}_1]\times\mathbf{a}_5\cdot\mathbf{q}_5 \\ &= s\alpha_{23}s\alpha_{45}[(c\alpha_{12}\mathbf{a}_1\times\mathbf{q}_1 + s\alpha_{12}\mathbf{a}_1)\times\mathbf{a}_5]\cdot\mathbf{q}_5 \\ &= s\alpha_{23}s\alpha_{45}[c\alpha_{12}(\mathbf{a}_1\cdot\mathbf{a}_5)\mathbf{q}_1 - c\alpha_{12}(\mathbf{q}_1\cdot\mathbf{a}_5)\mathbf{a}_1 + s\alpha_{12}\mathbf{a}_1\times\mathbf{a}_5]\cdot\mathbf{q}_5 \\ &= s\alpha_{23}s\alpha_{45}[c\alpha_{12}c\alpha_{51}(\mathbf{q}_1\cdot\mathbf{q}_5) + s\alpha_{12}\mathbf{a}_1\times\mathbf{a}_5\cdot\mathbf{q}_5] \\ &= s\alpha_{23}s\alpha_{45}[c\alpha_{12}c\alpha_{51}c\theta_1 + s\alpha_{12}(-\mathbf{q}_5 s\alpha_{51})\cdot\mathbf{q}_5] \\ &= s\alpha_{23}s\alpha_{45}[c\alpha_{12}c\alpha_{51}c\theta_1 - s\alpha_{12}s\alpha_{51}] \end{aligned}$$

Here the procedure for deriving a_1 is displayed in detail. However, one can easily skip some of the steps. It is worth mentioning that the scalarizing procedure is quite flexible; for instance

$$[(\mathbf{a}_2\times\mathbf{q}_1)\times\mathbf{a}_5]\cdot\mathbf{q}_5=(\mathbf{a}_2\times\mathbf{q}_1)\cdot(\mathbf{a}_5\times\mathbf{q}_5)$$
$$=(\mathbf{a}_2\cdot\mathbf{a}_5)(\mathbf{q}_1\cdot\mathbf{q}_5)-(\mathbf{a}_2\cdot\mathbf{q}_5)(\mathbf{q}_1\cdot\mathbf{a}_5)$$
$$=(c\alpha_{51}c\alpha_{12}-s\alpha_{51}s\alpha_{12}c\theta_1)c\theta_1-(s\theta_1 s\alpha_{12})(s\alpha_{51}s\theta_1)$$
$$=(c\alpha_{51}c\alpha_{12}c\theta_1-s\alpha_{51}s\alpha_{12})$$

9.4. Analysis of the R_0-$RCCR$ mechanism

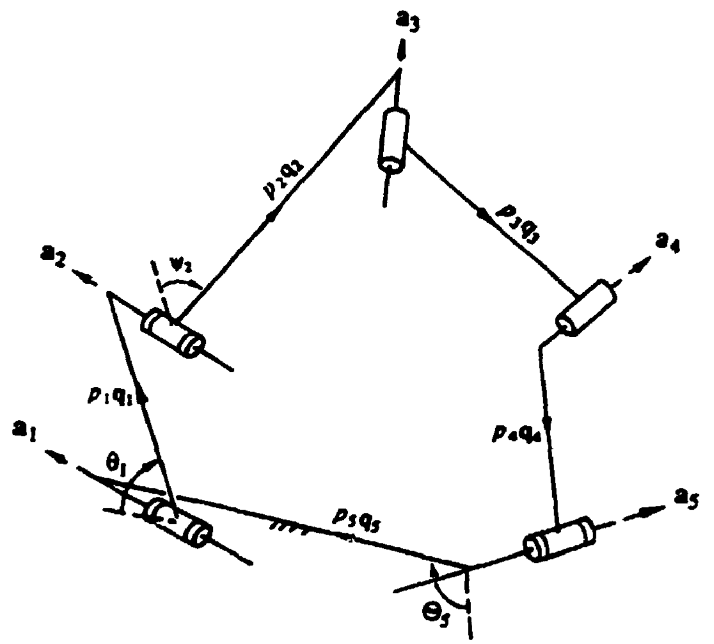


Fig. 9.3

In this case, $k=3$ and $l=4$. The diagram of this mechanism is shown in Fig. 9.3. The loop equation is

$$-x_3\mathbf{a}_3-x_4\mathbf{a}_4+\mathbf{K}+\mathbf{L}=\mathbf{F} \tag{9.26}$$

$$\begin{cases}\mathbf{L}=p_2\mathbf{q}_2 & (1)\\ \mathbf{K}=\mathbf{I}+\mathbf{J} & (2)\\ \mathbf{J}=p_4\mathbf{q}_4 & (3)\\ \mathbf{I}=(p_1\mathbf{q}_1-S_2\mathbf{a}_2) & \\ \quad+(p_5\mathbf{q}_5-S_5\mathbf{a}_5-S_1\mathbf{a}_1) & (4)\\ \mathbf{F}=-p_3\mathbf{q}_3 & (5)\end{cases} \tag{9.26a}$$

$$\begin{cases}\mathbf{a}_3=c\alpha_{23}\mathbf{a}_2+s\alpha_{23}\mathbf{q}_2\times\mathbf{a}_2 & (1)\\ \mathbf{q}_2=c\psi_2\mathbf{q}_1+s\psi_2\mathbf{a}_2\times\mathbf{q}_1 & (2)\\ \mathbf{q}_3=\csc\alpha_{34}\mathbf{a}_3\times\mathbf{a}_4 & (3)\\ \mathbf{q}_4=c\Theta_5\mathbf{q}_5-s\Theta_5\mathbf{a}_5\times\mathbf{q}_5 & (4)\end{cases} \tag{9.26b}$$

The first equation, relating θ_1, Θ_5 and ψ_2, is exactly the same as (9.9) to (9.11a).

Step 2. *Derivation of the second equation relating* θ_1, Θ_5 and ψ_2.

The scalar product of $\mathbf{a}_3\times\mathbf{a}_4$ with both sides of (9.26) yields

$$(\mathbf{K}+\mathbf{L})\cdot\mathbf{a}_3\times\mathbf{a}_4 = \mathbf{F}\cdot\mathbf{a}_3\times\mathbf{a}_4 \tag{9.27}$$

Substituting (9.26a-1) and (9.26b-1) into (9.27) yields

$$\mathbf{U}'\cdot\mathbf{q}_2 = V' \tag{9.28}$$

$$\begin{cases} \mathbf{U}' = s\alpha_{23}\,\mathbf{a}_2\times(\mathbf{a}_4\times\mathbf{K}) + p_2 c\alpha_{23}(\mathbf{a}_2\times\mathbf{a}_4) \\ V' = c\alpha_{23}\,\mathbf{a}_2\cdot\mathbf{a}_4\times\mathbf{K} - p_2 s\alpha_{23}(\mathbf{a}_2\cdot\mathbf{a}_4) + p_3 s\alpha_{34} \end{cases} \tag{9.28a}$$

Substituting (9.26b-2) into (9.28) yields

$$A'\cos\psi_2 + B'\sin\psi_2 = C' \tag{9.29}$$

$$\begin{cases} A' = \mathbf{U}'\cdot\mathbf{q}_1 \\ B' = \mathbf{U}'\cdot\mathbf{a}_2\times\mathbf{q}_1 \\ C' = V' \end{cases} \tag{9.29a}$$

The remaining derivation is identical to the one given in section 9.3.

9.5. Analysis of the R_0-$RRCC$ mechanism

In this case, $k=4$ and $l=5$. The diagram of this mechanism is shown in Fig. 9.4. The loop equation is

$$-x_4\mathbf{a}_4 - x_5\mathbf{a}_5 + \mathbf{K} + \mathbf{L} = \mathbf{F} \tag{9.30}$$

$$\begin{cases} \mathbf{L} = p_2\mathbf{q}_2 - S_3\mathbf{a}_3 & (1) \\ \mathbf{K} = \mathbf{I} + \mathbf{J} & (2) \\ \mathbf{J} = p_4\mathbf{q}_4 & (3) \\ \mathbf{I} = (p_1\mathbf{q}_1 - S_2\mathbf{a}_2) + (p_5\mathbf{q}_5 - S_1\mathbf{a}_1) & (4) \\ \mathbf{F} = -p_3\mathbf{q}_3 & (5) \end{cases} \tag{9.30a}$$

$$\begin{cases} \mathbf{a}_3 = c\alpha_{23}\mathbf{a}_2 + s\alpha_{23}\mathbf{q}_2\times\mathbf{a}_2 & (1) \\ \mathbf{q}_2 = c\psi_2\mathbf{q}_1 + s\psi_2\mathbf{a}_2\times\mathbf{q}_1 & (2) \\ \mathbf{q}_3 = \csc\alpha_{34}\,\mathbf{a}_3\times\mathbf{a}_4 & (3) \\ \mathbf{a}_4 = c\alpha_{45}\mathbf{a}_5 - s\alpha_{45}\mathbf{q}_4\times\mathbf{a}_5 & (4) \\ \mathbf{q}_4 = c\Theta_5\mathbf{q}_5 - s\Theta_5\mathbf{a}_5\times\mathbf{q}_5 & (5) \end{cases} \tag{9.30b}$$

The first equation, relating θ_1, Θ_5 and ψ_2, is exactly the same as the results derived and presented from (9.9) to (9.11a).

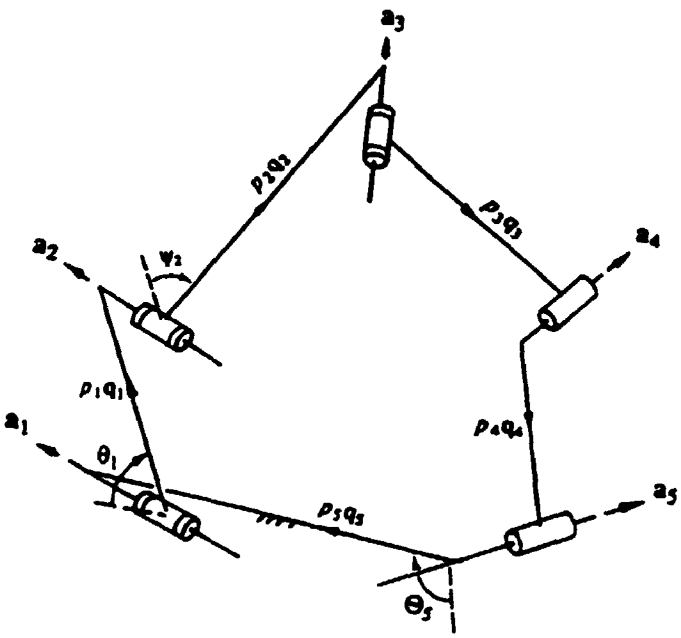


Fig. 9.4

Step 2. *Derivation of the second equation relating* θ_1, θ_5 and ψ_2.

The scalar product of $\mathbf{a}_4 \times \mathbf{a}_5$ with both sides of (9.30) yields

$$\mathbf{a}_4 \times \mathbf{a}_5 \cdot (\mathbf{K} + p_2 \mathbf{q}_2 - S_3 \mathbf{a}_3) = \mathbf{a}_4 \times \mathbf{a}_5 \cdot (-p_3 \mathbf{q}_3) \tag{9.31}$$

Substituting (9.30b-1) and (9.30b-3) into (9.31) yields

$$\mathbf{U}' \cdot \mathbf{q}_2 = V' \tag{9.32}$$

$$\begin{cases} \mathbf{U}' = p_2 \mathbf{a}_4 \times \mathbf{a}_5 - S_3 s\alpha_{23} \mathbf{a}_2 \times (\mathbf{a}_4 \times \mathbf{a}_5) - p_3 s\alpha_{23} csc\alpha_{34} \mathbf{a}_2 \times \mathbf{a}_5 \\ V' = \mathbf{K} \cdot \mathbf{a}_4 \times \mathbf{a}_5 + S_3 c\alpha_{23} \mathbf{a}_2 \cdot (\mathbf{a}_4 \times \mathbf{a}_5) \\ \qquad + p_3 c\alpha_{23} csc\alpha_{34} \mathbf{a}_2 \cdot \mathbf{a}_5 - p_2 cot\,\alpha_{34} c\alpha_{45} \end{cases} \tag{9.32a}$$

Substituting (9.30b-2) into (9.32) yields

$$A' \cos\psi_2 + B' \sin\psi_2 = C' \tag{9.33}$$

$$\begin{cases} A' = \mathbf{U}' \cdot \mathbf{q}_1 \\ B' = \mathbf{U}' \cdot \mathbf{a}_2 \times \mathbf{q}_1 \\ C' = V' \end{cases} \tag{9.33a}$$

The remaining derivation is identical to the one presented in section 9.3.

9.6. Analysis of the R_0–$CRRC$ mechanism

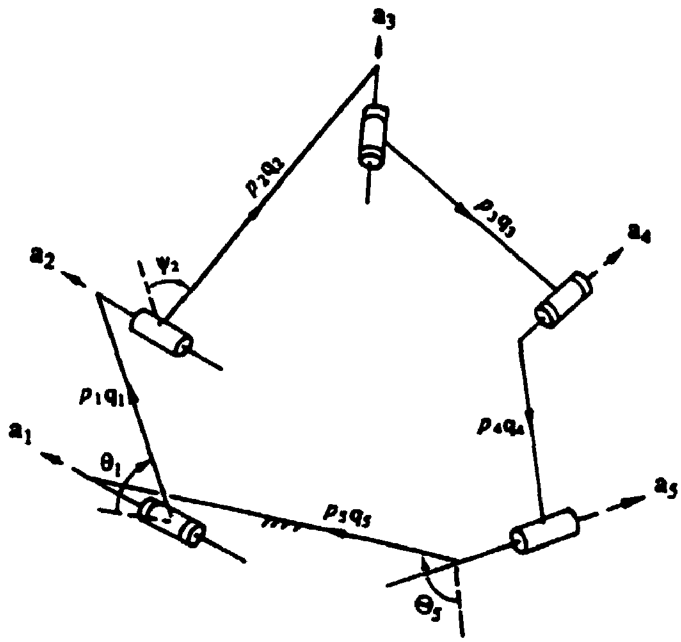


Fig. 9.5

In this case, $k=2$ and $l=5$. The diagram of this mechanism is shown in Fig. 9.5. The loop equation is

$$-x_2\mathbf{a}_2 - x_5\mathbf{a}_5 + \mathbf{K} + \mathbf{L} = \mathbf{F} \tag{9.34}$$

$$\begin{cases} \mathbf{L} = p_2\mathbf{q}_2 - S_3\mathbf{a}_3 & (1) \\ \mathbf{K} = \mathbf{I} + \mathbf{J} & (2) \\ \mathbf{J} = p_4\mathbf{q}_4 - S_4\mathbf{a}_4 & (3) \\ \mathbf{I} = p_1\mathbf{q}_1 + (p_5\mathbf{q}_5 - S_1\mathbf{a}_1) & (4) \\ \mathbf{F} = -p_3\mathbf{q}_3 & (5) \end{cases} \tag{9.34a}$$

$$\begin{cases} \mathbf{a}_3 = c\alpha_{23}\mathbf{a}_2 + s\alpha_{23}\mathbf{q}_2\times\mathbf{a}_2 & (1) \\ \mathbf{q}_2 = c\psi_2\mathbf{q}_1 + s\psi_2\mathbf{a}_2\times\mathbf{q}_1 & (2) \\ \mathbf{q}_3 = csc\alpha_{34}\mathbf{a}_3\times\mathbf{a}_4 & (3) \\ \mathbf{q}_4 = c\Theta_5\mathbf{q}_5 - s\Theta_5\mathbf{a}_5\times\mathbf{q}_5 & (4) \end{cases} \tag{9.34b}$$

The first equation, relating θ_1, Θ_5 and ψ_2, is the same as the results derived and presented from (9.7) to (9.11a).

Step 2. *Derivation of the second equation relating* θ_1, Θ_5 and ψ_2.

The scalar product of $\mathbf{a}_2\times\mathbf{a}_5$ with both sides of (9.34) yields

$$(\mathbf{K} + \mathbf{L})\cdot\mathbf{a}_2\times\mathbf{a}_5 = \mathbf{F}\cdot\mathbf{a}_2\times\mathbf{a}_5 \tag{9.35}$$

Substituting (9.34a-1) and (9.34b-1) into (9.35) yields

$$\mathbf{U}'\cdot\mathbf{q}_2 = V' \tag{9.36}$$

$$\begin{cases} \mathbf{U}' = [p_3 s\alpha_{23} \csc\alpha_{34} (\mathbf{a}_2 \cdot \mathbf{r}_4) + p_2](\mathbf{a}_2 \times \mathbf{a}_5) \\ \qquad + S_3 s\alpha_{23} (\mathbf{a}_2 \times \mathbf{a}_5) \times \mathbf{a}_2 \\ V' = \{p_3 c\alpha_{23} \csc\alpha_{34} (\mathbf{a}_2 \times \mathbf{a}_4) - \mathbf{K}\} \cdot (\mathbf{a}_2 \times \mathbf{a}_5) \end{cases} \tag{9.36a}$$

Substituting (9.34b-2) into (9.36) yields

$$A' \cos\psi_2 + B' \sin\psi_2 = C' \tag{9.37}$$

$$\begin{cases} A' = \mathbf{U}' \cdot \mathbf{q}_1 \\ B' = \mathbf{U}' \cdot \mathbf{a}_2 \times \mathbf{q}_1 \\ C' = V' \end{cases} \tag{9.37a}$$

The remaining derivation is identical to the one given in section 9.3.

9.7. Analysis of the R_0–$CRCR$ mechanism

In this case, $k=2$ and $l=4$. The diagram of this mechanism is shown in Fig. 9.6. The loop equation is

$$-x_2 \mathbf{a}_2 - x_4 \mathbf{a}_4 + \mathbf{K} + \mathbf{L} = \mathbf{F} \tag{9.38}$$

$$\begin{cases} \mathbf{L} = p_2 \mathbf{q}_2 - S_3 \mathbf{a}_3 & (1) \\ \mathbf{K} = \mathbf{I} + \mathbf{J} & (2) \\ \mathbf{J} = p_4 \mathbf{q}_4 & (3) \\ \mathbf{I} = p_1 \mathbf{q}_1 + (p_5 \mathbf{q}_5 - S_5 \mathbf{a}_5 - S_1 \mathbf{a}_1) & (4) \\ \mathbf{F} = -p_3 \mathbf{q}_3 & (5) \end{cases} \tag{9.38a}$$

$$\begin{cases} \mathbf{a}_3 = c\alpha_{23} \mathbf{a}_2 + s\alpha_{23} \mathbf{q}_2 \times \mathbf{a}_2 & (1) \\ \mathbf{q}_2 = c\psi_3 \mathbf{q}_2 + s\psi_3 \mathbf{a}_2 \times \mathbf{q}_1 & (2) \\ \mathbf{q}_3 = \csc\alpha_{34} \mathbf{a}_3 \times \mathbf{a}_4 & (3) \\ \mathbf{a}_4 = c\alpha_{45} \mathbf{a}_5 - s\alpha_{45} \mathbf{q}_4 \times \mathbf{a}_5 & (4) \\ \mathbf{q}_4 = c\Theta_5 \mathbf{q}_5 - s\Theta_5 \mathbf{a}_5 \times \mathbf{q}_5 & (5) \end{cases} \tag{9.38b}$$

Step 1. *Derivation of the first equation relating* θ_1, Θ_5 and ψ_2.

The derivation and results are exactly the same as from (9.7) to (9.11a).

Step 2. *Derivation of the second equation relating* θ_1, Θ_5 and ψ_2.

The scalar product of $\mathbf{a}_2 \times \mathbf{a}_4$ with both sides of (9.38) yields

$$\mathbf{a}_2 \times \mathbf{a}_4 \cdot [\mathbf{K} + (p_2 \mathbf{q}_2 - S_3 \mathbf{a}_3)] = \mathbf{a}_2 \times \mathbf{a}_4 \cdot (-p_3 \csc\alpha_{34} \mathbf{a}_3 \times \mathbf{a}_4) \tag{9.39}$$

Substituting (9.38b-1) into (9.39) yields

$$\mathbf{U}' \cdot \mathbf{q}_2 = V' \tag{9.40}$$

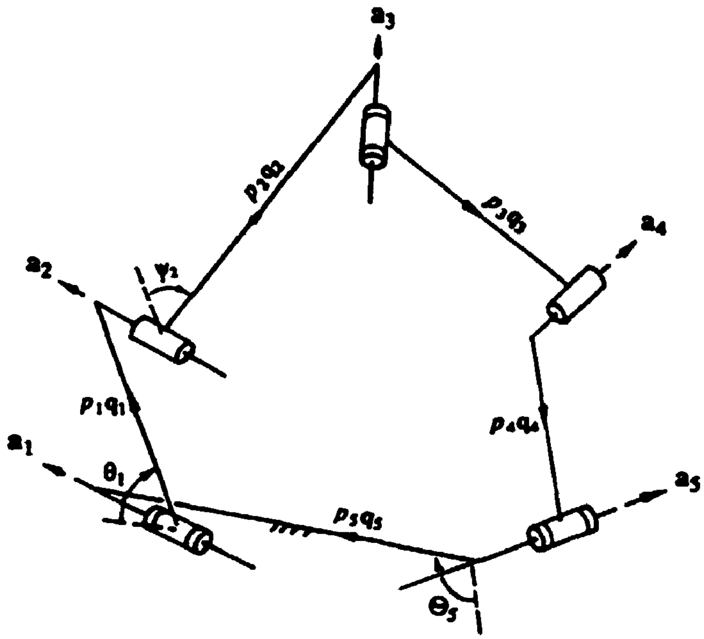


Fig. 9.6

$$\begin{cases} \mathbf{U}' = p_2\mathbf{a}_2\times\mathbf{a}_4 + S_3 s\alpha_{23}(\mathbf{a}_2\times\mathbf{a}_4)\times\mathbf{a}_2 \\ V' = p_3\csc\alpha_{34}[c\alpha_{34}(\mathbf{a}_2\cdot\mathbf{a}_4) - c\alpha_{23}] - \mathbf{a}_2\times\mathbf{a}_4\cdot\mathbf{K} \end{cases} \tag{9.40a}$$

Substituting (9.38b-2) into (9.40) yields

$$A'\cos\psi_2 + B'\sin\psi_2 = C' \tag{9.41}$$

$$\begin{cases} A' = \mathbf{U}'\cdot\mathbf{q}_1 \\ B' = \mathbf{U}'\cdot\mathbf{a}_2\times\mathbf{q}_1 \\ C' = V' \end{cases} \tag{9.41a}$$

Step 3. *Derivation of the I/O displacement equation.*

From (9.11) and (9.41) we can obtain

$$\begin{cases} \cos\psi_2 = -Q_2/Q_3 \\ \sin\psi_2 = Q_1/Q_3 \end{cases} \tag{9.42}$$

$$\begin{cases} Q_1 = (AC' - CA') \\ Q_2 = (BC' - CB') \\ Q_3 = (AB' - BA') \end{cases} \tag{9.42a}$$

Considering $\cos^2\psi_2 + \sin^2\psi_2 = 1$ and (9.42) yields

$$Q_1^2 + Q_2^2 = Q_3^2 \tag{9.43}$$

The following derivation is different from the corresponding one given in previous sections; however, the basic ideas are the same.

Substituting (9.11a) and (9.41a) into (9.42a) we get

$$\begin{cases} Q_1 = (V'\mathbf{U} - V\mathbf{U}')\cdot\mathbf{q}_1 \\ Q_2 = (V'\mathbf{U} - V\mathbf{U}')\cdot\mathbf{a}_2\times\mathbf{q}_1 \\ Q_3 = (\mathbf{U}\times\mathbf{U}')\cdot\mathbf{a}_2 \end{cases} \tag{9.43a}$$

Substituting (9.43a) into (9.43), and taking into account of the second equation of Eq.(2.10), we obtain

$$(V'\mathbf{U} - V\mathbf{U}')^2 - [(V'\mathbf{U} - V\mathbf{U}')\cdot\mathbf{a}_2]^2 = (\mathbf{U}\times\mathbf{U}'\cdot\mathbf{a}_2)^2 \tag{9.44}$$

Checking the expressions for $\mathbf{U}$ and $\mathbf{U}'$ in (9.10a) and (9.40a), respectively, we can see that both $\mathbf{U}$ and $\mathbf{U}'$ are perpendicular to $\mathbf{a}_2$, thus (9.44) becomes

$$(V'\mathbf{U} - V\mathbf{U}')^2 = (\mathbf{U}\times\mathbf{U}'\cdot\mathbf{a}_2)^2 \tag{9.45}$$

Expanding the left side of (9.45) yields

$$V'^2\mathbf{U}^2 + V^2\mathbf{U}'^2 - 2V'V(\mathbf{U}\cdot\mathbf{U}') = (\mathbf{U}\times\mathbf{U}'\cdot\mathbf{a}_2)^2 \tag{9.46}$$

Substituting (9.10a) and (9.40a) into (9.46), yields

$$(s\alpha_{23}V' - p_2V)^2 + (S_3 s\alpha_{23}V)^2 - S_3^2 s^4\alpha_{23} = 0 \tag{9.47}$$

where

$$\begin{cases} (s\alpha_{23}V' - p_2V) = \mathbf{W}_1\cdot\mathbf{q}_4 + Z_1 \\ (S_3 s\alpha_{23}V) = \mathbf{W}_2\cdot\mathbf{q}_4 + Z_2 \end{cases} \tag{9.47a}$$

$$\begin{cases} \mathbf{W}_1 = s\alpha_{45}(\rho_1\mathbf{a}_2 + s\alpha_{23}\mathbf{a}_2\times\mathbf{I})\times\mathbf{a}_5 \\ Z_1 = c\alpha_{45}(\rho_1\mathbf{a}_2 + s\alpha_{23}\mathbf{a}_2\times\mathbf{I})\cdot\mathbf{a}_5 + \rho_2 \\ \mathbf{W}_2 = S_3 c\alpha_{23} s\alpha_{23} s\alpha_{45}\mathbf{a}_2\times\mathbf{a}_5 \\ Z_2 = S_3 c\alpha_{23} s\alpha_{23} c\alpha_{45}(\mathbf{a}_2\cdot\mathbf{a}_5) - S_3 s\alpha_{23} c\alpha_{34} \end{cases} \tag{9.47b}$$

$$\begin{cases} \rho_1 = p_3 s\alpha_{23} \cot\alpha_{34} - p_4 s\alpha_{23} \cot\alpha_{45} \\ \rho_2 = p_2 c\alpha_{34} - p_3 c\alpha_{23} s\alpha_{23} \csc\alpha_{34} - p_4 s\alpha_{23} \csc\alpha_{45}(\mathbf{a}_2\cdot\mathbf{a}_5) \end{cases} \tag{9.47c}$$

Substituting (9.47a) into (9.47) yields

$$\begin{aligned} &(\mathbf{W}_1\cdot\mathbf{q}_4)^2 + (\mathbf{W}_2\cdot\mathbf{q}_4)^2 + 2Z_1(\mathbf{W}_1\cdot\mathbf{q}_4) + 2Z_2(\mathbf{W}_2\cdot\mathbf{q}_4) \\ &\quad + Z_1^2 + Z_2^2 - S_3^2 s^4\alpha_{23} = 0 \end{aligned} \tag{9.48}$$

where only $\mathbf{q}_4$ is a function of the output angle Θ_5. Substituting (9.38b-5) into (9.48) yields

$$\mu_1 c^2\Theta_5 + \mu_2 s^2\Theta_5 + \mu_3 c\Theta_5 s\Theta_5 + \mu_4 c\Theta_5 + \mu_5 s\Theta_5 + \mu_6 = 0 \tag{9.49}$$

$$\begin{cases} \mu_1 = (\mathbf{W}_1 \cdot \mathbf{q}_5)^2 + (\mathbf{W}_2 \cdot \mathbf{q}_5)^2 \\ \mu_2 = (\mathbf{W}_1 \cdot \mathbf{q}_5 \times \mathbf{a}_5)^2 + (\mathbf{W}_2 \cdot \mathbf{q}_5 \times \mathbf{a}_5)^2 \\ \mu_3 = 2(\mathbf{W}_1 \cdot \mathbf{q}_5)(\mathbf{W}_1 \cdot \mathbf{q}_5 \times \mathbf{a}_5) + 2(\mathbf{W}_2 \cdot \mathbf{q}_5)(\mathbf{W}_2 \cdot \mathbf{q}_5 \times \mathbf{a}_5) \\ \mu_4 = 2Z_1(\mathbf{W}_1 \cdot \mathbf{q}_5) + 2Z_2(\mathbf{W}_2 \cdot \mathbf{q}_5) \\ \mu_5 = 2Z_1(\mathbf{W}_1 \cdot \mathbf{q}_5 \times \mathbf{a}_5) + 2Z_2(\mathbf{W}_2 \cdot \mathbf{q}_5 \times \mathbf{a}_5) \\ \mu_6 = Z_1^2 + Z_2^2 - S_3^2 s^4 \alpha_{23} \end{cases} \tag{9.49a}$$

Let $y = \tan(\Theta_5/2)$, then from (9.49)

$$\sum_{i=0}^{4} v_i y^{4-i} = 0 \tag{9.50}$$

$$\begin{cases} v_0 = \mu_1 \\ v_1 = -2\mu_3 \\ v_2 = -2\mu_1 + 4\mu_2 - \mu_4 + \mu_6 \\ v_3 = 2\mu_3 + 2\mu_5 \\ v_4 = \mu_1 + \mu_4 + \mu_6 \end{cases} \tag{9.50a}$$

Solving (9.50) we obtain y, and then $\Theta_5 = 2\tan^{-1} y$.

9.8. Analysis of the R_0-$CRCR$ mechanism

This mechanism has been analyzed in section 9.7. In the following we will present a slightly different but simpler approach. The diagram of this mechanism is shown in Fig. 9.7. In this case let $\theta_3 = \psi_3$ be the auxiliary angle, the angle to be eliminated from the simultaneous equations. Thus the loop equation can be written as

$$-x_2 \mathbf{a}_2 - x_4 \mathbf{a}_4 + \mathbf{K} = \mathbf{F} \tag{9.51}$$

$$\begin{cases} \mathbf{K} = \mathbf{I} + \mathbf{J} & (1) \\ \mathbf{J} = p_4 \mathbf{q}_4 & (2) \\ \mathbf{I} = p_1 \mathbf{q}_1 + (p_5 \mathbf{q}_5 - S_5 \mathbf{a}_5 - S_1 \mathbf{a}_1) & (3) \\ \mathbf{F} = -p_2 \mathbf{q}_2 + S_3 \mathbf{a}_3 - p_3 \mathbf{q}_3 & (4) \end{cases} \tag{9.51a}$$

$$\begin{cases} \mathbf{a}_2 = c\alpha_{23} \mathbf{a}_3 - s\alpha_{23} \mathbf{q}_2 \times \mathbf{a}_3 & (1) \\ \mathbf{q}_2 = c\psi_3 \mathbf{q}_3 - s\psi_3 \mathbf{a}_3 \times \mathbf{q}_3 & (2) \\ \mathbf{a}_3 = c\alpha_{34} \mathbf{a}_4 - s\alpha_{34} \mathbf{q}_3 \times \mathbf{a}_4 & (3) \\ \mathbf{a}_4 = c\alpha_{45} \mathbf{a}_5 - s\alpha_{45} \mathbf{q}_4 \times \mathbf{a}_5 & (4) \\ \mathbf{q}_4 = c\Theta_5 \mathbf{q}_5 - s\Theta_5 \mathbf{a}_5 \times \mathbf{q}_5 & (5) \end{cases} \tag{9.51b}$$

Step 1. *Derivation of the first equation relating* θ_1, Θ_5 and ψ_3.

Equating $(\mathbf{a}_2 \cdot \mathbf{a}_4)$ for the floating and the fixed parts of the vector loop yields

$$(\mathbf{a}_2 \cdot \mathbf{a}_4)|_\psi = (\mathbf{a}_2 \cdot \mathbf{a}_4)|_\Theta \tag{9.52}$$

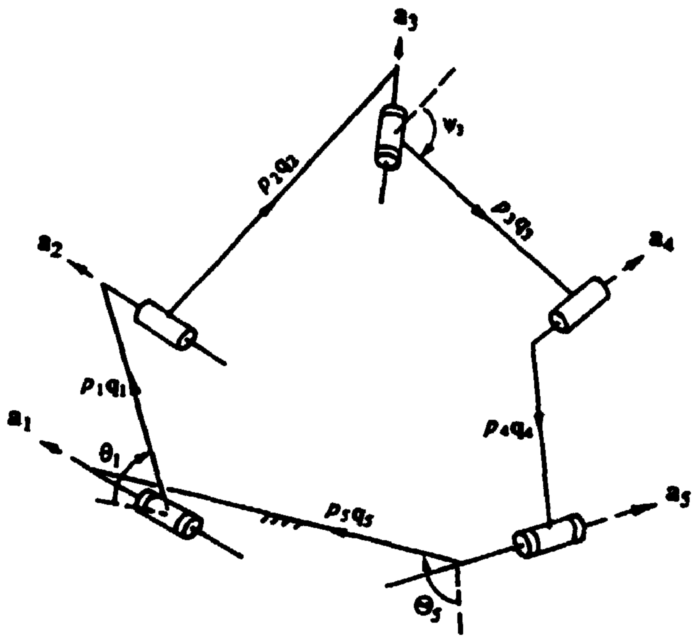


Fig. 9.7

Substituting (9.51b-2) and (9.51b-3) into (9.51b-1), then substituting (9.51b-1) into the left side of (9.52); at the same time, substituting (9.51b-4) into the right side of (9.52), we get

$$c\alpha_{23}c\alpha_{34} - s\alpha_{23}s\alpha_{34}c\psi_3 = \mathbf{a}_2\cdot(c\alpha_{45}\mathbf{a}_5 - s\alpha_{45}\mathbf{q}_4\times\mathbf{a}_5) \tag{9.53}$$

Rearranging (9.53) yields

$$\cos\psi_3 = \mathbf{U}_3\cdot\mathbf{q}_4 + V_3 \tag{9.54}$$

$$\begin{cases} \mathbf{U}_3 = \csc\alpha_{23}\csc\alpha_{34}s\alpha_{45}(\mathbf{a}_5\times\mathbf{a}_2) \\ V_3 = \csc\alpha_{23}\csc\alpha_{34}[c\alpha_{23}c\alpha_{34} - c\alpha_{45}(\mathbf{a}_5\cdot\mathbf{a}_2)] \end{cases} \tag{9.54a}$$

Step 2. *Derivation of the second equation relating* θ_1, θ_5 and ψ_3.

The scalar product of $\mathbf{a}_2\times\mathbf{a}_4$ with both sides of (9.51) yields $\mathbf{a}_2\times\mathbf{a}_4\cdot(\mathbf{K}-\mathbf{F})=0$, i.e.

$$\mathbf{a}_2\times\mathbf{a}_4\cdot(\mathbf{I}+p_4\mathbf{q}_4+p_2\mathbf{q}_2-S_3\mathbf{a}_3+p_3\mathbf{q}_3)=0 \tag{9.55}$$

i.e.

$$\mathbf{a}_2\times\mathbf{a}_4\cdot(\mathbf{I}+p_4\mathbf{q}_4+p_2\csc\alpha_{23}\mathbf{a}_2\times\mathbf{a}_3-S_3\mathbf{a}_3+p_3\csc\alpha_{34}\mathbf{a}_3\times\mathbf{a}_4)=0 \tag{9.56}$$

i.e.

$$\begin{aligned}(\mathbf{I}\times\mathbf{a}_2)\cdot\mathbf{a}_4+(p_4\mathbf{a}_2)\cdot\mathbf{a}_4\times\mathbf{q}_4+p_2\csc\alpha_{23}(c\alpha_{34}-c\alpha_{23}[\mathbf{a}_2\cdot\mathbf{a}_4]) \\ +S_3s\alpha_{23}s\alpha_{34}s\psi_3+p_3\csc\alpha_{34}(c\alpha_{23}-c\alpha_{34}[\mathbf{a}_2\cdot\mathbf{a}_4])=0\end{aligned} \tag{9.57}$$

Rearranging (9.57) yields

$$S_3 s\alpha_{23} s\alpha_{34} s\psi_3 = \{(p_2 \cot\alpha_{23} + p_3 \cot\alpha_{34})\mathbf{a}_2 - \mathbf{l}\times\mathbf{a}_2\}\cdot\mathbf{a}_4$$

$$- (p_4\mathbf{a}_2)\cdot\mathbf{a}_4\times\mathbf{q}_4 - (p_2 \csc\alpha_{23} c\alpha_{34} + p_3 c\alpha_{23} \csc\alpha_{34}) \qquad (9.58)$$

Substituting (9.51b-4) into (9.58) yields

$$\sin\psi_3 = \mathbf{U}_3'\cdot\mathbf{q}_4 + V_3' \qquad (9.59)$$

$$\begin{cases} \mathbf{U}_3' = (s\alpha_{45}\mathbf{N} - p_2 c\alpha_{45}\mathbf{a}_2)\times\mathbf{a}_5 \\ V_3' = (c\alpha_{45}\mathbf{N} + p_2 s\alpha_{45}\mathbf{a}_2)\cdot\mathbf{a}_5 \\ \qquad -\csc\alpha_{23}(p_2 c\alpha_{34} + p_3 \csc\alpha_{34}) \\ \mathbf{N} = (p_2 \cot\alpha_{23} + p_3 \cot\alpha_{34})\mathbf{a}_2 - \mathbf{l}\times\mathbf{a}_2 \end{cases} \qquad (9.59a)$$

Step 3. *Derivation of the I/O displacement equation.*

Substituting (9.51b-5) into (9.54) yields

$$\cos\psi_3 = c_1 \cos\Theta_5 + c_2 \sin\Theta_5 + c_3 \qquad (9.60)$$

$$\begin{cases} c_1 = \mathbf{U}_3\cdot\mathbf{q}_5 \\ c_2 = \mathbf{U}_3\cdot\mathbf{q}_5\times\mathbf{a}_5 \\ c_3 = V_3 \end{cases} \qquad (9.60a)$$

Substituting (9.51b-5) into (9.59) yields

$$\sin\psi_3 = c_1' \cos\Theta_5 + c_2' \sin\Theta_5 + c_3' \qquad (9.61)$$

$$\begin{cases} c_1' = \mathbf{U}_3'\cdot\mathbf{q}_5 \\ c_2' = \mathbf{U}_3'\cdot\mathbf{q}_5\times\mathbf{a}_5 \\ c_3' = V_3' \end{cases} \qquad (9.61a)$$

From (9.60), (9.61) and $\cos^2\psi_3+\sin^2\psi_3=1$ we can get

$$\mu_1 c^2\Theta_5 + \mu_2 s^2\Theta_5 + \mu_3 c\Theta_5 s\Theta_5 + \mu_4 c\Theta_5 + \mu_5 s\Theta_5 + \mu_6 = 0 \qquad (9.62)$$

$$\begin{cases} \mu_1 = c_1^2 + c_1'^2 \\ \mu_2 = c_2^2 + c_2'^2 \\ \mu_3 = 2(c_1c_2 + c_1'c_2') \\ \mu_4 = 2(c_1c_3 + c_1'c_3') \\ \mu_5 = 2(c_2c_3 + c_2'c_3') \\ \mu_6 = c_3^2 + c_3'^2 - 1 \end{cases} \qquad (9.62a)$$

Let $y=\tan(\Theta_5/2)$, then from (9.62) we can obtain a fourth order polynomial equation which is exactly the same as (9.63) and (9.63a).

9.9. Analysis of the R_0-$RCRC$ mechanism

The analysis for this mechanism are similar to that for mechanism R_0-$CRCR$, as shown in sections 9.7 and 9.8.

9.10. Conclusion

Of all methods mentioned at the beginning of this chapter, the matrix method is the most popular one. In fact the methods used in references [84], [66], [50], [106], [59], [60], [43] and [44] can all be considered as matrix methods, although there are some differences between them. The matrix method and the spherical trigonometry based method [19-21, 106] are generally acknowledged to be the most efficient methods.

Using the matrix method, one usually performs a series of matrix multiplications (or matrix homogeneous transformations) and obtains a relation of the form

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{9.63}$$

i.e.

$$a_{ij} = b_{ij} \qquad (i=1\text{-}3,\ j=1\text{-}4) \tag{9.64}$$

For the R_C-$2R$-$2C$ mechanism, each of the 12 equations in (9.63) contains at least two unknown angular or translational variables, excluding the input angle. One [84, 66] to six [43] of the 12 equations are then actually used for deriving the I/O displacement equations. The calculation for the other six to eleven equations in (9.63) constitutes unavoidable extraneous work. However, using the approach presented in this chapter, one can obtain exactly what is needed directly, without anything extraneous attached to the procedure.

The matrix, in fact, is an "expanded" form of vectors. A $(k \times k)$ matrix can be regarded as being composed of k "expanded" vectors, or k vectors of k dimensions, namely, k row vectors: $r_i = (r_{i1}\ r_{i2}\ \cdots\ r_{ik})$ or k column vectors: $c_i = (c_{1i}\ c_{2i}\ \cdots c_{ki})$, where $(i=1\text{-}k)$. Using the matrix method, one has to "expand" everything from the very beginning. The resulting algebraic manipulation process of the displacement analysis is laborious and error prone.

Using the spherical trigonometry based method, "the derivation of the input-output equations for the inversions of the $RCRCR^{(2)}$ and $RCCRR^{(2)}$ mechanisms, namely the ($RRCRC^{(2)}$, $RCRRC^{(2)}$) and $RRRCC^{(2)}$ mechanisms is however further complicated, since the spatial loop equation may not contain the required input and output angular displacements, and it may contain more than one unwanted variable." [28](pp.227-228). However, in the new approach, the analyses for the *RCRCR* and *RRCRC* mechanisms (i.e. R_0-*CRCR* and R_0-*RCRC*) are identical in terms of the analysis procedure and the amount of calculation. This also applies to the *RCCRR*, *RCRRC* and *RRRCC* mechanisms, as shown in sections 9.3, 9.6 and 9.5, respectively. Moreover, the two equations derived in the first two steps of the proposed method always contain the desired input, output and (only one!) auxiliary angles.

Let us review section 3. In equations (9.10) and (9.15) everything unrelated to q_2, a function of the auxiliary angle ψ_2, are included in $\{U, V\}$ and $\{U', V'\}$, respectively. Similarly, in equations (9.11) and (9.16) those terms unrelated to ψ_2 are also collected into $\{A, B, C\}$ and $\{A', B', C'\}$. Since we do not need to expand (i.e. scalarize) everything at the beginning, we manipulate only those vectors directly relating to what we need, i.e. q_2 and ψ_2; thus, we are able to obtain equations (9.11) and (9.16) by only one transformation and a few lines of derivation, respectively. In the process of expanding (9.11a) and (9.16a), once again, we do not need to expand everything right away, and we simply manipulate those vectors directly relating to the output angle θ_3. This makes it possible to express the coefficients $\{a_i, b_i, c_i\}$ and $\{a_i', b_i', c_i'\}$ $(i = 1 - 3)$ of (9.17) and (9.18) in standard compact forms, as shown in equations (9.17a-d) and (9.18a-d), respectively. Comparing equations (9.11) to (9.12a) with equations (9.15) to (9.16a), and considering that the analyses for mechanisms in the other sections are either identical or similar, the uniformity, symmetry and simplicity of the proposed method are readily apparent.

Since the analysis steps and expressions are standardized, the proposed method offers a convenient way to write computer programs for these mechanisms.

"Polynomial displacement equations have been derived for virtually all single-loop spatial mechanisms. It remains to examine and to search for patterns in the coefficients of the polynomials, such as symmetry, which could lead to simplification. This recommendation is not original: it was suggested to me by Ferdinand Freudenstein some 20 years ago." This remark was recently made by Professor Joseph Duffy in Ref [29] (page 154). We believe that this chapter has offered a positive answer to the 20 years old issue, for the ideas of the proposed approach are readily applicable to all other spatial mechanisms.

CHAPTER 10. THE R_0–3R–P–C MECHANISM

10.1 Introduction

Yuan ([96], 1971) was the first to study one of the R_0–3R–C–P mechanisms. He obtained a 16th order polynomial displacement equation for the R_0–PRCRR mechanism by using the *method of line coordinates*. Since the correct order is eight, Yuan's solution involves eight extraneous (or unwanted) roots.

In 1974 Duffy and Rooney [23] analyzed the R_0–CRPRR, R_0–CRRPR and R_0–RRPCR mechanisms using *spherical trigonometry method*. They stated, "the derivation of the input-output displacement equation for each mechanism is different and, therefore, warrants special attention. The most difficult result to obtain was the degree eight equation for the spatial six-link RRRPCR mechanism"([23] page 706).

In 1980 Zhang [106] tackled the R_0–RPRCR mechanism using *direction cosine matrix method*. He expressed a similar viewpoint that the derivation of the input-output displacement equations for the R_0–PRRCR and R_0–RRPCR mechanisms was more complicated as compared to that for the R_0–RPRCR mechanism ([106] page 263).

In this chapter the input-output displacement equations for spatial six-link mechanisms R_0–RRPCR, R_0–RCRPR and R_0–RPRRC are derived as eighth order polynomials by using *vector algebraic method*. The *star product operation* * is introduced, which is a supplement to the existing four basic operations, i.e. $\{+, -, \times, \cdot\}$, that vectors may have. The star product operation can substantially simplify the vector algebraic expressions in the analysis of complex mechanisms. As compared to previous pertinent works, the proposed approach is that the derivation for any one of the 20 variant mechanisms of R_0–3R–P–C is *identical* and, therefore, warrants no special attention. Moreover, the proposed approach is characterized by its standardized analysis steps and simplicity.

10.2. Star product operation

Star product operation is introduced to facilitate the vector algebraic derivation in the analysis of complex spatial mechanisms. It is composed of the inseparable couple: the *star product*, * , and the *star product operator*, Φ. Their implication, function and relation can be defined by the following three identities,

$$\Phi * \mathbf{a} = \mathbf{a} * \Phi = \mathbf{a} \tag{10.1}$$

$$\Phi*(a\times b) = (\Phi\times a)*b \tag{10.2}$$

$$\Phi\times a = -a\times\Phi \tag{10.3}$$

where a and b are two arbitrary vectors.

In the displacement analysis of complex spatial mechanisms and serial robots, we may often encounter mathematic expressions which are the sum of the following vector items,

$$q, \quad a\times q, \quad (b\times q)\times c, \quad [(q\times d)\times e]\times f$$

The key point is that every item above contains the same vector q and can be expressed as

$$\begin{cases} q = \Phi*q \\ a\times q = \Phi*(a\times q) = (\Phi\times a)*q \\ (b\times q)\times c = [(c\times\Phi)\times b]*q \\ [(q\times d)\times e]\times f = \{d\times[e\times(f\times\Phi)]\}*q \end{cases} \tag{10.4}$$

For instance, we have the equation,

$$U = \beta_1 q + \beta_2 a\times q + \beta_3 (b\times q)\times c + \beta_4 [(q\times d)\times e]\times f \tag{10.5}$$

where $\{\beta_i\}$ $(i = 1-4)$ are known scalar parameters. Then U can be re-expressed as follows by using (10.4),

$$U = D(\Phi)*q \tag{10.6}$$

$$D(\Phi) = \beta_1\Phi + \beta_2(\Phi\times a) + \beta_3(c\times\Phi)\times b + \beta_4 d\times[e\times(f\times\Phi)] \tag{10.6a}$$

Following is an important formula,

$$\boxed{[D(\Phi)*q]\cdot k = D(k)\cdot q} \tag{10.7}$$

where k is an arbitrary vector.

Using the star product operation, the vector algebraic derivations in the analysis of complex spatial mechanisms can be substantially simplified.

10.3. Analysis of the R_0–$RRPCR$ mechanism

The structure diagram of the mechanism is shown in Fig.10.1. The input angle is θ_1; Let $\theta_6 = \Theta_6$ be the "output angle", the first rotary variable to be determined. Let $\theta_2 = \psi_2$ be the "auxiliary angle", the variable to be eliminated from simultaneous equations. The vector loop equation can now be written as,

$$-x_4 a_4 - x_5 a_5 + K + L = F \tag{10.8}$$

$$\begin{cases} \mathbf{L} = \mathbf{L}(\psi_2) = p_2\mathbf{q}_2 - S_3\mathbf{a}_3 & (1) \\ \mathbf{K} = \mathbf{J} + \mathbf{I} & (2) \\ \mathbf{J} = \mathbf{J}(\theta_6) = p_5\mathbf{q}_5 & (3) \\ \mathbf{I} = (p_1\mathbf{q}_1 - S_2\mathbf{a}_2) + \mathbf{I}_0 & (4) \\ \mathbf{I}_0 = -S_6\mathbf{a}_6 + p_6\mathbf{q}_6 - S_1\mathbf{a}_1 & (5) \\ \mathbf{F} = -p_3\mathbf{q}_3 + S_4\mathbf{a}_4 - p_4\mathbf{q}_4 + S_5\mathbf{a}_5 & (6) \end{cases} \qquad (10.8a)$$

$$\begin{cases} \mathbf{a}_3 = c\alpha_{23}\mathbf{a}_2 + s\alpha_{23}\mathbf{q}_2\times\mathbf{a}_2 & (1) \\ \mathbf{q}_2 = c\psi_2\mathbf{q}_1 + s\psi_2\mathbf{a}_2\times\mathbf{q}_1 & (2) \\ \mathbf{a}_2 = c\alpha_{12}\mathbf{a}_1 + s\alpha_{12}\mathbf{q}_1\times\mathbf{a}_1 & (3) \\ \mathbf{q}_1 = c\theta_1\mathbf{q}_6 + s\theta_1\mathbf{a}_1\times\mathbf{q}_6 & (4) \end{cases} \qquad (10.8b)$$

$$\begin{cases} \mathbf{a}_5 = c\alpha_{56}\mathbf{a}_6 - s\alpha_{56}\mathbf{q}_5\times\mathbf{a}_6 & (5) \\ \mathbf{q}_5 = c\theta_6\mathbf{q}_6 - s\theta_6\mathbf{a}_6\times\mathbf{q}_6 & (6) \end{cases} \qquad (10.8b)$$

$\mathbf{I}$ is the *input vector* of the mechanism. It is the sum of those vectors in the loop of the mechanism which are given or known at the beginning. $\mathbf{J}$ is the *output vector* of the mechanism. It is the sum of those constant-magnitude vectors in the loop of the mechanism that can be expressed as a function of the output angle (θ_6 in this case). $\mathbf{L}$ is the *auxiliary vector*. It is the sum of those constant-magnitude vectors in the loop of the mechanism that can be expressed as a function of the auxiliary angle (ψ_2 in this case). $\mathbf{F}$ is the *floating vector*. Cutting the loop at the two ends of the floating vector, we obtain two separate chains, where one is fixed to the ground and the other is *floating*. For detailed definitions of these vectors see Chapter 2.

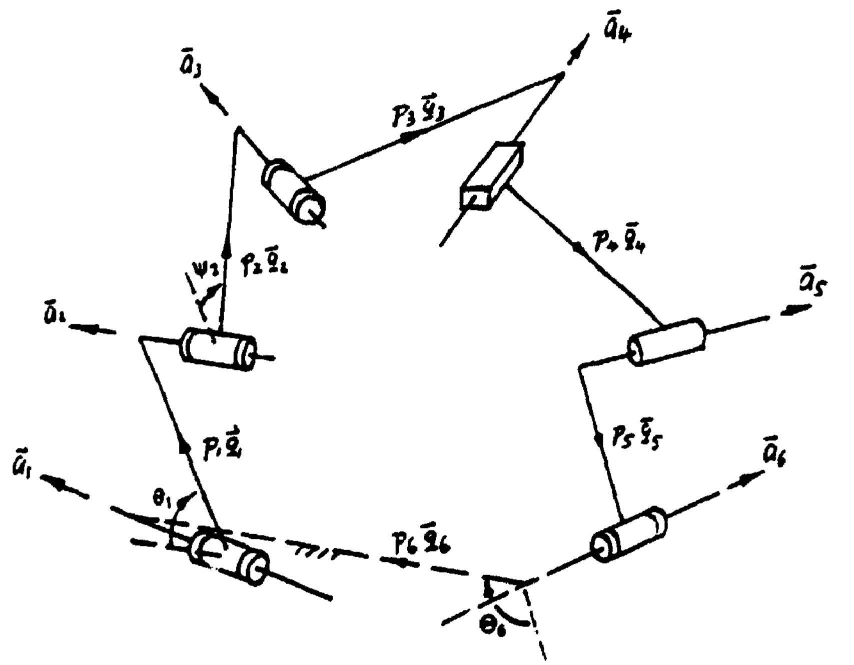


Fig. 10.1

Vectors I, J, K, L and F can be easily obtained from the vector loop diagram of the mechanism. The vector equations in Eq.(10.8b), called *direction equations*, specify the relative direction of any individual (unit) vector with respect to its two adjacent (unit) vectors in the vector loop.

The set of Eq.(10.8b) can also be expressed as $\mathbf{a}_i \times \mathbf{a}_{i+1} = \mathbf{q}_i \sin\alpha_{i\,i+1}$ $(i = 1 - 5)$ and $\mathbf{a}_6 \times \mathbf{a}_1 = \mathbf{q}_6 \sin\alpha_{61}$. However, representing the direction equations in the form of Eq.(10.8b) is more convenient for use.

(1). *Derivation of the first equation relating* θ_1, Θ_6 and ψ_2.

From the structure of the mechanism we have

$$\mathbf{a}_3(\psi_2) \cdot \mathbf{a}_5(\Theta_6) = (\mathbf{a}_3 \cdot \mathbf{a}_5) \tag{10.9}$$

Substituting (10.8b-1) into (10.9) yields

$$\mathbf{U} \cdot \mathbf{q}_2 = V \tag{10.10}$$

$$\begin{cases} \mathbf{U} = s\alpha_{23}\, \mathbf{a}_5 \times \mathbf{a}_2 \\ V = c\alpha_{23}(\mathbf{a}_5 \cdot \mathbf{a}_2) - (\mathbf{a}_3 \cdot \mathbf{a}_5) \end{cases} \tag{10.10a}$$

Substituting (10.8b-2) into (10.10) yields

$$A \cos\psi_2 + B \sin\psi_2 = C \tag{10.11}$$

$$\begin{cases} A = \mathbf{U} \cdot \mathbf{q}_1 \\ B = \mathbf{U} \cdot \mathbf{a}_2 \times \mathbf{q}_1 \\ C = V \end{cases} \tag{10.11a}$$

Substituting (10.8b-5) into (10.10a) yields

$$\begin{cases} \mathbf{U} = \mathbf{D} * \mathbf{q}_5 + \mathbf{d} \\ V = \mathbf{G} \cdot \mathbf{q}_5 + g \end{cases} \tag{10.12}$$

$$\begin{cases} \mathbf{D}(\Phi) = -s\alpha_{23}\, s\alpha_{56}\, \mathbf{a}_6 \times (\mathbf{a}_2 \times \Phi) \\ \mathbf{d} = s\alpha_{23}\, c\alpha_{56}(\mathbf{a}_6 \times \mathbf{a}_2) \\ \mathbf{G} = -c\alpha_{23}\, s\alpha_{56}(\mathbf{a}_6 \times \mathbf{a}_2) \\ g = c\alpha_{23}\, c\alpha_{56}(\mathbf{a}_6 \cdot \mathbf{a}_2) - (\mathbf{a}_3 \cdot \mathbf{a}_5) \end{cases} \tag{10.12a}$$

Substituting (10.12) into (10.11a) yields

$$\begin{cases} A = (\mathbf{D} * \mathbf{q}_1) \cdot \mathbf{q}_5 + (\mathbf{d} \cdot \mathbf{q}_1) \\ B = (\mathbf{D} * \mathbf{a}_2 \times \mathbf{q}_1) \cdot \mathbf{q}_5 + (\mathbf{d} \cdot \mathbf{a}_2 \times \mathbf{q}_1) \\ C = \mathbf{G} \cdot \mathbf{q}_5 + g \end{cases} \tag{10.13}$$

Substituting (10.8b-6) into (10.13) yields

$$\begin{cases} A = a_1\cos\Theta_6 + a_2\sin\Theta_6 + a_3 \\ B = b_1\cos\Theta_6 + b_2\sin\Theta_6 + b_3 \\ C = c_1\cos\Theta_6 + c_2\sin\Theta_6 + c_3 \end{cases} \tag{10.14}$$

$$\begin{cases} a_1 = (\mathbf{D}*\mathbf{q}_1)\cdot\mathbf{q}_6 \\ a_2 = (\mathbf{D}*\mathbf{q}_1)\cdot\mathbf{q}_6\times\mathbf{a}_6 \\ a_3 = (\mathbf{d}\cdot\mathbf{q}_1) \end{cases} \tag{10.14a}$$

$$\begin{cases} b_1 = (\mathbf{D}*\mathbf{a}_2\times\mathbf{q}_1)\cdot\mathbf{q}_6 \\ b_2 = (\mathbf{D}*\mathbf{a}_2\times\mathbf{q}_1)\cdot\mathbf{q}_6\times\mathbf{a}_6 \\ b_3 = (\mathbf{d}\cdot\mathbf{a}_2\times\mathbf{q}_1) \end{cases} \tag{10.14b}$$

$$\begin{cases} c_1 = \mathbf{G}\cdot\mathbf{q}_6 \\ c_2 = \mathbf{G}\cdot\mathbf{q}_6\times\mathbf{a}_6 \\ c_3 = g \end{cases} \tag{10.14c}$$

(2). *Derivation of the second equation relating* θ_1, Θ_6 and ψ_2.

Since $\mathbf{a}_4\times\mathbf{a}_5 = \mathbf{q}_4\sin\alpha_{45}$, hence the scalar product of $\mathbf{a}_4\times\mathbf{a}_5$ with both sides of (8) is the same as the scalar product of $\mathbf{q}_4$ with both sides of (10.8),

$$\mathbf{q}_4\cdot(\mathbf{K} + \mathbf{L}) = \mathbf{q}_4\cdot\mathbf{F} \tag{10.15}$$

Let

$$\mathbf{q}_4 = \rho_1\mathbf{a}_3 + \rho_2\mathbf{a}_5 + \rho_3\mathbf{a}_3\times\mathbf{a}_5 \tag{10.16}$$

The scalar product of $\{\mathbf{a}_5\times(\mathbf{a}_3\times\mathbf{a}_5),\ (\mathbf{a}_3\times\mathbf{a}_5)\times\mathbf{a}_3,\ \mathbf{a}_3\times\mathbf{a}_5\}$ with both sides of (10.16) yields

$$\begin{cases} \rho_1 = (\mathbf{a}_3\cdot\mathbf{q}_4)N \\ \rho_2 = -(\mathbf{a}_3\cdot\mathbf{q}_4)(\mathbf{a}_3\cdot\mathbf{a}_5)N \\ \rho_3 = \csc\alpha_{45}\,[c\alpha_{34} - c\alpha_{45}(\mathbf{a}_3\cdot\mathbf{a}_5)]N \end{cases} \tag{10.16a}$$

$$\begin{cases} N = 1/[1-(\mathbf{a}_3\cdot\mathbf{a}_5)^2] \\ (\mathbf{a}_3\cdot\mathbf{q}_4) = s\alpha_{34}\,s\theta_4 \\ (\mathbf{a}_3\cdot\mathbf{a}_5) = c\alpha_{34}\,c\alpha_{45} - s\alpha_{34}\,s\alpha_{45}\,c\theta_4 \end{cases} \tag{10.16b}$$

Substituting {(10.16), (10.8a-1), (10.8a-6)} into (10.15) yields

$$\begin{aligned} &(\rho_2 p_2\mathbf{a}_5)\cdot\mathbf{q}_2 + (\rho_1\mathbf{K} - \rho_2 S_3\mathbf{a}_5 + \rho_3\mathbf{a}_5\times\mathbf{K})\cdot\mathbf{a}_3 \\ &\quad + (\rho_3 p_2\mathbf{a}_5)\cdot\mathbf{q}_2\times\mathbf{a}_3 = -\rho_2(\mathbf{a}_5\cdot\mathbf{K}) + (\rho_1 S_3 - \rho_3 c\theta_4 - p_4) \end{aligned} \tag{10.17}$$

Substituting (10.8b-1) into (10.17) yields

$$\mathbf{U}'\cdot\mathbf{q}_2 = V' \tag{10.18}$$

$$\begin{cases} \mathbf{U}' = \mathbf{a}_2\times[s\alpha_{23}(\rho_1\mathbf{K}-\rho_2 S_3\mathbf{a}_5+\rho_3\mathbf{a}_5\times\mathbf{K})+\rho_3 p_2 c\alpha_{23}\mathbf{a}_5] \\ V' = \mathbf{a}_2\cdot[-c\alpha_{23}(\rho_1\mathbf{K}-\rho_2 S_3\mathbf{a}_5+\rho_3\mathbf{a}_5\times\mathbf{K})+\rho_3 p_2 s\alpha_{23}\mathbf{a}_5] \\ \qquad -\rho_2(\mathbf{a}_5\cdot\mathbf{K})+(\rho_1 S_3-p_3 c\theta_4-p_4) \end{cases} \tag{10.18a}$$

Substituting (10.8b-2) into (10.18) yields

$$A'\cos\psi_2 + B'\sin\psi_2 = C' \tag{10.19}$$

$$\begin{cases} A' = \mathbf{U}'\cdot\mathbf{q}_1 \\ B' = \mathbf{U}'\cdot\mathbf{a}_2\times\mathbf{q}_1 \\ C' = V' \end{cases} \tag{10.19a}$$

Expressing $\mathbf{a}_5$ and $\mathbf{K}$ of (10.18a) in terms of $\mathbf{q}_5$ yields

$$\begin{cases} \mathbf{U}' = \mathbf{D}'*\mathbf{q}_5 + \mathbf{d}' \\ V' = \mathbf{G}'\cdot\mathbf{q}_5 + g' \end{cases} \tag{10.20}$$

$$\begin{cases} \mathbf{D}'(\Phi) = (\rho_2 p_2 s\alpha_{56}\mathbf{a}_6 + \rho_1 p_5 s\alpha_{23}\mathbf{a}_2)\times\Phi \\ \qquad +\rho_4(\Phi\times\mathbf{a}_6)\times\mathbf{a}_2+\rho_3 s\alpha_{23}s\alpha_{56}[(\Phi\times\mathbf{a}_6)\times\mathbf{I}]\times\mathbf{a}_2 \\ \mathbf{d}' = \rho_2 p_2 c\alpha_{56}\mathbf{a}_6 + \rho_1 s\alpha_{23}\mathbf{a}_2\times\mathbf{I} \\ \qquad +\rho_5\mathbf{a}_6\times\mathbf{a}_2-\rho_3 s\alpha_{23}c\alpha_{56}[(\Phi\times\mathbf{a}_6)\times\mathbf{I}]\times\mathbf{a}_2 \end{cases} \tag{10.20a}$$

$$\begin{cases} \mathbf{G}' = \rho_2 s\alpha_{56}\mathbf{a}_6\times\mathbf{I} - \rho_1 p_5 c\alpha_{23}\mathbf{a}_2 \\ \qquad +\rho_6\mathbf{a}_6\times\mathbf{a}_2+\rho_3 c\alpha_{23}s\alpha_{56}\mathbf{a}_6\times(\mathbf{I}\times\mathbf{a}_2) \\ g' = -(\rho_2 c\alpha_{56}\mathbf{a}_6 + \rho_1 c\alpha_{23}\mathbf{a}_2)\cdot\mathbf{I} \\ \qquad +\rho_7\mathbf{a}_6\cdot\mathbf{a}_2-\rho_3 c\alpha_{23}c\alpha_{56}(\mathbf{a}_6\times\mathbf{I})\cdot\mathbf{a}_2+(\rho_1 S_3-p_3 c\theta_4-p_4) \end{cases} \tag{10.20a}$$

$$\begin{cases} \rho_4 = s\alpha_{23}(\rho_2 S_3 s\alpha_{56}+\rho_3 p_5 \cdot\alpha_{56})+\rho_3 p_2 c\alpha_{23}s\alpha_{56} \\ \rho_5 = s\alpha_{23}(\rho_2 S_3 c\alpha_{56}+\rho_3 p_5 s\alpha_{56})-\rho_3 p_2 s\alpha_{23}c\alpha_{56} \\ \rho_6 = c\alpha_{23}(\rho_2 S_3 s\alpha_{56}+\rho_3 p_5 c\alpha_{56}) \\ \rho_7 = c\alpha_{23}(\rho_2 S_3 c\alpha_{56}+\rho_3 p_5 s\alpha_{56}) \end{cases} \tag{10.20b}$$

Substituting (10.20) into (10.19a) yields

$$\begin{cases} A' = (\mathbf{D}'*\mathbf{q}_1)\cdot\mathbf{q}_5 + (\mathbf{d}'\cdot\mathbf{q}_1) \\ B' = (\mathbf{D}'*\mathbf{a}_2\times\mathbf{q}_1)\cdot\mathbf{q}_5 + (\mathbf{d}'\cdot\mathbf{a}_2\times\mathbf{q}_1) \\ C' = \mathbf{G}'\cdot\mathbf{q}_5 + g' \end{cases} \tag{10.21}$$

Substituting (10.8b-6) into (10.21) yields

$$\begin{cases} A' = a_1'\cos\Theta_6 + a_2'\sin\Theta_6 + a_3' \\ B' = b_1'\cos\Theta_6 + b_2'\sin\Theta_6 + b_3' \\ C' = c_1'\cos\Theta_6 + c_2'\sin\Theta_6 + c_3' \end{cases} \tag{10.22}$$

$$\begin{cases} a_1' = (\mathbf{D}'*\mathbf{q}_1)\cdot\mathbf{q}_6 \\ a_2' = (\mathbf{D}'*\mathbf{q}_1)\cdot\mathbf{q}_6\times\mathbf{a}_6 \\ a_3' = (\mathbf{d}'\cdot\mathbf{q}_1) \end{cases} \tag{10.22a}$$

$$\begin{cases} b_1' = (\mathbf{D}'*\mathbf{a}_2\times\mathbf{q}_1)\cdot\mathbf{q}_6 \\ b_2' = (\mathbf{D}'*\mathbf{a}_2\times\mathbf{q}_1)\cdot\mathbf{q}_6\times\mathbf{a}_6 \\ b_3' = (\mathbf{d}'\cdot\mathbf{a}_2\times\mathbf{q}_1) \end{cases} \tag{10.22b}$$

$$\begin{cases} c_1' = \mathbf{G}'\cdot\mathbf{q}_6 \\ c_2' = \mathbf{G}'\cdot\mathbf{q}_6\times\mathbf{a}_6 \\ c_3' = g' \end{cases} \tag{10.22c}$$

Solving (10.11) and (10.19) we can obtain the input-output equation $f(\theta_1, \theta_6)=0$ and the solution for $\{\theta_6, \psi_2\}$, as shown in Section 2.6.4. The determination for other variables is omitted here, for it is much easier.

10.4. Analysis of the R_0-$RCRPR$ mechanism

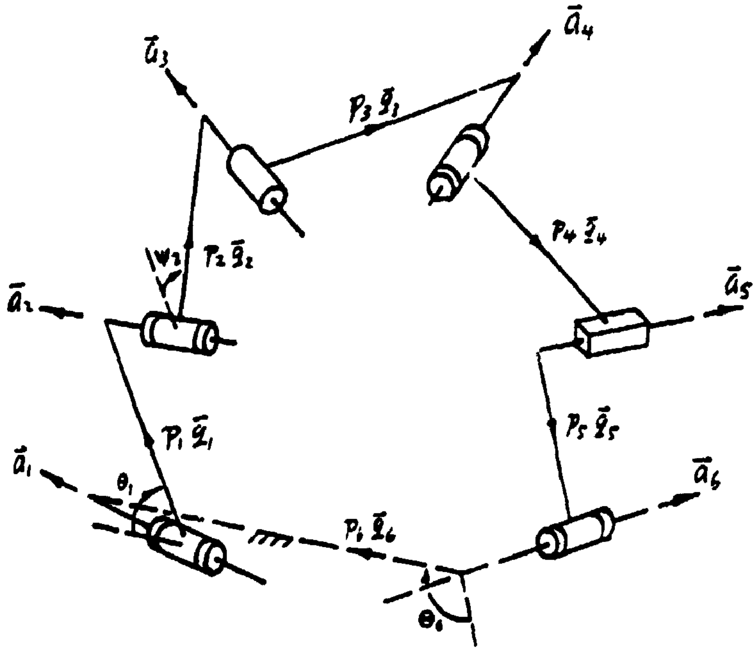


Fig. 10.2

The structure diagram of the mechanism is shown in Fig. 10.2. The input angle is θ_1; the output angle is θ_6, i.e. $\theta_6=\Theta_6$; The auxiliary angle is θ_2, i.e. $\theta_2=\psi_2$. The vector loop equation can be written as,

$$-x_3\mathbf{a}_3 - x_5\mathbf{a}_5 + \mathbf{K} + \mathbf{L} = \mathbf{F} \tag{10.23}$$

$$\begin{cases} \mathbf{L} = p_2\mathbf{q}_2 & (1) \\ \mathbf{K} = \mathbf{J} + \mathbf{I} & (2) \\ \mathbf{J} = p_5\mathbf{q}_5 + p_4\mathbf{q}_4 - S_4\mathbf{a}_4 & (3) \\ \mathbf{I} = (p_1\mathbf{q}_1 - S_2\mathbf{a}_2) + \mathbf{I}_0 & (4) \\ \mathbf{I}_0 = -S_6\mathbf{a}_6 + p_6\mathbf{q}_6 - S_1\mathbf{a}_1 & (5) \\ \mathbf{F} = -p_3\mathbf{q}_3 & (6) \end{cases} \tag{10.23a}$$

$$\begin{cases} \mathbf{a}_3 = c\alpha_{23}\mathbf{a}_2 + s\alpha_{23}\mathbf{q}_2\times\mathbf{a}_2 & (1) \\ \mathbf{q}_2 = c\psi_2\mathbf{q}_1 + s\psi_2\mathbf{a}_2\times\mathbf{q}_1 & (2) \\ \mathbf{a}_2 = c\alpha_{12}\mathbf{a}_1 + s\alpha_{12}\mathbf{q}_1\times\mathbf{a}_1 & (3) \\ \mathbf{q}_1 = c\theta_1\mathbf{q}_6 + s\theta_1\mathbf{a}_1\times\mathbf{q}_6 & (4) \\ \mathbf{q}_3 = csc\alpha_{34}\mathbf{a}_3\times\mathbf{a}_4 & (5) \end{cases} \tag{10.23b}$$

$$\begin{cases} \mathbf{a}_4 = c\alpha_{45}\mathbf{a}_5 - s\alpha_{45}\mathbf{q}_4\times\mathbf{a}_5 & (6) \\ \mathbf{q}_4 = c\theta_5\mathbf{q}_5 - s\theta_5\mathbf{a}_5\times\mathbf{q}_5 & (7) \\ \mathbf{a}_5 = c\alpha_{56}\mathbf{a}_6 - s\alpha_{56}\mathbf{q}_5\times\mathbf{a}_6 & (8) \\ \mathbf{q}_5 = c\Theta_6\mathbf{q}_6 - s\Theta_6\mathbf{a}_6\times\mathbf{q}_6 & (9) \end{cases} \tag{10.23b}$$

(1). *Derivation of the first equation relating* θ_1, Θ_6 and ψ_2.

From the structure of the mechanism we can write,

$$\mathbf{a}_3(\psi_2)\cdot\mathbf{a}_4(\Theta_6) = c\alpha_{34} \tag{10.24}$$

Substituting (10.23b-1) into (10.24) yields

$$\mathbf{U}\cdot\mathbf{q}_2 = V \tag{10.25}$$

$$\begin{cases} \mathbf{U} = s\alpha_{23}\mathbf{a}_4\times\mathbf{a}_2 \\ V = c\alpha_{23}(\mathbf{a}_4\cdot\mathbf{a}_2) - c\alpha_{34} \end{cases} \tag{10.25a}$$

Substituting (10.23b-2) into (10.25) yields

$$A\cos\psi_2 + B\sin\psi_2 = C \tag{10.26}$$

$$\begin{cases} A = \mathbf{U}\cdot\mathbf{q}_1 \\ B = \mathbf{U}\cdot\mathbf{a}_2\times\mathbf{q}_1 \\ C = V \end{cases} \tag{10.26a}$$

Using (10.23b-6), (10.23b-7) and (10.23b-8) we can express $\mathbf{a}_4$ in terms of $\mathbf{q}_5$, $\mathbf{a}_6$ and $\mathbf{q}_5\times\mathbf{a}_6$ as follow,

$$\mathbf{a}_4 = \rho_1 \mathbf{q}_5 + \rho_2 \mathbf{q}_5 \times \mathbf{a}_6 + \rho_3 \mathbf{a}_6 \tag{10.27}$$

$$\begin{cases} \rho_1 = s\alpha_{45} s\theta_5 \\ \rho_2 = c\alpha_{45} s\alpha_{56} + s\alpha_{45} c\alpha_{56} c\theta_5 \\ \rho_3 = c\alpha_{45} c\alpha_{56} - s\alpha_{45} s\alpha_{56} c\theta_5 \end{cases} \tag{10.27a}$$

Substituting (10.27) into (10.25a) yields

$$\begin{cases} \mathbf{U} = \mathbf{D} * \mathbf{q}_5 + \mathbf{d} \\ V = \mathbf{G} \cdot \mathbf{q}_5 + g \end{cases} \tag{10.28}$$

$$\begin{cases} \mathbf{D}(\Phi) = s\alpha_{23}(\rho_1 \Phi + \rho_2 \Phi \times \mathbf{a}_6) \times \mathbf{a}_2 \\ \mathbf{d} = s\alpha_{23}(\rho_3 \mathbf{a}_6) \times \mathbf{a}_2 \\ \mathbf{G} = c\alpha_{23}(\rho_1 \mathbf{a}_2 + \rho_2 \mathbf{a}_6 \times \mathbf{a}_2) \\ g = c\alpha_{23}(\rho_3 \mathbf{a}_6) \cdot \mathbf{a}_2 - c\alpha_{34} \end{cases} \tag{10.28a}$$

Substituting (10.28) into (10.26a) yields

$$\begin{cases} A = (\mathbf{D} * \mathbf{q}_1) \cdot \mathbf{q}_5 + (\mathbf{d} \cdot \mathbf{q}_1) \\ B = (\mathbf{D} * \mathbf{a}_2 \times \mathbf{q}_1) \cdot \mathbf{q}_5 + (\mathbf{d} \cdot \mathbf{a}_2 \times \mathbf{q}_1) \\ C = \mathbf{G} \cdot \mathbf{q}_5 + g \end{cases} \tag{10.29}$$

Substituting (10.23b-9) into (10.29) yields

$$\begin{cases} A = a_1 \cos\Theta_6 + a_2 \sin\Theta_6 + a_3 \\ B = b_1 \cos\Theta_6 + b_2 \sin\Theta_6 + b_3 \\ C = c_1 \cos\Theta_6 + c_2 \sin\Theta_6 + c_3 \end{cases} \tag{10.30}$$

$$\begin{cases} a_1 = (\mathbf{D} * \mathbf{q}_1) \cdot \mathbf{q}_6 \\ a_2 = (\mathbf{D} * \mathbf{q}_1) \cdot \mathbf{q}_6 \times \mathbf{a}_6 \\ a_3 = (\mathbf{d} \cdot \mathbf{q}_1) \end{cases} \tag{10.30a}$$

$$\begin{cases} b_1 = (\mathbf{D} * \mathbf{a}_2 \times \mathbf{q}_1) \cdot \mathbf{q}_6 \\ b_2 = (\mathbf{D} * \mathbf{a}_2 \times \mathbf{q}_1) \cdot \mathbf{q}_6 \times \mathbf{a}_6 \\ b_3 = (\mathbf{d} \cdot \mathbf{a}_2 \times \mathbf{q}_1) \end{cases} \tag{10.30b}$$

$$\begin{cases} c_1 = \mathbf{G} \cdot \mathbf{q}_6 \\ c_2 = \mathbf{G} \cdot \mathbf{q}_6 \times \mathbf{a}_6 \\ c_3 = g \end{cases} \tag{10.30c}$$

(2). *Derivation of the second equation relating* θ_1, θ_6 and ψ_2.

The scalar product of $\mathbf{a}_3 \times \mathbf{a}_5$ with both sides of (10.23) yields

$$\mathbf{a}_3 \times \mathbf{a}_5 \cdot (\mathbf{K} + \mathbf{L} - \mathbf{F}) = 0 \tag{10.31}$$

i.e.

$$\mathbf{a}_3 \times \mathbf{a}_5 \cdot (\mathbf{K} + \mathbf{L} - p_3 \csc\alpha_{34} \mathbf{a}_3 \times \mathbf{a}_4) = 0 \tag{10.31a}$$

i.e.

$$\mathbf{a}_3 \cdot (\mathbf{a}_5 \times \mathbf{K} + p_3 \cot\alpha_{34} \mathbf{a}_5) + (\mathbf{L} \times \mathbf{a}_3) \cdot \mathbf{a}_5 = p_3 \csc\alpha_{34} c\alpha_{45} \tag{10.31b}$$

Substituting (10.23a-1) and (10.23b-1) into (10.31b) yields

$$\mathbf{U}' \cdot \mathbf{q}_2 = V' \tag{10.32}$$

$$\begin{cases} \mathbf{U}' = \mathbf{M}_1 \times \mathbf{a}_2 \\ V' = \mathbf{M}_2 \cdot \mathbf{a}_2 - p_3 \csc\alpha_{34} c\alpha_{45} \end{cases} \tag{10.32a}$$

$$\begin{cases} \mathbf{M}_1 = s\alpha_{23} \mathbf{a}_5 \times \mathbf{K} + m_1 \mathbf{a}_5 \\ \mathbf{M}_2 = c\alpha_{23} \mathbf{a}_5 \times \mathbf{K} + m_2 \mathbf{a}_5 \end{cases} \tag{10.32b}$$

$$\begin{cases} m_1 = (p_3 s\alpha_{23} \cot\alpha_{34} + p_2 c\alpha_{23}) \\ m_2 = (p_3 c\alpha_{23} \cot\alpha_{34} - p_2 s\alpha_{23}) \end{cases} \tag{10.32c}$$

Substituting (10.23b-1) into (10.32) yields

$$A' \cos\psi_2 + B' \sin\psi_2 = C' \tag{10.33}$$

$$\begin{cases} A' = \mathbf{U}' \cdot \mathbf{q}_1 \\ B' = \mathbf{U}' \cdot \mathbf{a}_2 \times \mathbf{q}_1 \\ C' = V' \end{cases} \tag{10.33a}$$

From (10.32b) we obtain

$$\begin{cases} \mathbf{M}_1 = \mathbf{D}_1 * \mathbf{q}_5 + \mathbf{d}_1 \\ \mathbf{M}_2 = \mathbf{D}_2 * \mathbf{q}_5 + \mathbf{d}_2 \end{cases} \tag{10.34}$$

$$\begin{cases} \mathbf{D}_1(\Phi) = s\alpha_{23} [\beta_2 \Phi - s\alpha_{56} (\Phi \times \mathbf{I}) \times \mathbf{a}_6] + \sigma_1 \Phi \times \mathbf{a}_6 \\ \mathbf{D}_2(\Phi) = c\alpha_{23} [\beta_2 \Phi - s\alpha_{56} (\Phi \times \mathbf{I}) \times \mathbf{a}_6] + \sigma_2 \Phi \times \mathbf{a}_6 \\ \mathbf{d}_1 = s\alpha_{23} c\alpha_{56} \mathbf{a}_6 \times \mathbf{I} + \sigma_3 \mathbf{a}_6 \\ \mathbf{d}_2 = c\alpha_{23} c\alpha_{56} \mathbf{a}_6 \times \mathbf{I} + \sigma_4 \mathbf{a}_6 \end{cases} \tag{10.34a}$$

$$\begin{cases} \sigma_1 = m_1 s\alpha_{56} + \beta_1 s\alpha_{23} c\alpha_{56} \\ \sigma_2 = m_2 s\alpha_{56} + \beta_1 c\alpha_{23} c\alpha_{56} \\ \sigma_3 = m_1 c\alpha_{56} - \beta_2 s\alpha_{23} s\alpha_{56} \\ \sigma_4 = m_2 c\alpha_{56} - \beta_2 c\alpha_{23} s\alpha_{56} \end{cases} \tag{10.34b}$$

Substituting (10.34) into (10.32a), and then substituting (10.32a) into (10.33a), yields

$$\begin{cases} A' = [\mathbf{D}_1 * \mathbf{a}_2 \times \mathbf{q}_1] \cdot \mathbf{q}_5 + \mathbf{d}_1 \cdot \mathbf{a}_2 \times \mathbf{q}_1 \\ B' = [\mathbf{D}_1 * (-\mathbf{q}_1)] \cdot \mathbf{q}_5 + \mathbf{d}_1 \cdot (-\mathbf{q}_1) \\ C' = (\mathbf{D}_2 * \mathbf{a}_2) \cdot \mathbf{q}_5 + (\mathbf{d}_2 \cdot \mathbf{a}_2) - p_3 \csc\alpha_{34} c\alpha_{45} \end{cases} \tag{10.35}$$

Substituting (10.23b-9) into (10.35) yields

$$\begin{cases} A' = a_1' \cos\Theta_6 + a_2' \sin\Theta_6 + a_3' \\ B' = b_1' \cos\Theta_6 + b_2' \sin\Theta_6 + b_3' \\ C' = c_1' \cos\Theta_6 + c_2' \sin\Theta_6 + c_3' \end{cases} \tag{10.36}$$

$$\begin{cases} a_1' = (\mathbf{D}_1 * \mathbf{a}_2 \times \mathbf{q}_1) \cdot \mathbf{q}_6 \\ a_2' = (\mathbf{D}_1 * \mathbf{a}_2 \times \mathbf{q}_1) \cdot \mathbf{q}_6 \times \mathbf{a}_6 \\ a_3' = (\mathbf{d}_1 \cdot \mathbf{a}_2 \times \mathbf{q}_1) \end{cases} \tag{10.36a}$$

$$\begin{cases} b_1' = [\mathbf{D}_1 * (-\mathbf{q}_1)] \cdot \mathbf{q}_6 \\ b_2' = [\mathbf{D}_1 * (-\mathbf{q}_1)] \cdot \mathbf{q}_6 \times \mathbf{a}_6 \\ b_3' = \mathbf{d}_1 \cdot (-\mathbf{q}_1) \end{cases} \tag{10.36b}$$

$$\begin{cases} c_1' = (\mathbf{D}_2 * \mathbf{a}_2) \cdot \mathbf{q}_6 \\ c_2' = (\mathbf{D}_2 * \mathbf{a}_2) \cdot \mathbf{q}_6 \times \mathbf{a}_6 \\ c_3' = (\mathbf{d}_2 \cdot \mathbf{a}_2) - p_3 \csc\alpha_{34} c\alpha_{45} \end{cases} \tag{10.36c}$$

10.5. Analysis of the R_0-$RPRRC$ mechanism

The structure diagram of the mechanism is shown in Fig.10.3. The input angle is θ_1; the output angle is θ_6, i.e. $\theta_6 = \Theta_6$; The auxiliary angle is θ_2, i.e. $\theta_2 = \psi_2$. The vector loop equation can be written as,

$$-x_3 \mathbf{a}_3 - x_6 \mathbf{a}_6 + \mathbf{K} + \mathbf{L} = \mathbf{F} \tag{10.37}$$

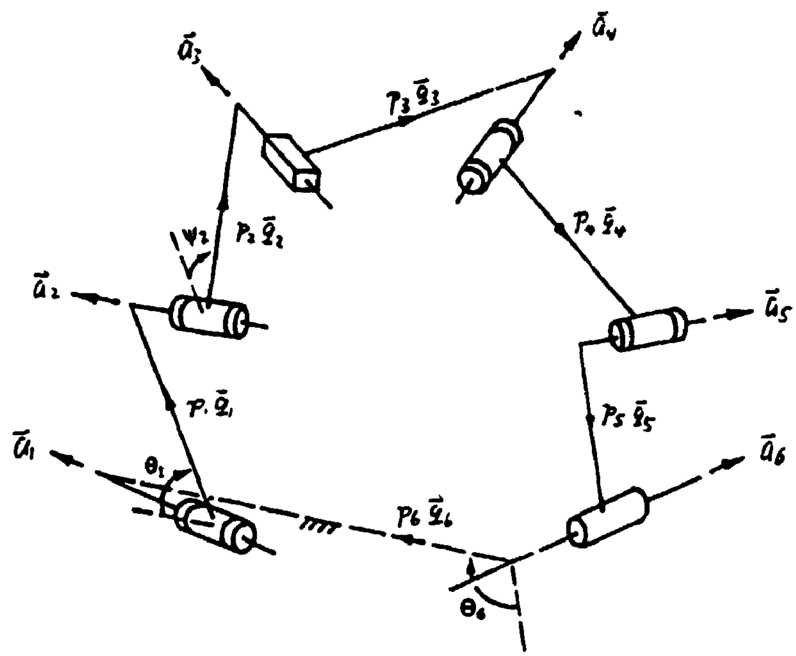


Fig. 10.3

$$\begin{cases} \mathbf{L} = p_2\mathbf{q}_2 + p_3\mathbf{q}_3 - S_4\mathbf{a}_4 & (1) \\ \mathbf{K} = \mathbf{J} + \mathbf{I} & (2) \\ \mathbf{J} = p_5\mathbf{q}_5 - S_5\mathbf{a}_5 & (3) \\ \mathbf{I} = (p_1\mathbf{q}_1 - S_2\mathbf{a}_2) + \mathbf{I}_0 & (4) \\ \mathbf{I}_0 = p_6\mathbf{q}_6 - S_1\mathbf{a}_1 & (5) \\ \mathbf{F} = -p_4\mathbf{q}_4 = -p_4 \csc\alpha_{45}\,\mathbf{a}_4\times\mathbf{a}_5 & (6) \end{cases} \qquad (10.37a)$$

$$\begin{cases} \mathbf{a}_4 = c\alpha_{34}\mathbf{a}_3 + s\alpha_{34}\mathbf{q}_3\times\mathbf{a}_3 & (1) \\ \mathbf{q}_3 = c\theta_3\mathbf{q}_2 + s\theta_3\mathbf{a}_3\times\mathbf{q}_2 & (2) \\ \mathbf{a}_3 = c\alpha_{23}\mathbf{a}_2 + s\alpha_{23}\mathbf{q}_2\times\mathbf{a}_2 & (3) \\ \mathbf{q}_2 = c\psi_2\mathbf{q}_1 + s\psi_2\mathbf{a}_2\times\mathbf{q}_1 & (4) \\ \mathbf{a}_2 = c\alpha_{12}\mathbf{a}_1 + s\alpha_{12}\mathbf{q}_1\times\mathbf{a}_1 & (5) \\ \mathbf{q}_1 = c\theta_1\mathbf{q}_6 + s\theta_1\mathbf{a}_1\times\mathbf{q}_6 & (6) \end{cases} \qquad (10.37b)$$

$$\begin{cases} \mathbf{a}_5 = c\alpha_{56}\mathbf{a}_6 - s\alpha_{56}\mathbf{q}_5\times\mathbf{a}_6 & (7) \\ \mathbf{q}_5 = c\Theta_6\mathbf{q}_6 - s\Theta_6\mathbf{a}_6\times\mathbf{q}_6 & (8) \end{cases} \qquad (10.37b)$$

(1). *Derivation of the first equation relating* θ_1, Θ_6 and ψ_2.

From the structure of the mechanism we can write,

$$\mathbf{a}_4(\psi_2)\cdot\mathbf{a}_5(\Theta_6) = c\alpha_{45} \qquad (10.38)$$

Using (10.30b-1), (10.30b-2) and (10.31b-3) we can express $\mathbf{a}_4$ in terms of $\mathbf{q}_2$, $\mathbf{a}_2$ and $\mathbf{q}_2\times\mathbf{a}_2$ as follow,

$$\mathbf{a}_4 = \rho_1\mathbf{q}_2 + \rho_2\mathbf{q}_2\times\mathbf{a}_2 + \rho_3\mathbf{a}_2 \qquad (10.39)$$

$$\begin{cases} \rho_1 = s\theta_3 s\alpha_{34} \\ \rho_2 = s\alpha_{23} c\alpha_{34} + c\alpha_{23} s\alpha_{34} c\theta_3 \\ \rho_3 = c\alpha_{23} c\alpha_{34} - s\alpha_{23} s\alpha_{34} c\theta_3 \end{cases} \tag{10.39a}$$

Substituting (10.39) into (10.38) yields

$$\mathbf{U} \cdot \mathbf{q}_2 = V \tag{10.40}$$

$$\begin{cases} \mathbf{U} = \rho_1 \mathbf{a}_5 + \rho_2 \mathbf{a}_2 \times \mathbf{a}_5 \\ V = -\rho_3 (\mathbf{a}_2 \cdot \mathbf{a}_5) + c\alpha_{45} \end{cases} \tag{10.40a}$$

Substituting (10.37b-4) into (10.40) yields

$$A \cos\psi_2 + B \sin\psi_2 = C \tag{10.41}$$

$$\begin{cases} A = \mathbf{U} \cdot \mathbf{q}_1 \\ B = \mathbf{U} \cdot \mathbf{a}_2 \times \mathbf{q}_1 \\ C = V \end{cases} \tag{10.41a}$$

Substituting (10.37b-7) into (10.40a) yields

$$\begin{cases} \mathbf{U} = \mathbf{D} * \mathbf{q}_5 + \mathbf{d} \\ V = \mathbf{G} \cdot \mathbf{q}_5 + g \end{cases} \tag{10.42}$$

$$\begin{cases} \mathbf{D}(\Phi) = s\alpha_{56} [\, \rho_1 \Phi \times \mathbf{a}_6 + \rho_2 a 6 \times (\mathbf{a}_2 \times \Phi)] \\ \mathbf{d} = c\alpha_{56} (\rho_1 \mathbf{a}_6 + \rho_2 \mathbf{a}_2 \times \mathbf{a}_6) \\ \mathbf{G} = s\alpha_{56} \rho_3 (\mathbf{a}_6 \times \mathbf{a}_2) \\ g = -c\alpha_{56} \rho_3 (\mathbf{a}_6 \cdot \mathbf{a}_2) + c\alpha_{45} \end{cases} \tag{10.42a}$$

Substituting (10.42) into (10.41a) yields

$$\begin{cases} A = (\mathbf{D} * \mathbf{q}_1) \cdot \mathbf{q}_5 + (\mathbf{d} \cdot \mathbf{q}_1) \\ B = (\mathbf{D} * \mathbf{a}_2 \times \mathbf{q}_1) \cdot \mathbf{q}_5 + (\mathbf{d} \cdot \mathbf{a}_2 \times \mathbf{q}_1) \\ C = \mathbf{G} \cdot \mathbf{q}_5 + g \end{cases} \tag{10.43}$$

Substituting (10.37b-8) into (10.43) yields

$$\begin{cases} A = a_1 \cos\Theta_6 + a_2 \sin\Theta_6 + a_3 \\ B = b_1 \cos\Theta_6 + b_2 \sin\Theta_6 + b_3 \\ C = c_1 \cos\Theta_6 + c_2 \sin\Theta_6 + c_3 \end{cases} \tag{10.44}$$

The expressions for $\{a_i,\ b_i,\ c_i\}$ $(i = 1 - 3)$ are exactly the same as (10.14a-c).

(2). *Derivation of the second equation relating* θ_1, Θ_6 and ψ_2.

The scalar product of $a_3 \times a_6$ with both sides of (10.37) yields

$$a_3 \times a_6 \cdot (K + L - F) = 0 \tag{10.45}$$

i.e.

$$a_3 \times a_6 \cdot (K + L - p_4 \csc\alpha_{45}\, a_4 \times a_5) = 0 \tag{10.45a}$$

i.e.

$$\begin{aligned}&(a_6 \times K)\cdot a_3 + a_6 \cdot (L \times a_3)\\ &\quad - p_4 \csc\alpha_{45} [\, c\alpha_{34} c\alpha_{56} - (a_3 \cdot a_5)(a_4 \cdot a_6)] = 0\end{aligned} \tag{10.45b}$$

where

$$\begin{aligned}(a_3 \cdot a_5)(a_4 \cdot a_6) &= (a_3 \times a_4)\cdot(a_5 \times a_6) + (a_3 \cdot a_6)(a_4 \cdot a_5)\\ &= (\csc\alpha_{34} \csc\alpha_{56}\, q_5)\cdot q_3 - (c\alpha_{45}\, a_6)\cdot a_3\end{aligned} \tag{10.45c}$$

Substituting (10.45c) into (10.45b) and expressing a_3, q_3 and $L \times a_3$ in terms of q_2 yields

$$U' \cdot q_2 = V' \tag{10.46}$$

$$\begin{cases} U' = s\alpha_{23}\, a_2 \times (a_6 \times K) + \gamma_1 (\, c\theta_1 q_5 + c\alpha_{23} s\theta_3\, q_5 \times a_2) \\ \qquad + \gamma_2 (a_2 \times a_6) + \beta_1 a_6 \\ V' = -c\alpha_{23}\, a_2 \cdot (a_6 \times K) - \gamma_1 (\, s\alpha_{23} s\theta_3\, q_5 \cdot a_2) \\ \qquad + \gamma_3 (\, a_2 \cdot a_6) + p_4 c\alpha_{34} \csc\alpha_{45} c\alpha_{56} \end{cases} \tag{10.46a}$$

$$\begin{cases} \gamma_1 = p_4 \csc\alpha_{34} \csc\alpha_{45} \csc\alpha_{56} \\ \gamma_2 = \beta_2 - p_4 s\alpha_{23} \cot\alpha_{45} \\ \gamma_3 = -\beta_3 + p_4 c\alpha_{23} \cot\alpha_{45} \end{cases} \tag{10.46b}$$

$$\begin{cases} \beta_1 = p_2 (\, s\alpha_{23} c\theta_3 + s\theta_3) + S_4 c\theta_3 s\alpha_{34} \\ \beta_2 = c\alpha_{23} (\, p_2 + p_3 c\theta_3 - S_4 s\theta_3 s\alpha_{34} \\ \beta_3 = s\alpha_{23} (\, S_4 s\theta_3 s\alpha_{34} - p_2 s\alpha_{23}) \end{cases} \tag{10.46c}$$

Substituting (10.37b-4) into (10.46) yields

$$A' \cos\psi_2 + B' \sin\psi_2 = C' \tag{10.47}$$

$$\begin{cases} A' = U' \cdot q_1 \\ B' = U' \cdot a_2 \times q_1 \\ C' = V' \end{cases} \tag{10.47a}$$

From (10.46a) and expressing K in terms of q_5 yields

$$\begin{cases} U' = D'^{*} q_5 + d' \\ V' = D' \cdot q_5 + g' \end{cases} \tag{10.48}$$

$$\begin{cases} \mathbf{D}'(\Phi) = s\alpha_{23}\{p_5(\Phi\times\mathbf{a}_2)\times\mathbf{a}_6 - S_5 s\alpha_{56}[(\Phi\times\mathbf{a}_2)\times\mathbf{a}_6]\times\mathbf{a}_6\} \\ \qquad + \gamma_1(c\theta_3\Phi + c\alpha_{23}s\theta_3\mathbf{a}_2\times\Phi) \\ \mathbf{d}' = s\alpha_{23}\mathbf{a}_2\times[\mathbf{a}_6\times(\mathbf{I} - S_5 c\alpha_{56}\mathbf{a}_6)] + \gamma_2\mathbf{a}_2\times\mathbf{a}_6 + \beta_1\mathbf{a}_6 \\ \mathbf{G}' = -c\alpha_{23}\{p_5\mathbf{a}_2\times\mathbf{a}_6 - S_5 s\alpha_{56}(\mathbf{a}_2\times\mathbf{a}_6)\times\mathbf{a}_6\} \\ \qquad - \gamma_1(s\alpha_{23}s\theta_3\mathbf{a}_2) \\ g' = \gamma_3(\mathbf{a}_2\cdot\mathbf{a}_6) + p_4 c\alpha_{34}\csc\alpha_{45}c\alpha_{56} \end{cases} \tag{10.48a}$$

Substituting (10.48) into (10.47a) yields

$$\begin{cases} A' = (\mathbf{D}'*\mathbf{q}_1)\cdot\mathbf{q}_5 + \mathbf{d}'\cdot\mathbf{q}_1 \\ B' = (\mathbf{D}'*\mathbf{a}_2\times\mathbf{q}_1)\cdot\mathbf{q}_5 + \mathbf{d}'\cdot\mathbf{a}_2\times\mathbf{q}_1 \\ C' = \mathbf{G}'\cdot\mathbf{q}_5 + g' \end{cases} \tag{10.49}$$

Substituting (10.37b-8) into (10.49) yields

$$\begin{cases} A' = a_1'\cos\Theta_6 + a_2'\sin\Theta_6 + a_3' \\ B' = b_1'\cos\Theta_6 + b_2'\sin\Theta_6 + b_3' \\ C' = c_1'\cos\Theta_6 + c_2'\sin\Theta_6 + c_3' \end{cases} \tag{10.50}$$

where the expressions for $\{a_i',\ b_i',\ c_i'\}$ $(i=1-3)$ are the same as (10.22a-c).

10.6. Conclusion

The analysis steps of Duffy's *spherical trigonometry method* in Ref.[23] can be summarized as follows:

- The 1st step: Derive the equation containing pair variables $\{\theta_6, \theta_1, \theta_2\}$;
- The 2nd step: Derive the first equation containing $\{\theta_6, \theta_1, \theta_2, \theta_4\}$;
- The 3rd step: Derive the second equation containing $\{\theta_6, \theta_1, \theta_2, \theta_4\}$;
- The 4th step: Eliminate θ_4 from the two equations in steps 2 and 3, the second equation containing $\{\theta_6, \theta_1, \theta_2\}$ is obtained;
- The 5th step: Eliminate θ_2 from the two equations in steps 1 and 4, the input-output displacement equation $f(\theta_6, \theta_1)=0$ is obtained.

The analysis steps of Zhang's *direction cosine matrix method* in Ref.[106] can be summarized as follows:

- The 1st step: Derive the first equation containing pair variables $\{\theta_6, \theta_1, \theta_2, \theta_4\}$;
- The 2nd step: Derive the second equation containing $\{\theta_6, \theta_1, \theta_2, \theta_4\}$;
- The 3rd step: Derive the equation containing $\{\theta_6, \theta_1, \theta_4\}$;
- The 4th step: Eliminate θ_4 from the equation in step 2 by substituting the two equations obtained in steps 1 and 3, the first equation containing $\{\theta_6, \theta_1, \theta_2\}$ is

obtained;

- The 5th step: Derive the second equation containing $\{\theta_6, \theta_1, \theta_2\}$;
- The 6th step: Eliminate θ_2 from the two equations in steps 4 and 5, the input-output displacement equation $f(\theta_6, \theta_1)=0$ is obtained.

The analysis steps of the ***vector algebraic method*** in this paper can be summarized as follows:

- The 1st step: The scalar product of $\mathbf{a}_k \times \mathbf{a}_l$ with both sides of the loop equation yields the 1st equation containing $\{\theta_6, \theta_1, \theta_2\}$;
- The 2nd step: Using $\mathbf{a}_m \cdot \mathbf{a}_n = const$, we directly obtain the second equation containing $\{\theta_6, \theta_1, \theta_2\}$;
- The 3rd step: Eliminate θ_2 from the two equations obtained in steps 1 and 2, the input-output displacement equation $f(\theta_6, \theta_1)=0$ is derived.

CHAPTER 11. ANALYSIS OF TWO 6R INDUSTRIAL ROBOTS

11.1. Introduction

In the past ten years, significant progress has been made in the area of kinematic analysis of serial robots. The most popular approaches use 4x4 homogeneous transformation matrices ([58] Paul 1981) and 3x3 direction cosine matrices ([106] Zhang 1980), based on the Denavit-Hartenberg convention ([10] 1955). These matrices are the combination of decomposed vectors. When using a matrix method, one has to express any vector, such as the direction of a link or an axis, in terms of *three* or *four* scalar elements. This expands and complicates the algebraic expressions; thus, laborious algebraic manipulation is often required for deriving a solution and the process is error prone. Even for a robot whose structure is of medium complexity, the analysis procedure of the matrix methods becomes quite difficult.

This chapter presents the forward and inverse kinematic analysis of two 6R robots, based on the *vector algebraic method*. The method uses only vector operations and, as a result, the algebraic expressions are compact. In addition, the new analysis procedure is standardized and straightforward, and it offers better geometric insight into the problems than the matrix approach.

The first 6R robot analyzed in this chapter has a spherical wrist, i.e. the last three joint axes intersecting at a point. The second 6R robot does not have a spherical wrist, but has three consecutive parallel joint axes. Using the vector algebraic method, the analysis procedures are almost the same for the two robots.

11.2. Analysis of the first *6R* robot

(1) *The Forward Kinematic Analysis.*

The configuration of the first 6R serial robot is shown in Fig. 11.1. where $\{\mathbf{a}_i\}$ $(i=0\text{-}8)$ are unit vectors and $\mathbf{a}_i \cdot \mathbf{a}_{i+1} = 0$. This robot has six degrees of freedom corresponding to six rotary controllable variables, denoted as θ_i $(i=1,2,4,5,6,7)$, where $\theta_i = (\mathbf{a}_{i-1} \hat{,} \mathbf{a}_{i+1})$ $(i=1\text{-}7)$ is the right-hand-rotation angle from $\mathbf{a}_{i-1}$ to $\mathbf{a}_{i+1}$ about $\mathbf{a}_i$. From the geometry of the robot it is known that $\mathbf{a}_2 = \mathbf{a}_4$, hence $\theta_3 = 0$.

Now the problem can be stated as follows. Given: $\{\mathbf{a}_0, \mathbf{a}_1, S_1, S_3, S_5, S_7, \theta_i\}$, $(i=1\text{-}7)$; Find: $\{\mathbf{a}_7, \mathbf{a}_8, \mathbf{R}\}$.

From Fig. 11.1, we can write the vector loop equation and the structure constraint equations directly,

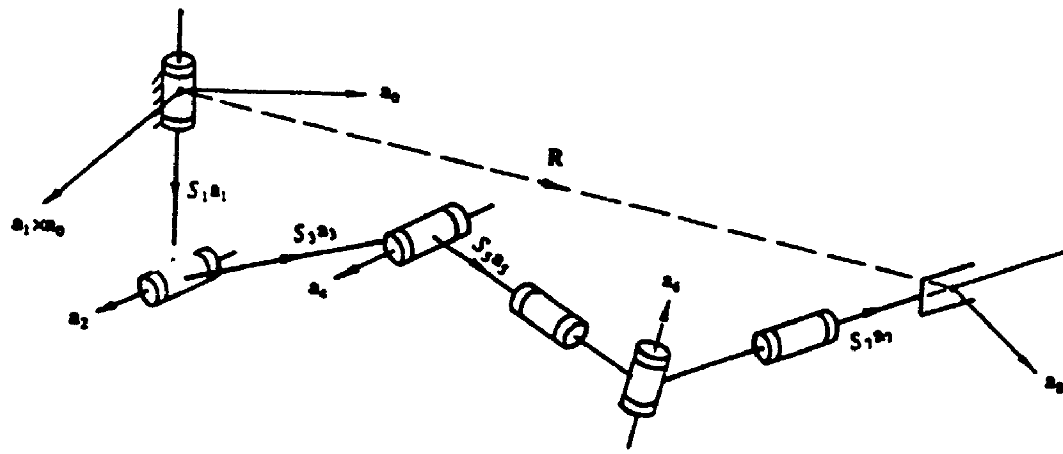


Fig. 11.1

$$\mathbf{R} = S_1\mathbf{a}_1 + S_3\mathbf{a}_3 + S_5\mathbf{a}_5 + S_7\mathbf{a}_7 \tag{11.1}$$

$$\mathbf{a}_{i+1} = \cos\theta_i\, \mathbf{a}_{i-1} + \sin\theta_i\, \mathbf{a}_i \times \mathbf{a}_{i-1} \quad (i = 1 - 7) \tag{11.1a}$$

From (11.1a) and $\mathbf{a}_4 = \mathbf{a}_2$ we get

$$\begin{cases} \mathbf{a}_2 = \cos\theta_1 \mathbf{a}_0 + \sin\theta_1 \mathbf{a}_1 \times \mathbf{a}_0 & (1) \\ \mathbf{a}_3 = \cos\theta_2 \mathbf{a}_1 + \sin\theta_2 \mathbf{a}_2 \times \mathbf{a}_1 & (2) \\ \mathbf{a}_5 = \cos\theta_4 \mathbf{a}_3 + \sin\theta_4 \mathbf{a}_2 \times \mathbf{a}_3 & (3) \\ \mathbf{a}_6 = \cos\theta_5 \mathbf{a}_2 + \sin\theta_5 \mathbf{a}_5 \times \mathbf{a}_2 & (4) \\ \mathbf{a}_7 = \cos\theta_6 \mathbf{a}_5 + \sin\theta_6 \mathbf{a}_6 \times \mathbf{a}_5 & (5) \\ \mathbf{a}_8 = \cos\theta_7 \mathbf{a}_6 + \sin\theta_7 \mathbf{a}_7 \times \mathbf{a}_6 & (6) \end{cases} \tag{11.1b}$$

$\mathbf{a}_0$, $\mathbf{a}_1 \times \mathbf{a}_0$ and $\mathbf{a}_1$ are fixed mutually perpendicular (unit) vectors, which are chosen as reference. Vectors $\mathbf{a}_7$ and $\mathbf{a}_8$ specify the direction and orientation of the end-effector (or hand) and they can be obtained iteratively from equations (11.1a). After $\mathbf{a}_3$, $\mathbf{a}_5$ and $\mathbf{a}_7$ are determined, i.e. expressed in terms of $\mathbf{a}_0$, $\mathbf{a}_1$ and the known parameters, $\mathbf{R}$ can be obtained from (11.1). The detailed calculation is as follow:

Substituting (11.1b-1) into (11.1b-2) yields

$$\mathbf{a}_3 = s\theta_1 s\theta_2 \mathbf{a}_0 - c\theta_1 s\theta_2 \mathbf{a}_1 \times \mathbf{a}_0 + c\theta_2 \mathbf{a}_1 \tag{11.2}$$

Substituting {(11.2), (11.1b-1)} into (11.1b-3) yields

$$\mathbf{a}_5 = \delta_1 \mathbf{a}_0 + \delta_2 \mathbf{a}_1 \times \mathbf{a}_0 + \delta_3 \mathbf{a}_1 \qquad \bullet(11.3)$$

† The equation number specified by a bullet •( ) corresponds to solution.

$$\begin{cases} \delta_1 = s\theta_1(s\theta_2 c\theta_4 + c\theta_2 s\theta_4) \\ \delta_2 = -c\theta_1(s\theta_2 c\theta_4 + c\theta_2 s\theta_4) \\ \delta_3 = (c\theta_2 c\theta_4 - s\theta_2 s\theta_4) \end{cases} \tag{11.3a}$$

Substituting {(11.3), (11.1b-1)} into (11.1b-4) yields

$$\mathbf{a}_6 = \alpha_1 \mathbf{a}_0 + \alpha_2 \mathbf{a}_1 \times \mathbf{a}_0 + \alpha_3 \mathbf{a}_1 \tag{11.4}$$

$$\begin{cases} \alpha_1 = c\theta_1 c\theta_5 - \delta_3 s\theta_1 s\theta_5 \\ \alpha_2 = s\theta_1 c\theta_5 + \delta_3 c\theta_1 s\theta_5 \\ \alpha_3 = \delta_1 s\theta_1 s\theta_5 - \delta_2 c\theta_1 s\theta_5 \end{cases} \tag{11.4a}$$

Substituting {(11.3), (11.4)} into (11.1b-5) yields

$$\mathbf{a}_7 = \beta_1 \mathbf{a}_0 + \beta_2 \mathbf{a}_1 \times \mathbf{a}_0 + \beta_3 \mathbf{a}_1 \tag{11.5}$$

$$\begin{cases} \beta_1 = \delta_1 c\theta_6 + (\alpha_2\delta_3 - \alpha_3\delta_2) s\theta_6 \\ \beta_2 = \delta_2 c\theta_6 + (\alpha_3\delta_1 - \alpha_1\delta_3) s\theta_6 \\ \beta_3 = \delta_3 c\theta_6 + (\alpha_1\delta_2 - \alpha_2\delta_1) s\theta_6 \end{cases} \tag{11.5a}$$

Substituting {(11.4), (11.5)} into (11.1b-6) yields

$$\mathbf{a}_8 = \rho_1 \mathbf{a}_1 + \rho_2 \mathbf{a}_1 \times \mathbf{a}_0 + \rho_3 \mathbf{a}_1 \tag{11.6}$$

$$\begin{cases} \rho_1 = \alpha_1 c\theta_7 + (\beta_2\alpha_3 - \beta_3\alpha_2) s\theta_7 \\ \rho_2 = \alpha_2 c\theta_7 + (\beta_3\alpha_1 - \beta_1\alpha_3) s\theta_7 \\ \rho_3 = \alpha_3 c\theta_7 + (\beta_1\alpha_2 - \beta_2\alpha_1) s\theta_7 \end{cases} \tag{11.6a}$$

Substituting {(11.2), (11.3), (11.5)} into (11.1) yields

$$\mathbf{R} = \sigma_1 \mathbf{a}_0 + \sigma_2 \mathbf{a}_1 \times \mathbf{a}_0 + \sigma_3 \mathbf{a}_1 \tag{11.7}$$

$$\begin{cases} \sigma_1 = S_5\delta_1 + S_7\beta_1 + S_3 s\theta_1 s\theta_2 \\ \sigma_2 = S_5\delta_{\_} + S_7\beta_2 - S_3 c\theta_1 s\theta_2 \\ \sigma_3 = S_5\delta_3 + S_7\beta_3 + S_3 c\theta_2 + S_1 \end{cases} \tag{11.7a}$$

(2) *The Inverse Kinematic Analysis.*

Given: $\{\mathbf{R}, \mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_7, \mathbf{a}_8, \theta_3 = 0, S_j\}$ $(j = 1,3,5,7)$; Find: $\{\theta_i\}$, $(i = 1,2,4,5,6,7)$.

Let $\theta_1 = \Theta_1$ be the "output angle", the first rotational variable to be determined. Then the vector loop equation (11.1) of the robot can be rewritten as,

$$\mathbf{I} = \mathbf{F} \tag{11.8}$$

$$\begin{cases} \mathbf{I} = S_1\mathbf{a}_1 + S_7\mathbf{a}_7 - \mathbf{R} & (1) \\ \mathbf{F} = -(S_3\mathbf{a}_3 + S_5\mathbf{a}_5) & (2) \end{cases} \tag{11.8a}$$

where $\mathbf{I}$ is defined as the *input vector* (of the robot loop). It is the summation of those vectors on the robot loop which are given or known at the beginning. $\mathbf{F}$ is the *floating vector* (of the robot loop).

The scalar product of $\mathbf{a}_2$ with both sides of (3.8) yields

$$\mathbf{I}\cdot\mathbf{a}_2 = 0 \tag{11.9}$$

Substituting (11.1b-1) into (11.9) yields

$$A_1\cos\Theta_1 + B_1\sin\Theta_1 = C_1 \tag{11.10}$$

$$\begin{cases} A_1 = (\mathbf{a}_0\cdot\mathbf{I}) \\ B_1 = (\mathbf{a}_1\times\mathbf{a}_0\cdot\mathbf{I}) \\ C_1 = 0 \end{cases} \tag{11.10a}$$

From (11.10) we get

$$\Theta_1 = \tan^{-1}(-A_1/B_1) \tag{11.11}$$

Squaring both sides of (11.8) and taking into account of (11.8a-2) yields

$$2S_3S_5(\mathbf{a}_3\cdot\mathbf{a}_5) = \mathbf{I}^2 - (S_3^2 + S_5^2) \tag{11.12}$$

Substituting (11.1b-3) into (11.12) yields

$$A_4\cos\theta_4 + B_4\sin\theta_4 = C_4 \tag{11.13}$$

$$\begin{cases} A_4 = 2S_3S_5 \\ B_4 = 0 \\ C_4 = [\mathbf{I}^2 - (S_3^2 + S_5^2)] \end{cases} \tag{11.13a}$$

From (11.13) we get

$$\theta_4 = \cos^{-1}(C_4/A_4) \tag{11.14}$$

At this point, θ_1 and θ_4 are known. From Fig. 11.1 we can see that both $\mathbf{a}_3$ and $\mathbf{a}_5$ can be expressed in terms of one unknown, θ_2. Consequently. we can say that the loop equation (11.8) involves only one unknown, i.e. θ_2. The scalar product of $\{\mathbf{a}_1, \mathbf{a}_2\times\mathbf{a}_1\}$ with both sides of (11.8) yields

$$\begin{cases} (S_3\mathbf{a}_3 + S_5\mathbf{a}_5)\cdot\mathbf{a}_1 = -(\mathbf{I}\cdot\mathbf{a}_1) \\ (S_3\mathbf{a}_3 + S_5\mathbf{a}_5)\cdot\mathbf{a}_2\times\mathbf{a}_1 = -(\mathbf{I}\cdot\mathbf{a}_2\times\mathbf{a}_1) \end{cases} \tag{11.15}$$

Substituting (11.1b-3) into (11.15) yields

$$\begin{cases} \mathbf{U}_2\cdot\mathbf{a}_3 = V_2 \\ \mathbf{U}_2'\cdot\mathbf{a}_3 = V_2' \end{cases} \tag{11.16}$$

$$\begin{cases} \mathbf{U}_2=(S_3+S_5)\mathbf{a}_1-S_5 s\theta_4 \mathbf{a}_2\times\mathbf{a}_1 \\ \mathbf{U}_2'=(S_3+S_5)\mathbf{a}_2\times\mathbf{a}_1+S_5 s\theta_4\mathbf{a}_1 \\ V_2=-(\mathbf{I}\cdot\mathbf{a}_1) \\ V_2'=-[\mathbf{I}\cdot(\mathbf{a}_2\times\mathbf{a}_1)] \end{cases} \tag{11.16a}$$

Substituting (11.1b-2) into (11.16) yields

$$\begin{cases} A_2\cos\theta_2 + B_2\sin\theta_2 = C_2 \\ A_2'\cos\theta_2+B_2'\sin\theta_2=C_2' \end{cases} \tag{11.17}$$

$$\begin{cases} A_2 = \mathbf{U}_2\cdot\mathbf{a}_1 \\ B_2 = \mathbf{U}_2\cdot\mathbf{a}_2\times\mathbf{a}_1 \\ C_2 = V_2 \end{cases} \tag{11.17a}$$

$$\begin{cases} A_2' = \mathbf{U}_2'\cdot\mathbf{a}_1 \\ B_2' = \mathbf{U}_2'\cdot\mathbf{a}_2\times\mathbf{a}_1 \\ C_2' = V_2' \end{cases} \tag{11.17a}$$

Expanding (11.17a), namely, substituting (11.16a) into (11.17a) yields

$$\begin{cases} A_2 = (S_3 + S_5 c\theta_4) \\ B_2 = - S_5 s\theta_4 \\ C_2 = S_7(\mathbf{a}_1\cdot\mathbf{a}_7)+S_1-\sigma_3 \end{cases} \tag{11.17b}$$

$$\begin{cases} A_2'= S_5 s\theta_4 \\ B_2'= (S_3 + S_5 c\theta_4) \\ C_2'=S_7[s\theta_1(\mathbf{a}_0\cdot\mathbf{a}_7)-c\theta_1(\mathbf{a}_1\times\mathbf{a}_0\cdot\mathbf{a}_7)] \\ \qquad -(\sigma_1 s\theta_1-\sigma_2 c\theta_1) \end{cases} \tag{11.17b}$$

From (11.17) we obtain θ_2:

$$\begin{cases} \cos\theta_2=(C_2B_2'-B_2C_2')/(A_2B_2'-B_2A_2') \\ \sin\theta_2=(A_2C_2'-C_2A_2')/(A_2B_2'-B_2A_2') \end{cases} \tag{11.18}$$

Because θ_1, θ_2 and θ_4 have all been determined, $\mathbf{a}_5$ can now be found from equation (11.3). From Fig. 11.1 we can see that $\mathbf{a}_6$ is perpendicular to $\mathbf{a}_7$, i.e.

$$\mathbf{a}_6(\theta_5)\cdot\mathbf{a}_7 = 0 \tag{11.19}$$

where $\mathbf{a}_6$ can be expressed in terms of θ_5. Substituting (11.1b-4) into (11.19) yields

$$A_5\cos\theta_5 + B_5\sin\theta_5 = C_5 \tag{11.20}$$

$$\begin{cases} A_5 = \mathbf{a}_2\cdot\mathbf{a}_7 \\ B_5 = \mathbf{a}_5\times\mathbf{a}_2\cdot\mathbf{a}_7 \\ C_5 = 0 \end{cases} \tag{11.20a}$$

From (11.20) we get

$$\theta_5 = \tan^{-1}(-A_5/B_5) \tag{11.21}$$

The scalar product of $\{\mathbf{a}_i,\ \mathbf{a}_{i+1}\times\mathbf{a}_i\}$ with both sides of (3.1b-i) $(i=5,6)$ yields θ_{i+1}:

$$\begin{cases} \cos\theta_{i+1} = \mathbf{a}_i\cdot\mathbf{a}_{i+2} \\ \sin\theta_{i+1} = \mathbf{a}_{i+1}\times\mathbf{a}_i\cdot\mathbf{a}_{i+2} \end{cases} \quad (i=5,6) \tag{11.22}$$

Because each of the angles θ_1, θ_4 and θ_5 may have two solutions, as determined from (11.11), (11.14) and (11.21), we can conclude that this robot can have a maximum of eight closures or inverse solutions.

11.3. Analysis of the second 6*R* robot

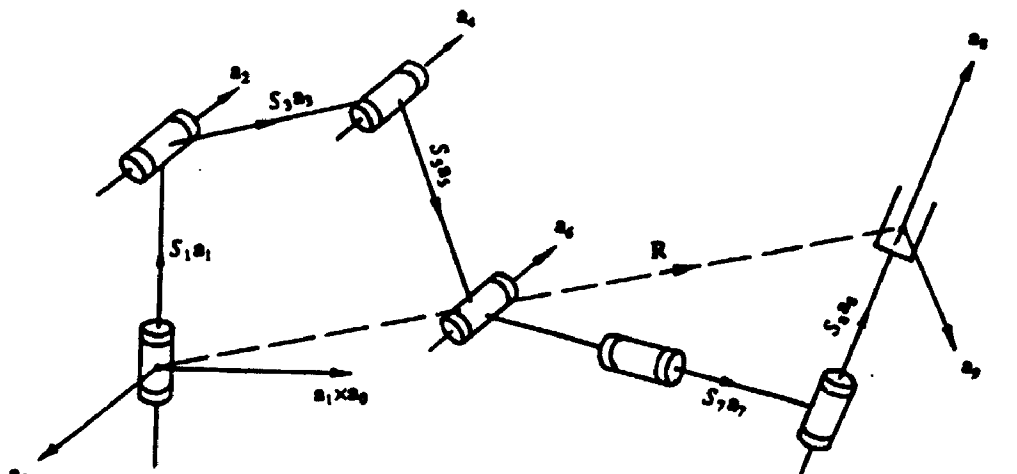


Fig. 11.2

(1) *The Forward Kinematic Analysis.*

The configuration of the second 6R robot is shown in Fig. 11.2, where $\{\mathbf{a}_i\}$ $(i=0\text{-}9)$ are unit vectors and $\mathbf{a}_i\cdot\mathbf{a}_{i+1}=0$. Let $\theta_i = (\mathbf{a}_{i-1}\hat{,}\mathbf{a}_{i+1})$ $(i=1\text{-}8)$, be the right handed rotation from $\mathbf{a}_{i-1}$ to $\mathbf{a}_{i+1}$ about $\mathbf{a}_i$. This robot has six rotary controllable variables, i.e.

$$\begin{cases} \theta_i & (i=1,2,4,6,7,8) \\ \theta_j=0 & (j=3,5) \end{cases}$$

Now the forward analysis problem can be stated as follow. Given: $\{\mathbf{a}_0, \mathbf{a}_1, \theta_i, S_k\}$, $(i=1\text{-}8,\ k=1,3,5,7,8)$; Find: $\{\mathbf{a}_8, \mathbf{a}_9, \mathbf{R}\}$.

From Fig. 11.2, we can write the vector loop equation and structure constraint equations directly,

$$\mathbf{R} = S_1\mathbf{a}_1 + S_3\mathbf{a}_3 + S_5\mathbf{a}_5 + S_7\mathbf{a}_7 + S_8\mathbf{a}_8 \tag{11.23}$$

$$\mathbf{a}_{i+1} = \cos\theta_i\,\mathbf{a}_{i-1} + \sin\theta_i\,\mathbf{a}_i\times\mathbf{a}_{i-1} \quad (i=1\text{-}8) \tag{11.23a}$$

From (11.23a) and considering that $\theta_3=\theta_5=0$, i.e. $(\mathbf{a}_2=\mathbf{a}_4=\mathbf{a}_6)$ we get

$$\begin{cases} \mathbf{a}_2 = c\theta_1\mathbf{a}_0 + s\theta_1\mathbf{a}_1\times\mathbf{a}_0 & (1) \\ \mathbf{a}_3 = c\theta_2\mathbf{a}_1 + s\theta_2\mathbf{a}_2\times\mathbf{a}_1 & (2) \\ \mathbf{a}_5 = c\theta_4\mathbf{a}_3 + s\theta_4\mathbf{a}_2\times\mathbf{a}_3 & (3) \\ \mathbf{a}_7 = c\theta_6\mathbf{a}_5 + s\theta_6\mathbf{a}_2\times\mathbf{a}_5 & (4) \\ \mathbf{a}_8 = c\theta_7\mathbf{a}_2 + s\theta_7\mathbf{a}_7\times\mathbf{a}_2 & (5) \\ \mathbf{a}_9 = c\theta_8\mathbf{a}_7 + s\theta_8\mathbf{a}_8\times\mathbf{a}_7 & (6) \end{cases} \tag{11.23b}$$

Substituting (11.23b-1) into (11.23b-2) yields

$$\mathbf{a}_3 = s\theta_1 s\theta_2\mathbf{a}_0 - c\theta_1 s\theta_2\mathbf{a}_1\times\mathbf{a}_0 + c\theta_2\mathbf{a}_1 \tag{11.24}$$

Substituting {(11.24), (11.23b-1)} into (11.23b-3) yields

$$\mathbf{a}_5 = \delta_1\mathbf{a}_0 + \delta_2\mathbf{a}_1\times\mathbf{a}_0 + \delta_3\mathbf{a}_1 \tag{11.25}$$

$$\begin{cases} \delta_1 = \ \ s\theta_1(s\theta_2 c\theta_4 + c\theta_2 s\theta_4) \\ \delta_2 = -c\theta_1(s\theta_2 c\theta_4 + c\theta_2 s\theta_4) \\ \delta_3 = \ (c\theta_2 c\theta_4 - s\theta_2 s\theta_4) \end{cases} \tag{11.25a}$$

Substituting {(11.25), (11.23b-1)} into (11.23b-4) yields

$$\mathbf{a}_7 = \alpha_1\mathbf{a}_0 + \alpha_2\mathbf{a}_1\times\mathbf{a}_0 + \alpha_3\mathbf{a}_1 \tag{11.26}$$

$$\begin{cases} \alpha_1 = \delta_1 c\theta_6 + \delta_3 s\theta_1 s\theta_6 \\ \alpha_2 = \delta_2 c\theta_6 - \delta_3 c\theta_1 s\theta_6 \\ \alpha_3 = \delta_3 c\theta_6 + (\delta_2 c\theta_1 - \delta_1 s\theta_1) s\theta_6 \end{cases} \tag{11.26a}$$

Substituting {(11.26), (11.23b-1)} into (11.23b-5) yields

$$\mathbf{a}_8 = \beta_1\mathbf{a}_0 + \beta_2\mathbf{a}_1\times\mathbf{a}_0 + \beta_3\mathbf{a}_1 \tag{11.27}$$

$$\begin{cases} \beta_1 = c\theta_1 c\theta_7 - \alpha_3 s\theta_1 s\theta_7 \\ \beta_2 = s\theta_1 c\theta_7 + \alpha_3 c\theta_1 s\theta_7 \\ \beta_3 = \alpha_1 s\theta_1 s\theta_7 - \alpha_2 c\theta_1 s\theta_7 \end{cases} \tag{11.27a}$$

Substituting {(11.26), (11.27)} into (11.23b-6) yields

$$\mathbf{a}_9 = \rho_1\mathbf{a}_0 + \rho_2\mathbf{a}_1\times\mathbf{a}_0 + \rho_3\mathbf{a}_1 \tag{11.28}$$

$$\begin{cases} \rho_1 = \alpha_1 c\theta_8 + (\beta_2\alpha_3 - \beta_3\alpha_2)s\theta_8 \\ \rho_2 = \alpha_2 c\theta_8 + (\beta_3\alpha_1 - \beta_1\alpha_3)s\theta_8 \\ \rho_3 = \alpha_3 c\theta_8 + (\beta_1\alpha_2 - \beta_2\alpha_1)s\theta_8 \end{cases} \tag{11.28a}$$

Substituting {(11.24), (11.25), (11.26), (11.27)} into (11.23) yields

$$\mathbf{R} = \sigma_1 \mathbf{a}_0 + \sigma_2 \mathbf{a}_1 \times \mathbf{a}_0 + \sigma_3 \mathbf{a}_1 \tag{11.29}$$

$$\begin{cases} \sigma_1 = S_8\beta_1 + S_7\alpha_1 + S_5\delta_1 + S_3 s\theta_1 s\theta_2 \\ \sigma_2 = S_8\beta_2 + S_7\alpha_2 + S_5\delta_2 - S_3 c\theta_1 s\theta_2 \\ \sigma_3 = S_8\beta_3 + S_7\alpha_3 + S_5\delta_3 + S_3 c\theta_2 + \delta_1 \end{cases} \tag{11.29a}$$

(2) *The Inverse Kinematic Analysis.*

Given: $\{\mathbf{R}, \mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_8, \mathbf{a}_9, S_i\}$ $(i = 1,3,5,7,8)$; Find: $\{\theta_i\}$, $(i = 1,2,4,6,7,8)$.

Let $\theta_1 = \Theta_1$ be the "output angle", the first rotary variable to be determined. Then the vector loop equation of the robot can be rewritten as,

$$\mathbf{I} = \mathbf{F} \tag{11.30}$$

$$\begin{cases} \mathbf{I} = (S_1\mathbf{a}_1 + S_8\mathbf{a}_8 - \mathbf{R}) & (1) \\ \mathbf{F} = -(S_3\mathbf{a}_3 + S_5\mathbf{a}_5 + S_7\mathbf{a}_7) & (2) \end{cases} \tag{11.30a}$$

$$\begin{cases} \mathbf{a}_7 = c\theta_8\mathbf{a}_9 - s\theta_8\mathbf{a}_8 \times \mathbf{a}_9 & (1) \\ \mathbf{a}_6 = c\theta_7\mathbf{a}_8 - s\theta_7\mathbf{a}_7 \times \mathbf{a}_8 & (2) \end{cases} \tag{11.30b}$$

From the scalar product of $\mathbf{a}_2$ with both sides of (11.30) and the robot geometry we have

$$\mathbf{I} \cdot \mathbf{a}_2 = 0 \tag{11.31}$$

Substituting (11.23b-1) into (11.31) yields

$$A_1 \cos\theta_1 + B_1 \sin\theta_1 = C_1 \tag{11.32}$$

$$\begin{cases} A_1 = \mathbf{I} \cdot \mathbf{a}_0 \\ B_1 = \mathbf{I} \cdot \mathbf{a}_1 \times \mathbf{a}_0 \\ C_1 = 0 \end{cases} \tag{11.32a}$$

From (11.32) we get

$$\Theta_1 = \tan^{-1}(-A_1/B_1) \tag{11.33}$$

From $\mathbf{a}_6 \cdot \mathbf{a}_7 = 0$ and $\mathbf{a}_6 = \mathbf{a}_2$ we have

$$\mathbf{a}_2 \cdot \mathbf{a}_7 = 0 \tag{11.34}$$

Substituting (11.30b-1) into (11.34) yields

$$A_8\cos\theta_8 + B_8\sin\theta_8 = C_8 \tag{11.35}$$

$$\begin{cases} A_8 = \mathbf{a}_2\cdot\mathbf{a}_9 \\ B_8 = -\mathbf{a}_2\cdot\mathbf{a}_8\times\mathbf{a}_9 \\ C_8 = 0 \end{cases} \tag{11.35a}$$

From (11.35) we get

$$\theta_8 = \tan^{-1}(-A_8/B_8) \tag{11.36}$$

From (11.30b-2) and $\mathbf{a}_6=\mathbf{a}_2$ we get

$$\mathbf{a}_2 = c\theta_7\mathbf{a}_8 - s\theta_7\mathbf{a}_7\times\mathbf{a}_8 \tag{11.36}$$

The scalar product of $\{\mathbf{a}_2,\ \mathbf{a}_2\times\mathbf{a}_1\}$ with both sides of (11.36) yields

$$\begin{cases} (\mathbf{a}_2\cdot\mathbf{a}_8)c\theta_7 - (\mathbf{a}_2\cdot\mathbf{a}_7\times\mathbf{a}_8)s\theta_7 = 1 \\ (\mathbf{a}_2\times\mathbf{a}_1\cdot\mathbf{a}_8)c\theta_7 - (\mathbf{a}_2\times\mathbf{a}_1\cdot\mathbf{a}_7\times\mathbf{a}_8)s\theta_7 = 0 \end{cases} \tag{11.37}$$

Solving (11.37) yields θ_7:

$$\begin{cases} \cos\theta_7 = [(\mathbf{a}_1\times\mathbf{a}_2)\cdot(\mathbf{a}_7\times\mathbf{a}_8)]/(\mathbf{a}_1\cdot\mathbf{a}_7) \\ \sin\theta_7 = [(\mathbf{a}_1\times\mathbf{a}_2)\cdot\mathbf{a}_8]/(\mathbf{a}_1\cdot\mathbf{a}_7) \end{cases} \tag{11.38}$$

Where $\mathbf{a}_7$ is given by (11.30b-1) and (11.36). Now the vector loop equation of (11.30) can be rearranged as:

$$S_3\mathbf{a}_3 + S_5\mathbf{a}_5 = -(\mathbf{I} + S_7\mathbf{a}_7) \tag{11.39}$$

Squaring both sides of (11.39) yields

$$2S_3S_5(\mathbf{a}_3\cdot\mathbf{a}_5) = (\mathbf{I}+S_7\mathbf{a}_7)^2 - (S_3^2 + S_5^2) \tag{11.40}$$

Substituting (11.23b-3) into (11.39) yields

$$A_4\cos\theta_4 + B_4\sin\theta_4 = C_4 \tag{11.41}$$

$$\begin{cases} A_4 = 2S_3S_5 \\ B_4 = 0 \\ C_4 = [(\mathbf{I}+S_7\mathbf{a}_7)^2 - (S_3^2 + S_5^2)] \end{cases} \tag{11.41a}$$

From (11.41) we get

$$\theta_4 = \cos^{-1}(C_4/A_4) \tag{11.42}$$

Substituting (11.23b-3) into (11.39) yields

$$(S_3+S_5c\theta_4)\mathbf{a}_3 + S_5s\theta_4\mathbf{a}_2\times\mathbf{a}_3 = -(\mathbf{I}+S_7\mathbf{a}_7) \tag{11.43}$$

The scalar product of $\{\mathbf{a}_1, \mathbf{a}_2\times\mathbf{a}_1\}$ with both sides of (11.43) yields

$$\begin{cases} \mathbf{U}_2\cdot\mathbf{a}_3 = V_2 \\ \mathbf{U}_2'\cdot\mathbf{a}_3 = V_2' \end{cases} \tag{11.44}$$

$$\begin{cases} \mathbf{U}_2 = (S_3+S_5c\theta_4)\mathbf{a}_1 - S_5s\theta_4\mathbf{a}_2\times\mathbf{a}_1 \\ \mathbf{U}_2' = (S_3+S_5c\theta_4)\mathbf{a}_2\times\mathbf{a}_1 + S_5s\theta_4\mathbf{a}_1 \\ V_2 = -(\mathbf{l}+S_7\mathbf{a}_7)\cdot\mathbf{a}_1 \\ V_2' = -(\mathbf{l}+S_7\mathbf{a}_7)\cdot\mathbf{a}_2\times\mathbf{a}_1 \end{cases} \tag{11.44a}$$

Substituting (11.23b-2) into (11.44) yields

$$\begin{cases} A_2\cos\theta_2 + B_2\sin\theta_2 = C_2 \\ A_2'\cos\theta_2 + B_2'\sin\theta_2 = C_2' \end{cases} \tag{11.45}$$

$$\begin{cases} A_2 = \mathbf{U}_2\cdot\mathbf{a}_1 \\ B_2 = \mathbf{U}_2\cdot\mathbf{a}_2\times\mathbf{a}_1 \\ C_2 = V_2 \end{cases} \tag{11.45a}$$

$$\begin{cases} A_2' = \mathbf{U}_2'\cdot\mathbf{a}_1 \\ B_2' = \mathbf{U}_2'\cdot\mathbf{a}_2\times\mathbf{a}_1 \\ C_2' = V_2' \end{cases} \tag{11.45a}$$

From (11.45) we get θ_2:

$$\begin{cases} \cos\theta_2 = (C_2B_2'-B_2C_2')/(A_2B_2'-B_2A_2') \\ \sin\theta_2 = (A_2C_2'-C_2A_2')/(A_2B_2'-B_2A_2') \end{cases} \tag{11.46}$$

The scalar product of $\{\mathbf{a}_5, \mathbf{a}_2\times\mathbf{a}_5\}$ with both sides of (11.23b-4) yields θ_6:

$$\begin{cases} \cos\theta_6 = \mathbf{a}_5\cdot\mathbf{a}_7 \\ \sin\theta_6 = \mathbf{a}_2\times\mathbf{a}_5\cdot\mathbf{a}_7 \end{cases} \tag{11.47}$$

Because each of the angles θ_1, θ_8 and θ_4 may have two solutions, as determined from (11.33), (11.36) and (11.42), we conclude that this robot has at most eight closures or inverse solutions.

11.4. Conclusion

It is demonstrated in the above analysis that simplicity is an intrinsic property of the vector algebraic method.

The forward kinematic analysis is straightforward and warrants no additional comments. As for the inverse kinematic analysis, after the first controllable variable is determined, there is a lot of flexibility in deciding which variable is determined next. However, the analysis steps are similar and the expressions are standardized.

Although the two robots considered here have special geometries, i.e. a spherical wrist or three parallel axes, the method is general and can be utilized to analyze any robot. The two robots considered in this chapter are frequently used in industry and their matrix analyses have appeared in many textbooks on robotics. The reason they were chosen for analysis in this chapter is to serve as a reference for comparison.

CHAPTER 12. TWO ROBOTS RRPRRR AND RPRPRR

12.1. Introduction

Two six degree-of-freedom serial robots RRPRRR and RPRPRR are kinematically analyzed using vector algebraic method.

12.2. Analysis of the *RRPRRR* robot

(1). *The Forward Kinematic Analysis.*

The diagram of the robot RRPRRR are shown in Fig. 12.1, where $\{\mathbf{a}_i\}$ $(i=1\text{-}6)$ are all unit vectors. This robot has one translational and five rotational controllable variables, which will be denoted as x_3 and θ_i $(i=1\text{-}5)$, where $\theta_i = (\mathbf{a}_{i-1} \hat{,} \mathbf{a}_{i+1})$ $(i=1\text{-}5)$ being measured by the right rotation of $\mathbf{a}_{i-1}$ to $\mathbf{a}_{i+1}$ about $\mathbf{a}_i$.

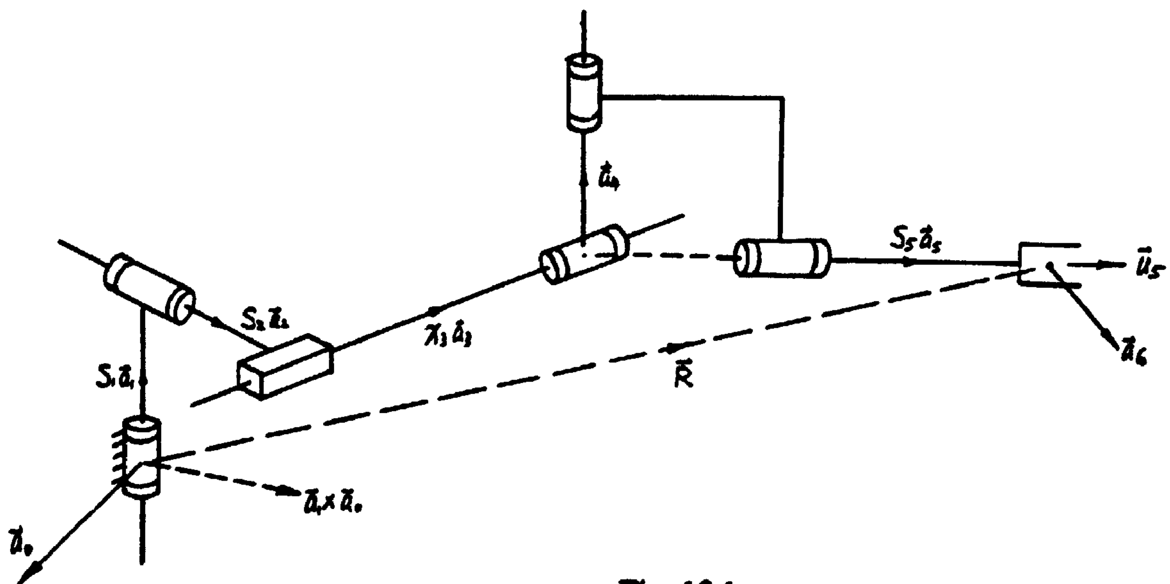


Fig. 12.1

Now the problem can be stated as follow. Given: $\{\mathbf{a}_0, \mathbf{a}_1, S_1, S_2, S_3, x_3, \theta_i\}$, $(i=1\text{-}5)$; Unknown: $\{\mathbf{a}_5, \mathbf{a}_6, \mathbf{R}\}$.

From Fig. 12.1, we can write the vector loop equation and structure constraint equations directly,

$$\mathbf{R} = S_1\mathbf{a}_1 + S_2\mathbf{a}_2 + x_3\mathbf{a}_3 + S_5\mathbf{a}_5 \tag{12.1}$$

$$\begin{cases} \mathbf{a}_2 = \cos\theta_1\,\mathbf{a}_0 + \sin\theta_1\,\mathbf{a}_1\times\mathbf{a}_0 & (1) \\ \mathbf{a}_3 = \cos\theta_2\,\mathbf{a}_1 + \sin\theta_2\,\mathbf{a}_2\times\mathbf{a}_1 & (2) \\ \mathbf{a}_4 = \cos\theta_3\,\mathbf{a}_2 + \sin\theta_3\,\mathbf{a}_3\times\mathbf{a}_2 & (3) \\ \mathbf{a}_5 = \cos\theta_4\,\mathbf{a}_3 + \sin\theta_4\,\mathbf{a}_4\times\mathbf{a}_3 & (4) \\ \mathbf{a}_6 = \cos\theta_5\,\mathbf{a}_4 + \sin\theta_5\,\mathbf{a}_5\times\mathbf{a}_4 & (5) \end{cases} \tag{12.1a}$$

$\mathbf{a}_0$ and $\mathbf{a}_1$ are fixed (unit) vectors, which can be chosen as reference. Vectors $\mathbf{a}_5$ and $\mathbf{a}_6$ expressed in terms of $\mathbf{a}_0$ and $\mathbf{a}_1$ can be obtained directly from iterations of the equations of (12.1a). After $\mathbf{a}_3$ and $\mathbf{a}_5$ are determined, i.e. expressed in terms of $\mathbf{a}_0$, $\mathbf{a}_1$ and the joint angle parameters, $\mathbf{R}$ can be obtained from (12.1).

Substituting (12.1a-1) into (12.1a-2) yields

$$\mathbf{a}_3 = s\theta_1 s\theta_2\,\mathbf{a}_0 - c\theta_1 s\theta_2\,\mathbf{a}_1\times\mathbf{a}_0 + c\theta_2\,\mathbf{a}_1 \tag{12.2}$$

Substituting {(12.1a-1), (12.1a-2), (12.1a-3)} into (12.1a-4) yields

$$\mathbf{a}_5 = \beta_1\,\mathbf{a}_0 + \beta_2\,\mathbf{a}_1\times\mathbf{a}_0 + \beta_3\,\mathbf{a}_1 \tag{12.3}$$

$$\begin{cases} \beta_1 = s\theta_4(c\theta_1 s\theta_3 + s\theta_1 c\theta_2 c\theta_3) + s\theta_1 s\theta_2 c\theta_4 \\ \beta_2 = s\theta_4(s\theta_1 s\theta_3 - c\theta_1 c\theta_2 c\theta_3) - c\theta_1 s\theta_2 c\theta_4 \\ \beta_3 = c\theta_2 c\theta_4 - s\theta_2 c\theta_3 s\theta_4 \end{cases} \tag{12.3a}$$

Similarly,

$$\mathbf{a}_6 = \rho_1\,\mathbf{a}_0 + \rho_2\,\mathbf{a}_1\times\mathbf{a}_0 + \rho_3\,\mathbf{a}_1 \tag{12.4}$$

$$\begin{cases} \rho_1 = c\theta_5(c\theta_1 c\theta_3 - s\theta_1 c\theta_2 s\theta_3) + s\theta_1 s\theta_2 s\theta_4 s\theta_5 \\ \qquad - c\theta_4 s\theta_5(c\theta_1 s\theta_3 + s\theta_1 c\theta_2 c\theta_3) \\ \rho_2 = c\theta_5(s\theta_1 c\theta_3 + c\theta_1 c\theta_2 s\theta_3) - c\theta_1 s\theta_2 s\theta_4 s\theta_5 \\ \qquad - c\theta_4 s\theta_5(s\theta_1 s\theta_3 - c\theta_1 c\theta_2 c\theta_3) \\ \rho_3 = s\theta_5(c\theta_2 s\theta_4 + s\theta_2 c\theta_3 c\theta_4) + s\theta_2 s\theta_3 c\theta_5 \end{cases} \tag{12.4a}$$

Substituting {(12.1a-1), (12.2), (12.3)} into (12.1) yields

$$\mathbf{R} = \gamma_1\,\mathbf{a}_0 + \gamma_2\,\mathbf{a}_1\times\mathbf{a}_0 + \gamma_3\,\mathbf{a}_1 \tag{12.5}$$

$$\begin{cases} \gamma_1 = S_2 c\theta_1 + x_3 s\theta_1 s\theta_2 + S_5\beta_1 \\ \gamma_2 = S_2 s\theta_1 - x_3 c\theta_1 s\theta_2 + S_5\beta_2 \\ \gamma_3 = S_1 + x_3 c\theta_2 + S_5\beta_3 \end{cases} \tag{12.5a}$$

(2). *Inverse Kinematic Analysis.*

Given: $\{\mathbf{R}, \mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_5, \mathbf{a}_6\}$; Unknown: $\{x_3, \theta_i\}$, $(i=1\text{-}5)$.

Without any loss of generality, let $\theta_1 = \Theta_1$, i.e. θ_1 is taken as the "output angle", which is the first rotational variables to be determined. Then the vector loop equation of the robot can be rewritten as,

$$x_3\mathbf{a}_3 + \mathbf{I} + \mathbf{J} = 0 \tag{12.6}$$

$$\begin{cases} \mathbf{J} = \mathbf{J}(\Theta_1) = S_2\mathbf{a}_2 & (1) \\ \mathbf{I} = S_1\mathbf{a}_1 + S_5\mathbf{a}_5 - \mathbf{R} & (2) \end{cases} \tag{12.6a}$$

Where $\mathbf{I}$ is defined as the *input vector* (of the robot loop), it is the summation of those vectors on the robot loop which are given or known at the very beginning. $\mathbf{J}$ is defined as the *output vector* (of the robot loop), it is the summation of those vectors on the robot loop which can be expressed in terms of the output angle, in this case it is Θ_1.

The scalar product of $\mathbf{a}_2$ with both sides of (12.6) yields

$$\mathbf{a}_2\cdot(\mathbf{I} + \mathbf{J}) = 0 \tag{12.7}$$

Substituting (12.6a-1) into (12.7) yields

$$\mathbf{I}\cdot\mathbf{a}_2 = -S_2 \tag{12.8}$$

Substituting (12.1a-1) into (12.8) yields

$$A\cos\Theta_1 + B\sin\Theta_1 = C \tag{12.9}$$

$$\begin{cases} A = \mathbf{I}\cdot\mathbf{a}_0 \\ B = \mathbf{I}\cdot\mathbf{a}_1\times\mathbf{a}_0 \\ C = -S_2 \end{cases} \tag{12.9a}$$

Let $y = \tan(\Theta_1/2)$, then we have,

$$\begin{cases} \cos\Theta_1 = (1 - y^2, / (1 + y^2) \\ \sin\Theta_1 = 2y / (1 + y^2) \end{cases} \tag{12.10}$$

Substituting (12.10) into (12.9) yields

$$(A + C)y^2 - 2By + (C - A) = 0 \tag{12.11}$$

Solving (12.11) yields y, then $\Theta_1 = 2\tan^{-1}y$, i.e.

$$\Theta_1 = 2\tan^{-1}\{(B \pm \sqrt{A^2 + B^2 - C^2}) / (A + C)\} \qquad \bullet(12.12)$$

Now x_3 and $\mathbf{a}_3$ are the only unknowns in (12.6), from (12.6) =>

$$x_3 = |\mathbf{I} + \mathbf{J}| = \sqrt{(\mathbf{I}^2 + 2\mathbf{I}\cdot\mathbf{J} + \mathbf{J}^2)} \qquad \bullet(12.13)$$

$$\mathbf{a}_3 = -(\mathbf{I} + \mathbf{J}) / x_3 \tag{12.14}$$

From Fig. 12.1 we can see clearly that,

$$\mathbf{a}_4 = \pm (\mathbf{a}_3 \times \mathbf{a}_5) \, / \, |\mathbf{a}_3 \times \mathbf{a}_5| \tag{12.15}$$

Substituting (12.14) into (12.15) and

$$|\mathbf{a}_3 \times \mathbf{a}_5| = \sqrt{(\mathbf{a}_3 \times \mathbf{a}_5)^2} = \sqrt{1 - (\mathbf{a}_3 \cdot \mathbf{a}_5)^2}$$

we obtain

$$\mathbf{a}_4 = \pm [(\mathbf{I} + \mathbf{J}) \times \mathbf{a}_5] \, / \, \sqrt{x_3^2 - [(\mathbf{I} + \mathbf{J}) \cdot \mathbf{a}_5]^2} \tag{12.16}$$

Now $\{\mathbf{a}_i\}$ $(i=1\text{-}6)$ are all known vectors, where $\mathbf{a}_2$, $\mathbf{a}_3$ and $\mathbf{a}_4$ are given by (12.1a-1), (12.14) and (12.16), respectively. $\{\theta_i\}$ $(i=1\text{-}5)$ can be easily obtained from (12.1a-2) to (12.1a-5).

The scalar product of $\{\mathbf{a}_{i-1}, \, \mathbf{a}_i \times \mathbf{a}_{i-1}\}$ with both sides of (12.1a-i), $(i=2\text{-}5)$ yields θ_i:

$$\begin{cases} \cos\theta_i = (\mathbf{a}_{i-1} \cdot \mathbf{a}_{i+1}) \\ \sin\theta_i = (\mathbf{a}_i \times \mathbf{a}_{i-1} \cdot \mathbf{a}_{i+1}) \end{cases} \qquad (i = 2, 3, 4, 5) \tag{12.17}$$

Because θ_1 and $\mathbf{a}_4$ both have two solutions, this can be seen from (12.12) and (12.16), therefore, we conclude that this robot has four closures.

12.3. Analysis of the *RPRPRR* robot

(1). *The Forward Kinematic Analysis.*

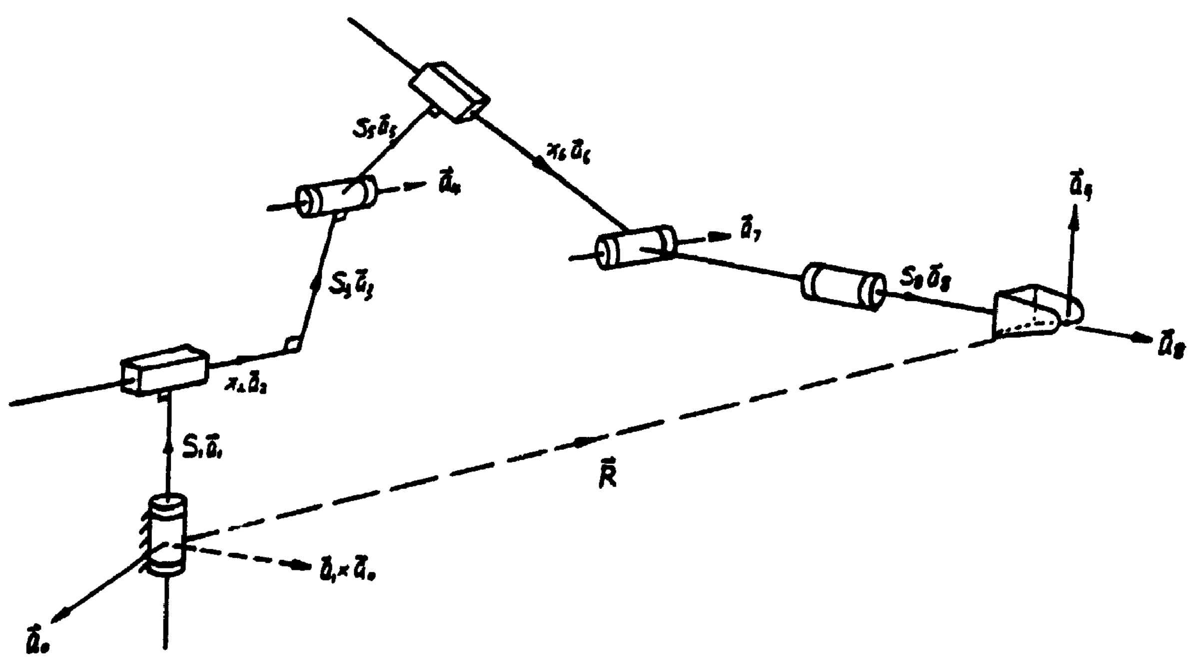


Fig. 12.2

The diagram of the robot RPRPRR are shown in Fig. 12.2, where $\{\mathbf{a}_i\}$ $(i=1\text{-}9)$ are all unit vectors. Denote $\theta_i = (\mathbf{a}_{i-1}\hat{,}\mathbf{a}_{i+1})$ $(i=1\text{-}8)$ being measured by the right rotation of $\mathbf{a}_{i-1}$ to $\mathbf{a}_{i+1}$ about $\mathbf{a}_i$. This robot has four rotational and two translational controllable variables, i.e.

$$\begin{cases} \theta_i & (i=1,4,7,8) \\ x_j & (j=2,6) \end{cases} \tag{12.18}$$

See Fig. 12.2, the following results relating the structure parameters are given:

$$\begin{cases} \theta_2 = (\mathbf{a}_1\hat{,}\mathbf{a}_3) = const. \\ \theta_3 = 0 \\ \theta_5 = 3\pi/4 \\ \theta_6 = 3\pi/4 \end{cases} \tag{12.19}$$

Now the forward displacement problem can be stated as follow. Given: $\{\mathbf{a}_0, \mathbf{a}_1, \theta_i, x_j, S_k\}$, $(i=1\text{-}8,\ j=2,6,\ k=1,3,5,8)$; Unknown: $\{\mathbf{a}_8, \mathbf{a}_9, \mathbf{R}\}$.

From Fig. 12.2, we can write the vector loop equation and structure constraint equations directly,

$$\mathbf{R} = S_1\mathbf{a}_1 + x_2\mathbf{a}_2 + S_3\mathbf{a}_3 + S_5\mathbf{a}_5 + x_6\mathbf{a}_6 + S_8\mathbf{a}_8 \tag{12.20}$$

$$\mathbf{a}_{i+1} = \cos\theta_i\,\mathbf{a}_{i-1} + \sin\theta_i\,\mathbf{a}_i\times\mathbf{a}_{i-1} \quad (i=1\text{-}8) \tag{12.20a}$$

From (12.20a) and considering (12.19) and $(\mathbf{a}_2=\mathbf{a}_4=\mathbf{a}_7)$ we get

$$\begin{cases} \mathbf{a}_2 = c\theta_1\mathbf{a}_0 + s\theta_1\mathbf{a}_1\times\mathbf{a}_0 & (1) \\ \mathbf{a}_3 = c\theta_2\mathbf{a}_1 + s\theta_2\mathbf{a}_2\times\mathbf{a}_1 & (2) \\ \mathbf{a}_5 = c\theta_4\mathbf{a}_3 + s\theta_4\mathbf{a}_2\times\mathbf{a}_3 & (3) \\ \mathbf{a}_6 = \mathbf{a}_2\times\mathbf{a}_5 & (4) \\ \mathbf{a}_8 = c\theta_7\mathbf{a}_6 + s\theta_7\mathbf{a}_2\times\mathbf{a}_6 & (5) \\ \mathbf{a}_9 = c\theta_8\mathbf{a}_2 + s\theta_8\mathbf{a}_8\times\mathbf{a}_2 & (6) \end{cases} \tag{12.20b}$$

Substituting (12.20b-1) into (12.20b-2) yields

$$\mathbf{a}_3 = s\theta_1 s\theta_2\mathbf{a}_0 - c\theta_1 s\theta_2\mathbf{a}_1\times\mathbf{a}_0 + c\theta_2\mathbf{a}_1 \tag{12.21}$$

Substituting {(12.20b-1), (12.20b-2)} into (12.20b-3) yields

$$\mathbf{a}_5 = \delta_1\mathbf{a}_0 + \delta_2\mathbf{a}_1\times\mathbf{a}_0 + \delta_3\mathbf{a}_1 \tag{12.22}$$

$$\begin{cases} \delta_1 = s\theta_1(s\theta_2 c\theta_4 + c\theta_2 s\theta_4) \\ \delta_2 = -c\theta_1(s\theta_2 c\theta_4 + c\theta_2 s\theta_4) \\ \delta_3 = (c\theta_2 c\theta_4 - s\theta_2 s\theta_4) \end{cases} \tag{12.22a}$$

Substituting {(12.22), (12.20b-1)} into (12.20b-4) yields

$$\mathbf{a}_6 = \delta_3 s\theta_1\mathbf{a}_0 - \delta_3 c\theta_1\mathbf{a}_1\times\mathbf{a}_0 + (\delta_2 c\theta_1 - \delta_1 s\theta_1)\mathbf{a}_1 \tag{12.23}$$

Substituting {(12.23), (12.20b-1)} into (12.20b-5) yields

$$\mathbf{a}_8 = \beta_1 \mathbf{a}_0 + \beta_2 \mathbf{a}_1 \times \mathbf{a}_0 + \beta_3 \mathbf{a}_1 \tag{12.24}$$

$$\begin{cases} \beta_1 = s\theta_1(m\,c\theta_2 - n\,s\theta_2) \\ \beta_2 = -c\theta_1(m\,c\theta_2 - n\,s\theta_2) \\ \beta_3 = -(m\,s\theta_2 + n\,c\theta_2) \end{cases} \tag{12.24a}$$

$$\begin{cases} m = (c\theta_4 c\theta_7 - s\theta_4 s\theta_7) \\ n = -(c\theta_4 s\theta_7 + s\theta_4 c\theta_7) \end{cases} \tag{12.24b}$$

Substituting {(12.24), (12.20b-1)} into (12.20b-6) yields

$$\mathbf{a}_9 = \rho_1 \mathbf{a}_0 + \rho_2 \mathbf{a}_1 \times \mathbf{a}_0 + \rho_3 \mathbf{a}_1 \tag{12.25}$$

$$\begin{cases} \rho_1 = c\theta_1 c\theta_8 - \beta_3 s\theta_1 s\theta_8 \\ \rho_2 = s\theta_1 c\theta_8 + \beta_3 c\theta_1 s\theta_8 \\ \rho_3 = \beta_1 s\theta_1 s\theta_8 - \beta_2 c\theta_1 s\theta_8 \end{cases} \tag{12.25a}$$

Substituting {(12.20b-1), (12.21), (12.22), (12.23), (12.24)} into (12.20) yields

$$\mathbf{R} = \sigma_1 \mathbf{a}_0 + \sigma_2 \mathbf{a}_1 \times \mathbf{a}_0 + \sigma_3 \mathbf{a}_1 \tag{12.26}$$

$$\begin{cases} \sigma_1 = x_2 c\theta_1 + S_3 s\theta_1 s\theta_2 + S_5\delta_1 + S_6\delta_3 s\theta_1 + S_8\beta_1 \\ \sigma_2 = x_2 s\theta_1 - S_3 c\theta_1 s\theta_2 + S_5\delta_2 - S_6\delta_3 c\theta_1 + S_8\beta_2 \\ \sigma_3 = S_1 + S_3 c\theta_2 + S_5\delta_3 + S_6(\delta_2 c\theta_1 - \delta_1 s\theta_1) + S_8\beta_3 \end{cases} \tag{12.26a}$$

(2). *Inverse Kinematic Analysis.*

Given: $\{\mathbf{R}, \mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_8, \mathbf{a}_9\}$; Unknown: $\{x_2, x_6, \theta_i\}$, $(i = 1, 4, 7, 8)$.

Without any loss of generality, let $\theta_1 = \Theta_1$, i.e. θ_1 is taken as the "output angle", which is the first rotational variables to be determined. Then the vector loop equation of the robot can be rewritten as,

$$x_3\mathbf{a}_3 + x_6\mathbf{a}_6 + S_5\mathbf{a}_5 + \mathbf{I} + \mathbf{J} = 0 \tag{12.27}$$

$$\begin{cases} \mathbf{J} = \mathbf{J}(\Theta_1) = S_3\mathbf{a}_3 & (1) \\ \mathbf{I} = S_1\mathbf{a}_1 + S_8\mathbf{a}_8 - \mathbf{R} & (2) \end{cases} \tag{12.27a}$$

From $(\mathbf{a}_7 \cdot \mathbf{a}_8) = 0$ and $\mathbf{a}_2 = \mathbf{a}_7$ we get

$$\mathbf{a}_8 \cdot \mathbf{a}_2 = 0 \tag{12.27}$$

Substituting (12.20b-1) into (12.28) we get

$$A_1 \cos\theta_1 + B_1 \sin\theta_1 = C_1 \tag{12.29}$$

$$\begin{cases} A_1 = \mathbf{a}_0 \cdot \mathbf{a}_8 \\ B_1 = \mathbf{a}_1 \times \mathbf{a}_0 \cdot \mathbf{a}_8 \\ C_1 = 0 \end{cases} \tag{12.29a}$$

From (12.29) we get

$$\theta_1 = \tan^{-1}[-A_1/B_1] \tag{12.30}$$

The scalar product of $\mathbf{a}_2$ with both sides of (12.27) yields

$$x_2 + \mathbf{a}_2 \cdot (\mathbf{I} + \mathbf{J}) = 0 \tag{12.31}$$

Substituting {(12.27a), (12.20b-1)} into (12.31) yields

$$x_2 = (c\theta_1 \mathbf{a}_0 + s\theta_1 \mathbf{a}_1 \times \mathbf{a}_0) \cdot (\mathbf{R} - S_8 \mathbf{a}_8) \tag{12.32}$$

Now (12.27) can be written as,

$$x_6 \mathbf{a}_6 + S_5 \mathbf{a}_5 = \mathbf{M} \tag{12.33}$$

$$\mathbf{M} = -(x_2 \mathbf{a}_2 + \mathbf{I} + \mathbf{J}) \tag{12.33a}$$

where $\mathbf{M}$ is known vector. Squaring both sides of (12.33) yields

$$x_6 = \sqrt{(\mathbf{M}^2 - S_5^2)} \tag{12.34}$$

Now $\mathbf{a}_5$ and $\mathbf{a}_6$ are the only unknowns in (12.33) and both of them can be expressed in terms of θ_4 and known vectors. Substituting (12.20b-3) into (12.20b-4) yields

$$\mathbf{a}_6 = c\theta_4 \mathbf{a}_2 \times \mathbf{a}_3 - s\theta_4 \mathbf{a}_3 \tag{12.35}$$

Substituting {(12.20b-3), (12.35)} into (12.33) yields

$$(x_6 c\theta_4 + S_5 s\theta_4)\mathbf{a}_2 \times \mathbf{a}_3 - (x_6 s\theta_4 - S_5 c\theta_4)\mathbf{a}_3 = \mathbf{M} \tag{12.36}$$

The scalar product of $\mathbf{a}_3$ with both sides of (12.36) yields

$$A_4 \cos\theta_4 + B_4 \sin\theta_4 = C_4 \tag{12.37}$$

$$\begin{cases} A_4 = S_5 \\ B_4 = -x_6 \\ C_4 = (\mathbf{M} \cdot \mathbf{a}_3) \end{cases} \tag{12.37a}$$

The form of (12.37) is the same as (12.9), therefore, using (12.12) we have

$$\theta_4 = 2\tan^{-1}\{[B_4 \pm \sqrt{A_4^2 + B_4^2 - C_4^2}\,]/(A_4 + B_4)\} \tag{12.38}$$

The scalar product of $\{\mathbf{a}_6, \mathbf{a}_2 \times \mathbf{a}_6\}$ with both sides of (12.20b-5) yields θ_7:

$$\begin{cases} \cos\theta_7 = (\mathbf{a}_6 \cdot \mathbf{a}_8) \\ \sin\theta_7 = (\mathbf{a}_2 \times \mathbf{a}_6 \cdot \mathbf{a}_8) \end{cases} \tag{12.39}$$

The scalar product of $\{\mathbf{a}_2, \mathbf{a}_8 \times \mathbf{a}_2\}$ with both sides of (12.20b-6) yields θ_8:

$$\begin{cases} \cos\theta_8 = (\mathbf{a}_2 \cdot \mathbf{a}_9) \\ \sin\theta_8 = (\mathbf{a}_8 \times \mathbf{a}_2 \cdot \mathbf{a}_9) \end{cases} \tag{12.40}$$

Because θ_1 and θ_4 both have two solutions, as seen from (12.30) and (12.38), we conclude that this robot also has four closures.

12.4. Conclusion

Most industrial robots have special geometric conditions, such as

- The adjacent pair axes are parallel to each other; or
- The pair axes are perpendicularly intersecting with each other; (this situation can also be stated as "zero link length", by the way, why?);

The special geometric conditions usually render a robot easy to analyze. Using the *vector algebraic method* it is very easy to take such advantages. What we should do is simply to check the diagram of the robot, identify the special geometric conditions, and then use one of them to establish an algebraic displacement equation. There is no rigid rule on which joint variable should be determined first and which one should be the next. It is quite flexible!

CHAPTER 13. ANALYSIS OF THE GENERAL 4R AND 5R ROBOTS

13.1. Introduction

In order to arbitrarily position and orient a robot end effector in space, a robot with at least 6 degrees of freedom (DoF) is required. However, for many industrial tasks such as welding and assembly, robots with 4 or 5 DoF are sufficient. These simpler robots have the advantages of being less expensive and easier to control.

Most industrial robots are designed with parallel and/or intersecting joint axes, resulting in "special" kinematic cases which are relatively easy to analyze. The kinematic analysis of the general case, in which all axes are skewed and nonintersecting, is substantially more difficult. In this chapter, the general 4R and 5R robots (4 DoF and 5 DoF robots with rotary joints) are kinematically analyzed using *vector algebraic method*.

The general 4R robot was first analyzed by Manseur and Doty [55,56], using the Denavit-Hartenberg matrix transformation method [10]. The displacement equations are obtained as first order polynomials for the general 4R robot, and as second order polynomials for ten 4R robots with special geometries.

Using the Denavit-Hartenberg matrix method, the analysis consists of first performing a series of matrix transformations, and then equating the corresponding matrix elements. To this stage, a set of simultaneous equations containing 1 to 3 joint variables is obtained; these are then used to derive displacement equations containing only one joint variable. Since less than half of the equations obtained in the second step are useful, there is always some unavoidable extraneous work in the first step —— this is an intrinsic disadvantage of the matrix method. However, the *vector algebraic method* is free of this drawback.

The general 5R robot was first analyzed by Sugimoto and Duffy [67]. In their analysis, they joined the end effector to the first or grounded link of the arm by a pair of hypothetical joints and links, thus forming a hypothetical closed-loop spatial mechanism with mobility one. Sugimoto and Duffy outlined the analysis procedure for the resulting mechanism using the Spherical Trigonometry Method [28], but they did not show the detailed analysis for the general 5R robot. By introducing a pair of hypothetical joints and links, the 5R robot problem is transformed into 6R robot problem, which complicates the 5R robot problem substantially. We will show that the 5R robot can be solved directly using the vector algebraic method, without the need for additional hypothetical links and joints.

By 1983, the best result obtained for the displacement equation of a 6R robot was a 32nd order polynomial [27]. This solution was replaced by a better one, a 16th order polynomial, in 1988 [49]. The order of the polynomial equation which could be derived from Ref.[67] remains unknown.

The general 5R robot was recently re-analyzed by Manseur and Doty [57], and solved numerically using the Newton-Raphson technique after first formulating the problem using the Denavit-Hartenberg method. They did not derive the solution in closed form, i.e., as polynomial displacement equations.

In this chapter, the displacement equations of the general 4R and 5R robots are obtained as first and eighth order polynomials, respectively. To our knowledge, this is the first time an eighth order polynomial solution for the 5R robot has been obtained. The analysis for those 4R and 5R robots with special geometries is omitted, for they are easier to analyze than their corresponding general cases.

13.2. Several vector formulae

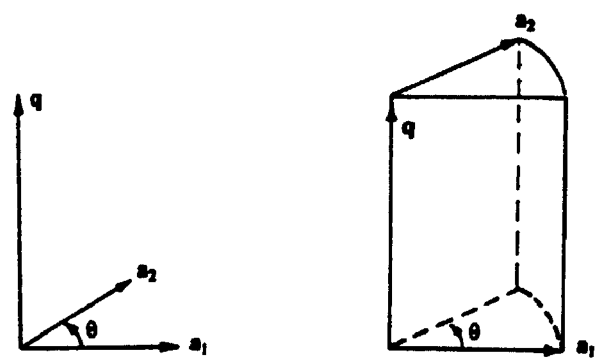


Fig. 13.1

The vector algebraic method is described in detail in chapter 2. The method makes extensive use of well known vector formulas. Of particular importance is the representation of rotations. In Fig. 13.1, unit vector q is perpendicular to both unit vectors a_1 and a_2, and angle θ is the right-hand rotation angle from a_1 to a_2. The vector formulas representing this rotation are

$$a_1 \times a_2 = q \sin\theta \tag{13.1}$$

$$a_2 = \cos\theta\, a_1 + \sin\theta\, q \times a_1 \tag{13.2}$$

$$a_1 = \cos\theta\, a_2 - \sin\theta\, q \times a_2 \tag{13.3}$$

Furthermore, extensive use is made of the following well known vector formulas, which are listed below for convenience:

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\,\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\,\mathbf{c} \tag{13.4}$$

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = (\mathbf{a}\cdot\mathbf{c})\,\mathbf{b} - (\mathbf{b}\cdot\mathbf{c})\,\mathbf{a} \tag{13.5}$$

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c}) \tag{13.6}$$

13.3. The forward kinematic analysis of the 4R robot

The configuration of the general 4R serial robot is shown in Fig. 13.2, where $\{\mathbf{a}_i, \mathbf{q}_j\}$ $(i=1-5;\ j=0-5)$ are all unit vectors. This robot has four degrees of freedom corresponding to four rotary controllable variables, which will be denoted as θ_i $(i=1-4)$, where $\theta_i = (\mathbf{q}_{i-1}\hat{,}\mathbf{q}_i)$ $(i=1-5)$ is the right-hand-rotation angle from $\mathbf{q}_{i-1}$ to $\mathbf{q}_i$ about $\mathbf{a}_i$. The perpendicular distance and twist angle between successive joint axes $\mathbf{a}_i$ and $\mathbf{a}_{i+1}$ are p_i and $\alpha_{i,i+1}$ respectively. The mutual perpendicular distance (or offset) between successive links $p_i\mathbf{q}_i$ and $p_{i+1}\mathbf{q}_{i+1}$ is denoted by S_i.

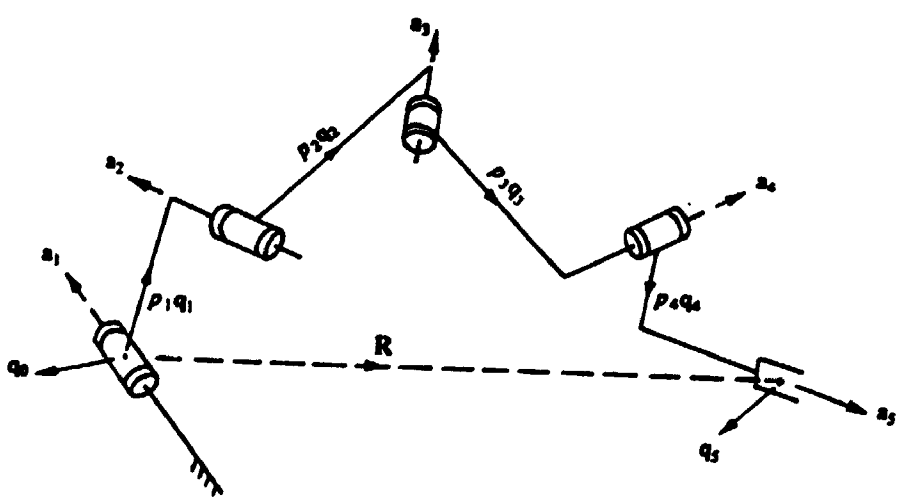


Fig. 13.2 General 4R Robot

Now the forward kinematic problem can be stated as follows: Given $\{\mathbf{q}_0, \mathbf{a}_1, p_i, S_{i+1}, \alpha_{i,i+1}\}$ $(i=1-4)$, determine $\{\mathbf{a}_5, \mathbf{q}_5, \mathbf{R}\}$.

From Fig. 13.2, we can write the vector loop equation directly.

$$\mathbf{R} = p_1\mathbf{q}_1 - S_2\mathbf{a}_2 + p_2\mathbf{q}_2 - S_3\mathbf{a}_3 + p_3\mathbf{q}_3 - S_4\mathbf{a}_4 + p_4\mathbf{q}_4 + S_5\mathbf{a}_5 \tag{13.7}$$

Using (13.2), we can state the relative direction equations as

$$\begin{cases} \mathbf{a}_{i+1} = \cos\alpha_{i,i+1}\,\mathbf{a}_i + \sin\alpha_{i,i+1}\,\mathbf{q}_i\times\mathbf{a}_i \\ \mathbf{q}_{i+1} = \cos\theta_i\,\mathbf{q}_i + \sin\theta_i\,\mathbf{a}_{i+1}\times\mathbf{q}_i \end{cases} \quad (i=1\text{-}4) \tag{13.7a}$$

Using (3.7a) repeatedly, vectors $\mathbf{q}_i$, $\mathbf{a}_i$ $(i=1\text{-}5)$ can be expressed in terms of $\{\mathbf{a}_1, \mathbf{q}_0, \mathbf{a}_1\times\mathbf{q}_0\}$ as follows

$$\begin{cases} \mathbf{a}_i = x_i\,\mathbf{a}_1 + y_i\,\mathbf{q}_0 + z_i\,\mathbf{a}_1\times\mathbf{q}_0 \\ \mathbf{q}_i = u_i\,\mathbf{a}_1 + v_i\,\mathbf{q}_0 + w_i\,\mathbf{a}_1\times\mathbf{q}_0 \end{cases} \quad (i=1\text{-}5) \qquad \bullet(13.8)$$

$$\begin{cases} x_1 = 1 \\ y_1 = 0 \\ z_1 = 0 \end{cases} \tag{13.8a}$$

$$\begin{cases} u_1 = 0 \\ v_1 = c\theta_1 \\ w_1 = s\theta_1 \end{cases} \tag{13.8b}$$

$$\begin{cases} x_{i+1} = x_i\,c\alpha_{i,i+1} + (z_i v_i - y_i w_i)\,s\alpha_{i,i+1} \\ y_{i+1} = y_i\,c\alpha_{i,i+1} + (x_i w_i - z_i u_i)\,s\alpha_{i,i+1} \\ z_{i+1} = z_i\,c\alpha_{i,i+1} + (y_i u_i - x_i v_i)\,s\alpha_{i,i+1} \end{cases} \quad (i=1\text{-}4) \tag{13.8c}$$

$$\begin{cases} u_{i+1} = u_i\,c\theta_{i+1} + (w_i y_{i+1} - v_i z_{i+1})\,s\theta_{i+1} \\ v_{i+1} = v_i\,c\theta_{i+1} + (u_i z_{i+1} - w_i x_{i+1})\,s\theta_{i+1} \\ w_{i+1} = w_i\,c\theta_{i+1} + (v_i x_{i+1} - u_i y_{i+1})\,s\theta_{i+1} \end{cases} \quad (i=1\text{-}4) \tag{13.8d}$$

After $\{x_i, y_i, z_i, u_i, v_i, w_i\}$ $\{i=1\text{-}5\}$ are all calculated from (13.8a-d), the orientation vectors $\mathbf{a}_5$, $\mathbf{q}_5$ are known. The end effector position is found by substituting (13.8) into (13.7) to get

$$\mathbf{R} = r_1\,\mathbf{a}_1 + r_2\,\mathbf{q}_0 + r_3\,\mathbf{a}_1\times\mathbf{q}_0 \qquad \bullet(13.9)$$

$$\begin{cases} r_1 = \sum_{i=1}^{4}(p_i u_i) - \sum_{i=2}^{4} S_i x_i + S_5 x_5 \\ r_2 = \sum_{i=1}^{3}(p_i v_i) - \sum_{i=2}^{4} S_i y_i + S_5 y_5 \\ r_3 = \sum_{i=1}^{3}(p_i w_i) - \sum_{i=2}^{4} S_i z_i + S_5 z_5 \end{cases} \tag{13.9a}$$

13.4. The inverse kinematic analysis of the 4*R* robot

The inverse kinematics problem can be stated as follows: Given $\{\mathbf{q}_0, \mathbf{a}_1\}$, $\{\mathbf{R}, \mathbf{a}_5, \mathbf{q}_5\}$ and $\{p_i, S_{i+1}, \alpha_{i,i+1}, \theta_5\}$ $(i=1\text{-}4)$, determine $\{\theta_i\}$ $(i=1\text{-}4)$.

† The equation numbers indicated by bullets •( ) correspond to solutions.

Let $\theta_4 = \Theta_4$ be the "output angle", the first rotational variable to be determined. Let $\theta_2 = \psi_2$ be the "auxiliary angle", the variable to be eliminated from simultaneous equations. Then the vector loop equation (13.7) of the robot can be rewritten as,

$$\mathbf{I} + \mathbf{J} = \mathbf{F} \tag{13.10}$$

I is defined as the *input vector* of the robot loop. It is the sum of those vectors in the robot loop which are given or known at the beginning. J is the *output vector* of the robot loop. It is the sum of those vectors in the robot loop which can be expressed as a function of the output angle (Θ_4 in this case). F is the *floating vector*. Cutting off the loop at the two ends of the floating vector, we get two separate chains, where one part is fixed on the ground and the other part is *floating*. This is how floating vector was named. The strict definitions of the three vectors are given in chapter 2.

Given these definitions, the vectors I, J, and F can be determined from

$$\begin{cases} \mathbf{J} = p_3\mathbf{q}_3 - S_3\mathbf{a}_3 & (1) \\ \mathbf{I} = -S_4\mathbf{a}_4 + p_4\mathbf{q}_4 + S_5\mathbf{a}_5 - \mathbf{R} & (2) \\ \mathbf{F} = -p_1\mathbf{q}_1 + S_2\mathbf{a}_2 - p_2\mathbf{q}_2 & (3) \end{cases} \tag{13.10a}$$

$$\begin{cases} \mathbf{a}_3 = c\alpha_{34}\mathbf{a}_4 - s\alpha_{34}\mathbf{q}_3 \times \mathbf{a}_4 & (1) \\ \mathbf{q}_3 = c\theta_4\mathbf{q}_4 - s\theta_4\mathbf{a}_4 \times \mathbf{q}_4 & (2) \\ \mathbf{a}_4 = c\alpha_{45}\mathbf{a}_5 - s\alpha_{45}\mathbf{q}_4 \times \mathbf{a}_5 & (3) \\ \mathbf{q}_4 = c\theta_5\mathbf{q}_5 - s\theta_5\mathbf{a}_5 \times \mathbf{q}_5 & (4) \end{cases} \tag{13.10b}$$

Substituting (13.10a) into (13.10), we get

$$\mathbf{I} + (p_3\mathbf{q}_3 - S_3\mathbf{a}_3) = -p_1\mathbf{q}_1 + S_2\mathbf{a}_2 - p_2\mathbf{q}_2 \tag{13.10$'$}$$

The analysis proceeds by performing certain basic operations as defined in chapter 2. For this type of mechanism, the basic operations are: (i). dot product $\mathbf{a}_1$ and $\mathbf{a}_3$ with both sides of the vector loop equation; (ii). square both sides of the vector loop equation; (iii). evaluate $\mathbf{a}_1 \cdot \mathbf{a}_3$ from the floating part and ground part of the loop equation.

The vector algebraic method leads to certain standard vector equations, which have been solved in chapter 2.

In the first stage of the analysis that follows, we obtain two linear equations in $\cos\Theta_4$ and $\sin\Theta_4$, from which we can obtain Θ_4. It is always possible to find two pairs of equations in Θ_4 and one auxiliary variable ψ_2. The necessary linear equations can then be obtained by eliminating ψ_2.

Step 1. *Derivation of the first equation containing* Θ_4.

Dot product $\mathbf{a}_1$ with both sides of (13.10)' to get

$$\mathbf{a}_1\cdot\mathbf{I}+(p_3\mathbf{a}_1)\cdot\mathbf{q}_3-(S_3\mathbf{a}_1)\cdot\mathbf{a}_3=S_2c\alpha_{12}-p_2s\alpha_{12}s\psi_2 \tag{13.11}$$

Dot product $\mathbf{a}_3$ with both sides of (13.10)' to get

$$\mathbf{I}\cdot\mathbf{a}_3-S_3=S_2c\alpha_{23}-p_1s\alpha_{23}s\psi_2 \tag{13.12}$$

Eliminating ψ_2 from (13.11) and (13.12) we get

$$\begin{aligned}(p_1p_3s\alpha_{23}\mathbf{a}_1)\cdot\mathbf{q}_3-(p_1S_3s\alpha_{23}\mathbf{a}_1+p_2s\alpha_{12}\mathbf{I})\cdot\mathbf{a}_3\\ =p_1s\alpha_{23}(S_2c\alpha_{12}-\mathbf{a}_1\cdot\mathbf{I})-p_2s\alpha_{12}(S_3+S_2c\alpha_{23})\end{aligned} \tag{13.13}$$

Substituting (13.10b-1) into (13.13) we get

$$\mathbf{U}\cdot\mathbf{q}_3=V \tag{13.14}$$

$$\begin{cases}\mathbf{U}=p_1p_3s\alpha_{23}\mathbf{a}_1+s\alpha_{23}\mathbf{a}_4\times(p_1S_3s\alpha_{23}\mathbf{a}_1+p_2s\alpha_{12}\mathbf{I})\\ V=p_1s\alpha_{23}(S_2c\alpha_{12}-\mathbf{a}_1\cdot\mathbf{I})-p_2s\alpha_{12}(S_3+S_2c\alpha_{23})\\ \qquad+c\alpha_{34}\mathbf{a}_4\cdot(p_1S_3s\alpha_{23}\mathbf{a}_1+p_2s\alpha_{12}\mathbf{I})\end{cases} \tag{13.14a}$$

Substituting (13.10b-2) into (13.14) we get

$$A\cos\Theta_4+B\sin\Theta_4=C \tag{13.15}$$

$$\begin{cases}A=\mathbf{U}\cdot\mathbf{a}_4\\ B=\mathbf{U}\cdot\mathbf{q}_4\times\mathbf{a}_4\\ C=V\end{cases} \tag{13.15a}$$

Step 2. *Derivation of the second equation containing* Θ_4.

Equating $\mathbf{a}_1\cdot\mathbf{a}_3$ evaluated for the floating part and ground part of the vector loop, we get

$$\mathbf{a}_1\cdot\mathbf{a}_3=c\alpha_{12}c\alpha_{23}-s\alpha_{12}s\alpha_{23}c\psi_2 \tag{13.16}$$

Squaring both sides of (13.10)' we get

$$\begin{aligned}(\mathbf{I}^2+p_3^2+S_3^2)+(2p_3\mathbf{I})\cdot\mathbf{q}_3-(2S_3\mathbf{I})\cdot\mathbf{a}_3\\ =(p_1^2+p_2^2+S_2^2)+2p_1p_2c\psi_2\end{aligned} \tag{13.17}$$

Eliminating ψ_2 from (13.16) and (13.17) we get

$$\begin{aligned}(2p_3s\alpha_{12}s\alpha_{23}\mathbf{I})\cdot\mathbf{q}_3+2(p_1p_2\mathbf{a}_1-S_3s\alpha_{12}s\alpha_{23}\mathbf{I})\cdot\mathbf{a}_3\\ =2p_1p_2c\alpha_{12}c\alpha_{23}-s\alpha_{12}s\alpha_{23}(\mathbf{I}^2+p_3^2+S_3^2-p_1^2-p_2^2-S_2^2)\end{aligned} \tag{13.18}$$

Substituting (13.10b-1) into (13.18) we get

$$\mathbf{U}'\cdot\mathbf{q}_3=V' \tag{13.19}$$

$$\begin{cases} \mathbf{U}' = 2p_3 s\alpha_{12} s\alpha_{23} \mathbf{I} - 2s\alpha_{34} \mathbf{a}_4 \times (p_1 p_2 \mathbf{a}_1 - S_3 s\alpha_{12} s\alpha_{23} \mathbf{I}) \\ V' = 2p_1 p_2 c\alpha_{12} c\alpha_{23} - 2c\alpha_{34} \mathbf{a}_4 \cdot (p_1 p_2 \mathbf{a}_1 - S_3 s\alpha_{12} s\alpha_{23} \mathbf{I}) \\ \qquad - s\alpha_{12} s\alpha_{23} (\mathbf{I}^2 - p_3^2 + S_3^2 - p_1^2 - p_2^2 - S_2^2) \end{cases} \tag{13.19a}$$

Substituting (13.10b-2) into (13.19) we get

$$A' \cos\theta_4 + B' \sin\theta_4 = C' \tag{13.20}$$

$$\begin{cases} A' = \mathbf{U}' \cdot \mathbf{a}_4 \\ B' = \mathbf{U}' \cdot \mathbf{q}_4 \times \mathbf{a}_4 \\ C' = V' \end{cases} \tag{13.20a}$$

Step 3. *Determination of* θ_4.

Solving the two linear equations (13.15) and (13.20) we get

$$\begin{cases} \cos\theta_4 = -Q_2 / Q_3 \\ \sin\theta_4 = Q_1 / Q_3 \end{cases} \tag{13.21}$$

$$\begin{cases} Q_1 = (AC' - A'C) \\ Q_2 = (BC' - B'C) \\ Q_3 = (AB' - A'B) \end{cases} \tag{13.21a}$$

Step 4. *Determination of* θ_2.

From (13.16) and (13.12) we get

$$\begin{cases} \cos\theta_2 = (c\alpha_{12} c\alpha_{23} - \mathbf{a}_1 \cdot \mathbf{a}_3) / (s\alpha_{12} s\alpha_{23}) \\ \sin\theta_2 = (S_3 + S_2 c\alpha_{23} - \mathbf{I} \cdot \mathbf{a}_3) / (p_1 s\alpha_{23}) \end{cases} \tag{13.22}$$

Since θ_4 is known, $\mathbf{a}_5$ of (4.13) is also known. Using (13.10b), $\mathbf{a}_3$ can be expressed in terms of $\{\theta_4, \mathbf{a}_5, \mathbf{q}_5\}$ and other known quantities as follows

$$\mathbf{a}_3 = \rho_1 \mathbf{a}_5 + \rho_2 \mathbf{q}_5 + \rho_3 \mathbf{a}_5 \times \mathbf{q}_5 \tag{13.23}$$

$$\begin{cases} \rho_1 = (c\alpha_{34} c\alpha_{45} - s\alpha_{34} c\theta_4 s\alpha_{45}) \\ \rho_2 = (c\alpha_{34} s\alpha_{45} + s\alpha_{34} c\theta_4 c\alpha_{45}) s\theta_5 + s\alpha_{34} s\theta_4 c\theta_5 \\ \rho_3 = (c\alpha_{34} s\alpha_{45} + s\alpha_{34} c\theta_4 c\alpha_{45}) c\theta_5 - s\alpha_{34} s\theta_4 s\theta_5 \end{cases} \tag{13.23a}$$

Step 5. *Determination of* θ_1.

We can express $\mathbf{a}_3$ in terms of unknown variable θ_1 and known quantities θ_2, $\mathbf{q}_0$, $\mathbf{a}_1$, etc. as follows,

$$\mathbf{a}_3 = \mathbf{m}_1 \cos\theta_1 + \mathbf{m}_2 \sin\theta_1 + \beta_3 \mathbf{a}_1 \tag{13.24}$$

$$\begin{cases} \mathbf{m}_1=\beta_1\mathbf{q}_0+\beta_2\mathbf{q}_0\times\mathbf{a}_1 \\ \mathbf{m}_2=\beta_2\mathbf{q}_0-\beta_1\mathbf{q}_0\times\mathbf{a}_1 \\ \beta_1=s\theta_2 s\alpha_{23} \\ \beta_2=s\alpha_{12}c\alpha_{23}+c\theta_2 c\alpha_{12}s\alpha_{23} \\ \beta_3=c\alpha_{12}c\alpha_{23}-c\theta_2 s\alpha_{12}s\alpha_{23} \end{cases} \quad (13.24a)$$

Combining (13.24) and (13.23) we get

$$\mathbf{m}_1\cos\theta_1+\mathbf{m}_2\sin\theta_1=\mathbf{m} \quad (13.25)$$

$$\mathbf{m} = \rho_1\mathbf{a}_5+\rho_2\mathbf{q}_5+\rho_3\mathbf{a}_5\times\mathbf{q}_5-\beta_3\mathbf{a}_1 \quad (13.25a)$$

Taking the dot product of $\mathbf{m}_1$ and $\mathbf{m}_2$ with both sides of (13.25) we get

$$\begin{cases} \cos\theta_1 = (\mathbf{m}_1\cdot\mathbf{m})/(\beta_1^2+\beta_2^2) \\ \sin\theta_1 = (\mathbf{m}_2\cdot\mathbf{m})/(\beta_1^2+\beta_2^2) \end{cases} \quad \bullet(13.26)$$

Step 6. *Determination of* θ_3.

From (13.7a) or directly from Fig. 13.2 we can write

$$\mathbf{q}_3 = c\theta_3\mathbf{q}_2+s\theta_3\mathbf{a}_3\times\mathbf{q}_2 \quad (13.27)$$

Taking the dot product of $\mathbf{q}_2$ and $\mathbf{a}_3\times\mathbf{q}_2$ with both sides of (13.27) we get

$$\begin{cases} \cos\theta_3 = \mathbf{q}_3\cdot\mathbf{q}_2 \\ \sin\theta_3 = \mathbf{q}_3\cdot\mathbf{a}_3\times\mathbf{q}_2 \end{cases} \quad \bullet(13.28)$$

where $\mathbf{q}_3$ and $\mathbf{a}_3$ can be expressed as functions of θ_4, and $\mathbf{q}_2$ can be expressed as a function of θ_1 and θ_2.

13.5. The inverse kinematic analysis of the *5R* robot

The forward kinematic analysis of the general 5R robot is omitted, for it is exactly the same as that of the 4R robot discussed in section 13.3. The configuration of the general 5R serial robot is shown in Fig. 13.3. The symbolic system is similar to that of Fig. 13.2.

The problem can be stated as follows: Given $\{\mathbf{q}_0, \mathbf{a}_1\}$, $\{\mathbf{R}, \mathbf{a}_6, \mathbf{q}_6\}$ and $\{p_i, S_{i+1}, \alpha_{i,i+1}, \theta_6\}$ $(i=1-5)$, determine $\{\theta_i\}$ $(i=1-5)$.

Now we would like to reformulate the problem slightly. After $\{\mathbf{R}, \mathbf{a}_6, \mathbf{q}_6\}$ are given, the position and orientation of the 5th joint are also specified. Therefore, solving the problem of Fig. 13.3 is equivalent to solving the problem of Fig. 13.4, which can be formulated as follows: Given $\{\mathbf{a}_1, p_5\mathbf{q}_5, \mathbf{a}_5\}$ and $\{p_i, S_i, \alpha_{j,j+1}, \alpha_{51}\}$ $(i=1-5,\ j=-4)$, determine $\{\theta_i\}$ $(i=1-5)$. Here in Fig. 13.4 the angles θ_1 and θ_5 are different from the

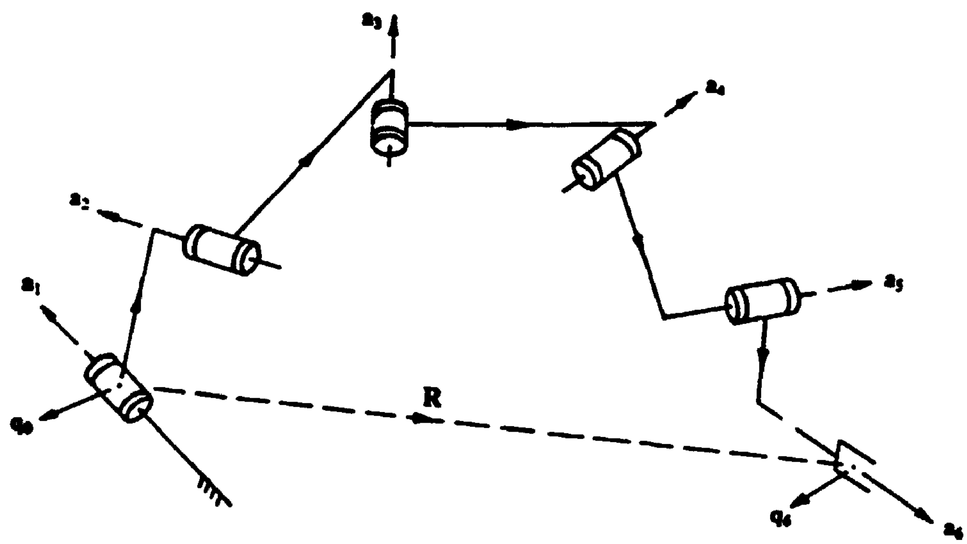


Fig. 13.3 General 5R Robot

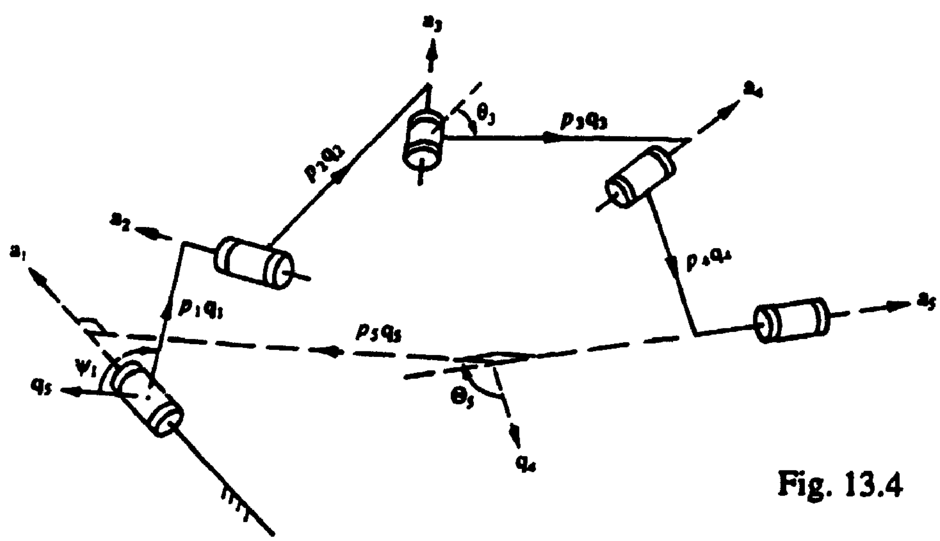


Fig. 13.4

corresponding ones shown in Fig. 13.3; however, their relationship can be easily determined.

Let $\theta_5 = \Theta_5$ be the "output angle", the first rotational variable to be determined. Let $\theta_1 = \psi_1$ be the "auxiliary angle", the variable to be eliminated from simultaneous equations. Then the vector loop equation and the relative direction equations of the robot can be written as,

$$\mathbf{I}+\mathbf{J}+\mathbf{L}=\mathbf{F} \tag{13.29}$$

where $\mathbf{L}$ is defined as *auxiliary vector*. It is the sum of those vectors in the robot loop which can be expressed as a function of the auxiliary angle, which is ψ_1 in this case. Using this definition, and the previously stated definition for $\mathbf{I}$, $\mathbf{J}$ and $\mathbf{F}$, we obtain

$$\begin{cases} \mathbf{L}=(p_1\mathbf{q}_1-S_2\mathbf{a}_2)=p_1\mathbf{q}_1-S_2 s\alpha_{12}\mathbf{q}_1\times\mathbf{a}_5-S_2 c\alpha_{12}\mathbf{a}_1 & (1)\\ \mathbf{J}=(p_4\mathbf{q}_4-S_4\mathbf{a}_4)=p_4\mathbf{q}_4+S_4 s\alpha_{45}\mathbf{q}_4\times\mathbf{a}_5-S_4 c\alpha_{45}\mathbf{a}_5 & (2)\\ \mathbf{I}=-S_5\mathbf{a}_5+p_5\mathbf{q}_5-S_1\mathbf{a}_1 & (3)\\ \mathbf{F}=-p_2\mathbf{q}_2+S_3\mathbf{a}_3-p_3\mathbf{q}_3 & (4) \end{cases} \tag{13.29a}$$

$$\begin{cases} \mathbf{a}_4=c\alpha_{45}\mathbf{a}_5-s\alpha_{45}\mathbf{q}_4\times\mathbf{a}_5 & (1)\\ \mathbf{q}_4=c\Theta_5\mathbf{q}_5-s\Theta_5\mathbf{a}_5\times\mathbf{q}_5 & (2)\\ \mathbf{a}_2=c\alpha_{12}\mathbf{a}_1+s\alpha_{12}\mathbf{q}_1\times\mathbf{a}_1 & (3)\\ \mathbf{q}_1=c\psi_1\mathbf{q}_5+s\psi_1\mathbf{a}_1\times\mathbf{q}_5 & (4)\\ \mathbf{a}_1=c\alpha_{51}\mathbf{a}_5+s\alpha_{51}\mathbf{q}_5\times\mathbf{a}_5 & (5) \end{cases} \tag{13.29b}$$

Step 1. *Derivation of the first equation relating* ψ_1 and Θ_5.

Dot product $\mathbf{a}_2$ with both sides of (13.29) to get

$$\mathbf{a}_2\cdot\mathbf{I}+\mathbf{a}_2\cdot\mathbf{J}-S_2=S_3c\alpha_{23}-p_3s\alpha_{23}s\theta_3 \tag{13.30}$$

Dot product $\mathbf{a}_4$ with both sides of (13.29) to get

$$\mathbf{a}_4\cdot\mathbf{I}-S_4+\mathbf{a}_4\cdot\mathbf{L}=-p_2s\alpha_{34}s\theta_3+S_3c\alpha_{34} \tag{13.31}$$

Eliminating θ_3 from (13.30) and (13.31) we get

$$(p_3s\alpha_{23}\mathbf{a}_4)\cdot\mathbf{L}-p_2s\alpha_{34}(\mathbf{I}+\mathbf{J})\cdot\mathbf{a}_2=(p_3s\alpha_{23}\mathbf{I})\cdot\mathbf{a}_4+h_1 \tag{13.32}$$

$$h_1=(S_4+S_3c\alpha_{34})p_3s\alpha_{23}-(S_2+S_3c\alpha_{23})p_2s\alpha_{34} \tag{13.32a}$$

Substituting (13.29a-1) and (13.29b-3) into (13.32) we get

$$\mathbf{U}\cdot\mathbf{q}_1=V \tag{13.33}$$

$$\begin{cases} \mathbf{U}=\gamma_1\mathbf{a}_4+\gamma_2\mathbf{q}_4-\gamma_3\mathbf{a}_1\times(\mathbf{I}+\mathbf{J})\\ V=\mathbf{m}_1\cdot\mathbf{a}_4+\mathbf{m}_2\cdot(\mathbf{I}+\mathbf{J})+h_1 \end{cases} \tag{13.33a}$$

$$\begin{cases} \gamma_1=p_1p_3s\alpha_{23}\\ \gamma_2=p_3S_2s\alpha_{12}s\alpha_{23}s\alpha_{45}\\ \gamma_3=p_2s\alpha_{12}s\alpha_{34}\\ \mathbf{m}_1=p_3s\alpha_{23}\mathbf{I}+S_2c\alpha_{12}s\alpha_{23}\mathbf{a}_1\\ \mathbf{m}_2=p_2c\alpha_{12}s\alpha_{34}\mathbf{a}_1 \end{cases} \tag{13.33b}$$

Substituting (13.29b-4) into (13.33) we get

$$A\cos\psi_1 + B\sin\psi_1 = C \tag{13.34}$$

$$\begin{cases} A = \mathbf{U}\cdot\mathbf{q}_5 \\ B = \mathbf{U}\cdot\mathbf{a}_1\times\mathbf{q}_5 \\ C = V \end{cases} \tag{13.34a}$$

Expanding (13.34a) we get

$$\begin{cases} A = a_1\cos\Theta_5 + a_2\sin\Theta_5 + a_3 \\ B = b_1\cos\Theta_5 + b_2\sin\Theta_5 + b_3 \\ C = c_1\cos\Theta_5 + c_2\sin\Theta_5 + c_3 \end{cases} \tag{13.35}$$

where

$$\begin{cases} a_1 = \mathbf{D}(\mathbf{q}_5)\cdot\mathbf{q}_5 \\ a_2 = \mathbf{D}(\mathbf{q}_5)\cdot\mathbf{q}_5\times\mathbf{a}_5 \\ a_3 = \mathbf{d}\cdot\mathbf{q}_5 \end{cases} \tag{13.35a1}$$

$$\begin{cases} b_1 = \mathbf{D}(\mathbf{a}_1\times\mathbf{q}_5)\cdot\mathbf{q}_5 \\ b_2 = \mathbf{D}(\mathbf{a}_1\times\mathbf{q}_5)\cdot\mathbf{q}_5\times\mathbf{a}_5 \\ b_3 = \mathbf{d}\cdot\mathbf{a}_1\times\mathbf{q}_5 \end{cases} \tag{13.35a2}$$

$$\begin{cases} c_1 = \mathbf{G}\cdot\mathbf{q}_5 \\ c_2 = \mathbf{G}\cdot\mathbf{q}_5\times\mathbf{a}_5 \\ c_3 = g \end{cases} \tag{13.35a3}$$

$$\begin{cases} \mathbf{D}(\Phi) = \gamma_1 s\alpha_{45}\Phi\times\mathbf{a}_5 + \gamma_2\Phi - \gamma_3 p_4\Phi\times\mathbf{a}_1 - \gamma_3 S_4 s\alpha_{45}\mathbf{a}_5\times(\Phi\times\mathbf{a}_1) \\ \mathbf{d} = \gamma_1 c\alpha_{45}\mathbf{a}_5 - \gamma_3 S_4 c\alpha_{45} s\alpha_{51}\mathbf{q}_5 - \gamma_3\mathbf{a}_1\times\mathbf{I} \\ \mathbf{G} = s\alpha_{45}(\mathbf{m}_1 - S_4\mathbf{m}_2)\times\mathbf{a}_5 + p_4\mathbf{m}_2 \\ g = c\alpha_{45}(\mathbf{m}_1 - S_4\mathbf{m}_2)\cdot\mathbf{a}_5 + \mathbf{m}_2\cdot\mathbf{I} + h_1 \end{cases} \tag{13.35b}$$

Step 2. *Derivation of the second equation relating* ψ_1 and Θ_5.

Equating $\mathbf{a}_2\cdot\mathbf{a}_4$ evaluated for the floating and ground parts of the vector loop, we get

$$\mathbf{a}_4\cdot\mathbf{a}_2 = c\alpha_{23}c\alpha_{34} - s\alpha_{23}s\alpha_{34}c\theta_3 \tag{13.36}$$

Squaring both sides of (13.29) we get

$$2(\mathbf{I}+\mathbf{J})\cdot\mathbf{L} + (\mathbf{I}+\mathbf{J})^2 + \mathbf{L}^2 = p_2 p_3 c\theta_3 + (p_2^2 + S_3^2 + p_3^2) \tag{13.37}$$

Eliminating θ_3 from (13.36) and (13.37) we get

$$2s\alpha_{23}s\alpha_{34}(\mathbf{I}+\mathbf{J})\cdot\mathbf{L} + p_2 p_3\mathbf{a}_4\cdot\mathbf{a}_2 = -(2s\alpha_{23}s\alpha_{34}\mathbf{I})\cdot\mathbf{J} + h_2 \tag{13.38}$$

$$h_2 = s\alpha_{23}s\alpha_{34}[(p_2^2+S_3^2+p_3^2) - (p_1^2+S_2^2+S_4^2+p_4^2+S_5^2+p_5^2+S_1^2+2S_1S_5c\alpha_{51})] \tag{13.38a}$$

Substituting (13.29a-1) and (13.29b-3) into (13.38) we get

$$\mathbf{U}'\cdot\mathbf{q}_1 = V' \tag{13.39}$$

$$\begin{cases} \mathbf{U}' = p_1\mathbf{n} - S_2 s\alpha_{12}\mathbf{a}_5\times\mathbf{n} + p_2p_3 s\alpha_{12}\mathbf{a}_1\times\mathbf{a}_4 \\ V' = c\alpha_{12}(S_2\mathbf{a}_1\cdot\mathbf{n} - p_2p_3\mathbf{a}_1\cdot\mathbf{a}_4) - 2s\alpha_{23}s\alpha_{34}\mathbf{I}\cdot\mathbf{J} + h_2 \\ \mathbf{n} = 2s\alpha_{23}s\alpha_{34}(\mathbf{I}+\mathbf{J}) \end{cases} \tag{13.39a}$$

Substituting (13.29b-4) into (13.39) we get

$$A'\cos\psi_1 + B'\sin\psi_1 = C' \tag{13.40}$$

$$\begin{cases} A' = \mathbf{U}'\cdot\mathbf{q}_5 \\ B' = \mathbf{U}'\cdot\mathbf{a}_1\times\mathbf{q}_5 \\ C' = V' \end{cases} \tag{13.40a}$$

Expanding (13.40a) we get

$$\begin{cases} A' = a_1'\cos\Theta_5 + a_2'\sin\Theta_5 + a_3' \\ B' = b_1'\cos\Theta_5 + b_2'\sin\Theta_5 + b_3' \\ C' = c_1'\cos\Theta_5 + c_2'\sin\Theta_5 + c_3' \end{cases} \tag{13.41}$$

where

$$\begin{cases} a_1' = \mathbf{D}'(\mathbf{q}_5)\cdot\mathbf{q}_5 \\ a_2' = \mathbf{D}'(\mathbf{q}_5)\cdot\mathbf{q}_5\times\mathbf{a}_5 \\ a_3' = \mathbf{d}'\cdot\mathbf{q}_5 \end{cases} \tag{13.41a1}$$

$$\begin{cases} b_1' = \mathbf{D}'(\mathbf{a}_1\times\mathbf{q}_5)\cdot\mathbf{q}_5 \\ b_2' = \mathbf{D}'(\mathbf{a}_1\times\mathbf{q}_5)\cdot\mathbf{q}_5\times\mathbf{a}_5 \\ b_3' = \mathbf{d}'\cdot\mathbf{a}_1\times\mathbf{q}_5 \end{cases} \tag{13.41a2}$$

$$\begin{cases} c_1' = \mathbf{G}'\cdot\mathbf{q}_5 \\ c_2' = \mathbf{G}'\cdot\mathbf{q}_5\times\mathbf{a}_5 \\ c_3' = g' \end{cases} \tag{13.41a3}$$

$$\begin{cases} \mathbf{D}'(\Phi) = t_1\Phi + t_2\mathbf{a}_5\times\Phi - t_3\mathbf{a}_5\times(\Phi\times\mathbf{a}_1) \\ \mathbf{d}' = 2s\alpha_{23}s\alpha_{34}[p_1(\mathbf{I} - S_4c\alpha_{45}\mathbf{a}_5) + S_2s\alpha_{12}\mathbf{I}\times\mathbf{a}_5] \\ \qquad + p_2p_3s\alpha_{12}c\alpha_{45}\mathbf{a}_1\times\mathbf{a}_5 \\ \mathbf{G}' = c\alpha_{12}s\alpha_{45}s\alpha_{51}(2S_2S_4s\alpha_{23}s\alpha_{34} + p_2p_3)\mathbf{q}_5 \\ \qquad + 2p_4S_2c\alpha_{12}s\alpha_{23}s\alpha_{34}\mathbf{a}_1 - 2s\alpha_{23}s\alpha_{34}(p_4\mathbf{I} + S_4s\alpha_{45}\mathbf{a}_5\times\mathbf{I}) \\ g' = 2s\alpha_{23}s\alpha_{34}[S_2c\alpha_{12}(-S_4c\alpha_{45}c\alpha_{51} + \mathbf{a}_1\cdot\mathbf{I}) \\ \qquad + S_4c\alpha_{45}\mathbf{a}_5\cdot\mathbf{I}] - p_2p_3c\alpha_{12}c\alpha_{45}c\alpha_{51} + h_2 \end{cases} \tag{13.41b}$$

$$\begin{cases} t_1 = 2s\alpha_{23}s\alpha_{34}(p_1p_4 - S_2S_4s\alpha_{12}s\alpha_{45}) \\ t_2 = 2s\alpha_{23}s\alpha_{34}(p_1S_4s\alpha_{45} + p_4S_2s\alpha_{12}) \\ t_3 = p_2p_3s\alpha_{12}s\alpha_{45} \end{cases} \tag{13.41c}$$

Step 3. ***Derivation of the equation containing*** θ_5.

Equations (13.34) and (13.40) comprise a set of canonical equations for which a standard solution has been derived in chapter 2. The analysis procedure is summarized below.

Solving the linear equations (13.34) and (13.40) we get

$$\begin{cases} \cos\psi_1 = -Q_2 / Q_3 \\ \sin\psi_1 = Q_1 / Q_3 \end{cases} \tag{13.42}$$

$$\begin{cases} Q_1 = (AC' - CA') \\ Q_2 = (BC' - CB') \\ Q_3 = (AB' - BA') \end{cases} \tag{13.42a}$$

From $\cos^2\psi_1 + \sin^2\psi_1 = 1$ and (13.42) we get

$$Q_1^2 + Q_2^2 = Q_3^2 \tag{13.43}$$

Letting $y = \tan(\theta_5/2)$, we obtain

$$\begin{cases} \cos\theta_5 = (1 - y^2) / (1 + y^2) \\ \sin\theta_5 = 2y / (1 + y^2) \end{cases} \tag{13.44}$$

Substituting (13.44) into (13.35) and (13.41), then substituting into (13.42a) and finally substituting into (13.43) we get an eighth order polynomial. The solutions has already been derived in chapter 2, and it can be simply restated here as

$$\sum_{i=0}^{8} \delta_i y^{8-i} = 0 \tag{13.45}$$

The coefficients $\{\delta_i\}$ $(i = 1 - 8)$ of (13.45) are the same as shown in Section 2.6.4 from Eq.(2.35a) to (2.35e).

We can obtain y from (13.45) using standard numerical techniques for finding roots of polynomials. Since a 5R robot has a unique solution, only one of the eight roots of (13.45) is a valid solution. Thus, each distinct real root must be tested by substituting into $\theta_5 = 2\tan^{-1}y$ and checking for consistency.

ψ_1 can be uniquely determined from (13.42). θ_3 can be obtained uniquely from {(13.30), (13.36)} or from {(13.31), (13.37)}. The other two joint angles can be obtained in a way similar to step 5 and step 6 of section 13.4.

13.6. Conclusions

The forward kinematic analysis is straightforward and warrants no more comments. As for the inverse kinematic analysis, after the first controllable variable is determined, there is a lot of flexibility in deciding which controllable variable should be selected as the next one to be solved. However, the analysis steps are similar and the expressions are standardized.

Since the general 5R robot can be considered as an over-constrained mechanism for the purpose of inverse kinematic analysis, there should be a single unique solution, i.e., the displacement equations should be first order polynomials. An eighth order polynomial is much better than nothing, but the first order polynomial remains the ultimate goal.

CHAPTER 14. CLASSIFICATION OF SPATIAL MECHANISMS

14.1. Introduction

A single-loop spatial linkage mechanism is generally characterized by the total number of its links, the number of its passive constraints, and the number and type of its kinematic pairs used in the kinematic loop ([36]Harrisberger, 1965; [16]Dobrjanskyj and Freudenstein, 1967; [1]Alizade, et al. 1985). These classifications are useful, although they do not provide all the information needed about the mechanism. They only disclose the number and type of kinematic pairs that constitute the mechanism and whether or not the mechanism is over-constrained.

One of the main problems in designing spatial mechanisms is the kinematic analysis and synthesis. Once the mechanism's components (i.e. pairs) are specified, it would be very desirable to determine the maximum number of possible configurations such a kinematic loop can be assembled together for a proper set of input structural parameters, and the maximum order of the algebraic equation representing the output (displacement) of the mechanism, which is free of extraneous or unwanted roots. In connection with this, the following questions arise: does the constitution of the components of a spatial mechanism alone really provide sufficient information about some general kinematic features of the output motion of its loop and can it help determine how to derive the input-output displacement?

Rooney and Duffy ([62] 1972) analyzed the maximum number of closures of some spatial mechanisms by using a geometric approach and verified corresponding results obtained by other authors using analytical approaches (i.e. algebraic methods). However, a criterion disclosing the maximum number of closures of spatial mechanisms was not provided.

Duffy ([28] 1980, pp.141-144) proposed another idea by relating some of the spatial mechanisms to certain models of spherical mechanisms. This work can be regarded as the first attempt to classify spatial mechanisms from the *kinematic* point of view. However, except for the group (1) mechanisms of [28], all other groups of mechanisms are not kinematically equivalent to their corresponding spherical mechanisms. In addition, the four analogous models of spherical polygons cover only a portion of spatial mechanisms. Therefore, the models proposed in [28] can not be used as a criterion to predict the maximum number of closures of spatial mechanisms.

In this chapter, a new scheme of classifying spatial mechanisms is proposed, which provides a method to determine the maximum number of closures of the mechanism and the maximum order of the input-output displacement equation, which is free of extraneous roots, of spatial mechanisms.

14.2. On the closure of mechanisms

Let us see Fig. 14.1(a-b). When the robot hand reaches the object, there are two distinct configurations for the robot arm relative to the frame (or ground). Similarly in Fig. 14.2, for any given input θ_1, there may exist two distinct configuration for the mechanism, or two distinct solutions for the output angle θ_4. The number of *closure* is a very important concept. It has been used in some literatures, however, its implication has never been rationally defined before.

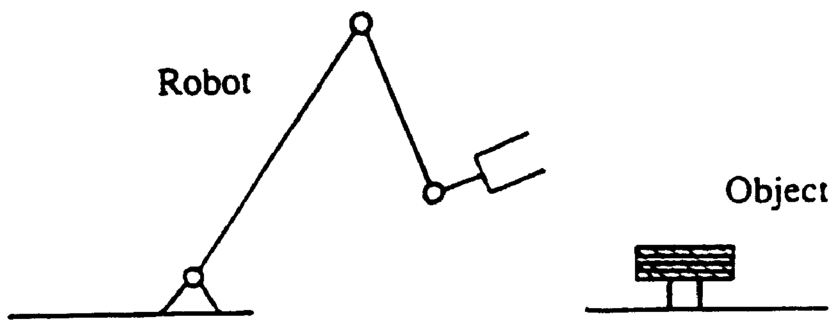


Fig. 14.1(a)

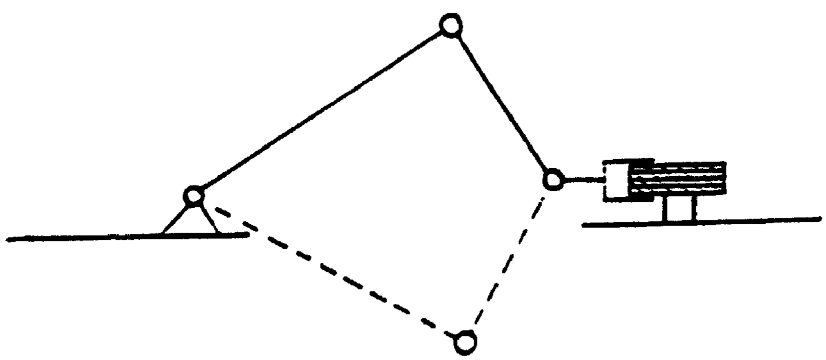


Fig. 14.1(b)

Definition. *The Number of Closure of Mechanism.*

For a given mechanism, let $\{\theta_i\}=\{\theta_1, \theta_2, \cdots, \theta_\lambda\}$, $(0\le\lambda\le6)$, be the angular variables of its *basic pair variables*. Then the *number of closure* of the mechanism is the maximum number of the distinctive set of solution of $\{\theta_1, \theta_2, \cdots, \theta_\lambda\}$, $(0\le\lambda\le6)$. If $\{\theta_i\}=\varnothing$ (empty set), i.e. $\lambda=0$, the number of closure of the mechanism is defined to be one.

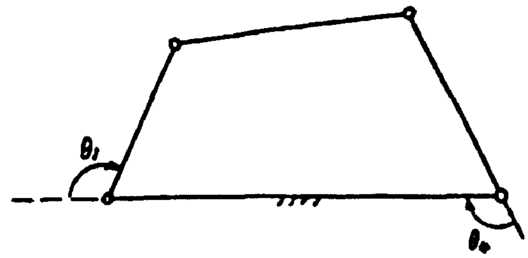


Fig. 14.2

Theorem 14.1. *On the Closure of Planar and Spherical Mechanisms.*

Let $\hat{f}$ be the number of the *active* rotary axes of a planar mechanism perpendicular to the plane of motion, and of a spherical mechanism passing through the centre of the sphere, we have the conclusions:

(1). If $\hat{f} \le 2$, then, the number of closure is one;

(2). If $\hat{f} = 3$, then, the number of closure is two ;

(3). If $\hat{f} \ge 4$, then, the mechanism has $(f_1 - 3)$ R-IDoF-2 (i.e. Rotary Idle Degree of Freedom of the second kind), which renders infinite number of closure.

Proposition 14.1. *On the Closure of Spatial Mechanisms.*

All RM and HMs (excluding those with S_{SH} or S_{GH} pairs) can be classified into six groups and all O_iRM and O_iHMs $(i = 1 - 5)$ (excluding those with S_{SH} or S_{GH} pairs) can be classified into $(6 - i)$ groups as shown in Table 14.1 in terms of Λ,

$$\Lambda = \lambda - \delta \qquad (14.1)$$

$$\delta = \begin{cases} 1, & \text{if one of the three conditions is true:} \\ & (i).\ \sum_{i=1}^{n} \tilde{N}_T^i = 0,\ (2 \le n \le 7); \\ & (ii).\ \lambda = 2 \text{ or } 3, \text{ and } (\lambda + \sum_{i=1}^{n} \tilde{N}_T^i) = f; \\ & (iii).\ \lambda = 4, \text{ and one of the pair's} \\ & \quad \textit{active DoF} \text{ is (120) or (220).} \\ 0, & \text{Otherwise.} \end{cases} \qquad (14.1a)$$

where $f = \sum_{i=1}^{n} (\tilde{N}_R^i + \tilde{N}_T^i + \tilde{N}_H^i),\ (2 \le n \le 7).$

The Λ of (14.1) is called ***big lambda***. Table 14.1 also gives the maximum finite number of closures a mechanism with a proper set of structure parameters can have.

Proposition 14.2. *On the Closure of Mechanisms* (containing s_{SH} or s_{GH} pairs).

In order to classify mechanisms with s_{SH} or s_{GH} pairs and use the results presented in Table 14.1, the following modifications must be imposed.

(1). for those mechanisms with an s_{SH} pair, if the pair's 3-digit number of active DoF is $(\tilde{N}_R\tilde{N}_T\tilde{N}_H)=(201)$, then we must use (210) as its 3-digit number of active DoF instead of (201) for calculating λ and Λ, moreover, $\delta = 1$. i.e.

$$s_{SH}:\ (\tilde{N}_R\tilde{N}_T\tilde{N}_H) = (201)\ \rightarrow\ (210);\ \ \delta = 1 \tag{14.2}$$

(2). for those mechanisms with an s_{GH} pair if the pair's 3-digit number of active DoF is $(\tilde{N}_R\tilde{N}_T\tilde{N}_H)=(301)$, then we must use (310) as its 3-digit number of active DoF instead of (301) for calculating λ and Λ, moreover, $\delta = 0$. i.e.

$$s_{GH}:\ (\tilde{N}_R\tilde{N}_T\tilde{N}_H) = (301)\ \rightarrow\ (310);\ \ \delta = 0 \tag{14.3}$$

The results in Propositions 14.1 and 14.2 are summarized from an extensive literature review and are also supported by exhaustive verification using *vector algebraic method*.

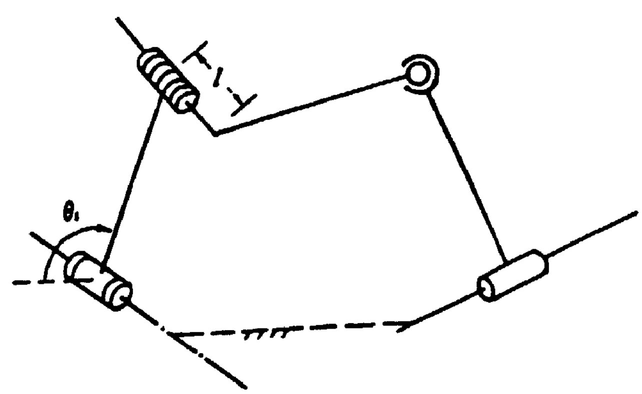


Fig. 14.3

The results in Table 14.1 showing the number of closures of HM or OHM based on the above two propositions were obtained on the basis that the motion range of each helical freedom is confined within one full period, i.e. the motion range of the rotational part of the helical freedom is confined within a range of 2π, or the motion range of the translational part of the helical freedom is confined within one full pitch. The reason can be explained by an HM, R_0-HSC, as shown in Fig. 14.3. Depending on the initial assembly condition, the helical off-set, l, may have multiple feasible assembly lengths. However, once the mechanism is assembled together, within any "one full period" of the helical pair, the maximum number of closure of the mechanism is two.

Definition. *Degeneration of Closure.*

Given a mechanism, if its maximum (finite) number of closure is derived, by Theorems 14.1, 14.2 and Propositions 14.1 and 14.2, as k, but its actual number of closure is smaller than k, then, this is called ***degeneration of closure***. If the closure of a mechanism is degenerated, then the mechanism is called a ***degenerated mechanism***.

Degeneration of closure is usually caused by special geometric conditions of a mechanism.

The generally held opinion that "with increasing of the number of links in spatial mechanisms, the displacement analysis becomes more difficult" is not true. For instance, the analyses for X_0-*CCT*, X_0-*RRRS*, X_0-*PRPRC* and X_0-*PRPPRR* become easier with increased number of links.

Proposition 14.3. *On the Degree of Complexity*

The *degree of complexity* of linkage mechanisms is defined as the sum of the *Small* λ and the *Big* Λ:

$$\varepsilon = \lambda + \Lambda \tag{14.4}$$

Generally speaking, the bigger the ε, the more complex the structure and the motion of the mechanism. From the definitions of λ and Λ, we have

$$\lambda = \Lambda \text{ or } \Lambda + 1; \quad (\Lambda = 0, 1, 2, 3, 4, 5) \tag{14.5}$$

therefore, there are 11 degrees of complexity, i.e. $\{\varepsilon\} = \{1, 2, \cdots, 11\}$.

Proposition 14.4. *On the Degree of Complexity of the Same Group of Mechanisms.*

With increasing the value of the ε defined in (14.4), the displacement analysis for the mechanisms *within* each of the 13 groups $\{RM, ORM, H_jM\ (j=1\text{-}6)$ and $OH_kM\ (k=1\text{-}5)\}$ of spatial mechanisms becomes more complicated.

For $\Lambda \le 2$, the difference on the degree of complexity for the analyses of the mechanisms corresponding to $\lambda = \Lambda + 1$ and $\lambda = \Lambda$ is not significant.

Proposition 14.5. *On the Degree of Complexity of Different Groups of Mechanisms.*

- For an *ORM* and an *RM* of the same ε, the *RM* is more difficult to analyze, i.e. the structure or the motion of the *RM* is more complex;
- For an OH_jM and an H_jM of the same ε and $j\ (j=1\text{-}5)$, the H_jM is more difficult to analyze;

- For an OH_jM and an $OH_{j+1}M$ of the same ε and specified j $(j=1-4)$, the $OH_{j+1}M$ is more difficult to analyze;
- For a H_jM and a $H_{j+1}M$ of the same ε and specified j $(j=1-5)$, the $H_{j+1}M$ is more difficult to analyze;
- OHM and HM are more difficult to analyze than their corresponding ORM and RM, which are obtained by assuming that the pitches of all the helical freedoms of the OHM and HM are zero.

Generally speaking, the more the *translational axes* involved in a mechanism, the simpler the stucture and the motion of the mechanism. It is clear that if the number of the translational axes exceeds certain limit or the directions of two translational axes are parallel, the T-IDoF-2 will occur. Now we want to know the exact conditions that render the T-IDoF-2. But first let us introduce a useful concept.

Definition. *The Number of Parallel-couples.*

The *number of parallel-couple* for the following three kinds of spatial line set is respectively defined as

$$p_1 = u-1; \quad p_2 = v; \quad p_3 = 0.$$

(1). Given a set of spatial lines that are muturally parallel, $\{a_i\}$ $(i=1-u;\ u\geq 3)$;

(2). Given a set of spatial lines, $\{b_i\}=\{(d_1,e_1),(d_2,e_2),\cdots(d_{2v},e_{2v})\}$, $(i=1-v;\ v\geq 1)$, where each of (d_i,e_i) is an *independent* parallel-couple and (d_i,d_j) $(i\neq j)$ is not a paralle-couple;

(3). Given a set of spatial lines that contains no parallel-couple, $\{c_i\}=\{c_1,c_2,\cdots,c_w\}$.

It is clear that any spatial line set $\{l_i\}$ is composed of the three kinds of line set. Hence, the *number of parallel-couple* for any spatial line set can be expressed as

$$p = \sum p_{1i} + \sum p_{2i} = \sum_1^m (u_i - 1) + \sum_1^n v_i \qquad (14.5)$$

where m and n are the numbers of the 1st and the 2nd kind of line set.

Theorem 14.2. *On the T-IDoF-2 of Mechanisms.*

(1). **For planar mechanism:**

Let ρ be the number of the *independant* T-IDoF-2 that are coplanar with the plane of motion; let $\tilde{t}$ and $\tilde{t}_0$ be the numbers of the *active* translational axes coplanar with and perpendicular to the plane of motion, respectively; and p be the *number of the parallel-couple* of the *active* translational axes coplanar with the plane of motion.

- If $\tilde{i}_0 \leq 1$ and $\tilde{i} \leq 2$, then, $\rho = p$;
- If $\tilde{i}_0 \leq 1$ and $\tilde{i} \geq 3$, then, $(\tilde{i} - 3 \leq \rho \leq (\tilde{i} - 1)$;
- If $\tilde{i}_0 \geq 2$, the mechanism is no longer a planar mechanism, for there would be $(\tilde{i}_0 - 1)$ T-IDoF-2 in the direction perpendicular to the plane of motion.

(2). **For spatial mechanism;**

Let ρ be the number of the *independant* T-IDoF-2 of the mechanism; let $\tilde{i} = \sum \tilde{N}_T^i$, and p the *number of the parallel-couple* of the *active* translational axes.

- If $\tilde{i} \leq 3$, then, $\rho = p$;
- If $\tilde{i} \geq 4$, then, $(\tilde{i} - 4) \leq \rho \leq (\tilde{i} - 1)$.

The proofs for Theorems 14.1 and 14.2 can be performed in a way similar to the proof of Theorem 1.2.

Examples of application

Example 1.

Given the two-loop spatial mechanism shown in Fig. 1.10, the input pair is a C pair at a; the actuator controls either a rotary or a translational motion along vector e_3. Describe the mechanism and study the structure of each loop.

Solution:

The first loop can be considered as *(o-a-b-c-d-o)*. The motion of the R pair at d uniquely determines the position and direction of the axis of the R pair at e, which is the pair directly connected with the second loop; therefore, we can consider the R pair at d as the input pair of the second loop which is then becomes *(d-e-f-g-h-i-d)*.

(1) The first loop if the actuator controls only a rotary motion:

- Symbolic representation: $C_{(010)(100)}$-RSR or $C_{(010)}$-RSR;
- $f = \sum(\tilde{N}_R^i + \tilde{N}_T^i + \tilde{N}_H^i) = 6$, $\sum \tilde{N}_H^i = 0$, so this is a regular mechanism (i.e. P.M);
- $\lambda = 3$, $\Lambda = (\lambda - \delta) = 3 - 0 = 3$, accordingly the maximum number of closures of the first loop is 4. The maximum order of the input-output displacement equation, free of extraneous roots, for this loop is also 4. If there are special geometric conditions involved in the length of the links or the relative direction of the pair

axes, the number of closures and the order of the input-output displacement equation may decrease to 2 .

(2) The first loop if the actuator controls only a translational motion:

- Symbolic representation: $C_{(100)}$–RSR ;
- $f=6$, $\sum \tilde{N}_H^i = 0$, so this is an RM ;
- $\lambda=4$, $\delta=1$ and $\Lambda=(\lambda-\delta)=3$, therefore, the number of closures and the order of the input-output displacement equation are the same as above.

(3) The second loop:

- Symbolic representation: $R_{(000)}$–$RSRP$ or R_0–$RSRP$;
- This is an RM ;
- $\lambda=3$, $\Lambda=3$, the inference is the same as above for each input on the R pair at d. Hence, for each input on the C pair at a, the second loop may have 4, 8 or 16 assembly configurations, depending on the geometric condition of the mechanism.

(4 Additional comments about the second loop:

There is nothing wrong in regarding the R pair at e as the input pair of the second loop. In this case, the second loop can be considered as *(e-f-g-h-i-e)* and symbolically denoted as $X_{(100)}$–SRP , or $R_{(100)}$–SRP or R_1–SRP . Symbol $X_{(100)}$ is more general as compared to $R_{(100)}$. $X_{(100)}$ includes not only $R_{(100)}$ but also $C_{(100)}$, $T_{(100)}$, et al.

Example 2. Given a robot which is composed of five revolute pairs (5R), study its structure and the number of closures.

Solution:

$f=5<6$ and $\sum \tilde{N}_H=0$, so this robot can be regarded as an O_1RM for the inverse kinematic analysis. According to Proposition 14.1 and $\sum \tilde{N}_T=0$, we have $\delta=1$, $\lambda=5$ and $\Lambda=4$. We can now conclude that generally this robot has only one closure; for special geometries, it may have either two or infinite closures. For instance, if the axes of four consecutive R pairs are parallel, then the robot will have at least one A-IDoF-2 which renders infinite solution for the configuration of the robot.

Example 3.

Determine the number of closures for an X_0–SS_{SH} mechanism.

Solution:

$f=6$ and $\sum \tilde{N}_H=1$, so this is a H_1M. Because this mechanism contains an S_{SH} pair and its 3-digit number of active DoF is (201), according to Proposition 14.2, $\delta=1$ and we use (210) as the 3-digit number of active DoF for the S_{SH} pair and we have

$\lambda=3$ and $\delta=1$, which give $\Lambda=2$. Therefore, we conclude that the maximum number of closures of the mechanism is 2.

14.3 Conclusion

All displacement analysis problems for ORM and RM can be solved satisfactorily by algebraic methods (i.e. analytical methods). But this is not the case for OHM and HM. We believe that it is worthwhile to pay more attention to this area.

The motion of an H pair with a short pitch is more practical mechanically than that with a long pitch, because the friction torque arising from the two contacting surfaces of the two elements of a short-pitch H pair is generally easier to be kept small than that of a long-pitch H pair. Consequently, a short-pitch H pair is more popular than a long-pitch H pair. It is necessary to point out that no matter how long the pitch of the helix is we can classify HM and O_iHM $(i=1-5)$ respectively into six and $(6-i)$ groups, based on the value of Λ defined in theorem 1, however, care must be taken when applying the results of Table 14.1 upon those HM and OHM with long-pitch H pairs. The reason is quite obvious: the kinematic feature of a short-pitch H pair is "close" to that of an R pair, whereas a long-pitch H pair is "close" to that of a P pair. How long is too long is difficult to specify quantitatively, for it depends on the overall geometrical size of the mechanism.

An open kinematic chain can be regarded as a closed kinematic loop for the purpose of analysis. It is apparent that all serial robot arms with a degree of freedom less than or equal to six can also be included in the four basic groups of SSMs. Therefore, for any SSM of the four basic groups and also for any serial robot arms, we can easily determine their maximum finite number of closures, and their maximum order of the input-output displacement equations which is free of extraneous or unwanted roots.

As a convenient reference, all regular mechanisms (RM) and over-constrained regular mechanisms (O_iRM) $(i=1-5)$ are listed in Tables 14.2 - 14.7 and Tables 14.2(a) - 14.7(a) by class and type symbols of their kinematic pairs, respectively. Information such as the number of variant mechanisms of each basic mechanism, the value of the first structure criterion, λ, and the value of the second structure criterion, Λ, are also displayed. From Tables 14.2(b) - 14.7(b), we can see that the total numbers of RM and O_iRM $(i=1-5)$ are 1121, 373, 120, 37, 10 and 2, respectively. The tables of the H_jM and O_iH_jM expressed by class and type symbols of their kinematic pairs can be compiled similarly, although they are not displayed in this paper.

This chapter can serve as a general guideline in the search for and design of more useful spatial mechanisms.

Table 14.1. The relationship of Λ and the maximum number of closures

	Λ=0	Λ=1	Λ=2	Λ=3	Λ=4	Λ=5
Maximum (finite) number of closure of RM and HM • in general => • for special geometries =>	1 (1)	1 (1)	2 (1)	4 (2)	8 (6) (4)	16 (12) (8) (6)
Maximum order of input-output displacement equation free of extraneous roots for RM & HM • in general => • for special geometries =>	1 (1)	1 (1)	2 (1)	4 (2)	8 (6) (4) (2)	16 (12) (8) (6) (4)
Maximum (finite) number of closure of O_iRM and O_iHM (i = 1-5), ($0 \le \Lambda \le 5-i$) • in general => • for special geometries =>	1 (1)	1 (1)	1 (2)	1 (2)	1 (2)	

Table 14.2. All RM expressed by the class symbols of their kinematic pairs

	X_0	X_1	X_2	X_3	X_4	X_5
2	$X_0-1p_1-1p_5$ $X_0-1p_2-1p_4$ X_0-2p_3	$X_{100}-1p_5$ $X_{010}-1p_5$	$X_{200}-1p_4$ $X_{110}-1p_4$ $X_{020}-1p_4$	$X_{300}-1p_3$ $X_{210}-1P_3$ $X_{120}-1p_3$	$X_{310}-1p_2$ $X_{220}-1p_2$	$X_{320}-1p_1$
3	$X_0-2p_1-1p_4$ $X_0-1p_1-1p_2-1p_3$ X_0-3p_2	$X_{100}-1p_1-1p_4$ $X_{010}-1p_1-1p_4$ $X_{100}-1p_2-1p_3$ $X_{010}-1p_2-1p_3$	$X_{200}-1p_1-1p_3$ $X_{110}-1p_1-1p_3$ $X_{020}-1p_1-1p_3$ $X_{200}-2p_2$ $X_{110}-2p_2$ $X_{020}-2p_2$	$X_{300}-1p_1-1p_2$ $X_{210}-1p_1-1p_2$ $X_{120}-1p_1-1p_2$	$X_{310}-2p_1$ $X_{220}-2p_1$	
4	$X_0-3p_1-1p_3$ $X_0-2p_1-2p_2$	$X_{100}-2p_1-1p_3$ $X_{010}-2p_1-1p_3$ $X_{100}-1p_1-1p_2$ $X_{010}-1p_1-1p_2$	$X_{200}-2p_1-1p_2$ $X_{110}-2p_1-1p_2$ $X_{020}-2p_1-1p_2$	$X_{300}-3p_1$ $X_{210}-3p_1$ $X_{120}-3p_1$		
5	$X_0-4p_1-1p_2$	$X_{100}-3p_1-1p_2$ $X_{010}-3p_1-1p_2$	$X_{200}-4p_1$ $X_{110}-4p_1$ $X_{020}-4p_1$			
6	X_0-6p_1	$X_{100}-5p_1$ $X_{010}-5p_1$				

Notes: The elements corresponding to the number i (i=2-6) of the first column represent those RM loops that they have just i pairs whose Active DoF are not equal to zero in every single loop; X_j (j=0-5) represents the input pair whose Active DoF equals j. p_k represents the pairs whose DoF equals k.

Table 14.2(b). Statistical data of RM

		Λ=0	Λ=1	Λ=2	Λ=3	Λ=4	Λ=5
2	50	0	13	23	11	3	0
3	201	0	32	74	52	27	18
4	406	0	72	130	39	100	65
5	336	1	70	104	0	101	60
6	128	2	42	40	0	30	14
Total number of RM = 1121		3	229	369	102	261	157

Notes: The number 201 (the second number of the second column) is the total number of those RM loops that they have just 3 (the second number of the first column) pairs whose Active DoF are not equal to zero in every single loop. The data in this table come from Table 14.2(a).

Table 14.3. All O_1RM expressed by the class symbols of their kinematic pair

	X_0	X_1	X_2	X_3	X_4	X_5
1	X_0-1p_5					
2	$X_0-1p_1-1p_4$ $X_0-1p_2-1p_3$	$X_{100}-1p_4$ $X_{010}-1p_4$	$X_{200}-1p_3$ $X_{110}-1p_3$ $X_{020}-1p_3$	$X_{300}-1p_2$ $X_{210}-1p_2$ $X_{120}-1p_2$	$X_{310}-1P_1$ $X_{220}-1p_1$	
3	$X_0-1p_1-2p_2$ $X_0-2p_1-1p_3$	$X_{100}-1p_1-1p_3$ $X_{010}-1p_1-1p_3$ $X_{100}-2p_2$ $X_{010}-2p_2$	$X_{200}-1p_1-1p_2$ $X_{110}-1p_1-1p_2$ $X_{020}-1p_1-1p_2$	$X_{300}-2p_1$ $X_{210}-2p_1$ $X_{120}-2p_1$		
4	$X_0-3p_1-1p_2$	$X_{100}-2p_1-1p_2$ $X_{010}-2p_1-1p_2$	$X_{200}-3p_1$ $X_{110}-3p_1$ $X_{020}-3p_1$			
5	X_0-5p_1	$X_{100}-4p_1$ $X_{010}-4p_1$				

Notes: The elements corresponding to the number i ($i=1$-5) of the first column represent those O_1RM loops that they have just i pair whose active DoF are not equal to zero in every single loop; X_j ($j=0$-5) represents the input pair whose active DoF equals j. p_k represents the pairs whose DoF equals k.

Table 14.3(b). Statistical data of O_1RM

		Λ=0	Λ=1	Λ=2	Λ=3	Λ=4	Λ=5
1	2	0	2	0	0	0	0
2	43	0	15	24	0	4	0
3	128	0	42	56	0	30	0
4	136	1	46	49	0	40	0
5	64	2	30	20	0	12	0
Total number of O_1RM=373		3	135	149	0	86	0

Notes: The number 128 (the third number of the second column) is the total number of those O_1RM loops that they have just 3 (the third number of the first column) pairs whose active DoF are not equal to zero in every single loop. The data in this table come from Table 14.3(a).

Table 14.4. All O_2RM expressed by the class symbols of their kinematic pair

	X_0	X_1	X_2	X_3	X_4	X_5
1	X_0-1p_4					
2	X_0-2p_2	$X_{100}-1p_3$ $X_{010}-1p_3$	$X_{200}-1p_2$ $X_{110}-1p_2$ $X_{020}-1p_2$	$X_{300}-1p_1$ $X_{210}-1p_1$ $X_{120}-1p_1$		
3	$X_0-2p_1-1p_2$	$X_{100}-1p_1-1p_2$ $X_{010}-1p_1-1p_2$	$X_{200}-2p_1$ $X_{110}-2p_1$ $X_{020}-2p_1$			
4	X_0-4p_1	$X_{100}-3p_1$ $X_{010}-3p_1$				

Table 14.4(b). Statistical data of O_2RM

		A=0	A=1	A=2	A=3	A=4	A=5
1	2	0	2	0	0	0	0
2	34	0	24	8	2	0	0
3	52	1	27	18	6	0	0
4	32	2	20	8	2	0	0
Total number of $O_2RM=120$		3	73	34	10	0	0

Table 14.5. All O_3RM expressed by the class symbols of their kinematic pair

	X_0	X_1	X_2	X_3	X_4	X_5
1	X_0-1p_3					
2	$X_0-1p_1-1p_2$	$X_{100}-1p_2$ $X_{010}-1p_2$	$X_{200}-1p_1$ $X_{110}-1p_1$ $X_{020}-1p_1$			
3	X_0-3p_1	$X_{100}-2p_1$ $X_{010}-2p_1$				

Table 14.5(b). Statistical data of O_3RM

		$\Lambda=0$	$\Lambda=1$	$\Lambda=2$	$\Lambda=3$	$\Lambda=4$	$\Lambda=5$
1	3	0	3	0	0	0	0
2	18	1	13	4	0	0	0
3	16	2	12	2	0	0	0
total number of $O_3RM=37$		3	28	6	0	0	0

Table 14.6. All O_4RM expressed by the class symbols of their kinematic pair

	X_0	X_1	X_2	X_3	X_4	X_5
1	X_0-1p_2					
2	X_0-2p_1	$X_{100}-1p_1$ $X_{010}-1p_1$				

Table 14.6(b). Statistical data of O_4RM

		$\Lambda=0$	$\Lambda=1$	$\Lambda=2$	$\Lambda=3$	$\Lambda=4$	$\Lambda=5$
1	2	0	2	0	0	0	0
2	8	2	6	0	0	0	0
Total number of $O_4RM=10$		2	8	0	0	0	0

Table 14.7. All O_5RM expressed by the class symbols of their kinematic pair

	X_0	X_1	X_2	X_3	X_4	X_5
1	X_0-1p_1					

Table 14.7(b). Statistical data of O_5RM

		$\Lambda=0$	$\Lambda=1$	$\Lambda=2$	$\Lambda=3$	$\Lambda=4$	$\Lambda=5$
1	2	1	1	0	0	0	0
Total number of $O_5RM=2$		1	1	0	0	0	0

Table 14.2(a). All RM expressed by the type symbols of their kinematic pair

I.---- Regular Mechanisms; II.--- Number of the mechanisms;
III.-- $\lambda - \sum N_i$; IV.--- The value of Λ.

I.	II.	III.	IV.
(TWO)			
$X_{100}-S_P$	1	2-2	2
$X_{010}-S_P$	1	1-3	1
$X_{100}-B_P$	1	2-2	2
$X_{010}-B_P$	1	1-3	1
$X_{200}-S_G$	1	3-1	3
$X_{110}-S_G$	1	2-2	2
$X_{020}-S_G$	1	1-3	1
$X_{200}-C_P$	1	4-2	3
$X_{110}-C_P$	1	3-3	2
$X_{020}-C_P$	1	2-4	1
$X_{300}-S$ †	1	2-0	1
$X_{210}-S$	1	3-1	3
$X_{120}-S$ †	1	2-2	2
$X_{300}-E$ †	1	2-2	2
$X_{210}-E$	1	3-3	2
$X_{120}-E$	1	2-4	1
$X_{300}-S_{SC}$	1	3-1	3
$X_{210}-S_{SC}$	1	4-2	4
$X_{120}-S_{SC}$	1	3-3	2
$X_{310}-C$	1	2-2	2
$X_{220}-C$	1	3-3	2
$X_{310}-T$	1	3-1	3
$X_{220}-T$	1	4-2	3
$X_{320}-R$	1	2-2	2
$X_{320}-P$	1	1-3	1
X_0-R-S_P	2	2-2	2
X_0-P-S_P	2	1-3	1
X_0-R-B_P	2	2-2	2
X_0-P-B_P	2	1-3	1
X_0-C-S_G	2	2-2	2
X_0-C-C_P	2	3-3	2
X_0-T-S_G	2	3-1	3
X_0-T-C_P	2	4-2	3
X_0-2S †	1	2-0	1
X_0-2E	1	2-4	1
X_0-2S_{SC}	1	4-2	4
X_0-S-E †	2	2-2	2
X_0-S-S_{SC}	2	3-1	3
X_0-E-S_{SC}	2	3-3	2
(THREE)			
$X_{100}-R-S_G$	2	3-1	3
$X_{100}-P-S_G$	2	2-2	2
$X_{010}-R-S_G$	2	2-2	2
$X_{010}-P-S_G$	2	1-3	1
$X_{100}-R-C_P$	2	4-2	3
$X_{100}-P-C_P$	2	3-3	2
$X_{010}-R-C_P$	2	3-3	2
$X_{010}-P-C_P$	2	2-4	1
$X_{200}-R-S$	2	4-0	3
$X_{200}-P-S$	2	3-1	3
$X_{110}-R-S$	2	3-1	3
$X_{110}-P-S$	2	2-2	2
$X_{020}-R-S$	2	2-2	2
$X_{020}-P-S$	2	1-3	1
$X_{200}-R-E$	2	4-2	3
$X_{200}-P-E$	2	3-3	2
$X_{110}-R-E$	2	3-3	2
$X_{110}-P-E$	2	2-4	1
$X_{020}-R-E$	2	2-4	1
$X_{020}-P-E$	2	1-5	1
$X_{200}-R-S_{SC}$	2	5-1	5
$X_{200}-P-S_{SC}$	2	4-2	4
$X_{110}-R-S_{SC}$	2	4-2	4
$X_{110}-P-S_{SC}$	2	3-3	2
$X_{020}-R-S_{SC}$	2	3-3	2
$X_{020}-P-S_{SC}$	2	2-4	1
$X_{200}-2C$	1	4-2	4
$X_{110}-2C$	1	3-3	2
$X_{020}-2C$	1	2-4	1
$X_{200}-2T$	1	6-0	5
$X_{110}-2T$	1	5-1	5
$X_{020}-2T$	1	4-2	4
$X_{200}-C-T$	2	5-1	5
$X_{110}-C-T$	2	4-2	4
$X_{020}-C-T$	2	3-3	2
$X_{200}-R-C$	2	3-1	3
$X_{200}-P-C$	2	2-2	2
$X_{110}-R-C$	2	4-2	4
$X_{110}-P-C$	2	3-3	2
$X_{020}-R-C$	2	3-3	2
$X_{020}-P-C$	2	2-4	1
$X_{200}-R-T$	2	4-0	3
$X_{200}-P-T$	2	3-1	3
$X_{110}-R-T$	2	5-1	5
$X_{110}-P-T$	2	4-2	4
$X_{020}-R-T$	2	4-2	3
$X_{020}-P-T$	2	3-3	2
$X_{300}-2R$	1	3-1	3
$X_{300}-R-P$	2	2-2	2
$X_{300}-2P$	1	1-3	1
$X_{210}-2R$	1	4-2	3
$X_{210}-R-P$	2	3-3	2
$X_{210}-2P$	1	2-4	1
X_0-2R-S_G	3	3-1	3
$X_0-R-P-S_G$	6	2-2	2
X_0-2P-S_G	3	1-3	1
X_0-2R-C_P	3	4-2	3
$X_0-R-P-C_P$	6	3-3	2
X_0-2P-C_P	3	2-4	1
$X_0-R-C-S$	6	3-1	3
$X_0-P-C-S$	6	2-2	2
$X_0-R-T-S$	6	4-0	3
$X_0-P-T-S$	6	3-1	3
$X_0-R-C-E$	6	3-3	2
$X_0-P-C-E$	6	2-4	1
$X_0-R-T-E$	6	4-2	3
$X_0-P-T-E$	6	3-3	2
$X_0-R-C-S_{SC}$	6	4-2	4
$X_0-P-C-S_{SC}$	6	3-3	2
$X_0-R-T-S_{SC}$	6	5-1	5
$X_0-P-T-S_{SC}$	6	4-2	4
X_0-3C	1	3-3	2
X_0-2C-T	3	4-2	4
X_0-C-2T	3	5-1	5
X_0-3T	1	6-0	5
(FOUR)			
$X_{100}-2R-S$	3	4-0	3
$X_{100}-R-P-S$	6	3-1	3
$X_{100}-2P-S$	3	2-2	2
$X_{010}-2R-S$	3	3-1	3
$X_{010}-R-P-S$	6	2-2	2
$X_{010}-2P-S$	3	1-3	1
$X_{100}-2R-E$	3	4-2	3
$X_{100}-R-P-E$	6	3-3	2
$X_{100}-2P-E$	3	2-4	1
$X_{010}-2R-E$	3	3-3	2
$X_{010}-R-P-E$	6	2-4	1
$X_{010}-2P-E$	3	1-5	1
$X_{100}-2R-S_{SC}$	2	5-1	5
$X_{100}-R-P-S_{SC}$	6	4-2	4
$X_{100}-2P-S_{SC}$	3	3-3	2
$X_{010}-2R-S_{SC}$	3	4-2	4
$X_{010}-R-P-S_{SC}$	6	3-3	2
$X_{010}-2P-S_{SC}$	3	2-4	1
$X_{100}-R-2C$	3	4-2	4
$X_{100}-P-2C$	3	3-3	2
$X_{010}-R-2C$	3	3-3	2
$X_{010}-P-2C$	3	2-4	1
$X_{100}-R-2T$	3	6-0	5
$X_{100}-P-2T$	3	5-1	5
$X_{010}-R-2T$	3	5-1	5
$X_{010}-P-2T$	3	4-2	4
$X_{100}-R-C-T$	6	5-1	5
$X_{100}-P-C-T$	6	4-2	4
$X_{010}-R-C-T$	6	4-2	4
$X_{010}-P-C-T$	6	3-3	2
$X_{200}-2R-C$	3	5-1	5
$X_{200}-R-P-C$	6	4-2	4
$X_{200}-2P-C$	3	3-3	2
$X_{110}-2R-C$	3	4-2	4
$X_{110}-R-P-C$	6	3-3	2
$X_{110}-2P-C$	3	2-4	1
$X_{020}-2R-C$	3	3-3	2
$X_{020}-R-P-C$	6	2-4	1
$X_{020}-2P-C$	3	1-5	1
$X_{200}-2R-T$	3	6-0	5
$X_{200}-R-P-T$	6	5-1	5
$X_{200}-2P-T$	3	4-2	4
$X_{110}-2R-T$	3	5-1	5
$X_{110}-R-P-T$	6	4-2	4
$X_{110}-2P-T$	3	3-3	2
$X_{020}-2R-T$	3	4-2	4
$X_{020}-R-P-T$	6	3-3	2
$X_{020}-2P-T$	3	2-4	1
$X_{300}-3R$	1	4-0	3
$X_{300}-2R-P$	3	3-1	3
$X_{300}-R-2P$	3	2-2	2
$X_{300}-3P$	1	1-3	1
$X_{210}-3R$	1	5-1	5
$X_{210}-2R-P$	3	4-2	4
$X_{210}-R-2P$	3	3-3	2
$X_{210}-3P$	1	2-4	1
$X_{120}-3R$	1	4-2	3
$X_{120}-2R-P$	3	3-3	2
$X_{120}-R-2P$	3	2-4	1
$X_{120}-3P$	1	1-5	1
X_0-3R-S	4	4-0	3
$X_0-2R-P-S$	12	3-1	3
$X_0-R-2P-S$	12	2-2	2
X_0-3P-S	4	1-3	1
X_0-3R-E	4	4-2	3
$X_0-2R-P-E$	12	3-3	2
$X_0-R-2P-E$	12	2-4	1
X_0-3P-E	4	1-5	1
X_0-3R-S_{SC}	4	5-1	5
$X_0-2R-P-S_{SC}$	12	4-2	4
$X_0-R-2P-S_{SC}$	12	3-3	2
X_0-3P-S_{SC}	4	2-4	1
$X_0-2R-2C$	6	4-2	4
$X_0-R-P-2C$	12	3-3	2
$X_0-2P-2C$	6	2-4	1
$X_0-2R-2T$	6	6-0	5
$X_0-R-P-2T$	12	5-1	5
$X_0-2P-2T$	6	4-2	4
$X_0-2R-C-T$	12	5-1	5
$X_0-R-P-C-T$	24	4-2	4
$X_0-2P-C-T$	12	3-3	2
(FIVE)			
$X_{100}-3R-C$	4	5-1	5
$X_{100}-2R-P-C$	12	4-2	4
$X_{100}-R-2P-C$	12	3-3	2
$X_{100}-3P-C$	4	2-4	1
$X_{010}-3R-C$	4	4-2	4
$X_{010}-2R-P-C$	12	3-3	2
$X_{010}-R-2P-C$	12	2-4	1
$X_{010}-3P-C$	4	1-5	1
$X_{100}-3R-T$	4	6-0	5
$X_{100}-2R-P-T$	12	5-1	5
$X_{100}-R-2P-T$	12	4-2	4
$X_{100}-3P-T$	4	3-3	2
$X_{010}-3R-T$	4	5-1	5
$X_{010}-2R-P-T$	12	4-2	4
$X_{010}-R-2P-T$	12	3-3	2
$X_{010}-3P-T$	4	2-4	1
$X_{200}-4R$	1	6-0	5
$X_{200}-3R-P$	4	5-1	5
$X_{200}-2R-2P$	6	4-2	4
$X_{200}-R-3P$	4	3-3	2
$X_{200}-4P$	1	2-4	1
$X_{110}-4R$	1	5-1	5
$X_{110}-3R-P$	4	4-2	4
$X_{110}-2R-2P$	6	3-3	2
$X_{110}-R-3P$	4	2-4	1
$X_{110}-4P$	1	1-5	1
$X_{020}-4R$	1	4-2	4
$X_{020}-3R-P$	4	3-3	2
$X_{020}-2R-2P$	6	2-4	1
$X_{020}-R-3P$	4	1-5	1
$X_{020}-4P$	1	0-6	0
X_0-4R-C	5	5-1	5
$X_0-3R-P-C$	20	4-2	4
$X_0-2R-2P-C$	30	3-3	2
$X_0-R-3P-C$	20	2-4	1
X_0-4P-C	5	1-5	1
X_0-4R-T	5	6-0	5
$X_0-3R-P-T$	20	5-1	5
$X_0-2R-2P-T$	30	4-2	4
$X_0-R-3P-T$	20	3-3	2
X_0-4P-T	5	2-4	1
(SIX)			
$X_{100}-5R$	1	6-0	5
$X_{100}-4R-P$	5	5-1	5
$X_{100}-3R-2P$	10	4-2	4
$X_{100}-2R-3P$	10	3-3	2
$X_{100}-R-4P$	5	2-4	1
$X_{100}-5P$	1	1-5	1
$X_{010}-5R$	1	5-1	5
$X_{010}-4R-P$	5	4-2	4
$X_{010}-3R-2P$	10	3-3	2
$X_{010}-2R-3P$	10	2-4	1
$X_{010}-R-4P$	5	1-5	1
$X_{010}-5P$	1	0-6	0
X_0-6R	1	6-0	5
X_0-5R-P	6	5-1	5
$X_0-4R-2P$	15	4-2	4
$X_0-3R-3P$	20	3-3	2
$X_0-2R-4P$	15	2-4	1
X_0-R-5P	6	1-5	1
X_0-6P	1	0-6	0

Note: There will always be an IDoF in each of the 5 mechanisms specified by †.

Table 14.3(a)-14.7(a). All O_iRM (i=1-5) expressed by the type symbols of their kinematic pair

I.---- O_iRM (i=1-5); II.--- Number of the mechanisms;

III.-- $\lambda - \sum_{i=1}^{i} N_T$; IV.--- The value of Λ.

Table 14.3(a)

I.	II.	III.	IV.
(O_1RM - 1)			
X_0-S_f	1	1-2	1
X_0-B_g	1	1-2	1
O_1RM - 2			
$X_{100}-S_G$	1	2-1	2
$X_{010}-S_G$	1	1-2	1
$X_{100}-C_P$	1	3-2	2
$X_{010}-C_P$	1	2-3	1
$X_{200}-S$	1	3-0	2
$X_{110}-S$	1	2-1	2
$X_{020}-S$	1	1-2	1
$X_{200}-E$	1	3-2	2
$X_{110}-E$	1	2-3	1
$X_{020}-E$	1	1-4	1
$X_{200}-S_{SC}$	1	4-1	4
$X_{110}-S_{SC}$	1	3-2	2
$X_{020}-S_{SC}$	1	2-3	1
$X_{300}-C$	1	2-1	2
$X_{210}-C$	1	3-2	2
$X_{120}-C$	1	2-3	1
$X_{300}-T$	1	3-0	2
$X_{210}-T$	1	4-1	4
$X_{120}-T$	1	3-2	2
$X_{310}-R$	1	2-1	2
$X_{310}-P$	1	1-2	1
$X_{220}-R$	1	3-2	2
$X_{220}-P$	1	2-3	1
X_0-R-S_G	2	2-1	2
X_0-P-S_G	2	1-2	1
X_0-R-C_P	2	3-2	2
X_0-P-C_P	2	2-3	1
X_0-C-S	2	2-1	2
X_0-C-E	2	2-3	1
X_0-C-S_{SC}	2	3-2	2
X_0-T-S	2	3-0	2
X_0-T-E	2	3-2	2
X_0-T-S_{SC}	2	4-1	4
(O_1RM - 3)			
$X_{100}-R-S$	2	3-0	2
$X_{100}-P-S$	2	2-1	2
$X_{010}-R-S$	2	2-1	2
$X_{010}-P-S$	2	1-2	1
$X_{100}-R-E$	2	3-2	2
$X_{100}-P-E$	2	2-3	1
$X_{010}-R-E$	2	2-3	1
$X_{010}-P-E$	2	1-4	1
$X_{100}-R-S_{SC}$	2	4-1	4
$X_{100}-P-S_{SC}$	2	3-2	2
$X_{010}-R-S_{SC}$	2	3-2	2
$X_{010}-P-S_{SC}$	2	2-3	1
$X_{100}-2C$	1	3-2	2
$X_{010}-2C$	1	2-3	1
$X_{100}-2T$	1	5-0	4
$X_{010}-2T$	1	4-1	4
$X_{100}-C-T$	2	4-1	4
$X_{010}-C-T$	2	3-2	2
$X_{200}-R-C$	2	4-1	4
$X_{200}-P-C$	2	3-2	2
$X_{110}-R-C$	2	3-2	2
$X_{110}-P-C$	2	2-3	1
$X_{020}-R-C$	2	2-3	1
$X_{020}-P-C$	2	1-4	1
$X_{200}-R-T$	2	5-0	4
$X_{200}-P-T$	2	4-1	4
$X_{110}-R-T$	2	4-1	4
$X_{110}-P-T$	2	3-2	2
$X_{020}-R-T$	2	3-2	2
$X_{020}-P-T$	2	2-3	1
$X_{300}-2R$	1	3-0	2
$X_{300}-R-P$	2	2-1	2
$X_{300}-2P$	1	1-2	1
$X_{210}-2R$	1	4-1	4
$X_{210}-R-P$	2	3-2	2
$X_{210}-2P$	1	2-3	1
$X_{120}-2R$	1	3-2	2
$X_{120}-R-P$	2	2-3	1
$X_{120}-2P$	1	1-4	1
X_0-R-2C	3	3-2	2
X_0-P-2C	3	2-2	1
$X_0-R-C-T$	6	4-1	4
$X_0-P-C-T$	6	3-2	2
X_0-R-2T	3	5-0	4
X_0-P-2T	3	4-1	4
X_0-2R-S	3	3-0	2
$X_0-R-P-S$	6	2-1	2
X_0-2P-S	3	1-2	1
X_0-2R-E	3	3-2	2
$X_0-R-P-E$	6	2-3	1
X_0-2P-E	3	1-4	1
X_0-2R-S_{SC}	3	4-1	4
$X_0-R-P-S_{SC}$	6	3-2	2
X_0-2P-S_{SC}	3	2-3	1
(O_1RM - 4)			
$X_{100}-2R-C$	3	4-1	4
$X_{100}-R-P-C$	6	3-2	2
$X_{100}-2P-C$	3	2-3	1
$X_{010}-2R-C$	3	3-2	2
$X_{010}-R-P-C$	6	2-3	1
$X_{010}-2P-C$	3	1-4	1
$X_{100}-2R-T$	3	5-0	4
$X_{100}-R-P-T$	6	4-1	4
$X_{100}-2P-T$	3	3-2	2
$X_{010}-2R-T$	3	4-1	4
$X_{010}-R-P-T$	6	3-2	2
$X_{010}-2P-T$	3	2-3	1
$X_{200}-3R$	1	5-0	4
$X_{200}-2R-P$	3	4-1	4
$X_{200}-R-2P$	3	3-2	2
$X_{200}-3P$	1	2-3	1
$X_{110}-3R$	1	4-1	4
$X_{110}-2R-P$	3	3-2	2
$X_{110}-R-2P$	3	2-3	1
$X_{110}-3P$	1	1-4	1
$X_{020}-3R$	1	3-2	2
$X_{020}-2R-P$	3	2-3	1
$X_{020}-R-2P$	3	1-4	1
$X_{020}-3P$	1	0-5	0
X_0-3R-C	4	4-1	4
$X_0-2R-P-C$	12	3-2	2
$X_0-R-2P-C$	12	2-3	1
X_0-3P-C	4	1-4	1
X_0-3R-T	4	5-0	4
$X_0-2R-P-T$	12	4-1	4
$X_0-R-2P-T$	12	3-2	2
X_0-3P-T	4	2-3	1
(O_1RM - 5)			
$X_{100}-4R$	1	5-0	4
$X_{100}-3R-P$	4	4-1	4
$X_{100}-2R-2P$	6	3-2	2
$X_{100}-R-3P$	4	2-3	1
$X_{100}-4P$	1	1-4	1
$X_{010}-4R$	1	4-1	4
$X_{010}-3R-P$	4	3-2	2
$X_{010}-2R-2P$	6	2-3	1
$X_{010}-R-3P$	4	1-4	1
$X_{010}-4P$	1	0-5	0
X_0-5R	1	5-0	4
X_0-4R-P	5	4-1	4
$X_0-3R-2P$	10	3-2	2
$X_0-2R-3P$	10	2-3	1
X_0-R-4P	5	1-4	1
X_0-5P	1	0-5	0
....	.	...	.

Table 14.4(a)

I.	II.	III.	IV.
(O_2RM - 1)			
X_0-S_G	1	1-1	1
X_0-C_P	1	2-2	1
(O_2RM - 2)			
$X_{100}-S$	1	2-0	1
$X_{010}-S$	1	1-1	1
$X_{100}-E$	1	2-2	1
$X_{010}-E$	1	1-3	1
$X_{100}-S_{SC}$	1	3-1	2
$X_{010}-S_{SC}$	1	2-2	1
$X_{200}-C$	1	3-1	2
$X_{110}-C$	1	2-2	1
$X_{020}-C$	1	1-3	1
$X_{200}-T$	1	4-0	3
$X_{110}-T$	1	3-1	2
$X_{020}-T$	1	2-2	1
$X_{200}-R$	1	2-0	1
$X_{200}-P$	1	1-1	1
$X_{110}-R$	1	3-1	2
$X_{110}-P$	1	2-2	1
$X_{020}-R$	1	2-2	1
$X_{020}-P$	1	1-3	1
X_0-R-S	2	2-0	1
X_0-P-S	2	1 •	1
X_0-R-E	2	1-3	1
X_0-P-E	2	1-3	1
X_0-R-S_{SC}	2	3-1	2
X_0-P-S_{SC}	2	2-2	1
X_0-2C	1	2-2	1
X_0-2T	1	4-0	3
X_0-C-T	2	3-1	2
(O_2RM - 3)			
$X_{100}-R-C$	2	3-1	2
$X_{100}-P-C$	2	2-2	1
$X_{010}-R-C$	2	2-2	1
$X_{010}-P-C$	2	1-3	1
$X_{100}-R-T$	2	4-0	3
$X_{100}-P-T$	2	3-1	2
$X_{010}-R-T$	2	3-1	2
$X_{010}-P-T$	2	2-2	1
$X_{200}-2R$	1	4-0	3
$X_{200}-R-P$	2	3-1	2
$X_{200}-2P$	1	2-2	1
$X_{110}-2R$	1	3-1	2
$X_{110}-R-P$	2	2-2	1
$X_{110}-2P$	1	1-3	1
$X_{020}-2R$	1	2-2	1
$X_{020}-R-P$	2	1-3	1
$X_{020}-2P$	1	0-4	0
X_0-2R-C	3	3-1	2
$X_0-R-P-C$	6	2-2	1
X_0-2P-C	3	1-3	1
X_0-2R-T	3	4-0	3
$X_0-R-P-T$	6	3-1	2
X_0-2P-T	3	2-2	1
(O_2RM - 4)			
$X_{100}-3R$	1	4-0	3
$X_{100}-2R-P$	3	3-1	2
$X_{100}-R-2P$	3	2-2	1
$X_{100}-3P$	1	1-3	1
$X_{010}-3R$	1	3-1	2
$X_{010}-2R-P$	3	2-2	1
$X_{010}-R-2P$	3	1-3	1
$X_{010}-3P$	1	0-4	0
X_0-4R	1	4-0	3
X_0-3R-P	4	3-1	2
$X_0-2R-2P$	6	2-2	1
X_0-R-3P	4	1-3	1
X_0-4P	1	0-4	0
....	.	...	.

Table 14.5(a)

I.	II.	III.	IV.
(O_3RM - 1)			
X_0-S	1	1-0	1
X_0-E	1	1-2	1
X_0-S_{SC}	1	2-1	1
(O_3RM - 2)			
$X_{100}-C$	1	2-1	1
$X_{010}-C$	1	1-2	1
$X_{100}-T$	1	3-0	2
$X_{010}-T$	1	2-1	1
$X_{200}-R$	1	3-0	2
$X_{200}-P$	1	2-1	1
$X_{110}-R$	1	2-1	1
$X_{110}-P$	1	1-2	1
$X_{020}-R$	1	1-2	1
$X_{020}-P$	1	0-3	0
X_0-R-C	2	2-1	1
X_0-P-C	2	1-2	1
X_0-R-T	2	3-0	2
X_0-P-T	2	2-1	1
(O_3RM - 3)			
$X_{100}-2R$	1	3-0	2
$X_{100}-R-P$	2	2-1	1
$X_{100}-2P$	1	1-2	1
$X_{010}-2R$	1	2-1	1
$X_{010}-R-P$	2	1-2	1
$X_{010}-2P$	1	0-3	0
X_0-3R	1	3-0	2
X_0-2R-P	3	2-1	1
X_0-R-2P	3	1-2	1
X_0-3P	1	0-3	0
....	.	...	.

Table 14.6(a)

I.	II.	III.	IV.
(O_4RM - 1)			
X_0-C	1	1-1	1
X_0-T	1	2-0	1
(O_4RM - 2)			
$X_{100}-R$	1	2-0	1
$X_{100}-P$	1	1-1	1
$X_{010}-R$	1	1-1	1
$X_{010}-P$	1	0-2	0
X_0-2R	1	2-0	1
X_0-R-P	2	1-1	1
X_0-2P	1	0-2	0
....	.	...	.

Table 14.7(a)

I.	II.	III.	IV.
(O_5RM - 1)			
X_0-R	1	1-0	1
X_0-P	1	0-1	0

CHAPTER 15. CORE-LOOPS OF SPATIAL MECHANISMS AND ROBOTS

15.1. Introduction

The concept of *inversion*, proposed by Reuleaux ([61] 1875) has long been considered a very important concept in the theory of mechanisms. It has been extensively used in the analysis of mechanisms. However, in reality, *inversion* only associates certain physically seemingly different mechanisms. As to the interrelationship of kinematic features among the associated mechanisms, this concept is often misleading. In addition, *inversion* is not a clearly defined concept. Its implication is vague and different scholars have different comprehension on what is really *inversion*.

Instead, a new concept, *core-loops* of spatial mechanisms and serial robots, is introduced in this chapter. It is proved to be a more useful concept in revealing the structural and the kinematical features of mechanisms and robots. Moreover, it provides a rational basis in counting the exact minimum number of the kinematically distinct structures of spatial mechanisms and serial robots.

15.2. Review on the concept of inversion

According to Shigley ([65] 1959, page 115), "Inversion is the fixing or grounding of a different link to the frame or earth. There are as many inversions as there are links in a mechanism." This definition is similar to the one given by Hartenberg and Denavit ([37] 1964, page 55), "A mechanism is derived from a closed kinematic chain by making one of its links stationary: by choosing different links as the stationary link or frame, the same closed chain will yield as many distinct mechanisms as it has links." "The process of fixing different links of a chain to create different mechanisms is called *kinematic inversion*."

According to Hartenberg in the discussion attached to [36] (Harrisberger, page 219, 1967), "*R-R-C-S* and *R-S-R-C* are not only two different chains but that, from each, four different mechanisms can be formed by inversion." It can be easily inferred that the four different mechanisms derived from the chain *R-R-C-S* are as follow,

$$RRCS,\ RCSR,\ RSCR,\ RRSC; \qquad (15.1)$$

However, for chain *R-S-R-C*, we can find, in term of symbolic representation, only two distinct mechanisms,

$$RSRC \quad RCRS. \qquad (15.2)$$

Then why Hartenberg claimed that four different mechanisms can be derived from chain *R-S-R-C*? Let's see Fig. 15.1, if we fix link 4 and take the *R* pair at *a* as input pair, then we get an *RSRC*; if we fix link 3 and take the *R* pair at *c* as input pair, we also get an *RSRC*. Similarly, if we fix link 1 and link 3 respectively, we get two *RCRS* mechanisms. What is the difference of the two *RSRC* mechanisms, what is the difference of the two *RCRS*? Obviously, it's their geometries.

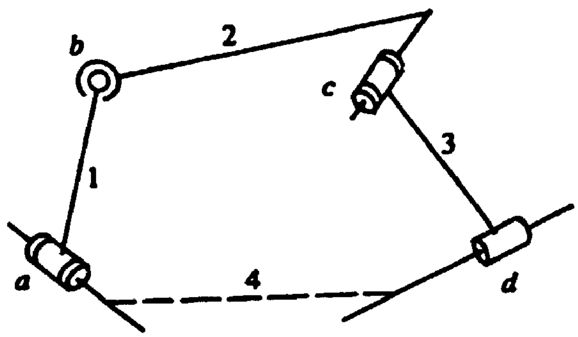


Fig. 15.1

We agree that there is nothing wrong to claim the two *RSRC* mechanisms are different. But what we would like to do here is to question the benefit we can get from emphasizing the difference of the two *RSRC* mechanisms. Based on the same rationale, we can make the following statements:

> Given a mechanism *RSRC* as shown in Fig. 15.1, we can get *infinite* number of *different* mechanisms by adjusting the geometries (such as the length of the links and the relative direction of the adjacent pair axes) of the mechanism.

From chain *R-S-R-C*, only *two* "different" *RSRC* mechanisms can be obtained by the method of inversion; however, by the method of adjusting the geometries of the mechanism, we can get *infinite* number of "different" *RSRC* mechanisms from chain *R-S-R-C*. Which method is more significant? There is no need to comment on the method of inversion, we can just say that the method of adjusting the geometries of mechanisms in order to get "different" mechanisms is meaningless, for it is self-evident.

According to Duffy ([28] p138, 1980), "in general, for any specified sequence of pairs a number of distinct mechanisms can be obtained by choosing various links as the frame. This process is known as *kinematic inversions* and the various mechanisms so obtained are simply inversions of one another". "For instance, the spatial five-link *RRCRC*$^{(2)}$ and *RCRRC*$^{(2)}$ mechanisms are inversions of the spatial *RCRCR*$^{(2)}$ mechanism". It is a well known fact that, for general geometries, the order of the algebraic equations (i.e. polynomials) governing the motions of the mechanisms *RRCRC* and *RCRCR* is four, whereas the order of the algebraic equation for *RCRRC* is eight ([84] Yang 1969, [95] Yuan 1971, [21] Duffy 1972, [110] Zhou, Buchal and Fenton, 1994). This obviously suggests that kinematically the mechanism *RCRRC* is different from both *RCRCR* and *RRCRC*. The kinematic features of a mechanism is determined by the structure of the mechanism. If the kinematic features of two mechanisms are different, it is quite natural for us to question the similarity or connection between the structures of the two mechanisms.

From the several scholars' treatments on inversion above , we can see that only revolute pair, i.e. R pair, has been chosen as input pair for those mechanisms. However, R pair is not the only pair usable for input pair. If the input pair is a *C* pair, the actuator of the input pair controls only the translational motion of the *C* pair, i.e. the input pair is C_{100}, being denoted by the new symbolic system introduced in chapter 1, then, how to define the inversion of the mechanism, say, for mechanism C_{100}*-HTHR*? This is a five link mechanism; how can we find five inversions of the mechanism? We can easily find many such examples where the concept of inversion appears awkward to apply.

Now the following questions naturally arise: given a closed kinematic chain or a mechanism, is there a clearly stated convention on which *inversions* of the chain or the mechanism can be derived? Since the number of *inversions* is not equal to the number of links of the chain or mechanism, then, how can we count the number of inversion of a chain or mechanism? Finally, what exactly is *inversion*? These are all unanswered questions. In our opinion, *inversion* is a concept which lacks rational basis and thus can never be clearly defined. Nevertheless, a new concept, the *core-loops* of spatial mechanisms and serial robots, can clarify all the confusion caused by the concept of inversion.

15.3. Core-loop

Given a mechanism R_0*-CRRC* as shown in Fig. 15.2, where the input is an angle θ_1, the unknown variables are $\{\theta_2, \theta_3, \theta_4, \theta_5, x_2, x_5\}$. For the same mechanism, let's replace the input pair, the *R* pair, by a *P* pair, to obtain P_0*-CRRC* as shown in Fig. 15.3. The input now becomes a length s_1 and the unknown variables are still the same. Without even proceeding to further detailed analysis, it is not difficult to perceive that the kinematic analysis for the two mechanisms can be expected to be the same. Why? Let's draw the common perpendicular line of $\mathbf{a}_2$ and $\mathbf{a}_5$ in Fig. 15.2 or Fig. 15.3, and disregard the other vector components related to the input pair, we get Fig. 15.4, where p_6 and angle $\hat{\mathbf{a}_5\mathbf{a}_2}$, being measured by a right rotation of $\mathbf{a}_5$ to $\mathbf{a}_2$ about $\mathbf{q}_6$, are functions of the controllable input. Since the controllable input can always be considered as fixed or invariable for the purpose of analysis, as a result p_6

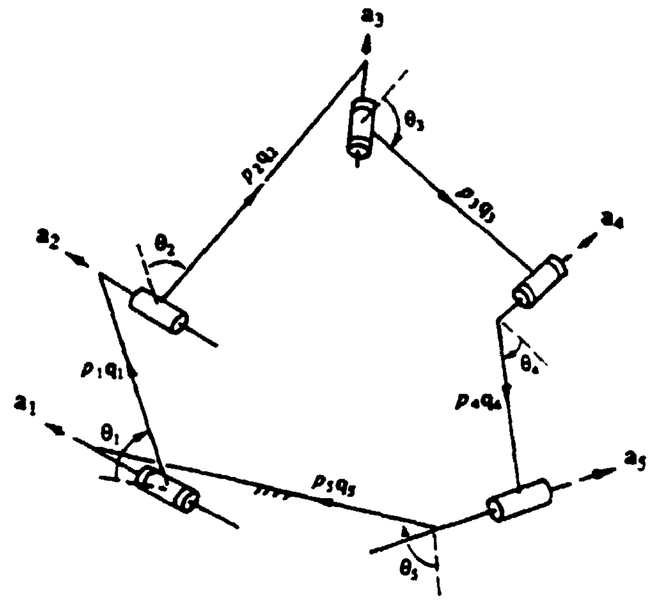


Fig. 15.2

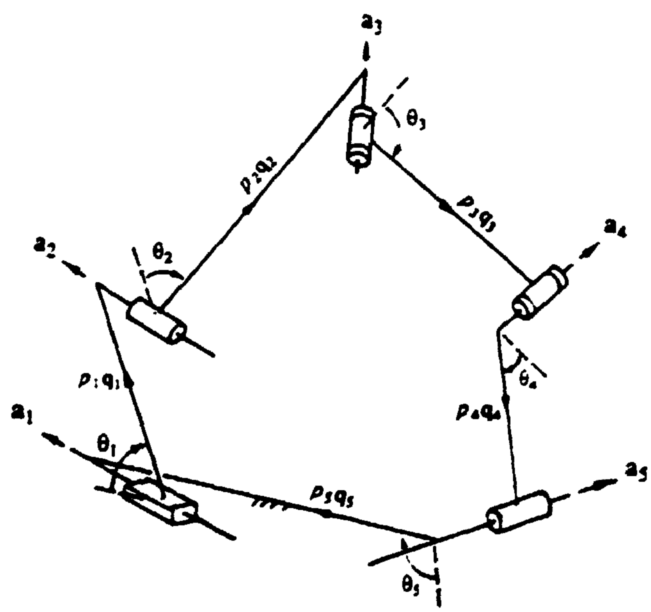


Fig. 15.3

and angle $(\widehat{a_5,a_2})$ can also be considered as known invariable. Now it becomes clear that the configuration analyses for both mechanisms of Fig. 15.2 and Fig. 15.3 are equivalent to the analysis of the structure of Fig. 15.4. Moreover, since the relative distance and direction of every two adjacent pair axes along the loop of Fig. 15.4 is fixed, i.e. the quantities $\{p_2, p_3, p_4, p_6, S_3, S_4, \alpha_{23}, \alpha_{34}, \alpha_{45}, \alpha_{52}\}$ are known and independent of the Earth (or the Sun) reference. Therefore, as long as the relative order (or position) of the four *active* pairs along the loop of Fig. 15.4 is unchanged, the analysis should be the same, i.e. the analysis for the four structures *CRRC*, *RRCC*, *RCCR* and

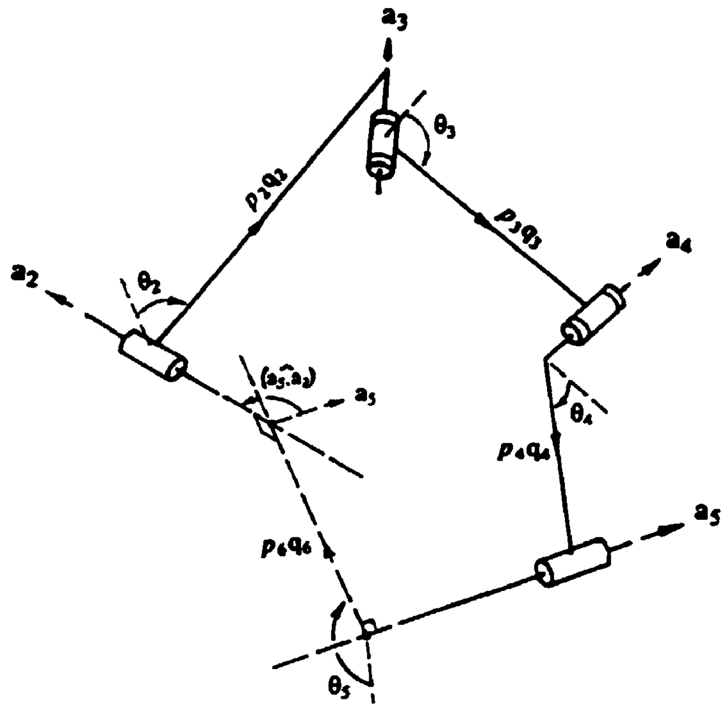


Fig. 15.4

CCRR should all be the same. This inference actually has been proved in chapter 9. If we add input pairs to the four structures, we can get various kinds of mechanisms such as X_0-*CRRC*, X_0-*RRCC*, X_{100}-*RCC*, X_{100}-*RCCR* and X_{110}-*CRR*, etc. where X_0 can be replaced by R_0, P_0, H_0, and C_0, etc.; X_{100} can be replaced by C_{100}, E_{100}, etc. The key point of all the five concrete mechanisms is that they all share the same structure of Fig. 15.4, which can be symbolically represented by any of the four forms:

$$CRRC,\quad RRCC,\quad RCCR,\quad CCRR \tag{15.3}$$

If we change the relative order (or position) of the four (active) pairs, we can get another distinct structure which symbolically can only be denoted by any of the two forms:

$$CRCR,\quad RCRC \tag{15.4}$$

Kinematically, the four structures of (15.3) are equivalent to each other, the two structures of (15.4) are also equivalent to each other. Now we have the following important concept:

Definition. *Core-loop*.

For any mechanism, if we disregard the controllable input, and count only the active pairs, we can get a loop which is composed of only active pairs. The loop so obtained is defined as the *core-loop* of the mechanism.

Any mechanism has one and only one core-loop; it is unique in terms of the arrangements of the relative order (or position) of the active pairs along the loop. By symbolical representation, it may appear in different forms, such as the two core-loops shown in (15.3) and (15.4), respectively. Therefore, it is possible that many seemingly different mechanisms or serial robots can be kinematically grouped together because of the same core-loop shared by each of them. As another example, let A, B, C and D be four different types of kinematic pairs, the core-loop of a mechanism is denoted as $ABCD$ as shown in Fig. 15.5, then the different representations of the same core-loop are as follow,

$$\begin{aligned} ABCD &= BCDA = CDAB = DABC \\ &= ADCB = BADC = CBAD = DCBA \end{aligned} \tag{15.5}$$

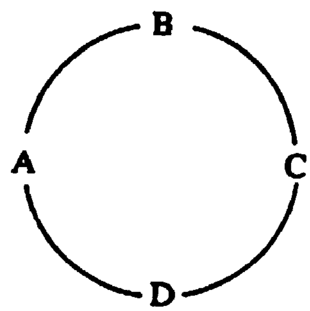


Fig. 15.5

By changing the relative position of the four pairs, we can easily find that there exist two more distinct core-loops, i.e. $ABDC$ and $ADCB$, and each of them may have other equivalent representations, similar to the way of (15.5).

Using the concept of core-loop, we can easily find out that it's not a surprise that the order of the displacement equations governing the mechanism R_0-$CRRC$ is different from that of R_0-$CRCR$ and R_0-$RCRC$.

15.4. Conclusion

As a convenient reference, the core-loops of all $\{RM \cdot O_iRM\}$ and $\{H_jM \cdot O_iH_jM\}$ $(j=1\text{-}6)$ are listed in Tables 15.1 - 15.7 and Tables 15.1(a) - 15.7(a) by class and type symbols of their kinematic pairs, respectively, where $(2 \le i+j \le 6)$. Information such as the number of permutation of each core-loop, the values of the *Small* λ and the *Big* Λ are also displayed.

The number of the distinct core-loops of the RM, O_iRM, H_jM and O_iH_jM are displayed in Tables 15.1(bi) - 15.6(bj) and 15.7(b), respectively. And the results are in turn be summarized in Tables 15.8. Reviewing the literatures on spatial mechanisms, we can often come across the following statements and its likes, "there are many spatial mechanisms" or "many thousands of spatial mechanisms, depending upon the type and arrangement of the pairing between links". How many? Nobody knew. This has always been a vague concept. Now adding all the numbers in Table 15.8 together, we get the total number of the distinct core-loops of spatial mechanisms, which is 560 !

Given a group of mechanisms: R_0-$2R$-$2C$, using the formulae in Eq.(1.6), we can easily find out that there is a total of 6 variant mechanisms. The number 6 is determined by 2-2, which is called the *pattern* of the mechanisms. Now can we ask how many distinct core-loops this group of mechanisms has? Obviously it is also determined by the *pattern*: 2-2. In Table 15.9, the numbers of the distinct core-loops corresponding to all possible *patterns* are displayed, from which we can see that the R_0-$2R$-$2C$ mechanisms have just 2 distinct core-loops.

This chapter can also serve as a general guideline for searching for and designing more useful mechanisms.

Table 15.1. All the core-loops of *RM* and O_iRM ($i=1$ - 5) expressed by the class symbols of kinematic pairs

0	1	2	3	4	5	6
RM		$1p_1-1p_5$ $1p_2-1p_4$	$1p_1-1p_2-1p_3$ $2p_1-1p_4$	$2p_1-2p_2$ $3p_1-1p_3$	$4p_1-1p_2$	$6P_1$
O_1RM	$1p_5$	$1p_1-1p_4$ $1p_2-1p_3$	$1p_1-2p_2$ $2p_1-1p_3$	$3p_1-1p_2$	$5p_1$	
O_2RM	$1p_4$	$1p_1-1p_3$ $2p_2$	$2p_1-1p_2$	$4p_1$		
O_3RM	$1p_3$	$1p_1-1p_2$	$3p_1$			
O_4RM	$1p_2$	$2p_1$				
O_5RM	$1p_1$					

Notes: The number i ($i=1$ - 6) in the first row means that the core-loops in column i are all composed of i active pairs. p_k represents the pairs whose DoF equals k.

Table 15.2. All the core-loops of H_1M and O_iH_1M ($i=1$ - 5) expressed by H, S_{SH}, S_{GH} and the class symbols of kinematic pairs

0	1	2	3	4	5	6
H_1M		$H-1p_5$ $S_{GH}-1p_2$ $S_{SH}-1p_3$ $2S_{SH}$	$H-1p_2-1p_3$ $H-1p_1-1p_4$ $S_{GH}-2p_1$ $S_{SH}-1p_1-1p_2$	$H-1p_1-2p_2$ $H-2p_1-1p_3$	$H-3p_1-1p_2$	$H-5p_1$
O_1H_1M		$H-1p_4$ $S_{GH}-1p_1$ $S_{SH}-1p_2$	$H-2p_2$ $H-1p_1-1p_3$ $S_{SH}-2p_1$	$H-2p_1-1p_2$	$H-4p_1$	
O_2H_1M	S_{GH}	$H-1p_3$ $S_{SH}-1p_1$	$H-1p_1-1p_2$	$H-3p_1$		
O_3H_1M	S_{SH}	$H-1p_2$	$H-2p_1$			
O_4H_1M		$H-1p_1$				
O_5H_1M	H					

Notes: The number i ($i=1$ - 6) in the first row means that the core-loops in column i are all composed of i active pairs.

Table 15.3. All the core-loops of H_2M and O_iH_2M ($i=1$ - 4) expressed by H, S_{SH}, S_{OH} and the class symbols of kinematic pairs

0	1	2	3	4	5	6
H_2M		$2S_{SH}$	$2H-1p_4$ $H-S_{OH}-1p_1$ $H-S_{SH}-1p_2$	$2H-2p_2$ $2H-1p_1-1p_3$ $H-S_{SH}-2p_1$	$2H-2p_1-1p_2$	$2H-4p_1$
O_1H_2M		$H-S_{OH}$	$2H-1p_3$ $H-S_{SH}-1p_1$	$2H-1p_1-1p_2$	$2H-3p_1$	
O_2H_2M		$H-S_{SH}$	$2H-1p_2$	$2H-2p_1$		
O_3H_2M			$2H-1p_1$			
O_4H_1M		$2H$				

Table 15.4. All the core-loops of H_3M and O_iH_3M ($i=1$ - 3) expressed by H, S_{SH}, S_{OH} and the class symbols of kinematic pairs

0	1	2	3	4	5	6
H_3M			$2H-S_{OH}$		$3H-1p_1-1p_2$	$3H-3p_1$
O_1H_3M			$2H-S_{SH}$	$3H-1p_2$	$3H-2p_1$	
O_2H_3M				$3H-1p_1$		
O_3H_3M			$3H$			

Table 15.5. All the core-loops of H_4M and O_iH_4M ($i=1$ - 2) expressed by H, S_{SH} and the class symbols of kinematic pairs

0	1	2	3	4	5	6
H_4M				$3H-S_{SH}$	$4H-1p_2$	$4H-2p_1$
O_1H_4M					$4H-1p_1$	
O_2H_4M				$4H$		

Table 15.6. All the core-loops of H_5M and O_1H_5M expressed by H and the class symbols of kinematic pairs

0	1	2	3	4	5	6
H_5M						$5H-1p_1$
O_1H_5M					$5H$	

Table 15.7. All the core-loops of H_6M expressed by H

0	1	2	3	4	5	6
H_6M						$6H$

Table 15.1(a). All RM and ORM core-loops expressed by the type symbols of kinematic pair

I.---- Core-loops; II.--- Number of permutation;
III.-- $\lambda - \sum_{i=1}^{\lambda-1} N_T$; IV.--- The value of Λ.

I.	II.	III.	IV.
(RM - 2)			
$R-S_P$	1	2-2	2
$R-B_B$	1	2-2	2
$P-S_P$	1	1-3	1
$P-B_B$	1	1-3	1
$C-S_G$	1	2-2	2
$C-C_P$	1	3-3	2
$T-S_G$	1	3-1	3
$T-C_P$	1	4-2	3
$S-S$†	1	2-0	1
$S-E$†	1	2-2	2
$S-S_{sc}$	1	3-1	3
$E-E$	1	2-4	1
$E-S_{sc}$	1	3-3	2
$S_{sc}-S_{sc}$	1	4-2	4
(RM - 3)			
$R-C-S$	1	3-1	3
$R-C-E$	1	3-3	2
$R-C-S_{sc}$	1	4-2	4
$R-T-S$	1	4-0	3
$R-T-E$	1	4-2	3
$R-T-S_{sc}$	1	5-1	5
$P-C-S$	1	2-2	2
$P-C-E$	1	2-4	1
$P-C-S_{sc}$	1	3-3	2
$P-T-S$	1	3-1	3
$P-T-E$	1	3-3	2
$P-T-S_{sc}$	1	4-2	4
$2R-S_G$	1	3-1	3
$2R-C_P$	1	4-2	3
$R-P-S_G$	1	2-2	2
$R-P-C_P$	1	3-3	2
$2P-S_G$	1	1-3	1
$2P-C_P$	1	2-4	1
3C	1	3-3	2
2C–T	1	4-2	4
C–2T	1	5-1	5
3T	1	6-0	5
(RM - 4)			
2R–2C	2	4-2	4
2R–C–T	2	5-1	5
2R–2T	2	6-0	5
R–P–2C	2	3-3	2
R–P–C–T	3	4-2	4
R–P–2T	2	5-1	5
2P–2C	2	2-4	1
2P–C–T	2	3-3	2
2P–2T	2	4-2	4
3R–S	1	4-0	3
3R–E	1	4-2	3
$3R-S_{sc}$	1	5-1	5
2R–P–S	2	3-1	3
2R–P–E	2	3-3	2
$2R-P-S_{sc}$	2	4-2	4
R–2P–S	2	2-2	2
R–2P–E	2	2-4	1
$R-2P-S_{sc}$	2	3-3	2
3P–S	1	1-3	1
3P–E	1	1-5	1
$3P-S_{sc}$	1	2-4	1
(RM - 5)			
4R–C	1	5-1	5
4R–T	1	6-0	5
3R–P–C	2	4-2	4
3R–P–T	2	5-1	5
2R–2P–C	3	3-3	2
2R–2P–T	3	4-2	4
R–3P–C	2	2-4	1
R–3P–T	2	3-3	2
4P–C	1	1-5	1
4P–T	1	2-4	1
(RM - 6)			
6R	1	6-0	5
5R–P	1	5-1	5
4R–2P	3	4-2	4
3R–3P	2	3-3	2
2R–4P	3	2-4	1
R–5P	1	1-5	1
6R	1	0-6	0
••••••••••	•	•••	•
$(O_1RM - 1)$			
S_P	1	1-2	1
B_B	1	1-2	1
$(O_1RM - 2)$			
$R-S_G$	1	2-1	2
$R-C_P$	1	3-2	2
$P-S_G$	1	1-2	1
$P-C_P$	1	2-3	1
C–S*	1	2-1	2
C–E	1	2-3	1
$C-S_{sc}$*	1	3-2	2
T–S	1	3-0	2
T–E	1	3-2	2
$T-S_{sc}$	1	4-1	4
$(O_1RM - 3)$			
R–2C	1	3-2	2
R–C–T	1	4-1	4
R–2T	1	5-0	4
P–2C	1	2-3	1
P–C–T	1	3-2	2
P–2T	1	4-1	4
2R–S	1	3-0	2
2R–E	1	3-2	2
$2R-S_{sc}$	1	4-1	4
R–P–S	1	2-1	2
R–P–E	1	2-3	1
$R-P-S_{sc}$	1	3-2	2
2P–S	1	1-2	1
2P–E	1	1-4	1
$2P-S_{sc}$	1	2-3	1
$(O_1RM - 4)$			
3R–C	1	4-1	4
3R–T	1	5-0	4
2R–P–C	2	3-2	2
2R–P–T	2	4-1	4
R–2P–C	2	2-3	1
R–2P–T	2	3-2	2
3P–C	1	1-4	1
3P–T	1	2-3	1
$(O_1RM - 5)$			
5R	1	5-0	4
4R–P	1	4-1	4
3R–2P	2	3-2	2
2R–3R	2	2-3	1
R–4P	1	1-4	1
5P	1	0-5	0
••••••••••	•	•••	•
$(O_2RM - 1)$			
S_G	1	1-1	1
C_P	1	2-2	1
$(O_2RM - 2)$			
R–S	1	2-0	1
R–E	1	2-2	1
$R-S_{sc}$	1	3-1	2
P–S	1	1-1	1
P–E	1	1-3	1
$P-S_{sc}$	1	2-2	1
2C	1	2-2	1
C–T	1	3-1	2
2T	1	4-0	3
$(O_2RM - 3)$			
2R–C	1	3-1	2
2R–T	1	4-0	3
R–P–C	1	2-2	1
R–P–T	1	3-1	2
2P–C	1	1-3	1
2P–T	1	2-2	1
$(O_2RM - 4)$			
4R	1	4-0	3
3R–P	1	3-1	2
2R–2P	2	2-2	1
R–3P	1	1-3	1
4P	1	0-4	0
••••••••••	•	•••	•
$(O_3RM - 1)$			
S	1	1-0	1
E	1	1-2	1
S_{sc}	1	2-1	1
$(O_3RM - 2)$			
R–C	1	2-1	1
P–C	1	1-2	1
R–T	1	3-0	2
P–T	1	2-1	1
$(O_3RM - 3)$			
3R	1	3-0	2
2R–P	1	2-1	1
R–2P	1	1-2	1
3P	1	0-3	0
••••••••••	•	•••	•
$(O_4RM - 1)$			
C	1	1-1	1
T	1	2-0	1
$(O_4RM - 2)$			
2R	1	2-0	1
R–P	1	1-1	1
2P	1	0-2	0
••••••••••	•	•••	•
$(O_5RM - 1)$			
R	1	1-0	1
P	1	0-1	0

Note: There will always be an IDoF in each of the 2 core loops specified by †.

Table 15.2(a). All H_1M and OH_1M core-loops expressed by the type symbols of kinematic pair

I.— H_1M core-loops; II.— Number of permutation;

III.— $\lambda - \sum_{i=1}^{j} N_T$; IV.— The value of Λ.

I.	II.	III.	IV.
(H_1M-2)			
$H-S_p$	1	2-2	2
$H-E_p$	1	2-2	2
$C-S_{om}$	1	2-2	2
$T-S_{om}$	1	3-1	3
$S-S_{sm}$	1	3-1	2
$E-S_{sm}$	1	3-3	2
$S_{sc}-S_{sm}$	1	4-2	3
(H_1M-3)			
$H-C-S$	1	3-1	3
$H-C-E$	1	3-3	2
$H-C-S_{sc}$	1	4-2	4
$H-R-S_g$	1	3-1	3
$H-R-C_p$	1	4-2	3
$H-P-S_g$	1	2-2	2
$H-P-C_p$	1	3-3	2
$2R-S_{om}$	1	3-1	3
$R-P-S_{om}$	1	2-2	2
$2P-S_{om}$	1	1-3	1
$R-C-S_{sm}$	1	4-2	3
$R-T-S_{sm}$	1	5-1	4
$P-C-S_{sm}$	1	3-3	2
$P-T-S_{sm}$	1	4-2	3
(H_1M-4)			
$H-R-2C$	2	4-2	4
$H-R-C-T$	3	5-1	5
$H-R-2T$	2	6-0	5
$H-P-2C$	2	3-3	2
$H-P-C-T$	3	4-2	4
$H-P-2T$	2	5-1	5
$H-2R-S$	2	4-0	3
$H-2R-E$	2	4-2	4
$H-2R-S_{sc}$	2	5-1	5
$H-R-P-S$	3	3-1	3
$H-R-P-E$	3	3-3	2
$H-R-P-S_{sc}$	3	4-2	4
$H-2P-S$	2	2-2	2
$H-2P-E$	2	2-4	1
$H-2P-S_{sc}$	2	3-3	2
(H_1M-5)			
$H-3R-C$	2	5-1	5
$H-3R-T$	2	6-0	5
$H-2R-P-C$	6	4-2	4
$H-2R-P-T$	6	5-1	5
$H-R-2P-C$	6	3-3	2
$H-R-2P-T$	6	4-2	4
$H-3P-C$	2	2-4	1
$H-3P-T$	2	3-3	2
(H_1M-6)			
$H-5R$	1	6-0	5
$H-4R-P$	3	5-1	5
$H-3R-2P$	6	4-2	4
$H-2R-3P$	6	3-3	2
$H-R-4P$	3	2-4	1
$H-5P$	1	1-5	1
••••••••••	•	•••	•
(O_1H_1M-2)			
$H-S_g$	1	2-1	2
$H-C_p$	1	3-2	2
$R-S_{om}$	1	2-1	2
$P-S_{om}$	1	1-2	1
$C-S_{sm}$	1	3-2	2
$T-S_{sm}$	1	4-1	3
(O_1H_1M-3)			
$H-2C$	1	3-2	2
$H-C-T$	1	4-1	4
$H-2T$	1	5-0	4
$H-R-S$	1	3-0	2
$H-R-E$	1	3-2	2
$H-R-S_{sc}$	1	4-1	4
$H-P-S$	1	2-1	2
$H-P-E$	1	2-3	1
$H-P-S_{sc}$	1	3-2	2
$2R-S_{sm}$	1	4-1	3
$R-P-S_{sm}$	1	3-2	2
$2P-S_{sm}$	1	2-3	1
(O_1H_1M-4)			
$H-2R-C$	2	4-1	4
$H-2R-T$	2	5-0	4
$H-R-P-C$	3	3-2	2
$H-R-P-T$	3	4-1	4
$H-2P-C$	2	2-3	1
$H-2P-T$	2	3-2	2
(O_1H_1M-5)			
$H-4R$	1	5-0	4
$H-3R-P$	2	4-1	4
$H-2R-2P$	3	3-2	2
$H-R-3P$	2	2-3	1
$H-4P$	1	1-4	1
••••••••••	•	•••	•
(O_2H_1M-1)			
S_{om}	1	1-1	1
(O_2H_1M-2)			
$H-S$	1	2-0	1
$H-E$	1	2-2	1
$H-S_{sc}$	1	3-1	2
$R-S_{sm}$	1	3-1	2
$P-S_{sm}$	1	2-2	1
(O_2H_1M-3)			
$H-R-C$	1	3-1	2
$H-R-T$	1	4-0	3
$H-P-C$	1	2-2	1
$H-P-T$	1	3-1	2
(O_2H_1M-4)			
$H-3R$	1	4-0	3
$H-2R-P$	2	3-1	2
$H-R-2P$	2	2-2	1
$H-3P$	1	1-3	1
••••••••••	•	•••	•
(O_3H_1M-1)			
S_{sm}	1	2-1	1
(O_3H_1M-2)			
$H-C$	1	2-1	1
$H-T$	1	3-0	2
(O_3H_1M-3)			
$H-2R$	1	3-0	2
$H-R-P$	1	2-1	1
$H-2P$	1	1-2	1
••••••••••	•	•••	•
(O_4H_1M-2)			
$H-R$	1	2-0	1
$H-P$	1	1-1	1
••••••••••	•	•••	•
(O_5H_1M-1)			
H	1	1-0	1

Table 15.3(a). All H_2M and OH_2M core-loops expressed by the type symbols of kinematic pair

I.---- Core-loops; II.--- Number of permutation;
III.-- $\lambda - \sum_{i=1}^{s-1} N_T$; IV.--- The value of Λ.

I.	II.	III.	IV.
(H_2M-2)			
$2S_{SH}$	1	4-2	3
(H_2M-3)			
$2H-S_O$	1	3-1	3
$2H-C_P$	1	4-2	3
$H-R-S_{OH}$	1	3-1	3
$H-P-S_{OH}$	1	2-2	2
$H-C-S_{SH}$	1	4-2	3
$H-T-S_{SH}$	1	5-1	4
(H_2M-4)			
$2H-2C$	2	4-2	4
$2H-C-T$	2	5-1	5
$2H-2T$	2	6-0	5
$2H-R-S$	2	4-0	3
$2H-R-E$	2	4-2	3
$2H-R-S_{SC}$	2	5-1	5
$2H-P-S$	2	3-1	3
$2H-P-E$	2	3-3	2
$2H-P-S_{SC}$	2	4-2	4
$H-2R-S_{SH}$	2	5-1	4
$H-R-P-S_{SH}$	3	4-2	3
$H-2P-S_{SH}$	2	3-3	2
(H_2M-5)			
$2H-2R-C$	6	5-1	5
$2H-2R-T$	6	6-0	5
$2H-R-P-C$	6	4-2	4
$2H-R-P-T$	6	5-1	5
$2H-2P-C$	6	3-3	2
$2H-2P-T$	6	4-2	4
(H_2M-6)			
$2H-4R$	3	6-0	5
$2H-3R-P$	6	5-1	5
$2H-2R-2P$	11	4-2	4
$2H-R-3P$	6	3-3	2
$2H-4P$	3	2-4	1
••••••••••	•	•••	•
(O_1H_2M-2)			
$H-S_{OH}$	1	2-1	2
(O_1H_2M-3)			
$2H-S$	1	3-0	2
$2H-E$	1	3-2	2
$2H-S_{SC}$	1	4-1	4
$H-R-S_{SH}$	1	4-1	3
$H-P-S_{SH}$	1	3-2	2
(O_1H_2M-4)			
$2H-R-C$	2	4-1	4
$2H-R-T$	2	5-0	4
$2H-P-C$	2	3-2	2
$2H-P-T$	2	4-1	4
(O_1H_2M-5)			
$2H-3R$	2	5-0	4
$2H-2R-P$	6	4-1	4
$2H-R-2P$	6	3-2	2
$2H-3P$	2	2-3	1
••••••••••	•	•••	•
(O_2H_2M-2)			
$H-S_{SH}$	1	3-1	2
(O_2H_2M-3)			
$2H-C$	1	3-1	2
$2H-T$	1	4-0	3
(O_2H_2M-4)			
$2H-2R$	2	4-0	3
$2H-R-P$	2	3-1	2
$2H-2P$	2	2-2	1
••••••••••	•	•••	•
(O_3H_2M-3)			
$2H-R$	1	3-0	2
$2H-P$	1	2-1	1
••••••••••	•	•••	•
(O_4H_2M-2)			
$2H$	1	2-0	1

Table 15.4(a)-15.7(a). All H_iM and OH_iM ($i = 3-6$) core-loops expressed by the type symbols of kinematic pair

I.---- Core-loops; II.--- Number of permutation;
III.-- $\lambda - \sum_{i=1}^{s-1} N_T$; IV.--- The value of Λ.

I.	II.	III.	IV.
Table 15.4(a). H_3M			
(H_3M-3)			
$2H-S_{OH}$	1	3-1	3
(H_3M-5)			
$3H-R-C$	2	5-1	5
$3H-R-T$	2	6-0	5
$3H-P-C$	2	4-2	4
$3H-P-T$	2	5-1	5
(H_3M-6)			
$3H-3R$	2	6-0	5
$3H-2R-P$	6	5-1	5
$3H-R-2P$	6	4-2	4
$3H-3P$	2	3-3	2
••••••••••	•	•••	•
(O_1H_3M-3)			
$2H-S_{SH}$	1	4-1	3
(O_1H_3M-4)			
$3H-C$	1	4-1	4
$3H-T$	1	5-0	4
(O_1H_3M-5)			
$3H-2R$	2	5-0	4
$3H-R-P$	2	4-1	4
$3H-2P$	2	3-2	2
••••••••••	•	•••	•
(O_2H_3M-4)			
$3H-R$	1	4-0	3
$3H-P$	1	3-1	2
••••••••••	•	•••	•
(O_3H_3M-3)			
$3H$	1	3-0	2
Table 15.5(a). H_4M . OH_4M			
(H_4M-4)			
$3H-S_{SH}$	1	5-1	4
(H_4M-5)			
$4H-C$	1	5-1	5
$4H-T$	1	6-0	5
(H_4M-6)			
$4H-2R$	3	6-0	5
$4H-R-P$	3	5-1	5
$4H-2P$	3	4-2	4
••••••••••	•	•••	•
(O_1H_4M-5)			
$4H-R$	1	5-0	4
$4H-P$	1	4-1	4
••••••••••	•	•••	•
(O_2H_4M-4)			
$4H$	1	4-0	3
Table 15.6(a). H_5M . OH_5M			
(H_5M-6)			
$5H-R$	1	6-0	5
$5H-P$	1	5-1	5
••••••••••	•	•••	•
(O_1H_5M-5)			
$5H$	1	5-0	4
Table 15.7(a). H_6M			
(H_6M-6)			
$6H$	1	6-0	5

Table 15.1(b1). Data on the core-loops of *RM*

		Λ=0	Λ=1	Λ=2	Λ=3	Λ=4	Λ=5
2	14	0	4	6	3	1	0
3	22	0	3	7	6	3	3
4	37	0	7	10	4	9	7
5	18	0	4	5	0	5	4
6	12	1	4	2	0	3	2
Total number 103		1	22	30	13	21	16

Notes: The number 37 (the third number of the second column) is the total number of those *RM* core-loops that they have just 4 (the third number of the first column) active pairs. The data in this table come frc~ Table 15.1(a).

Table 15.1(b2). Data on the core-loops of O_1RM

		Λ=0	Λ=1	Λ=2	Λ=3	Λ=4	Λ=5
1	2	0	2	0	0	0	0
2	10	0	3	6	0	1	0
3	15	0	5	6	0	4	0
4	12	0	4	4	0	4	0
5	8	1	3	2	0	2	0
Total number 47		1	17	18	0	11	0

Table 15.1(b3). Data on the core-loops of O_2RM

		Λ=0	Λ=1	Λ=2	Λ=3	Λ=4	Λ=5
1	2	0	2	0	0	0	0
2	9	0	6	2	1	0	0
3	6	0	3	2	1	0	0
4	6	1	3	1	1	0	0
Total number 23		1	14	5	3	0	0

Table 15.1(b4). Data on the core-loops of O_3RM

		Λ=0	Λ=1	Λ=2	Λ=3	Λ=4	Λ=5
1	3	0	3	0	0	0	0
2	4	0	3	1	0	0	0
3	4	1	2	1	0	0	0
Total number 11		1	8	2	0	0	0

Table 15.1(b5). Data on the core-loops of O_4RM

		Λ=0	Λ=1	Λ=2	Λ=3	Λ=4	Λ=5
1	2	0	2	0	0	0	0
2	3	1	2	0	0	0	0
Total number 5		1	4	0	0	0	0

Table 15.1(b6). Data on the core-loops of O_5RM

		Λ=0	Λ=1	Λ=2	Λ=3	Λ=4	Λ=5
1	2	1	1	0	0	0	0
Total number 2		1	1	0	0	0	0

Table 15.2(b1). Data on the core-loops of H_1M

		Λ=0	Λ=1	Λ=2	Λ=3	Λ=4	Λ=5
2	7	0	0	5	2	0	0
3	14	0	1	5	6	2	0
4	35	0	2	9	5	10	9
5	32	0	2	8	0	12	10
6	20	0	4	6	0	6	4
Total number 108		0	9	33	13	30	23

Notes: The number 35 (the third number of the second column) is the total number of those H_1M core-loops that they have just 4 (the third number of the first column) active pairs. The data in this table come from Table 15.2(a).

Table 15.2(b2). Data on the core-loops of O_1H_1M

		Λ=0	Λ=1	Λ=2	Λ=3	Λ=4	Λ=5
2	6	0	1	4	1	0	0
3	12	0	2	6	1	3	0
4	14	0	2	5	0	7	0
5	9	0	3	3	0	3	0
Total number 41		0	8	18	2	13	0

Table 15.2(b3). Data on the core-loops of O_2H_1M

		Λ=0	Λ=1	Λ=2	Λ=3	Λ=4	Λ=5
1	1	0	1	0	0	0	0
2	5	0	3	2	0	0	0
3	4	0	1	2	1	0	0
4	6	0	3	2	1	0	0
Total number 16		0	8	6	2	0	0

Table 15.2(b4). Data on the core-loops of O_3H_1M

		Λ=0	Λ=1	Λ=2	Λ=3	Λ=4	Λ=5
1	1	0	1	0	0	0	0
2	2	0	1	1	0	0	0
3	3	0	2	1	0	0	0
Total number 6		0	4	2	0	0	0

Table 15.2(b5). Data on the core-loops of O_4H_1M

		Λ=0	Λ=1	Λ=2	Λ=3	Λ=4	Λ=5
2	2	0	2	0	0	0	0
Total number 2		0	2	0	0	0	0

Table 15.2(b6). Data on the core-loops of O_3H_1M

		Λ=0	Λ=1	Λ=2	Λ=3	Λ=4	Λ=5
1	1	0	1	0	0	0	0
Total number 1		0	1	0	0	0	0

Table 15.3(b1). Data on the core-loops of H_2M

		Λ=0	Λ=1	Λ=2	Λ=3	Λ=4	Λ=5
2	1	0	0	0	1	0	0
3	6	0	0	1	4	1	0
4	25	0	0	4	9	6	6
5	36	0	0	6	0	12	18
6	29	0	3	6	0	11	9
Total number 97		0	3	17	14	30	33

Notes: The number 25 (the third number of the second column) is the total number of those H_2M core-loops that they have just 4 (the third number of the first column) active pairs.

Table 15.3(b2). Data on the core-loops of O_1H_2M

		Λ=0	Λ=1	Λ=2	Λ=3	Λ=4	Λ=5
2	1	0	0	1	0	0	0
3	5	0	0	3	1	1	0
4	8	0	0	2	0	6	0
5	16	0	2	6	0	8	0
Total number 30		0	2	12	1	15	0

Table 15.3(b3). Data on the core-loops of O_2H_2M

		Λ=0	Λ=1	Λ=2	Λ=3	Λ=4	Λ=5
2	1	0	0	1	0	0	0
3	2	0	0	1	1	0	0
4	6	0	2	2	2	0	0
Total number 9		0	2	4	3	0	0

Table 15.3(b4). Data on the core-loops of O_3H_2M

		Λ=0	Λ=1	Λ=2	Λ=3	Λ=4	Λ=5
3	2	0	1	1	0	0	0
Total number 2		0	1	1	0	0	0

Table 15.3(b5). Data on the core-loops of O_4H_2M

		Λ=0	Λ=1	Λ=2	Λ=3	Λ=4	Λ=5
2	1	0	1	0	0	0	0
Total number 1		0	1	0	0	0	0

Table 15.4(b1). Data on the core-loops of H_3M

		Λ=0	Λ=1	Λ=2	Λ=3	Λ=4	Λ=5
3	1	0	0	0	1	0	0
5	8	0	0	0	0	2	6
6	16	0	0	2	0	6	8
Total number 25		0	0	2	1	8	14

Notes: The number 16 (the third number of the second column) is the total number of those H_3M core-loops that they have just 6 (the third number of the first column) active pairs.

Table 15.4(b2). Data on the core-loops of O_1H_3M

		Λ=0	Λ=1	Λ=2	Λ=3	Λ=4	Λ=5
3	1	0	0	0	1	0	0
4	2	0	0	0	0	2	0
5	6	0	0	2	0	4	0
Total number 9		0	0	2	1	6	0

Table 15.4(b3). Data on the core-loops of O_2H_3M

		Λ=0	Λ=1	Λ=2	Λ=3	Λ=4	Λ=5
4	2	0	0	1	1	0	0
Total number 2		0	0	1	1	0	0

Table 15.4(b4). Data on the core-loops of O_3H_3M

		Λ=0	Λ=1	Λ=2	Λ=3	Λ=4	Λ=5
3	1	0	0	1	0	0	0
Total number 1		0	0	1	0	0	0

Table 15.5(b1). Data on the core-loops of H_4M

		$\Lambda=0$	$\Lambda=1$	$\Lambda=2$	$\Lambda=3$	$\Lambda=4$	$\Lambda=5$
4	1	0	0	0	0	1	0
5	2	0	0	0	0	0	2
6	9	0	0	0	0	3	6
Total number 12		0	0	0	0	4	8

Notes: The number 9 (the third number of the second column) is the total number of those H_4M core-loops that they have just 6 (the third number of the first column) active pairs.

Table 15.5(b2). Data on the core-loops of O_1H_4M

		$\Lambda=0$	$\Lambda=1$	$\Lambda=2$	$\Lambda=3$	$\Lambda=4$	$\Lambda=5$
5	2	0	0	0	0	2	(
Total number 2		0	0	0	0	2	0

Table 15.5(b3). Data on the core-loops of O_2H_4M

		$\Lambda=0$	$\Lambda=1$	$\Lambda=2$	$\Lambda=3$	$\Lambda=4$	$\Lambda=5$
4	1	0	0	0	1	0	0
Total number 1		0	0	0	1	0	0

Table 15.6(b1). Data on the core-loops of H_5M

		Λ=0	Λ=1	Λ=2	Λ=3	Λ=4	Λ=5
6	2	0	0	0	0	0	2
Total number 2		0	0	0	0	0	2

Notes: The number 2 (the first number of the second column) is the total number of those H_5M core-loops that they have just 6 (the third number of the first column) active pairs.

Table 15.6(b2). Data on the core-loops of O_1H_5M

		Λ=0	Λ=1	Λ=2	Λ=3	Λ=4	Λ=5
5	1	0	0	0	0	1	0
Total number 1		0	0	0	0	1	0

Table 15.7(b). Data on the core-loops of H_6M

		Λ=0	Λ=1	Λ=2	Λ=3	Λ=4	Λ=5
6	1	0	0	0	0	0	1
Total number 1		0	0	0	0	0	1

Table 15.8. Summary of the data in Tables 15.*i* (b*j*). ($i = 1 - 5$, $j = 1 - 6$).

RM 103	O_1RM 47	O_2RM 23	O_3RM 11	O_4RM 5	O_5RM 2
H_1M 108	O_1H_1M 41	O_2H_1M 16	O_3H_1M 6	O_4H_1M 2	O_5H_1M 1
H_2M 97	O_1H_2M 30	O_2H_2M 9	O_3H_2M 2	O_4H_2M 1	
H_3M 25	O_1H_3M 9	O_2H_3M 2	O_3H_3M 1		
H_4M 12	O_1H_4M 2	O_2H_4M 1			
H_5M 2	O_1H_5M 1				
H_6M 1					

Notes: The number 103 (located at the 1st row and the 1st column) represents the total number of the distinct core-loops of Regular Mechanisms.

Table 15.9. Core-loop pattern & the No. of distinct core-loops.

Number of the active pairs	Patterns of core-loops	Number of the distinct core-loops
1	1	1
2	2	1
	1-1	1
3	3	1
	2-1	1
	1-1-1	1
4	4	1
	3-1	1
	2-2	2
	2-1-1	2
	1-1-1-1	3
5	5	1
	1-4	1
	2-3	2
	1-1-3	2
	1-2-2	3
6	6	1
	1-5	1
	2-4	3
	3-3	2

GENERAL CONCLUSIONS

The theoretical framework developed in this thesis offered a new basis for kinematic analysis, synthesis and design of spatial mechanisms and robots.

The author believes that the *vector algebraic method* presented in this thesis will eventually replace the dominant status of the *matrix method* and the *spherical trigonometry method*.

It is natural that some readers may doubt the author's claim. In respect to this, the author's suggestion is that "*try it and compare it yourself*", for this is probably the best way to appreciate the difference.

The author also believes that eventually the *Theory of Spatial Mechanisms* will be widely taught in engineering schools, for this is not only desirable but also feasible.

SUGGESTIONS FOR FURTHER RESEARCH

The following research areas are worthy to be given more attention: Mobility analysis of spatial mechanisms; Force, torque and dynamic analysis of spatial mechanisms; Computer-aided design of spatial mechanisms; Computer-aided teaching of spatial mechanisms; Trajectory analysis of spatial mechanisms; Analysis of helical and over-constrained helical mechanisms; Analysis and design of multi-loop spatial mechanisms; etc.

There are many spatial mechanisms being used in engineering. It is certainly desirable to publish a book containing the figures of all these practical spatial mechanisms.

A comprehensive book should be written under the title: *Synthesis of Spatial Mechanisms*. Hartenberg and Denavit's book ([37] 1964), *Kinematic Synthesis of Linkages*, deals only with planar mechanisms. Zhang's book ([106] 1980), *Analysis and Synthesis of Spatial Mechanisms* (Volume 1), does not contain any content on synthesis of spatial mechanisms. I visited Professor Zhang in Beijing in August 1993. He told me that he had intended to write a book (i.e. Volume 2) dealing with the synthesis of spatial mechanisms. However, he had not been able to have the time and the mood to carry out this undertaking. Zhu and Liu's book ([111] 1986), *Analysis and Synthesis of Spatial Linkage Mechanisms*, contains 9 pages on synthesis. Angeles' book ([3] 1982), *Spatial Kinematic Chain: Analysis, Synthesis, Optimization*, is probably so far the best (book) reference on synthesis, for it contains 2 chapters (about 120 pages) on synthesis and optimal synthesis of spatial linkages. However, the content of Professor Angeles' book on synthesis is still at introductory level.

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APPENDIX : NEW TERMINOLOGIES

PART 1: The followings are new terminologies introduced in this thesis.

3-digit input control number	Generalized kinematic pair
3-digit number of active DoF	Ground vector
Adjacent angular variable	Helical mechanism
Auxiliary angle	
Auxiliary vector	I-axis
	Idle Degree of Freedom (i.e. IDoF)
Basic contacts	IDoF of the first kind
Basic contact lines	IDoF of the second kind
Basic contact surfaces	R-IDoF-2
Basic variable	T-IDoF-2
Ball-point	H-IDoF-2
Big Λ	Indefinite link
	Indefinite pair
Central line	Indefinite mechanism
Central vector polygon	Input vector
Central vector loop equation	Number of parallel-couple
Constrained axis, freedom	Output vector
Contact point-set	Over-constrained helical mechanism
Core-loop	Over-constrained regular mechanism
Definite link	
Definite pair	R-point
Definite mechanism	Regular mechanism
Degeneration of closure	Resultant input type
Degree of angular freedom	
Degree of active angular freedom	Small λ
Degree of complexity of mechanism	Small sphere approximation
Diameter of pair	Star product operation
Dimension of mechanism	Subauxiliary angle
End axis vector	Unconstrained rotary axis, freedom
Floating vector	Variant mechanism

PART 2: The followings are existing terminologies and their implications have been re-defined or amended in this thesis.

Closure	Mechanism
Displacement equation	Pair axis
Input-output displacement equation	Pair variable
Link	Serial linkage
Link length	Single-loop linkage
Linkage	

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