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Contextualism And Nonlocality In Quantum Mechanics

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by

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ABSTRACT

I describe the conceptual problems associated with the Kochen-Specker theorem including the presuppositions of the theorem and plausible interpretations of the conclusions motivated by the theorem. I describe an idealized quantum system which demonstrates both the Kochen-Specker theorem and the Bell argument for nonlocality. I present new findings about the mathematical structures which support a proof of the Kochen-Specker theorem.

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Chapter 1

Quantum mechanics and classical mechanics

In classical mechanics, all observables may be simultaneously assigned probabilities by atomic measures (concentrated on a single point in phase space), or by weighted averages of such measures. In quantum mechanics, however, there are pairs of observables which cannot be simultaneously assigned probabilities in this way, so that with respect to classical probability assignments, quantum mechanics appears to be incomplete. The idea of completing quantum mechanics through an extension of classical mechanics requires that somehow the simultaneous and comprehensive atomic probability measures may be recovered from 'hidden variables' at work in quantum systems.

The Kochen-Specker [24] argument demonstrates that it is not possible to assign hidden-values to the constituents of some quantum ensembles. This contradicts the classical idea that we may universally make attributions of truth about the properties of constituents of ensembles, even when we are not measuring them. In so far as quantum mechanics is concerned there is nothing novel in this conclusion, but for interpretations of quantum mechanics the conclusion poses a fundamental problem, which I will refer to as the problem of contextualism.

The interpretational problem is to explain the inconsistency between the statistical algorithm of quantum mechanics and the *counterfactual* attribution of properties to sets of unmeasured quantum states. The Kochen-Specker theorem proves that, under appropriate circumstances, counterfactual attributions must impose contradictory values to some state. The concept of 'contextualism' is an interpretation of quantum mechanics which avoids the contradiction implied by counterfactual value attributions by accepting that the properties of the constituents of ensembles depend upon the context in which we measure them.

For the purposes of this dissertation a *realist interpretation of quantum mechanics* is any set of explanatory hypotheses which is both consistent with quantum mechanics and which gives an account of the measurement correlations in quantum mechanical systems in terms of the actions of hidden-variables. There are a number of options available to the realist metaphysician with which to evade the difficulties imposed on hidden variable theories by the 'context dependence' of measurement. In the present thesis, these options will be described but no attempt at evaluation will be made. The arguments presented here will describe constructions which support a contextualist interpretation: we will not examine the options by which a contextualist conclusion might be avoided.

Contextualism is undesirable to the realist because it leads immediately to the following predicament: if truth values of propositions are to depend upon the context of measurement then it is not possible to counterfactually assign a complete set of truth values to yes/no propositions about certain physical systems and their properties. Thus realist interpretations which avoid the Kochen-Specker 'paradox' by accepting contextualism appear committed to giving up a central tenet of realism, namely, that the world has a determinate structure which underwrites causal explanations and our ability to make successful predictions independently of our knowledge of that structure.

Several alternatives for avoiding the contextualist argument are available. Among them is the idea, attributable to Bohm, that hidden values do not have to be consistent with quantum mechanics when they are not being measured. Bohm's idea may be rephrased as follows: if we cannot assign values to things we are not observing, then we need not demand that the things behave consistently with quantum mechanics when we are not measuring them. On this view there is no need to entertain the Kochen-Specker argument

A relationship between context dependent measurement (in the form of dependence on apparatus) and nonlocality was observed by Bell [2] in his remarks on the Bohm theory:

> So in this theory an explicit causal mechanism exists whereby the disposition of one piece of apparatus affects the results obtained with a distant piece. In fact the Einstein-Podolsky-Rosen paradox is resolved in the way which Einstein would have liked least.
>
> More generally, the hidden variable account of a given system becomes entirely different when we remember that it has undoubtedly interacted with numerous other systems in the past and that the total wave function will certainly not be factorable. The same effect complicates the hidden variable account of the theory of measurement, when it is desired to include part of the "apparatus" in the system.
>
> Bohm of course was well aware of these features of his scheme, and has given them much attention. However, it must be stressed that, to the present writer's knowledge, there is no *proof* that *any* hidden variable account of quantum mechanics *must* have this extraordinary character.

Of course we know that since the quoted passage was written a proof has been given (the Bell inequalities argument [1]) that indeed any hidden variable account of quantum mechanics must have the extraordinary nonlocal character exemplified by the Bohm scheme. Bell continues:

> It would therefore be interesting, perhaps, to pursue some further "impossibility proofs," replacing the arbitrary axioms objected to above by some conditions of locality, or of separability of distant systems.

This is reviewed by Mermin [29]. The arbitrary axioms mentioned by Bell are just those which attempt to circumvent the context dependence of quantum measurement through the introduction of some hidden mechanism.

We may distinguish two approaches on which a combined demonstration of nonlocality and contextualism may be given: one due to Stairs, Heywood, and Redhead, and one due to Mermin. The former approach involves the extension of the spin-1 system employed in the original Kochen-Specker argument to a composite system of two spin-1 particles. The Mermin approach exploits the eigenstates of the idealized system envisioned in the Greenberger, Horne, Zeilinger thought experiment [16] (abbreviated 'GHZ'). While the two approaches differ in form they both involve the same general techniques of proof, as I will show.

Both contextualist arguments and nonlocality proofs refute the hypothesis that hidden dispersion-free states can contain the information needed to restore classical mechanical explanations for otherwise mysterious contextual and nonlocal phenomena. The GHZ provides a circumstance in which simple arguments for contextualism may be brought together with straightforward arguments for nonlocality so as to expose a single mathematical object (which, for convenience, I will dub 'the GHZ-graph'). The object is a subspace of the Hilbert space associated with the eigenstates of the particles in the GHZ arrangement. I also describe a similar construction for the Bohm model of the Einstein, Podolsky, Rosen thought experiment which, while supporting the same argument for contextualism, fails to support the kind of state required to combine the argument for contextualism with the argument for nonlocality [29]. I will explain both of these arguments and show that they may be combined to construct a single mathematical object. The mathematical object is important to theorists who advocate a quantum logic, differing from classical logic, mirroring the change from classical to quantum mechanics. A leading approach to quantum logic exploits partial boolean algebras. However, it is not my intention to advocate quantum logic

The mathematical object constructed for the GHZ system is of intrinsic geometrical interest. In particular, the simple method of construction of such objects (together with their high degree of symmetry) shows Hilbert space to be rich with configurations of subspaces which satisfy (in the sense of Kochen and Specker [24]) classical contradictions.

Any scheme for a principled method of assigning definite values to operators of quantum systems must provide an account of how such simple and common mathematical objects are to be avoided. An interpretation of quantum mechanics is therefore hard pressed to provide a plausible general account of definite value ascription which does not make *ad hoc* definitions aimed at avoiding such 'Kochen-Specker configurations'. In this sense the results of the present dissertation place a new constraint on the interpretation of quantum mechanics by revealing that the geometry peculiar to contextualism is significantly more difficult to avoid than previous discussions of contextualism have suggested.

Mermin [29] maintains that the attribution of context-independent properties to quantum ensembles is badly motivated because it is not consistent with quantum mechanics. The motivation may be improved by replacing the noncontextualist presupposition with the presupposition of *Einstein locality*. Mermin observes that problematic questions about states of the apparatus become questions about Einstein locality when the apparata involved are sufficiently space-like separated. The technique used for demonstrating this argument involves applying the principles of the proof of contextualism to common eigenstates of entangled systems such as the EPR and GHZ. I will show that essentially the same technique is employed in a different form by both Penrose and by the original extension of the Kochen-Specker argument to composite systems developed by Stairs [46], Heywood and Redhead [19], and Brown and Svetlichny [8].

The differences in form are these: Mermin exploits simple relations among sets of operators to display the incoherence of counterfactual non-contextual assignments of predicted values for these operators; Penrose employs the Majorana representation of spin to exhibit the same sort of contradiction; the Stairs, Heywood, Redhead (SHR) construction exploits a map induced by correlated states of multiple systems, again with the aim of displaying the same kind of contradiction. That all three approaches should make the same argument in a different guise is made apparent through an application of the techniques developed by Peres which allow us to translate between operator constructions and constructions based on sets of rays in Hilbert space. These sets of rays may in turn be rendered as generators of projection lattices. Finally, the projection lattices are the subject of Gleason's theorem, and as von Neumann pointed out, projections correspond to propositions – a correspondence which may be developed to include within the range of our discussion questions concerning quantum logic and the possibility of a purely algebraic (semantic) proof of nonlocality as attempted by Demopoulos [12].

The contextuality of quantum measurement and quantum nonlocality pose important foundational and interpretive questions. There are numerous practical examples of the disparity between quantum mechanics and intuition. Of these, some deal solely with locality (Bell [2], Mermin [29], Hardy [17]); some with contextualism (Bell [1], Kochen and Specker [24], Peres [35]); and some with both locality and contextualism (Stairs [46], Heywood and Redhead [19], Mermin [29], Penrose [49]). Later portions of this thesis will give new examples of solely contextual and combined contextualist-nonlocality arguments, extending the work of Bell, Kochen and Specker, Stairs, Heywood and Redhead, Peres, Mermin, Penrose and Zimba.

With all of these results there remains the question as to the philosophical problems addressed by these efforts. Clearly the question of nonlocality is of interest for philosophical accounts of explanation and causality, so any progress with the formal expression of nonlocal phenomena will serve to clarify the problem of causation to some extent. Contextualism is not as clearly a philosophical problem - as is evidenced by the difficulty in motivating the concept without introducing quantum mechanics. Here I will maintain only that progress with the formal expression of logical and mathematical structural facts about physical theories aids in the interpretation of the transition from one physical theory to another. Finally, we need to remember with Bub [10] that:

> The transition from classical to quantum mechanics only poses a philosophically interesting problem because of the difficulties in the way of a realist interpretation of the theory. A proper resolution of these difficulties requires the interpretation of the non-Boolean structure of idempotent magnitudes of the theory as a possibility structure, or possibility structure of events, i.e. as yielding all combinatorial possibilities for molecular events or properties.

On this view it is possible to maintain a realist view of the structure of physical accounts of the world, provided all the combinatorial possibilities encompassed by a physical theory are given by the structure of the theory. The Kochen-Specker theorem tests the limits of realism because it shows that Boolean propositional structure does not suffice for the combinatorial possibilities of quantum mechanics. To account for the additional possibilities the structure must be non-Boolean, and since partial boolean algebras provide the required structure, it is plausible to argue that there is a kind of realism available to philosophers which interprets the models of partial boolean algebras as satisfiable by the physical world.

1.1 Nonlocality and contextualism

A nonlocal system is one in whicn measurements are correlated over large distances, so that a measurement on Earth, correlated with a measurement of an observable on Fomalhaut, allows someone on Earth to make a measurement and infer the value of the corresponding observable on Fomalhaut. The Special Theory of Relativity rules out instantaneous action at a distance. A 'peaceful coexistence' between nonlocal systems and special relativity is maintained because it is impossible to discern the correlation unless the data from Earth and Fomalhaut are brought together. This feature of quantum mechanics is analogous to the results of coin flips in different rooms. It may be that every head in room A corresponds to a tail in room B, but the correspondence is not of use for communicating between the rooms, since only when the transcripts of flips from the two rooms are brought together can the person in room B observe that, for example, each tail obtained there corresponded to a head thrown in room A.

To make the problem clear I use the idea of labelling states to 'encode' the hypothetical determining information concealed within states. If labels are instrinsic to states then the labels might be vehicles by which the nonlocal correlations could be explained in a way consistent with classical concepts, such as causality. The 'hypothetical hidden states' approach to explanation requires that there are intrinsic properties of systems which are "cryptodetermined": the labels already exist, so it is possible to counterfactually declare a complete list of the properties, such that if X were measured we would get result T. We call the assumption that there are such labels, *noncontextualism*. For the purposes of this thesis, I will take noncontextualism to be equivalent with the notion of 'counterfactual definiteness'.

Recall that EPR maintain that an element of reality corresponds to a physical quantity whenever the probability of predicting the value of the physical quantity is unity; providing of course that *what we need to do* to make the predictions does not disturb the system about which the prediction is made. Imagine a classical case in which a gambler predicts with certainty that Pittsburgh will lose the World Series. We have two possible situations: either the gambler knows with certainty that Pittsburgh will lose because it is possible to point to facts about players and coaches which suffice to make it physically impossible for Pittsburgh to win; or as a matter of natural law Pittsburgh will lose; clearly any gambler not aware of the second possibility should not be gambling. The difference is disguised by EPR's choice of phrasing in "we can predict with certainty," which is ambiguous between a strong and a weak sense of the phrase *can predict*. The strong sense is the gambler's second possibility: it is the sense on which a measurement result is already obtained and we may infer by a natural law what other predictions must be. The weak sense is that in which we have made no measurement, but are certain what the result will be because of facts about the particular system about which we make predictions. The ambiguity is resolved charitably by taking EPR to claim, consistently with quantum mechanics, that there are cases in which it is in principle impossible to directly measure all of the properties of a system. However, even in such cases physical laws allow us to predict with certainty that properties not *directly* measured hold (*cf.* [16], *ftn. 10*).

As an example of the EPR "paradox" consider Bohm's model for an electron and a positron that are separated decay products of some system with total spin zero. There is a simple correlation between spin components of the two particles, such that measuring any spin component of the positron allows an immediate determination of a corresponding spin-component of the electron. The EPR condition that a prediction may be made with certainty is satisfied and we may infer elements of reality associated with these spin components. The disagreement with quantum mechanics is now apparent, since quantum mechanics only permits a single spin component of the composite system (whose total spin is zero) to have a definite value (this is evident from the form of the singlet state of the composite spin zero system) [35].

1.2 A caveat due to Bell

For a classical statistical ensemble (such as a pile of rocks in free-fall) we may determine for each element X of the ensemble that X either has or lacks the property j. In quantum mechanics, however, the unqualified assumption that we are able to determine properties of elements of ensembles leads to contradictory predictions. Thus quantum mechanics does not allow all of the determinations made possible by classical statistical mechanics.

Another way to make the same point about the possible incompleteness of quantum mechanics was given by J.S. Bell [2]:

> To know the quantum mechanical state of a system implies, in general, only statistical restrictions on the results of measurements. It seems interesting to ask if this statistical element be thought of as arising, as in

> classical statistical mechanics, because the states in question are averages over better defined states for which individually the results would be quite determined. These hypothetical "dispersion free" states would be specified not only by the quantum mechanical state vector but also by additional "hidden variables" – "hidden" because if states with prescribed values of these variables could actually be prepared, quantum mechanics would be observably inadequate.

Bell [2] agreed that von Neumann [48] and Jauch & Piron [21] had given a *reductio ad absurdum* of the hypothesis that ensembles of dispersion free states "have all measurable properties of quantum mechanical states." However, he expressed dissatisfaction with the proofs, since the supposition against which the *reductio* is mounted lacks adequate motivation. In order to refute the hypothesis that dispersion free states have all the measurable properties of quantum mechanical states it is necessary, as Bell shows, to suppose that ensembles of dispersion-free states have

> ...certain other properties as well. These additional demands appear reasonable when results of measurements are loosely identified with properties of isolated systems. They are seen to be quite unreasonable when one remembers with Bohr "The impossibility of any sharp distinction between the behaviour of atomic objects and the interaction with the measuring instruments which serve to define the conditions under which the phenomena appear"

Bub [9] suggests that a "Bohrian dispositional interpretation" motivates Bell's disavowal of the additional properties. Bub maintains that Bohr's view that measurement results represent dispositions of the behaviour of systems (given conditions imposed by measuring instruments) underwrites "Bell's proposal that equivalence in the algebra of quantum magnitudes need not be preserved in a hidden variable theory."

We can identify two approaches to the question of the completeness of quantum mechanics. Either we say quantum *mechanics* is incomplete because it cannot answer all well-formed questions; or, we say there is an extension of quantum theory which answers all well-formed questions, but this theory must be such that its set of well-formed questions is not equivalent with the set of well-formed questions of the theory of classical statistical mechanics. Put this way the choice turns on what we permit as the well-formed questions of each theory. Bub's suggestion here is that our choice is contextual: if we follow Bohr we would question the motivation of a scheme for assigning values to incompatible magnitudes which adheres to restrictions imposed by dispersion-free measures on classical probability spaces. Only a quantum theory, one which attempts to honour such restrictions, could be incomplete.

Bell's caveat is against the assumption that algebraic functional relations must be satisfed by hidden variables. The proofs of von Neumann and Jauch & Piron demonstrate that where hidden variables are required to meet these demands it is possible to derive a contradiction. However, if this condition is relaxed the proofs do not hold. Bell seeks to leave open the possibility that there are hidden variable theories which do not presuppose the particular restrictions imposed on the values assigned to incompatible magnitudes by the functional relations of quantum mechanics. A theory which does not satisfy such relations is a contextual hidden variable theory [44]. An example is the Bohm theory [5]. We could summarize Bell's caveat as holding that, since quantum mechanics is *itself* a contextual theory, we have no reason to require that a hidden variable theory is not also contextual. The caveat is clearly consistent with the view I attributed to Bohm. Both Bohm and Bell agree that the values assigned to unmeasured observables should not be constrained by the functional relations of quantum mechanics.

1.3 Gleason's theorem

Von Neumann [48] had introduced an axiomatic construction for quantum mechanics. Gleason [15] describes how in attempting to weaken these axioms he derived what we now call Gleason's theorem:

Theorem 1 (Gleason) *Let μ be a measure on the closed subspaces of a separable (real or complex) Hilbert Space $\mathcal{H}$ of dimension at least three. There exists a positive* *semi-definite self-adjoint operator T of the trace class such that for all closed subspaces A of $\mathcal{H}$*

$$\mu(A) = \mathrm{Trace}(T\mathbf{P}_A), \tag{1.1}$$

where $\mathbf{P}_A$ is the orthogonal projection of $\mathcal{H}$ onto A.

An *eigenvector equation* describes the mechanical behaviour of systems. An operator (which must be Hermitian) corresponds to an observable for the system. The eigenvalues for the operator correspond to observational results. Gleason's theorem shows that, for any countably additive measure on the lattice of projections of Hilbert space, there is always an operator of trace class which induces it.

The contextual and nonlocal features of quantum mechanical descriptions of systems may be exhibited by instancing the failure of boolean relations among expectations. The boolean relations which fail can be picked out by assigning a valuation function to hypothetical 'elements of reality' and showing that the valuation is unsatisfiable for the eigenvectors of idealized systems such as those envisioned in the Einstein-Podolsky-Rosen (EPR) and Greenberger-Horne-Zeilinger (GHZ) thought experiments. The 'elements of reality' fail for the EPR and GHZ because their definition supposes 2-valued measures on lattices of projection operators, in contradiction with the non-2-valued measures which, with Gleason's theorem, are required for quantum mechanics.

A quantum mechanical system is associated with a state represented by a density operator acting on a complex Hilbert space. If $\mathcal{H}$ is a Hilbert space, every closed subspace of $\mathcal{H}$ is associated with a unique projection operator $\mathbf{P}$ onto the subspace. An operator W is a density operator if it is Hermitian; if it is positive definite (*i.e.* $\langle \phi | \mathrm{W} | \phi \rangle \geq 0$ for all $|\phi\rangle \in \mathcal{H}$); and if it has a well defined Trace such that $\mathrm{Tr(W)} = 1$. When W projects onto a one-dimensional subspace (or *ray*), so that $\mathrm{W} = |\phi\rangle\langle\phi|$, then the states $|\phi\rangle$ are called 'pure states'; otherwise the states are 'mixtures'.

Each projection $\mathbf{P}$ is associated with an observable $\mathcal{O}$, such that the expectation on the pure state $|\phi\rangle\langle\phi|$ for $\mathcal{O}$ is $\langle\phi|\mathbf{P}|\phi\rangle$. The expectation for $\mathcal{O}$ on a mixture

W is Tr(W$\mathbf{P}$). We can define a measure μ on the lattice of projections such that $\mu(\mathbf{P}) = \mathrm{Tr}(W\mathbf{P})$.

To obtain an orthocomplemented lattice of closed projections we can make the following associations for two projections $\mathbf{P}_1$ and $\mathbf{P}_2$: to obtain the lattice meet, let $\mathbf{P}_1 \wedge \mathbf{P}_2$ be the projection onto the subspace $\mathbf{P}_1(\mathcal{H}) \cap \mathbf{P}_2(\mathcal{H})$; to obtain the lattice join, let $\mathbf{P}_1 \vee \mathbf{P}_2$ be the projection onto the closed subspace spanned by $\mathbf{P}_1(\mathcal{H})$ and $\mathbf{P}_2(\mathcal{H})$ [40]. The orthocomplement $\mathbf{P}^\perp$ is associated with the subspace orthogonal to $\mathbf{P}(\mathcal{H})$.

For purposes of this thesis, the unitary symmetry of projections ensures that the configurations of subspaces in real Hilbert space to be introduced are unoriented, so that only their mutual orthogonality is required for our arguments, and not their scalar properties.

We may stipulate that $\mathbf{P}_1$ is orthogonal to $\mathbf{P}_2$ if, and only if, $\mathbf{P}_1(\mathbf{P}_2(x)) = 0$, for all $x \in \mathcal{H}$, so that $\mathbf{P}_1(\mathbf{P}_2(\mathcal{H}))$ is zero. The inner product exists on pairs of elements of $\mathcal{H}$, but here we are concerned with pairwise orthogonality among operators on $\mathcal{H}$ and the observables associated with them. Nevertheless, we should recall that the inner product is always present in Hilbert space, since this space is definable as a Banach space with an inner product. As well, we recall that in so far as Hilbert space is concerned, the general theory of measures on sets of orthonormal functions is interchangeable with the Lebesgue theory of integration.

The associations between the subspaces of a Hilbert space and a lattice of projections permit the definition of a measure over the projection lattice of closed subspaces of Hilbert space. Recall that $\mu(\mathbf{P})$, the measure on a projection, is given by Tr(W$\mathbf{P}$). We may write, for all $\mathbf{P}$, $\mu(\mathbf{P}^\perp) = 1 - \mu(\mathbf{P})$; and, where I is the identity operator and 0 is the null operator, $\mu(\mathrm{I}) = 1$ and $\mu(0) = 0$. Furthermore, if a set of projection operators $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n$ are such that any pair are orthogonal we may write:

$$\mu(\vee_{i=1}^{\infty} \mathbf{P}_i) = \sum_{i=1}^{\infty} \mu(\mathbf{P}_i). \quad (1.2)$$

The measure μ is therefore an ordinary additive measure when the operators form families of pairwise commuting projections [40]. In order to show that a candidate measure has countable additivity one need only show that it is additive up to n.

In a separable Hilbert space of dimension ≥ 3, a measure μ which satisfies the above considerations exists for a density matrix W. This fact is the subject of Gleason's theorem. Furthermore, the measure μ on closed projections in a Hilbert space $\mathcal{H}^{n\geq 3}$ must be non-2-valued, and we may take, except for certain pathological functions, the restriction of the space to the surface of the unit sphere.

The choice of the density matrix as the operationally appropriate formal device for the calculation of quantum expectations may be interpreted as carrying with it a restriction on the choice of measures, so that we are limited to just the non-2-valued measures if we choose to represent quantum states as closed subspaces, a fact which reveals the peculiar nature of the quantum world when the space of representation is a Hilbert space of dimension equal or greater than three.

With this in mind, we may understand the work of Bell and of Kochen and Specker in the following way: the operationally adequate quantum formalism implies that states of quantum systems support only non-2-valued measures on projection lattices – and this fact places limitations on interpretations of quantum mechanics; in particular, this fact rules out interpretations which would require 2-valued measures on the lattices which represent quantum states. Furthermore, the interpretation which holds that quantum mechanics may be 'completed' by the introduction of hidden variables must introduce 2-valued measures on projection lattices when the interpretation assumes that Boolean algebraic relations hold among the hidden variables.

Kochen and Specker took this interpretation of the implication of the Gleason theorem for hidden variable theories further by observing that the failure of Boolean algebraic relations over quantum states motivates the introduction of partial Boolean algebras. An analogy between Boolean and quantum algebras, the latter constructed from partial boolean algebras, may be extended to an analogy between Boolean propositional logic and quantum propositional logic (called 'quantum logic'). On this account, Kochen and Specker, by extending a consequence of the Gleason theorem, showed that some tautologies of Boolean propositional logic are, in a certain sense, refutable in quantum propositional logic.

Kochen and Specker provided a proof of a consequence of the Gleason theorem for a finite set of projections. I note that philosophers have tended to prefer the Kochen-Specker result both for its finitude and its manner of argument, which in form is well suited to the discussion of quantum logic. The Kochen-Specker result is in a clear sense a simplification of Bell's corollary to Gleason. This thesis is, in the same clear sense, a simplification of Kochen-Specker. In all cases we reduce the portion of the Hilbert space required for the proof. Bell's proof of the Gleason corollary depends on the fact that the surface of the unit sphere is dense. The Kochen-Specker argument may be given, without mention of the topology of the sphere, if we construct an axiomatic formal quantum theory from orthocomplemented lattices or partial boolean algebras.

The line of argument deriving quantum logic from the projective structure of experimental propositions is known as the Logico-Algebraic approach [20]. I demonstrate a construction which, while consistent with the Logico-Algebraic approach, is not as comprehensive in scope.

It is of interest to establish the minimal commitment that must be taken to advocate quantum logic as an avenue for avoiding the implication for realism imposed by the Kochen-Specker argument. Clearly this minimum commitment must include the claim that quantum logic salvages the determinacy of physical properties, for this is the point of trying to avoid the Kochen-Specker argument.

Chapter 2

The Kochen-Specker theorem

Gleason's theorem entails that all 2-valued measures on projection lattices are in contradiction with quantum mechanics. Bell's observation was that this entailment may be proved simply by noting the contradiction between an assumed additivity property of expectation values and the properties of projection operators. The geometric problem is to construct the smallest ensemble of projections which, when given a 2-valued measure, display a contradiction with quantum mechanics. I will show how 2-valued measures on projection lattices of real Hilbert space may be represented as 'colouring rules' on graphs. The result is the class of 'BKS-graphs' (named for Bell-Kochen-Specker) which display the distinction between classical mechanics and quantum mechanics. Colouring rules represent classical measures and a BKS-graph is a graph generated from idealized quantum states for which the colouring rule fails.

As an example of the difference between 2-valued and non-2-valued measures suppose I wish to distinguish features on the surface of the sphere. Let *blue* be one feature and *yellow* another. I want to keep *blue* and *yellow* distinct. Consider two points on the sphere. Let one of them be a *blue* point and the other a *yellow* point. Next, consider a point lying between the two points. Is it *blue* point or a *yellow* point? Let it be either, say a *blue* point. Next, pick another point lying between this new *blue* point and the original *yellow* point and repeat the process. Since the surface of the sphere is continuous and connected, the process of interpolating a new point between two old ones is endless and the boundary between *blue* and *yellow* regions must be an open one. Therefore two points may not be distinguished to arbitary degree on the surface of the sphere. That is to say, distinctions of features require a continuous nonconstant 2-valued map, but the topology of the surface of the sphere, being connected, does not support such a map.

The contention that it is possible to assign definite values to all the observables of all quantum states runs afoul of the connectedness property of the surface of the sphere. Bell [2] proved that the complete assignment of values to observables implies that there is a minimum angular difference with respect to the origin between distinct points that can be assigned different values (*e.g.* 1 or 0) on the surface of the sphere, and that therefore the assignment implies a 2-valued measure.

Kochen and Specker [24] showed that algebraic relations among projection operators cannot be preserved under a homomorphic mapping to a space of independent random variables. The result demonstrates with a finite explicit model that the statistical modelling developed for classical ensembles cannot be successfully employed in quantum mechanics. By the construction invented by Kochen and Specker we may find classical logical tautologies which are not satisfiable in quantum mechanics. Therefore the construction may be used to delineate instances of the distinction between classical and quantum mechanics. Recall that classical mechanics assumes that it is possible to employ atomic measures, or weights over atomic measures, to provide a complete description by inventory of the states of the constituents of any mechanical system. The Kochen-Specker argument demonstrates that such complete description by inventory is unavailable in quantum mechanics, even if it is maintained that hidden variables are available to restore the atomic measures.

It is worth stressing that there are interpretive and motivational differences between Kochen-Specker [24] and Bell [2] with regard to the properties necessary to a hidden variable account of quantum mechanics. The difference is clear from inspection of the following passage from the Kochen and Specker paper and comparison with the discussion of Bell's relation to Bohr in the previous chapter:

> The proposals in the literature for a classical reinterpretation usually introduce a phase space of hidden pure states in a manner reminiscent of statistical mechanics. The attempt is then shown to succeed in the sense that the quantum mechanical average of an ensemble is equal to the phase space average. However, this statistical condition does not take into account the algebraic structure of the quantum mechanical observables. A minimum such structure is given by the fact that some observables are functions of others. This structure is independent of the particular theory under consideration and should be preserved in a classical reinterpretation. That this is not provided for by the above statistical condition is easily shown by constructing a phase space in which the statistical condition is satisfied but the quantum mechanical observables become interpreted as independent random variables over the space.
>
> The algebraic structure to be preserved is formalised ... in the concept of a partial algebra. The set of quantum mechanical observables viewed as operators on Hilbert space form a partial algebra if we restrict the operations of sum and product to be defined only when the operators commute. A necessary condition then for the existence of hidden variables is that this partial algebra be embeddible in a commutative algebra (such as the algebra of all real-valued functions on a phase space). In sections III and IV it is shown that there exists a finite partial algebra of quantum mechanical observables for which no such embedding exists.

In section V of their paper Kochen and Specker show that their result is stronger than von Neumann's, as it must be given Bell's critique of von Neumann's proof. But it is still open to adopt Bell's caveat that the algebraic structure preserved in a partial algebra is not a necessary constraint on a hidden variable theory. Therefore, Bell could argue that Kochen and Specker demonstrate that quantum mechanics does not permit the embedding into commutative algebras of certain finite partial algebras formed from operators on Hilbert space (for a given set of observables), but this in no way repudiates the existence of hidden variables.

The Kochen-Specker argument against hidden variables does not require the topology of the surface of the sphere, if its presentation is limited to specific physical instances, such as the orthohelium atom I discuss below. That is to say, the Kochen-Specker argument does not require Gleason's theorem. However, the general principle at work in the physical cases to which the Kochen-Specker argument may be applied is the topological principle brought out by Gleason. The motivation for the generalization is to allow identification of the class of physical states which will support a Kochen-Specker argument, and this cannot be done case by case.

In the classical propositional calculus, any proposition may be paired by some logical connective with any other proposition. The notion of a partial propositional calculus of functions is suggested by propositions which may not be combined with other propositions. An example might be the two propositions expressing, respectively, the position and momentum of an electron. Since quantum mechanics precludes such pairings, a propositional calculus for quantum mechanics requires a generalization of the classical calculus. The generalization may be achieved by allowing functions on subsets of some set S and defining an appropriate equivalence relation among the functions. Where the set S is the set of pure states, an appropriate equivalence relation may be defined such that equivalence classes of functions correspond to observables. Kochen and Specker observed in [26] that:

> If $a \in S$ and if f is an element of the observable q, the $f(a)$ is the value of the observable q for the physical system in state a. In classical theories, every observable has a value for all states – the functions are defined for the whole set S; in quantum theory, an observable has a (fixed) value only for certain states – the functions are partial functions.

Following Kochen and Specker [25], partial Boolean algebras are structures $A = \langle B, \heartsuit, 0, 1, \perp, \vee \rangle$ such that B is non-empty; $\heartsuit$ is a binary relation on B (the relation 'commeasurable'); 0 and 1 are elements of B; $\perp$ is a unary function from B to B; and $\vee$ is a binary function whose domain is the set of ordered pairs $\langle a, b \rangle$ of $B \times B$ for which $\heartsuit(a, b)$.

Consider an α-dimensional Euclidean space E^{α}. We may let $\mathbf{B}(E^{\alpha})$ denote the partial boolean algebra of linear subspaces of E^{α}. The linear subspaces of E^3 (for example) form a structure A under the following [25] definitions: B is the set of linear subspaces of E^3; for subspaces a and b, $\heartsuit(a, b)$ *iff* there exists a basis of E^3

containing a basis of a and a basis of b; 0 is the 0-dimensional subspace of E^3 and 1 is the 3-dimensional subspace of E^3; $\perp$ is the orthogonal complement of a; and $a \vee b$ is the span of subspaces a and b, defined only for the pairs $\langle a, b\rangle$ for which $\heartsuit(a, b)$ is the case.

Kochen and Specker show that there is a finite partial boolean subalgebra D of $B(E^a)$ for which there is no homomorphism $h: D \rightarrow Z_2$. The elements of D must be shown to correspond to quantum mechanical observables. Kochen and Specker constructed a 117 vertex graph which generates the partial boolean subalgebra D. Their graph is constructed by iteration of the graph in Figure 2.1, which is a BKS-graph generator (since by repeated composition of the graph generator a full BKS-graph is obtained). The vertices of the graph correspond to rays in Hilbert space. If two vertices are connected by an edge, the rays so connected are orthogonal.

To obtain a partial boolean algebra from a BKS-graph, we complete the implicit orthogonal sets of which the edges in the graph are explicit members. Each edge represents an orthogonai pair of rays, and each orthogonal pair of rays is a subset of some orthonormal basis, but for proving the Kochen-Specker theorem we do not need all of these bases. Since the definition of a partial boolean algebra does require all the orthogonal bases we need to recognize that in a BKS-graph the full set of orthogonal bases is implicit. Since we can always complete the full set of bases by adding in the implicit rays, a BKS-graph is a generator of a partial boolean algebra. The graph in Figure C.2 is a smaller generator of a partial boolean algebra. In general, a BKS-graph represents the generator of a partial algebra that satisfies the variable D in the Kochen-Specker theorem:

Theorem 2 (Kochen-Specker) *The finite partial Boolean algebra D has no homomorphism onto Z_2.*

We will see with Peres and Mermin that BKS-graphs may be obtained directly from operators associated with observables of the EPR and GHZ experiments. The

Kochen-Specker theorem as given above is no longer needed in order to make the argument – since the contradiction with quantum mechanics may be demonstrated directly from properties of operators, and a generalized construction may be given for BKS-graphs from sets of mutually commuting eigenstates.

A Kochen-Specker 'contradiction' is entailed by any complete set of counterfactual value assignments to projection operators in Hilbert space which induce a 2-valued measure on the lattice of those projection operators. The result may be demonstrated with a simple physical system introduced by Kochen and Specker [24].

Suppose we have the following system: An atom of orthohelium in its lowest orbital state sits inside a set of six point charges. The point charges are arranged equidistantly, one in front, one behind, one above, another below, and one on each side of the atom. Thus the charges sit at the vertices of a three dimensional rhombus with the atom at the centre. Unperturbed, this system has some Hamiltonian H; but suppose we perturb the system by applying a small field. Since the point charges are rhombically arranged the field will have rhombic symmetry. The Hamiltonian of the perturbed system is then $H + H_S$, where H_S is called a spin-Hamiltonian. The spin-Hamiltonian corresponds to a physical observable. We observe the change in the energy of the lowest orbital state of the orthohelium atom due to the application of the field. This change may be observed in the spectrum.

Since the field has rhombic symmetry, the spin-Hamiltonian must have the form

$$H_S = aS_x^2 + bS_y^2 + cS_z^2, \tag{2.1}$$

with a, b, and c distinct in three dimensions. H_S is an operator associated with the observable change in energy level. The eigenvalues of the spin-Hamiltonian operator for this system are $a + b$, $b + c$, and $c + a$. Since a, b, c are distinct, so also are their sums. Since we have three distinct sums, the eigenvectors form a complete orthonormal set. The eigenvectors give the values we measure lying in the spectrum of H_S. If a measurement of H_S gives $a+b$, then it follows that S_x^2 and S_y^2 have value one and S_z^2 has value 0, and similarly for other measured values. Thus a measurement of H_S is a simultaneous measurement of all three of the spin angular momentum operators S_x^2, S_y^2 and S_z^2.

The atom of orthohelium in its lowest orbital state has total spin angular momentum $S = \sqrt{2}\hbar$. We recall that $s = 1$, and let us set $\hbar = 1$ to obtain

$$\left(S_x^2 + S_y^2 + S_z^2\right) | s \rangle = S^2 | s \rangle = s(s+1) | s \rangle = 2 | s \rangle, \tag{2.2}$$

where $| s \rangle$ is an arbitrary basis for the spin operator.

It follows that:

Fact 1 *Exactly one of the three components of spin angular momentum of the lowest orbital state of orthohelium is zero.*

Fact 1 is a *colouring rule.* It is predicted by quantum mechanics and it is always confirmed in measurement. For any set of orthogonal rays, if we assign value 1 to two of them the other must have value 0. We have obtained a colouring rule when we identify the values with colours, say red and green, such that any orthogonal set must have one ray green and the others red. I call the colouring rule of Fact 1 the BKS colouring rule. I use colours red and green to indicate a distinction between assigned values, instead of numbers 1 and 0, to avoid confusion with the numbers used to indicate coordinates. As well, the use of colours for valuations on graphs conforms with standard mathematical practice.

In a three dimensional vector space we may add to any pair of orthogonal vectors a third vector othogonal to each of the pair. If we have *explicitly* two vectors **A** and **B** then there is a third *implicit* vector which completes a mutually orthogonal set. Assume **A** and **B** are both red. Then by Fact 1 the implicit third vector must be green. If we assume that one of the two explicit vectors is green then by Fact 1 the implicit vector is red. It follows that any ray orthogonal to a green ray must be red. Thus we have:

Fact 2 *Any component vector orthogonal to a component vector with value zero must have value* $\hbar^2$

A Kochen-Specker argument may be obtained by finding a set of directions T in Hilbert space such that among them there are three orthogonal directions x, y, z which cannot all be simultaneously assigned values consistent with Fact 1. Given such a set T it can be shown that it is not possible to simultaneously predict all of the values of the components of spin angular momentum in the directions comprising T. For some α in T the value assignment $f[\alpha]$ must be wrong.

We may be add to the two facts above an assumption which captures the noncontextualist assumption:

Assumption 1 *The value assigned to a component vector is maintained by that component vector regardless of the order in which assignments are made.*

Without Assumption 1 no contradiction will ensue. The ray is permitted to have different values in different contexts. Thus the argument proceeds by explicitly demonstrating sets of directions for which Fact 1 and Assumption 1 cannot both be satisified. The denial of Assumption 1, while problematic to interpret, is easily seen to be a fair statement of the contextualist thesis; namely, it is the claim that if components have definite values, then these depend on the order in which the assignments are made, and the order of assignment is determined by the order in which measurements of the associated observables are made.

The colouring rule may be applied to graphs, where edges represent orthogonal relations among rays represented by vertices. Fact 1 then requires that any 3-clique has exactly one green vertex, say, and the other two being red. The generalization to n-dimensional systems is to apply the same rule to n-cliques, so that in any clique just one vertex may have a green colour, and all the others in the clique must be red. A colouring fails when it is not possible to distribute greens and reds among cliques consistently with the rule.

The original Kochen-Specker argument constructs a graph of 117 vertices from 15 graphs of eight vertices (we call these graphs 'nuggets'). Figure 2.1 shows a 10 vertex graph (which I will call a 'Specker graph'), originally defined by Kochen and Specker [24], in which the vertices labelled b and c are in addition to the required eight vertices, but are essential to the argument. Vertex b is an a-vertex for a second Specker graph, since the original Kochen-Specker set is formed by iterative composition of Specker graphs around an arbitrary basis. The details are evident from inspection of the original 117 vertex graph [24].

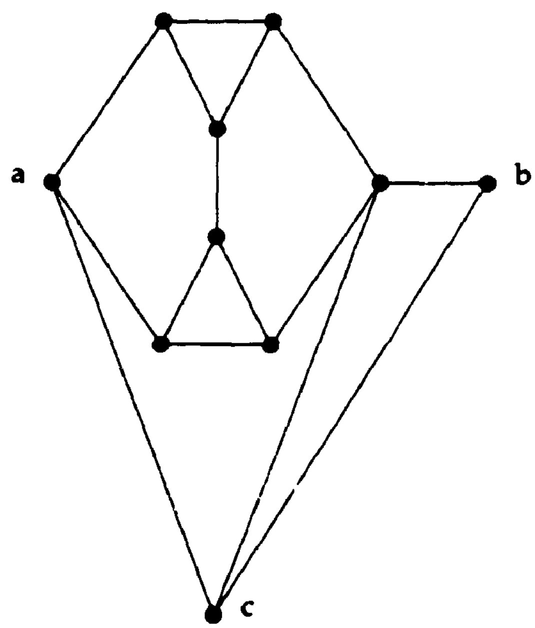

Figure 2.1: The Specker graph of 10 vertices. The colouring rule implies a dependency between vertices a and b.

In order that the orthogonal relations represented by the Specker graph be realizable in three dimensions it is sufficient that the angle between the rays represented by vertices a and b not exceed $\arcsin \frac{1}{3}$. In the original Kochen-Specker graph the angle is 18 degrees. For the Peres cube of Figure C.1 the angle is exactly $\arcsin \frac{1}{3}$, while for the Kochen-Conway cube (which has 37 vertices, six of which are not required for the proof of the Kochen-Specker theorem), the angle lies between the angles of the other two graphs. Thus there is a simple correlation between the number of vertices of a BKS-graph in three dimensional space and the angle between vertices a and b of the Specker graphs from which they are constructed. Nevertheless, the redundancy evidenced by the Kochen-Conway proof [35] counsels against ascribing any profound meaning to this correlation.

Mączyński [28] employs the direct limit of a partially ordered set of boolean algebras to construct sets of observables of quantum mechanics on Hilbert space which have boolean representations. We may interpret his results as restrictions on the range of subspaces of Hilbert space on which a BKS-graph is derivable. Mączyński proves, by construction, that there are boolean representations for any set of *compatible* observables, and for the set M of all maximal observables. The possibility of an extension of Mączyński's construction of boolean representable sets to include locally maximal observables has been explored in [46], [12], [19], and [42].

Chapter 3

Geometric aspects of contextualism

The mathematical construction of measures over projection lattices allows a geometrical demonstration of the difference between classical and quantum mechanics. As we have seen, context dependent measurement in quantum mechanics may be demonstrated, given the appropriate initial assumptions, by attempting to assign definite measurement values to quantum propositions. The configurations of subspaces of Hilbert space corresponding to these propositions may be identified by geometric relations. The geometry of Hilbert space determines what is, to borrow from Bell, speakable or unspeakable in quantum mechanics. One of the virtues of the Bell-Kochen-Specker graphical arguments is their capacity for providing simple geometrical representations of those mathematical objects which demonstrate contextualism and nonlocality. In this chapter I will describe the construction of these mathematical objects and their significance to the interpretation of quantum mechanics. We will see that various constructions share a general scheme: namely, assumptions derived from classical mechanics, and/or premises required to support a realist interpretation of the properties of constituents of ensembles, are shown to induce paradoxes when coupled with the quantum formalism. The paradox solved by adopting a contextual interpretation is the paradox demonstrated by the Kochen-Specker theorem. I will extend the considerations involved in this general scheme to include some demonstrations of nonlocality as well. We will see that the appropriate tools for this generalization are the geometrical and topological results concerning functions on orthonormal frames obtained by Gleason.

Stairs [46] gave a direct scheme for constructing a Kochen-Specker argument to demonstrate nonlocality. His scheme is given in detail by Brown and Svetlichny [8] and is essentially the same scheme exploited by Zimba and Penrose [49]. Stairs' method picks out a 2-valued measure entailed by locally maximal observables on a Hilbert space for a composite of two spin-1 systems. The method applies the BKS colouring rule to composite systems to demonstrate that certain properties attributed to hypothetical dispersion-free states in these systems are inconsistent with Quantum Mechanics, since these properties imply a 2-valued measure. The idea is informally referred to as 'cloning' and is, according to Shimony, properly attributed to Specker.

3.1 Eigenstates of the GHZ experiment

Greenberger, Horne, Shimony, and Zeilinger [16], in a detailed review of the earlier published GHZ thought experiment, demonstrated the Bell [1] theorem for more than two correlated spin-$\frac{1}{2}$ particles.

Mermin [30] demonstrates the GHZ argument for three spin states by developing an inconsistency implied by constraints on algebraic relations over operators with common eigenstates. He remarks:

> Kochen and Specker originally produced a set of 117 observables, associated with the squares of the components of the angular momentum operator along 117 directions. They demonstrate with a somewhat intricate geometrical argument that there is no way to assign values (0 or 1) to all these observables, consistent with the requirement that $v(A)+v(B)+v(C)=2$ for A, B, and C any subset of the observables associated with three mutually orthogonal directions. To the well trained quantum mechanician it must surely seem shocking that the direct refutation of so heretical an attempt should require so elaborate a counterexample, but that is where things have stood for almost 25 years.

Peres [36] introduced a proof of the Kochen-Specker theorem which exploits the operator formalism of quantum mechanics. He proves that quantum mechanics is not consistent with one or both of the propositions:

1. The result of the measurement of an operator A depends solely on A and on the system being measured;

2. If operators A and B commute, the result of a measurement of their product AB is the product of the results of separate measurements of A and of B.

In the singlet state for Bohm-EPR, any hypothetical attributed values must respect the perfect spin-correlations between the two single particle states. For convenience, measurements of spin angular momentum are taken in units of $\frac{\hbar}{2}$. The Pauli matrices then represent the spin operators for each particle. Consider the operators $\sigma_x^1, \sigma_x^2, \sigma_y^1$, and σ_y^2. The operators $\sigma_x^1\sigma_y^2$ and $\sigma_y^1\sigma_x^2$ are commuting products of these operators.[1] Any attribution of numerical values to these operators must respect the correlation between spin in one EPR-subsystem and the reverse of that spin in the other, so that $\vartheta(\sigma_x^2) = -\vartheta(\sigma_x^1)$ and $\vartheta(\sigma_y^2) = -\vartheta(\sigma_y^1)$. Here I use the notation $\vartheta(\cdot)$ to indicate a map $\vartheta: \mathcal{O} \rightarrow \{1, -1\}$.

Furthermore, since the products commute, the value assignments must preserve the product $\vartheta(\sigma_x^1\sigma_y^2)\vartheta(\sigma_y^1\sigma_x^2) = \vartheta(\sigma_z^1\sigma_z^2)$. We know that in the singlet state this value must be -1. However, the constraint thus derived contradicts proposition 2 above, since if that proposition were true then $\vartheta(\sigma_x^1\sigma_y^2) = \vartheta(\sigma_x^1)\vartheta(\sigma_y^2)$ and $\vartheta(\sigma_y^1\sigma_x^2) = \vartheta(\sigma_y^1)\vartheta(\sigma_x^2)$. It would then follow that $\vartheta(\sigma_x^1\sigma_y^2)\vartheta(\sigma_y^1\sigma_x^2) = \vartheta(\sigma_x^1)\vartheta(\sigma_y^2)\vartheta(\sigma_y^1)\vartheta(\sigma_x^2) = \vartheta(\sigma_z^1\sigma_z^2)$. The valuation must respect this product, but since it is a product of two pairs of operators in which the members of each pair have opposite values, the value assigned to the product must be $+1$. Thus, in the singlet state, the six operators offered by Peres demonstrate that proposition 2 is refutable in quantum mechanics. Proposition 1 is refutable as well, since the assumption that a value of a measurement may be attributed to an observable is dependent upon the exclusion, from the determination for that value, of any relation to other observables. Without this assumption we

[1]Note that whereas with 'cloning' the tensor product is taken between an operator and the unit matrix, here the tensor products among operators are considered.

could not have entertained the hypothesis that value attributions such as $\vartheta(\sigma_x^1)$ are the same regardless of whether we are measuring $\sigma_x^1\sigma_x^2$ or $\sigma_x^1\sigma_y^1$. That is, proposition 1 is a noncontextualist claim, and either it, or proposition 2, or both, are shown by Peres's argument to be refutable in quantum mechanics.

Mermin found that the Peres argument will apply for any state if Peres's set of six operators is supplemented by three additional operators. The resulting nine 'Mermin operators' may be arranged in a table such that operators commute if they are in the same row or if they are in the same column (Table 3.1).

σ_x^1	σ_x^2	$\sigma_x^1\sigma_x^2$
σ_y^2	σ_y^1	$\sigma_y^1\sigma_y^2$
$\sigma_x^1\sigma_y^2$	$\sigma_x^2\sigma_y^1$	$\sigma_z^1\sigma_z^2$.

Table 3.1: Mermin's nine operators which generalize Peres's proof of contextualism in quantum mechanics.

With the exception of the rightmost column, each column and row is such that the product of its constituents is +1. The product of the constituents of the rightmost column is −1. Hypothetical value attributions to products of observables cannot obey the second of Peres's two propositions given above since, if they did, the product of the three rows would agree with the product of the three columns. However, such agreement is impossible, since the product of rows must be +1 and the product of columns must be −1, because of the rightmost column.

The argument applies to any state for which the nine operators represent observables. Peres's six operators lack this generality only because they lack the operators comprising the rightmost column (although $\sigma_z^1\sigma_z^2$ is implied). Mermin has added the operators necessary for constructing a generalized eigenstate.

The nine Mermin operators allow a refutation of noncontextualism. But Mermin, like Bell, is not satisfied with a refutation of something already known to be impossible. GHZ extend the argument to prove that there are systems for which quantum mechanics entails nonlocal dynamics, by considering triple products of operators. The triple products succeed because they all are mutually commuting. The set of nine operators in Table 3.1, on the other hand, does not prove nonlocality because the five operator products among them are not mutually commuting. Hence there is no simultaneous eigenstate of the five for which a hypothetical nonlocal value attribution may be postulated (and refuted).

Mermin observes that while noncontextualism is a demand that in general is not well motivated, it is not unreasonable to look to noncontextual value assignments for local systems. Mermin [29] motivates this approach as follows:

> Suppose that the experiment that measures commuting observables $A, B, C, \ldots$ uses independent pieces of equipment far apart from one another, which separately register the values of $A, B, C, \ldots$. And suppose that the experiment to measure A with the commuting observables $L, M, \ldots$, not all of which commute with all of $B, C, \ldots$, requires changes in the complete apparatus that register the values of $B, C, \ldots$ with different pieces of equipment that register the values of $L, M, \ldots$. And suppose that all these changes of equipment are made far away from the unchanged piece of apparatus that registers the value of A. In the absence of action at a distance such changes in the complete disposition of the apparatus could hardly be expected to have an effect on the outcome of the A measurement on an individual system. In this case the problematic assumption of noncontextuality can be replaced by a straightforward assumption of locality.

As I observed in chapter one, the motivation outlined here reiterates the suggestion made by Bell [2].

The GHZ system exhibits an eight dimensional eigenstate for which there are four triple-product operators produced from six individual constituents. These, taken together, compose a set of ten commuting operators closed under products. The set is most effectively arranged as a star figure (Table 3.2, due to Peres). The operators composing each of the five legs of the star are mutually commuting.

		σ_y^1		
$\sigma_x^1\sigma_x^2\sigma_x^3$	$\sigma_y^1\sigma_y^2\sigma_x^3$		$\sigma_y^1\sigma_x^2\sigma_y^3$	$\sigma_x^1\sigma_y^2\sigma_y^3$
	σ_x^3		σ_y^3	
		σ_x^1		
σ_y^2				σ_x^2

Table 3.2: The ten operators for the GHZ system arranged to display their commuting products.

The three operators $\sigma_x^1\sigma_y^2\sigma_y^3$, $\sigma_y^1\sigma_x^2\sigma_y^3$, and $\sigma_y^1\sigma_y^2\sigma_x^3$ are composed of combinations of spin directions for three particles, such that where the x direction is indicated for one particle, the y is indicated for the other two. Of the six individual operators, all commute except those indicating different spins for the same particle. These pairs anticommute. Since any product formed from a pair of the triple product operators involves an even number of these anticommutations, all such pairwise products commute.

Any operator of the ten has eigenvalue $+1$ or -1 (since the ten are Pauli matrices or products of Pauli matrices). The three triple-product operators commute and therefore have simultaneous eigenstates. We can select, for convenience, the eigenstate in which all three have eigenvalue $+1$. Given the commutation relations among the operators, we may interpret them as simultaneous measurements of an x component of one particle and the y component of the other two. We see that the product of the eigenvalues of the three triple product operators must be $+1$, since this is the eigenvalue of the eigenstate for the three particle system. However, the product of the three triple products is given by the relation:

$$\sigma_x^1\sigma_x^2\sigma_x^3 = -(\sigma_x^1\sigma_y^2\sigma_y^3)(\sigma_y^1\sigma_x^2\sigma_y^3)(\sigma_y^1\sigma_y^2\sigma_x^3). \tag{3.1}$$

This demonstrates that the operator $\sigma_x^1\sigma_x^2\sigma_x^3$, which represents a simultaneous measurement of the x component of all three particles, is also an eigenstate of the system, though with eigenvalue -1. As can be seen in the star figure, this last operator commutes with the three product operators.

Now the argument against locality is straightforward: just as with the table of nine operators arranged in rows and columns, the star figure, with its arrangement along legs, contains an explicit example of 'hidden value' attributions which are unsatisfiable for the given set of commuting operators. Each operator is a member of two legs of the star. The hypothetical value attributions, if they are to respect the commutation relations given, require that values assigned to products of operators which share a leg agree for both of the legs to which each operator belongs. Yet this is impossible, since the product of operators along the horizontal leg is -1, while the products along the other four legs are all $+1$. It is clearly not possible to consistently satisfy this demand for these operators.

Since the operators are those applicable to the GHZ setup they are operators which exhibit perfect correlations among three spin-$\frac{1}{2}$ particles. These correlations permit prediction with certainty of spin properties of one particle from a measurement of the spin properties of another, even when the particles are spacelike separated. The BKS colouring argument demonstrates that the values predicted for such measurements either do not exist prior to measurement, or depend upon which particle is measured. Since the argument may be realized for a system with perfect correlations, we obtain nonlocal predictions simply because the context dependence is shared instantaneously over any distance. However, as Mermin [29] explains, if we use a locality assumption to motivate noncontextualism, then the refutation of noncontextualism is limited to those states which refute locality:

> This new version of Bell's theorem makes it clear that the use of a particular state is required to provide the information that is lost when one permits the assignment of noncontextual values only when noncontextuality is a consequence of locality.

The Peres-Mermin operator proof of the Kochen-Specker theorem represents a considerable gain in simplicity. However, the operators used in the proofs for EPR and GHZ carry redundancy, since it is possible to render the proofs in terms of relations among subsets of the eigenvectors of the operators. Peres recast the EPR operator argument as a Kochen-Specker 'impossible colouring' proof requiring 24 vectors. Later I show how this may be reduced to 20. Similarly, 40 vectors are required to generate the operators for the GHZ setup, but again, these can be shown to contain redundant information, since just 36 vectors are required to prove the proposition (*cf.* Appendix A).

Mermin [29] suggests that because we need not rely on a complicated geometrical argument, and because we can replace the noncontextualist motivation by a locality motivation, the EPR and GHZ operator proofs supersede the Kochen-Specker geometric argument. Mermin complains that Peres's vector representation complicates the proof by bringing the geometry back again. I point out against this view, that aside from the interesting geometry introduced, the vector representation does furnish information not obvious on the operator account. As I show below, fewer vectors are required to make the argument than are implied by the operators used in the GHZ and EPR operator proofs. I first discuss an exegetical advantage of the vector representation: namely, that with the vector representation we may clearly demonstrate the relation between the Kochen-Specker argument and the Gleason theorem. To see this we will need to review the manner by which the vectors may be derived from the operators.

Peres demonstrated that the operator proofs may be formulated as geometric demonstrations, in the manner in which Kochen and Specker had given a geometric demonstration of the Gleason theorem. He notes that in Table 3.1 each row and column is a set of commuting operators. We call this set complete because there is a single basis in which all of the operators of the set are diagonal and represent a simultaneous measurement of a single operator (note that operators are matrices). Peres observes that:

> If a matrix A is not degenerate, there is only one basis in which A is diagonal. That basis corresponds to a maximal quantum test which is equivalent to a *measurement* of the physical observable represented by the matrix A. If, on the other hand, A is degenerate, there are different bases in which A is diagonal. These bases correspond to *inequivalent physical procedures*, that we call "measurements of A." Therefore the word "measurement" is *ambiguous*.

To remove the ambiguity, Peres suggests we take "measurements" to be tests and a maximal test to be possible for commuting matrices, since we can always find a basis for commuting matrices in which both are diagonal. If the matrices do not commute then there is no possible maximal test, and the measurements, each represented by such a matrix, are not compatible with one another.

Peres [35] maintains that the assumption of "functional consistency" is consistent with quantum mechanics. Informally, the assumption is that since we can measure commuting operators simultaneously, we can also measure functions of commuting operators simultaneously.

Assumption 2 (Functional Consistency) *Where two operators* A *and* B *do not commute, but share a common eigenvector* ψ*, it is possible to prepare a state for* ψ*, so that* $\mathsf{A}\psi = \alpha\psi$ *and* $\mathsf{B}\psi = \beta\psi$*.*

If A and B share a common eigenvector ψ, then such a ψ permits functional consistency of all A and B, and in general $f(\mathsf{A},\mathsf{B})\psi = f(\alpha,\beta)\psi$, for any function $f(\mathsf{A},\mathsf{B})$ and shared eigenvector ψ. Note that Mermin's argument uses an eigenstate as ψ, while Stairs's uses the singlet state

The stronger assumption of "independence from context" is:

Postulate 1 (Independence from Context) ***Even if* ψ *is* not *an eigenstate of the commuting operators* A, B, *and* $f(\mathsf{A},\mathsf{B})$ *and even if these operators are not actually measured, one may still assume that the numerical results of their measurements (if these measurements were performed) would satisfy the same functional relationship as the operators.***

Peres maintains that together the assumption of functional consistency and the postulate of independence from context contradict quantum mechanics. The proof is demonstrated by the failure of the product relation to hold for the nine commuting operators in Table 3.1.

Since functional consistency is in principle consistent with quantum mechanics, even for non-commuting operators when there is a common eigenvector between them, the postulate of independence from context is refuted in quantum mechanics. The conclusion is equivalent to Bell's demonstration that a property of the statistical average of an ensemble, such as additivity, cannot be extended to the constituents of the ensemble in a way which is consistent with quantum mechanics, since such an extension presumes a 2-valued measure on the lattice of closed subspaces of the Hilbert Space.

3.2 A geometric construction due to Penrose

The techniques of proof and the simplifications of the BKS colouring argument I have introduced appear to suggest a purely geometrical demonstration of the failure of local deterministic hidden variable theories. Recall that from Bell and Kochen-Specker I have taken a colouring rule with which we may define configurations in Hilbert space that refute various classically-motivated assumptions. Noncontextualism is one assumption for which I may employ the colouring rule to show an inconsistency with quantum mechanics. If noncontextualism is implied by a hidden variable theory, we have an argument against such a theory. In particular, we can test the postulate of independence from context, and, where noncontextualism may be replaced by locality, we can test Einstein locality.

Von Neumann gave axioms for quantum mechanics based on a projective geometry. Gleason sought to solve an open problem arising from an axiomatization due to Mackey, who had looked to improve on von Neumann's axioms. Gleason found that quantum mechanics exhibits a rich structure of measures, and from these we must choose a standard. Trace is the preferred canonical measure because with it we can define functions which are linear over orthonormal frames and sum to a constant weight. We could choose a measure for which the weight is not constant, but only at the cost of losing the linear structure of weights over subspaces. However, the linear structure of the weights on subspaces is the mathematical fact by which we explain the outcomes of the the Stern-Gerlach experiment, or the stability of stars. The functions allowed by Trace are called *frame functions* and I will return to them in more detail shortly. I will maintain that the choice of Trace, as an axiomatic principle of quantum mechanics, is not made solely to preserve consistency with the formalism of density matrices and frame functions. The choice serves as well to give a foundational explanation of the contextualism exhibited in quantum mechanics.

The choice of Trace as canonical entails non-2-valued measures over lattices of projection operators in Hilbert spaces of dimension three or more. Since propositions are to projections as obervables are to operators, and since lattices of projection operators correspond one-to-one with lattices of subspaces of Hilbert spaces, we may construct lattices of projections with provably non-2-valued measures and employ these as counterexamples to a set of propositions whose algebraic structure implies a projection lattice with 2-valued measure.

We recall that Mermin motivates the Kochen-Specker theorem with the observation that there are circumstances in which "the problematic assumption of noncontextualism can be replaced by a straightforward assumption of locality" [29]. For instance, the GHZ system and its eigenstates are an example in which the locality assumption may be tested by the BKS colouring argument. Penrose couples Mermin's approach with the 'cloning' construction to find a geometric structure with which to test a locality assumption and prove that it is not consistent with quantum mechanics. He achieves this by showing that the Stairs construction permits a colouring rule identically expressed by relations among the Majorana representations of spin-$\frac{3}{2}$ systems.

Penrose introduces the Majorana map as a vehicle for representing configurations of rays in complex n-dimensional Hilbert space as unordered sets of $n-1$ points on the surface of the real unit sphere. For a four-dimensional Hilbert space of the EPR setup we may use the Majorana map to derive triples of points on the surface of the sphere in $\mathbb{R}^3$. In the GHZ case, the eight-dimensional Hilbert space of the GHZ eigenstates has a set of rays, picked out by eight-component coordinate vectors, which are in bijective correspondence with unordered seven-tuples of points on the real sphere $\mathbf{S}^2 \subset \mathbb{R}^3$ [49].

Penrose defines a Majorana map appropriate for the complex four dimensional Hilbert space $\mathbf{H}^4$ as follows: where d is a ray in $\mathbf{H}^4$, let $\mathbf{d}$ be a non-zero vector spanning d. For convenience in the construction, Penrose chooses a basis in $\mathbf{H}^4$ such that $\mathbf{d}$ has components:

$$\mathbf{d} = \begin{pmatrix} \Delta_0 \\ \sqrt{3}\Delta_1 \\ \sqrt{3}\Delta_2 \\ \Delta_3 \end{pmatrix}.$$

The Δ_i are complex numbers used to form a polynomial:

$$\Delta(z) = \Delta_3 z^3 + 3\Delta_2 z^2 + 3\Delta_1 z + \Delta_0.$$

The polynomial has three complex roots d_1, d_2, d_3 when $\Delta_3 \neq 0$ and, if the leading p coefficients are zero, then setting $d_1 = \ldots = d_p = \infty$ will give three roots in the Riemann sphere $\mathcal{R} \cong \mathbf{C} \cup \{\infty\}$ The stereographic projection from the Riemann sphere to the real 2-sphere $\mathbf{S}^2$ therefore permits the construction of an association between the ray d and an unordered triple of points (D_1, D_2, D_3) on the sphere. The triple of points may be denoted by $[d]$. The points $D_i \in \mathbf{S}^2$ project stereographically to the roots $d_i \in \mathcal{R}$ of $\Delta(z)$, and Penrose gives an informal diagram of the bijective Majorana map as

$$d \leftarrow - - \rightarrow \Delta(z) \leftarrow - - \rightarrow d_i \in \mathcal{R} \leftarrow - - \rightarrow D_i \in \mathbf{S}^2.$$

To explain why Penrose employs the Majorana map I must reiterate in more detail a construction employed in the proof of the the Gleason theorem. It is a corollary to Gleason's theorem that the non-2-valued nature of projection lattice measures may be expressed as a topological property of the sphere. The construction used by Gleason exploits the connectedness property of the sphere in the manner summarized in our earlier discussion of the topological properties of the surface of the unit sphere. Penrose exploits the Majorana map to render geometric properties of con 'ex four-dimensional space on the surface of the unit sphere of real three space, so as to exploit the same consequence of Gleason's theorem that Kochen and Specker exhibited in finite explicit form, namely, that the attribution of labels to directions in space must generate projection lattices with non-2-valued measure if they are to be consistent with quantum mechanics (recall that "hidden variables" and "independence from context" entail 2-valued measures). The demonstration that a set of hypothetical dispersion-free states entails a 2-valued measure may be made either by the geometrical colouring rule argument of Kochen and Specker, or by the operator arguments of Mermin and Peres. The operator arguments, however, are given for eigenstates of four and eight dimensional Hilbert spaces. On the other hand the Bell and Kochen-Specker proofs, and the Stairs extension-to-locality argument, all exploit the construction employed by Gleason to demonstrate the non-2-valı e condition for projection lattice measure as a property of $\mathbf{S}^2$. The Majorana map demonstrates relations between the complex n-dimensional Hilbert spaces for which the operator versions of the Kochen-Specker theorem are given and the topological properties of the unit sphere exploited by three-dimensional geometrical proofs of contextualism.

Penrose demonstrates that a colouring rule may be derived in a Majorana representation since the representation permits a unique condition under which rays are orthogonal. The Majorana representation (triples of points on the surface of the sphere) conserves orthogonal relations among the rays of $\mathcal{H}^n$. The BKS colouring argument may be constructed using properties of Majorana representations, and Penrose provides an explicit example using a regular dodecahedron inscribed within the unit sphere. The vertices of the polyhedron correspond to rays through the origin of the sphere. Where these rays intersect the surface of the sphere they have Majorana representation (x, y, z). The twenty such triples derived from the dodecahedron correspond to twenty rays in $\mathcal{H}^4$

The vertices of the dodecahedron label twenty eigenstates of a four dimensional Hilbert space. It turns out that these twenty eigenstates generate a projection lattice on S^2 to which we may apply the BKS colouring argument. That is, the eigenstates may be represented by a configuration of projections on Hilbert space which supports a Kochen-Specker argument.

For the example of an EPR pair of particles of spin-$\frac{3}{2}$, emitted in opposite directions from the singlet state, there exists an anticorrelation between measurements taken on each side. Penrose notes that this anticorrelation is realized on the Majorana representation in the following way: the measured value corresponds, by the eigenvalue equation, with an operator. We may take the operator that projects onto a ray ψ which has Majorana representation $[\psi]$. The anticorrelated measurement may also be taken as an operator that projects onto a ray which has as Majorana representation the antipodal point of $[\psi]$, which for convenience may be denoted by $-[\psi]$.

If we are to assign noncontextual values to these measurements, then the assigned values must be the same for $[\psi]$ and $-[\psi]$. In the EPR setup we separate the particles to be measured at a sufficient distance so that, consistently with special relativity, we can discount any transmission of information from one particle to the other. Accordingly, any noncontextual value must exist prior to the measurement (otherwise there could be no anticorrelation). This motivation reiterates Mermin's idea: a noncontextual value assignment is rendered as a locality condition, but since the value assignment derived contradicts quantum mechanics, so too must the locality condition. Once the hypothetical values are assigned to the rays with Majorana representations $[\psi]$ and $-[\psi]$, it is a straightforward matter to construct a set of rays whose Majorana representation retains the assigned values, but which cannot satisfy the requirements on its orthogonal relations stipulated by the colouring rule. Penrose constructs the dodecahedral set as an explicit example (*cf.* [34], §5.3).

In the course of demonstrating the orthogonality conditions for Majorana representations, Penrose proves that two rays are orthogonal if their Majorana representations are reflections of one another through the centre of the sphere. That is, antipodal points of the Majorana representation are orthogonal. We must take care, in light of this fact, about how we label Majorana representations, since, following Gleason, we see that frame functions do not change sign on passing to antipodes. The assignment of a negative sign to the antipodal point of $[\psi]$ should be taken as only a convenience to the presentation of the proof, since if we claim literally that the antipodal point of $[\psi]$ is $-[\psi]$ then the Majorana map cannot be a frame function. To see the importance of preserving consistency between the Majorana map and frame functions I review Gleason's introduction of frame functions.

Penrose's introduction of the Majorana representation, as an indicator for definite value assignments, is motivated by the elegance with which that representation encodes information about points on the sphere. However, since the Majorana representation and frame functions differ with respect to change of sign on passage to antipodal points, we must take care that the representational elegance does not obscure the central role of frame functions in the proof of contextualism and nonlocality in quantum mechanics. Furthermore, the Majorana representation is not necessary for the proof that Penrose's twenty eigenstates support a Kochen-Specker argument (*cf.* Peres [35], p. 212 for a succinct summary of such a proof).

Consider a separable Hilbert space $\mathcal{H}$ and the surface of the unit sphere of that space. Define a function f as real valued on the surface of the unit sphere of $\mathcal{H}$ so as to obtain a linear function over the orthonormal bases $\{x_i\}$ of $\mathcal{H}$, where i is the dimension of the space. The linear function will give a weight W and we can write:

$$W = f[x_1] + f[x_2] + \dots + f[x_i]. \tag{3.2}$$

The function f is a *frame function*. For all closed subspaces S of $\mathcal{H}$, a frame function on $\mathcal{H}$ is also a frame function on S by restriction. But since, in general, the restriction alters the $\{x_i\}$, the weight W will alter as well under the restriction. Where, for any unit vector x, there exists on $\mathcal{H}$ a self-adjoint operator T such that $f(x) = (Tx, x) = \langle x | T | x \rangle$, we call the frame function regular, and we say that only regular frame functions have self-adjoint operators associated with them. Gleason [15] proves that given certain additional assumptions every non-negative frame function in three or more dimensions is regular. He remarks:

> In dimension one it is obvious that every frame function is regular. In dimension two a frame function can be defined arbitrarily on a closed quadrant of the unit circle in the real case, and similarly in the complex case. In higher dimensions the orthonormal sets are intertwined and there is more to be said.

Among the additional hypotheses are those necessary to avoid unbounded frame functions induced under composition with pathological functions. The requirement is essentially satisfied for finite dimensional cases by the hypothesis that nothing is lost in proving the regularity of just the non-negative frame functions. We will see that the "intertwining" of orthonormal sets in higher dimensions is more than a metaphor.

We recall that a frame function is defined on the surface of the sphere and gives a constant sum over orthonormal bases $\{x_i\}$. The orthonormal bases may be given by the points at which their component basis vectors intersect the surface of the sphere. Therefore, we may write the $\{x_i\}$ in spherical coordinates and use angles θ and ϕ as decompositions of the unit vecors. This is the technique Gleason used to prove that every continuous frame function on the unit sphere in $\mathbb{R}^3$ is regular. The proof proceeds by expanding the frame function in terms of spherical harmonics and finding the conditions under which the expansion has constant terms. It turns out that odd harmonics cannot be frame functions because they induce a change of sign upon passing to the antipodal points (i.e., when θ is replaced by $\pi - \theta$ and ϕ by $\pi + \phi$, *cf.* Peres [35]). Frame functions do not change sign under such conditions, since passage to the antipodal point represents the extension of a vector in its opposite direction, and orthonormal bases are unaffected by this. That is, an orthonormal frame composed from vectors in directions north, east and up is the same frame as that composed of directions south, east and up. This feature of frame functions is an instance of the fact that it is the ray structure, and not the vector structure, which gives the essential features of quantum mechanical propositions and states. When we consider orthonormal bases we are considering rays through the centre of the sphere rather than just the vectors from the centre of the sphere. Finally, additional considerations lead to the rejection of all even harmonics greater than the second. The result is that all frame functions which are continuous on the unit sphere in $\mathbb{R}^3$ are regular, since the expansion in spherical harmonics reveals that any frame function which is continuous must be the restriction of a quadratic form. Peres makes this point clearly:

> We are thus finally left with spherical harmonics of order 0 or 2. These can be written as bilinear combinations of the Cartesian components of the unit vector $\mathbf{u}$, so that any frame function has the form
>
> $$f(\mathbf{u}) = \sum_{mn} \rho_{mn} u_m u_n, \qquad (3.3)$$

where ρ is a nonnegative matrix with unit trace.

The form given in 3.3 is clearly continuous. Throughout our discussion of the BKS colouring argument I have noted that noncontextualist hidden variable theories imply a 2-valued measure on the surface of the unit sphere. This is simply another way of saying that such theories entail functions on orthonormal bases which are not continuous frame functions. If we were to insist that a Majorana representation changes sign on passing to the antipodal point then the Majorana representation could not be a frame function on the unit sphere.

3.3 EPR, GHZ, and minimal Kochen-Specker proofs

Suppose we have commuting projection operators on a set of vectors such that the summation over the operators equals one. The operators may be given by the matrices formed by multiplying the vectors of the set by their adjoints. For a complete set of orthogonal vectors v_1 to v_n there will be N such matrices. If we assume that a value assignment to a vector which is a member of more than one orthogonal basis is the same regardless of the choice of other bases, then, for given sets of vectors in $\mathbb{R}^3$, a Bell-Kochen-Specker result arises. This general approach to the proof was used by Peres in constructing his 24 point set. However, I note that the procedure does not produce a minimal set of vectors. It is possible to remove vectors from four of the six orthogonal bases which make up Peres's set and still have the contradiction. But the fact of the removal renders the method ineffective for generating smallest sets - i.e., the method includes redundancy.

Recall that a BKS graph represents rays in Hilbert space and their mutual orthogonality relations. We call the subgraph representing the orthogonality relations among n mutually orthogonal rays an n-clique. Where the space is N-dimensional (we are concerned only with $N \geq 3$) then an N-clique represents an orthonormal basis for the space. Adopting the strong superposition principle [35], according to which every orthonormal basis represents a maximal test, we may infer that every n-clique represents a maximal test of a system with n degrees of freedom.[2]

Consider the EPR system: for spin angular momentum components in directions x, y, z there exists an eigenstate given by the operators in Table 3.1, each of which is a product of operators for given components we might measure (i.e. obtain test results for, where the results are expectation values, obtained from the probability amplitudes; though since we are taking the projections of the operators onto one dimensional rays of the space, all the results will be answers to yes/no questions of the sort "is there a component of spin in such and so a direction?"). Each operator is a product of two operators, for we may write σ_x as $\sigma_x \otimes 1$, since this is multiplication by unity. We may write each operator explicitly as a tensor product of Pauli matrices, so that, for example:

$$\sigma_x \otimes 1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

In this way Table 3.1 becomes a table of nine 4×4 matrices. Since the space is the four dimensional product space of the operators $\sigma_x, \sigma_y, \sigma_z$, the table is completely expressed by the 24 rays which comprise the six orthonormal bases corresponding with the matrices. Peres [35] obtained the representation in coordinates of $\mathbb{R}^4$ given in Table 3.3, where I have arranged the coordinates in bases; and since the only information relevant to our proofs concerns the orthogonality relations among rays, we lose nothing by calling them vectors.

It turns out that among the set of 24 vectors are 192 subsets, each of 20 vectors

[2] It could be objected that this procedure begs the question of the acceptance of superpositions as given, since I may consistently deny that superpositions are real without violation of any implication of the Gleason theorem. Rather, the objection might continue, the remarkable fact we are asked to explain and understand is that the Gleason proof shows that the continuity conditions must be satisfied, and that it is upon this fact that the applicability of the continuity conditions and the validity of the strong superpositional principle depend.

Eigenbasis	Mutually orthogonal vector set
A :	[1,0,0,0], [0,1,0,0], [0,0,1,0], [0,0,0,1]
B :	[-1,1,1,1], [1,-1,1,1], [1,1,-1,1], [1,1,1,-1]
C :	[0,1,1,0], [0,1,-1,0], [1,0,0,-1], [*1*, *0*, *0*, *1*]
D :	[0,0,1,1], [0,0,1,-1], [1,-1,0,0], [*1*, *1*, *0*, *0*]
E :	[1,0,1,0], [0,1,0,1], [0,1,0,-1], [*1*, *0*, $-$*1*, *0*]
F :	[1,1,1,1], [1,1,-1,-1], [1,-1,-1,1], [*1*, $-$*1*, *1*, $-$*1*]

Table 3.3: The coordinates of Peres's 24 vectors may be grouped in cliques each corresponding to an eigenbasis for Mermin's nine operators.

(three from four of the bases and four from two of the bases), which are sufficient to prove the Kochen-Specker theorem. That is, the claim that each ray may be consistently associated with a 'hidden value' of either one or zero is refutable in quantum mechanics. To demonstrate the refutation, I first assume an assignment of 'hidden values' to a specified set of rays. Then, by invoking a simple rule of quantum mechanics, I derive a contradiction. The simple rule, often called the quantum mechanical *sum rule*, is that given four mutually orthogonal rays $1, 2, 3, 4$ in $\mathbb{R}^4$ (a 4-clique), we may write $f(1) + f(2) + f(3) + f(4) = 1$, where $f()$ has value one or zero. Table 3.4 demonstrates a set of 20 vectors with 11 4-cliques for which the set of associated equations cannot be satisfied. I note in Table 3.4 that the sum of the LHS is *odd*, yet, since each vector contributes either one or zero an *even* number of times to the sum of the RHS, the equations are not simultaneously consistent.

We recall Mermin's observation that the operators in Table 3.1 do not permit an extension of the Kochen-Specker argument to a nonlocality proof because the four local operators do not all commute with the five nonlocal operators. To achieve a nonlocality proof we can perform the same argument as just given for the vectors which compose the eigenstates of the GHZ system. However, we know already that the result will contain the four triple product operators required for a proof that

$1 =$	$f(1,0,0,0) + f(0,1,0,0) + f(0,0,1,0) + f(0,0,0,1)$
$1 =$	$f(1,0,0,0) + f(0,1,0,0) + f(0,0,1,1) + f(0,0,1,-1)$
$1 =$	$f(1,0,0,0) + f(0,0,1,0) + f(0,1,0,1) + f(0,1,0,-1)$
$1 =$	$f(1,0,0,0) + f(0,0,0,1) + f(0,1,1,0) + f(0,1,-1,0)$
$1 =$	$f(-1,1,1,1) + f(1,-1,1,1) + f(1,1,-1,1) + f(1,1,1,-1)$
$1 =$	$f(-1,1,1,1) + f(1,1,-1,1) + f(1,0,1,0) + f(0,1,0,-1)$
$1 =$	$f(1,-1,1,1) + f(1,1,-1,1) + f(0,1,1,0) + f(1,0,0,-1)$
$1 =$	$f(1,1,-1,1) + f(1,1,1,-1) + f(0,0,1,1) + f(1,-1,0,0)$
$1 =$	$f(0,1,-1,0) + f(1,0,0,-1) + f(1,1,1,1) + f(1,-1,-1,1)$
$1 =$	$f(0,0,1,-1) + f(1,-1,0,0) + f(1,1,1,1) + f(1,1,-1,-1)$
$1 =$	$f(1,0,1,0) + f(0,1,0,1) + f(1,1,-1,-1) + f(1,-1,-1,1)$

Table 3.4: Inconsistent equations derived from mutually orthogonal rays. Each ray occurs twice or four times.

hidden values 1 and 0 may not be assigned without contradiction to the obervables associated with these operators. The complete demonstration of the Kochen-Specker theorem in 8 dimensions, corresponding to the Mermin construction for the GHZ common eigenstates, is given in Appendix A, which is a research paper jointly written by Kernaghan and Peres.

3.4 Minimal Kochen-Specker configurations

Several geometrical questions may be associated with the distinction between lattices of quantum propositions, which induce non-2-valued measures on lattices of subspaces of Hilbert space, and lattices of classical propositions, which require 2-valued measures. These questions concern the symmetries of the subsets of Hilbert space associated with projection lattices which do not permit a 2-valued measure, and the critical sizes of such subsets, where the critical size is the number of one-dimensional subspaces ot the Hilbert space giving the desired type of projection lattice. Any number of subspaces below the critical size supports a 2-valued measure, so the critical size is a lower bound on the number of subspaces required to prove the Kochen-Specker theorem.

The critical size clearly depends on the dimension of the Hilbert space. I note as well the important fact that there is not one critical size for each dimension, since the size is a measure only of a quantity of subsets, and there are distinct ways to arrange the same number of subsets and still prove the Kochen-Specker theorem. Critical size is also dependent upon the particular set of observables that a BKS argument proves to be contextual. This dependence of critical size upon a particular set of observables explains the multiplicity of examples of Kochen-Specker configurations. We have seen three systems for which the configuration of subsets supporting the BKS colouring argument are known: the three dimensional spin-1 system with rhombically symmetric point sources about atomic orthohelium (for which the Kochen-Conway example is of critical size); the EPR setup, for which the operator proofs of Mermin and Peres are not of critical size; and the GHZ setup, for which the subset of critical size is not given directly by the operator proofs. The configurations of subsets of Hilbert space of critical size for the EPR and GHZ systems are obtainable by computer calculation. Each of these three systems supports a BKS colouring argument because the eigenvectors of the systems support an arrangement of subsets whose corresponding projection lattice cannot have a 2-valued measure and be consistent with quantum mechanics. Thus we can specify a geometric configuration of projection operators in a given Hilbert space as a generator of a 'Kochen-Specker Contradiction' proof. That is, we can point to a set of vectors which prove that a given system must contradict expectations based on a noncontextualist, or in some cases local, account of quantum measurement.

The determination of the smallest configuration of critical size in three dimensions has been of interest for many years. Presently the smallest known configuration is the 31-ray set due to Kochen and Conway. I will outline an effective procedure by which a computer search of all graphs with 30 or fewer vertices will turn up any small Kochen-Specker configurations if they exist.

Before giving an outline for the effective search procedure for small Kochen-Specker configurations I will discuss some matters concerned with the symmetries of the configurations. In particular, it is of interest to evaluate the hypothesis that Kochen-Specker configurations in general correspond with regular polytopes. Certainly the dodecahedral configuration found by Penrose supports this hypothesis. As well, the 24 rays derived by Peres from the EPR states form a regular polytope in four dimensions called the Coxeter Simplex. However, as I have shown, the 24 rays contain 192 subsets which support a Kochen-Specker proof, and this result breaks the symmetry of the 24 ray set and the relation with the polytope is lost. Similarly, the Kochen-Conway 31 ray set may be obtained as a subset of a 37 ray set corresponding with a more symmetrical arrangement of rays. I will look in closer detail at the highly symmetric configuration of 33-rays found by Peres. It was observed by Penrose that the 33-ray configuration corresponds with the polytope obtained by an interpenetration of three cubes. However, it turns out not to be the case that the subsets of the 33-ray configuration correspond with the cubes from which the polytope is composed. I will show this failure of correspondence and maintain that it offers additional evidence that there is no systematic correspondence between Kochen-Specker configurations and regular polytopes, since, in addition to the existence of less symmetrical subsets, it appears not to the case that Kochen-Specker configurations may be built from smaller subsets corresponding with regular polytopes.

Figure 3.1 represents as a graph the orthogonality relations among thirteen vectors. The vectors span cubes, planes, and rays of $\mathbb{R}^3$. Where three vectors span a cube they are mutually orthogonal and form 3-cycles in the graph. In such a case we consider the three rays which the vectors span. We associate with each ray a projection operator. The eigenvalues of these operators must sum to unity since the rays span a cube, and a cube spans the space. For a spin-1 system with eigenvalues 0 and 1 we obtain the rule that exactly one vertex of a clique may be coloured green, where we have associated the eigenvalue 1 of a projection operator with the colour green of a vertex which represents the vector spanning the subpace (the ray) associated with the operator.

We can write, for vectors x, y, z, an equation $f[x] + f[y] + f[z] = 1$ to capture the associations made between a graph and the eigenvalues of the projection operators. The function $f[x]$ takes value either 1 or 0 according to the eigenvalue to be hypothetically made definite for the subspace picked out by the vector x. Clearly the given equation will have as solutions just the three combinations of one 1 and two 0s. Thus the equation corresponds to the graph colouring rule.

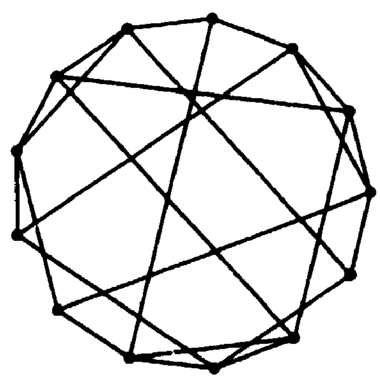

Figure 3.1: The Penrose BKS-graph generator, with four cliques and thirteen vertices. The graph is nonplanar.

The Penrose graph is a subgraph of the Peres33 graph (Figure C.2). Three Penrose graphs, sharing three rays which form a clique in each Penrose graph, comprise the Peres cube (Figure C.1). The shared clique is that formed by the rays through the centres of the faces of the Peres cube. We can show explictly the construction of the Peres33 set from three Penrose graphs.

The result of what I will call an Escher-gluing of three Penrose graphs is the full

Peres graph. This gluing corresponds to the three interpenetrating cubes shown in a picture by M. C. Escher [29]. Note the central triangle in figure C.2. The vertices at the corners of the triangle correspond with the rays through the centre of one of the cubes, and with the face centre points of the Peres cube. The second penetrating cube is obtained by a 90 degree rigid rotation of the first about an axis described from the centre of the original cube to the midpoint of an edge of that cube. The third penetrating cube is obtained by the same procedure applied to the second cube.

This relation may be seen on the Peres cube: imagine the ray which intersects the centre of the cube and one of the midpoints on an edge of the cube, say $(0, \sqrt{2}, \sqrt{2})$, and picture a rotation of the cube through 90 degrees around this ray. The result is a second cube, but we observe that the points painted on the first cube lie between the centre of the first cube and corresponding rotated points on the second cube. In this way it is demonstrated that the Escher cubes and the Peres cube are in one-one correspondence.

The graph of the Peres cube is nonplanar. This fact follows from its construction from three Penrose graphs, each of which itself is nonplanar. Note, however, that the graph of the orthogonality relations among rays through corner and mid-edge vertices of an ordinary cube is planar. Thus we should not say that the Penrose graph represents a cube, even though the orthogonality relations which compose it imply that a three dimensional space is spanned. Instead, we may say only that the Penrose graph is a set of thirteen projections determined only by the mutual orthorgonal relations given in the Penrose BKS-graph generator. It is equivalent under a unitary transformation to infinitely many sets of thirteen projections which have those orthogonal relations, as these are preserved under the unitary action. In other words, so long as the orthogonality relation remains invariant the set is freely orientable in (for this case) three dimensions. Given the appropriate relation among three Penrose projection sets a Kochen-Specker contradiction is obtainable (as represented by the 33 rays of the Peres BKS-graph).

The foregoing technical distinctions are aimed at highlighting the distinction between the Penrose polygon representations and the graph representations. It would be incautious to proceed in a search for BKS-graph generators by positing that they form regular polyhedra. The EPR, GHZ, and Kochen-Conway BKS-graphs demonstrate that polyhedral symmetry is not respected by Kochen-Specker directions.

Finally, I offer the conjecture that any (finite) graph of orthogonality relations that has no 2-valued homomorphisms onto a Boolean algebra must be nonplanar. The known examples are consistent with this conjecture, and if a proof may be had, a simple characteristic for such graphs may be given. This would be of benefit in narrowing the candidates for a computer search of graphs of order ≤ 30.

Kuratowski showed that a graph is nonplanar if and only if the graph contains a subgraph isomorphic with either of the elementary graphs K_5 or $K(3,3)$. There is a subgraph of the graph for Peres's 33 rays that is isomorphic with K_5 and a subgraph isomorphic with $K(3,3)$. By examination of the graph in Figure C.2 it is evident that there exists a subgraph homeomorphic with the graph K_5, the graph obtained by drawing edges in all possible ways among a set of five vertices. Note that there are algorithms for testing the planarity of graphs which are much more efficient than the method suggested by Kuratowski's proof.

To determine the existence or no of Kochen-Specker graphs in three dimensions of order ≤ 30 we may filter a large portion of the graphs of order ≤ 30 for the following requirements which characterize candidates:

- The graph is connected;
- The graph has no vertices with fewer than two edges;
- The graph can contain no 4-cycles;
- The graph contains at least one 3-cycle.

Rather than characterize realizability on the sphere as a filter it is simpler to include as filters such things as 'no 4-cycles' which are consequences of the fact that the graph must represent orthogonal relations, in this case in three dimensions. Also, this approach allows greater flexibility should a search of higher dimensional cases be desired.

It would of course help matters further to include 'no planar graphs'. It is possible to automate the search for lower order 'Kcchen-Specker' graphs by filtering the set of all graphs of a given order (for orders up to 16 this is simple, up to 24 is possible, or will be within the decade in all likelihood) and feeding the remaining graphs to an algorithm which checks colouring. Any result must be checked (in particular for realizability on the sphere). If there is such a graph this procedure will find it and if there are none then the lower bound is increased acordingly.

Chapter 4

Conclusions

The results given in this thesis extend the proofs of Mermin and Peres that both the EPR and the GHZ are systems in which a boolean representation for the observables is impossible. The extension consists in the demonstration that the common eigenstates of the operators for the EPR or the GHZ contain not just one noncontextual configuration of subspaces, but several hundred. I prove that there are 192 equivalent configurations of 20 rays available among the operators of the EPR system that support a proof of the Kochen-Specker theorem. In appendix A, Peres and I demonstrate the extension of these results to the eight dimensional real Hilbert space of the operators of the GHZ system. We found 1,280 configurations of 36 rays that support a Kochen-Specker proof.

I have also outlined a procedure to decide the least configurations required to prove the Kochen-Specker theorem in three dimensions. I show that the problem is entirely solvable by computer calculation, and I provide an effective procedure for sorting the configurations of Hilbert space that support a Kochen-Specker argument. I also introduce a conjecture that, if proved, would considerably shorten the calculations required.

One conclusion of this thesis is that the configurations of subspaces of Hilbert space that support the Kochen-Specker theorem are not characterizable as simple polytopes. On the basis of earlier results, it is plausible to suggest such a characterization.

However, I have shown that the polytope structures of configurations, such as the Coxeter 24-simplex associated with the eigenstates of the EPR, contain many smaller configurations sufficient to support a Kochen-Specker argument. This result reveals that it is not trivial to characterize the sets of rays of Hilbert space that support a Kochen-Specker theorem.

Mączyński showed that boolean representations may be given for sets of maximal observables, and we may use his results to demarcate the kinds of configurations of Hilbert space within which a hidden variable theory cannot be shown to contradict quantum mechanics.

More generally, hidden variable theories are free from contradiction with quantum mechanics only while the operators associated with the observables of such theories do not have eigenstates appropriately configured for the proof of a Kochen-Specker theorem. The arguments I have presented here show that a systematic selection of sets of observables, chosen to avoid families of observables that support a Kochen-Specker theorem, is implausible. Such a systematic selection is implausible because what must be ruled is quite ubiquitous. Earlier proofs of the Kochen-Specker theorem left open the possibility of families of observables that might support a hidden variable theory. The results obtained here apply to any interpretations that make definite claims about values assigned to operators. Typically, however, interpretations are not of that kind. Instead they give more abstract and general sets of criteria by which to decide what values might be possessed by various hidden parameters associated with observables. One example is the interpretation, due to van Fraassen, according to which Hermitian operators may be distinguished into 'maximal' and 'non-maximal' . This distinction departs from the standard quantum mechanical assumption that there is a one to one correspondence between operators and observables , since on van Fraassen's scheme, a non-maximal operator corresponds to many observables, while a maximal operator corresponds to just one. Van Fraassen's interpretation is 'Deoccamized', since the breaking of the one to one correspondence between operators and observables may be interpreted as supporting an ontological-contextualist interpretation of quantum mechanics. Contradictions are removed if non-maximal observables are allowed to correspond to many physically distinct observables. Avoiding the contradiction in this way leads to an interpretation whose ontology is, in some sense, rich enough to support context dependent measurements.

I have shown that any interpretation that offers a scheme for incorporating the EPR and GHZ observables within a hidden variable theory also must account for the multiplicity of configurations of subspaces, within the space of EPR or GHZ operators, that support a Kochen-Specker argument. I have maintained that an account of that multiplicity will be so complex as to be implausible. The simple and highly symmetric configurations of subsets of Hilbert space, such as the 33 directions of the Peres proof in three dimensions, must be systematically ruled out on any hidden variable theory, so that a system for providing a hidden variable theory as an interpretation becomes more complex than the quantum theory that is the subject of the interpretation.

The observation that the operator based proofs due to Mermin and Peres contain redundant information, in that we only need subsets of the operators' common eigenvectors to prove the Kochen-Specker theorem, has a significant relationship to experimental proposals made recently by Bernstein, Bertani, Reck, and Zeilinger [3]. They show that it is possible to realize, with optical devices, any finite-dimensional discrete unitary operator. They maintain that their contruction is the first to show that "any discrete unitary operator can be given operational meaning in the real world."

The basis for the experimental proposals is an entirely constructive proof that any unitary operator may be realized by optical devices in the laboratory. Thus, for example, it is now possible to make a direct measurement of any of the unitary matrices corresponding to the Hermitian operators appearing in Table 3.1. As well, the method for constructing the unitary matrices that correspond with matrix representations of the Hermitian operators is constructive (an explicit example of the construction is given in [3]). This fact absolves us of any worry that the constructions carried out in this thesis are not physically realistic, since any system of rays in Hilbert space may be given a physical realization. Furthermore, the proof that any configuration of rays is physically realizable is a purely constructive proof, so again we are absolved of any worry about arbitrary or hypothetical assumptions hidden in the formulation of the configurations that support a Kochen-Specker argument.

Clearly the experimental results obtained by Bernstein, Bertani, Reck, and Zeilinger carry the important implication, for this thesis, that any configuration of eigenvalues of unitary operators is physically realizable and these correspond with the Hermitian operators used in our proofs. However, the converse implication from the reduced proofs of Kochen-Specker to their experimental test of Kochen-Specker, is not so clear. Since the Kocken-Specker theorem may be proved for a subset of the eigenvectors of the Mermin operators it may be possible to arrange a reduction in the number of optical devices required to carry out the experiment. A similar observation may be made for the reduction of the Kochen-Specker proof given in Appendix A for an eight dimensional Hilbert space. However, since the reduced proofs do not remove the operators, but only subsets of their eigenvectors, it is not obvious that fewer optical devices may be sufficient. The possibility is nevertheless intriguing and should be investigated.

Appendix A

Kochen-Specker theorem for 8-dimensional space

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Abstract

A Kochen-Specker contradiction is produced with 36 vectors in a real 8-dimensional Hilbert space. These vectors can be combined into 30 distinct projection operators (14 of rank 2, and 16 of rank 1). A state-specific variant of this contradiction requires only 13 vectors, a remarkably low number for 8 dimensions.

The Kochen-Specker theorem [1] asserts that, in a Hilbert space with a finite number of dimensions, $d \geq 3$, it is possible to produce a set of n projection operators, representing yes-no questions about a quantum system, such that none of the 2^n possible answers is compatible with the sum rules of quantum mechanics. Namely, if a subset of mutually orthogonal projection operators sums up to the unit matrix, one and only one of the answers is yes. The physical meaning of this theorem is that there is no way of introducing noncontextual "hidden" variables [2] which would ascribe definite outcomes to these n yes-no tests. This conclusion holds irrespective of the quantum state of the system being tested.

It is also possible to formulate a "state-specific" version of this theorem, valid for systems which have been prepared in a known pure state. In that case, the projection operators are chosen in a way adapted to the known state. A smaller number of questions is then sufficient to obtain incompatibility with the quantum mechanical sum rules. An even smaller number is needed if strict sum rules are replaced by weaker probabilistic arguments [3,4].

The original proof by Kochen and Specker [1] involved projection operators over 117 vectors in a 3-dimensional real Hilbert space $\mathbf{R}^3$. A simple proof with 33 vectors was later given by Peres [5], who also reported an unpublished construction by Conway and Kochen, using only 31 vectors [6]. A proof with 20 vectors in $\mathbf{R}^4$ was recently given by Kernaghan [7]. Here, we consider the Kochen-Specker theorem in $\mathbf{R}^8$. A state-independent proof is produced with 36 vectors, which can be collected into 30 distinct projection operators. A state-specific proof involves only 13 vectors, and achieves the lowest value of the ratio n/d that has been obtained so far.

Our construction is based on Mermin's remark [8] that, for any three spin-$\frac{1}{2}$ particles, the four operators

$$A = \sigma_{1z} \otimes \sigma_{2z} \otimes \sigma_{3z}, \tag{A.1}$$

$$B = \sigma_{1z} \otimes \sigma_{2x} \otimes \sigma_{3x}, \tag{A.2}$$

$$C = \sigma_{1x} \otimes \sigma_{2z} \otimes \sigma_{3x}, \tag{A.3}$$

$$D = \sigma_{1x} \otimes \sigma_{2x} \otimes \sigma_{3z}, \tag{A.4}$$

commute. Moreover, their product is proportional to the unit matrix,

$$ABCD = -\mathbb{1}. \tag{A.5}$$

Mermin's three particle system is the simplest one that illustrates *both* Bell's theorems, on nonlocality [9] and contextuality [10].

Each one of the five equations above involves a complete set of commuting operators, and therefore determines a complete orthogonal basis in $\mathbf{R}^8$ (complex numbers are not needed here, because σ_x and σ_z are real matrices). The five orthogonal octads generated by the above equations are listed in the first column of Table 1. To simplify typography, each vector was given norm 2 (this avoids the use of fractions) and the symbol $\bar{1}$ means -1. The components of each vector are written as a horizontal array, rather than the usual "column vector." The basis used is the direct product of the bases where each σ_z is diagonal. For example, the vector with components 00002000 is the eigenvector for which σ_{1z} has eigenvalue -1, and σ_{2z} and σ_{3z} have eigenvalue 1. This is most easily seen by using binary digits, 0 and 1, for labelling the "up" and "down" components of a spinor, respectively, and combining them into binary numbers, 000, 001, ..., 111, for labelling components of vectors in $\mathbf{R}^8$. Thus, the vector 00002000 has the physical meaning stated above because its only nonvanishing component is the 100th one (that is, the fifth one, in binary notation).

It is easy to see that the 40 projection operators on these 40 vectors provide an example of the Kochen-Specker contradiction. Indeed, associating values 0 (no) and 1 (yes) to the vectors in a basis (with a single 1, of course) amounts to selecting one of these eigenvectors, and the latter indicates which ones of the four commuting operators which generate that basis have value 1, and which ones have value -1. Such a mapping of each operator on one of its eigenvalues cannot be done for all of them in

Eqs. (1–5): it would lead to an inconsistency, because of the minus sign in (5). This argument is readily generalized to a larger number of spin-$\frac{1}{2}$ particles, and provides a proof of the Kochen-Specker theorem in a real Hilbert space with 2^n dimensions. However, fewer than 2^n vectors are actually needed for the proof, as we shall see.

First, we note that the above set of 40 vectors has a high degree of symmetry. (Unfortunately, we have not been able to determine its invariance group. We asked several well known experts, who also did not find it.) A direct inspection, best done by computer, shows that each vector is orthogonal to 23 other ones (7 in the same basis, and 4 in each one of the four other bases), and it makes a 60° or 120° angle with each one of the 16 remaining vectors. It is possible to construct with the 40 vectors 25 distinct orthogonal octads (each vector appears in 5 octads). Eleven of these octads are listed in the remaining columns of Table 1 (the 14 other octads are not needed for the proof and have not been listed).

It is seen that there are four vectors, namely those with components 20000000, 00001111, $001\bar{1}001\bar{1}$, and $10\bar{1}0\bar{1}010$, that do not appear in the 11 octads (they appear of course in the 14 unlisted octads). These four vectors are mutually orthogonal, and they belong to four different of the original octads. It is easily seen that there are 1280 different ways of choosing four vectors with these properties (that is, of choosing which are the 11 octads, out of 25, that appear in Table 1).

It is also seen that each one of the 36 remaining vectors appears in Table 1 either twice or four times. Let us now consider the projection operators over these 36 vectors. According to quantum mechanics, each projection operator corresponds to a yes-no question: the eigenvalues 1 and 0 mean yes and no, respectively. Each orthogonal octad defines 8 commuting projection operators (that is, 8 compatible questions) which sum up to the unit matrix. This means that if an experimental test is actually performed for these 8 questions, the answer is yes to one, and only one of them. The value 1 is thereby associated with one of the eight vectors of each octad, and the value 0 with the others.

If nothing is known of the state of the system, quantum mechanics is unable to predict which vector will get the value 1. A natural question is whether there could be a more complete theory, such that the value associated with each vector would be determined by "hidden variables." Table 1 readily shows that this goal cannot be achieved. Indeed, the sum of values in each octad is always 1, therefore the sum of values for the 11 octads in the table is 11, which is an *odd* number. On the other hand, each vector (tentatively associated with a value which is either 0 or 1) appears either twice or four times in the table (with the *same* value), thus contributing 2 or 4 (an *even* number) to the sum of values. We have reached a contradiction. This is the proof of the Kochen-Specker theorem in $\mathbb{R}^8$.

Furthermore, our 36 incompatible propositions can be combined into a smaller number, namely 30 distinct ones, by using projection operators of rank 2 on the *planes* spanned by some pairs of vectors. Table 2 shows how 20 of the vectors can be combined into 14 planes. The vectors on each line are mutually orthogonal. Each plane is spanned by two adjacent vectors on the same line (for example, 02000000 and 00000002). When such a pair of vectors occurs in any of the 11 octads listed in Table 1 (that is, in one of the columns) this vector pair should be replaced by the corresponding plane. Once this is done, each column lists planes and unpaired vectors, all of which are mutually orthogonal. They correspond to commuting projection operators of rank 1 or 2, which sum up to $\mathbb{1}$. Therefore, if "experimental" (that is, counterfactual) values 0 or 1 are attributed to them, these values sum up to 11, exactly as before. On the other hand, each one of the remaining 16 vectors (those not used in Table 2) appears twice in the new version of Table 1, and each of the 14 planes appears 2 or 4 times – always an even number. We thus reach the same Kochen-Specker contradiction as before.

Until now, we assumed nothing about the state of the quantum system. If that state is known with certainty, it becomes possible to find a much smaller number of propositions which lead to a Kochen-Specker contradiction. For instance, let the quantum system be prepared in the pure state $100\bar{1}0\bar{1}\bar{1}0$ (this does not restrict the generality of this discussion in any way, because it is always possible to choose the basis in $\mathbf{R}^8$ in such a way that any given pure state be represented by a vector with these components). We can now discard from Table 1 that vector, whose associated value is 1, by definition, and all the vectors orthogonal to it, whose associated values are zero. Only 7 non-empty columns, with 13 different vectors, remain in Table 1. In each column, there are four mutually orthogonal vectors, which span a subspace containing the known pure state $100\bar{1}0\bar{1}\bar{1}0$ (because the complementary orthogonal subspace is also orthogonal to $100\bar{1}0\bar{1}\bar{1}0$). Explicitly, we have

$$2 \times 100\bar{1}0\bar{1}\bar{1}0 = 1010\bar{1}0\bar{1}0 + 10\bar{1}010\bar{1}0 - 00020000 - 00000200, \tag{A.6}$$

$$= 1100\bar{1}\bar{1}00 + 1\bar{1}001\bar{1}00 - 00020000 - 00000020, \tag{A.7}$$

$$= 11\bar{1}\bar{1}0000 + 1\bar{1}1\bar{1}0000 - 00000200 - 00000020, \tag{A.8}$$

$$= 11\bar{1}\bar{1}0000 + 00001\bar{1}\bar{1}1 + 1010\bar{1}0\bar{1}0 - 01010101, \tag{A.9}$$

$$= 1\bar{1}1\bar{1}0000 + 00001\bar{1}\bar{1}1 + 1100\bar{1}\bar{1}00 - 00110011, \tag{A.10}$$

$$= 1100\bar{1}\bar{1}00 + 1\bar{1}001\bar{1}00 + 001\bar{1}00\bar{1}1 - 00110011, \tag{A.11}$$

$$= 1100\bar{1}\bar{1}00 + 001\bar{1}00\bar{1}1 + 10\bar{1}010\bar{1}0 - 01010101. \tag{A.12}$$

Quantum mechanics asserts that, for each one of the above equations, there exists an experimental test which randomly selects one of the four vectors appearing on the right hand side of that equation. A hidden variable theory would claim that the selection is not random, and that there are preassigned values, 0 and 1, corresponding to all 13 vectors: on each line, one of the four vectors has value 1, the others have value 0. Moreover, we may demand that, if the same vector appears in different lines, the value associated to it is everywhere the same (that is, the result of an experimental test which would determine that value is not "contextual"). These demands then lead us to the same contradiction: there are 7 equations, so that the sum of values is 7. On the other hand, each vector appears 2 or 4 times, so that the sum of values is even. It is remarkable that no more than 13 propositions are needed to reach the contradiction. The proof entirely holds in these seven equations.

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20000000											
02000000	×							×			
00200000	×						×				
00020000							×	×			
00002000	×					×					
00000200						×		×			
00000020						×	×				
00000002	×					×	×	×			
11110000		×				×					
$11\bar{1}\bar{1}0000$						×				×	
$1\bar{1}1\bar{1}0000$						×			×		
$1\bar{1}\bar{1}10000$		×				×			×	×	
00001111											
$000011\bar{1}\bar{1}$		×								×	
$00001\bar{1}1\bar{1}$		×							×		
$00001\bar{1}\bar{1}1$									×	×	
11001100							×		×		
$1100\bar{1}\bar{1}00$			×				×		×		×
$1\bar{1}001\bar{1}00$			×				×				
$1\bar{1}00\bar{1}100$							×				×
00110011			×						×		
$001100\bar{1}\bar{1}$									×		×
$001\bar{1}001\bar{1}$											
$001\bar{1}00\bar{1}1$			×								×
10101010				×				×		×	×
$1010\bar{1}0\bar{1}0$								×		×	
$10\bar{1}010\bar{1}0$								×			×
$10\bar{1}0\bar{1}010$											
01010101				×				×			
$01010\bar{1}0\bar{1}$										×	×
$010\bar{1}010\bar{1}$				×						×	
$010\bar{1}0\bar{1}01$				×							×
$100101\bar{1}0$	×		×	×	×						
$100\bar{1}0110$	×	×	×		×						
$10010\bar{1}10$	×				×						
$100\bar{1}0\bar{1}\bar{1}0$	×	×		×	×						
$0110\bar{1}001$				×	×						
$01\bar{1}01001$		×	×	×	×						
$0\bar{1}101001$		×			×						
$0\bar{1}\bar{1}0\bar{1}001$			×		×						

Table A.1: Orthogonal octads used for proving the theorem.

02000000	00000002	00000020	
11110000	$1\bar{1}\bar{1}10000$	$00001\bar{1}\bar{1}1$	
$1\bar{1}001\bar{1}00$	$1100\bar{1}\bar{1}00$	$001100\bar{1}\bar{1}$	
01010101	10101010	$01010\bar{1}0\bar{1}$	
$10010\bar{1}10$	$100101\bar{1}0$	$01\bar{1}01001$	$0\bar{1}101001$
$0\bar{1}\bar{1}0\bar{1}001$	$100\bar{1}0110$	$100\bar{1}0\bar{1}\bar{1}0$	$0110\bar{1}001$

Table A.2: Construction of 14 planes from 20 vectors.

Appendix B

Computer program for testing Kochen-Specker colourings

Included here is an expansion of an algorithm devised by Peres [35] for testing a set of vectors firstly for their orthogonal relations and their mutually orthorgonal basis sets, and secondly for the consistency of these relations with the Kochen-Specker colouring rule. The expansion provides the same test for eight dimensional vectors, applying the rule that only one vector in each eight vector basis may have value one and the other seven must have value zero.

The program example included here records the series of tests made to reduce the 40 vectors implied by the common eigenstates of the operators in the Mermin-Peres star figure to a minimum configuration inconsistent with the colouring rule. The resulting set of vectors is a generator for a projection lattice which has a 2-valued measure and is by Gleason's theorem inconsistent with measures on closed subspaces of Hilbert space, subspaces correspondent with the projections in the lattice. The propositions associated with such lattices are therefore refutable in quantum mechanics. Since such a generator should exist for propositions associated with local measurement outcomes in a GHZ experiment we have obtained a refutation of locality within quantum mechanics, using only independent geometric facts consistent with the quantum formalism and the attribution to quantum states of hypothetical properties prior to measurement.

```
C  ** Kochen-Specker checker for eight dimensions **
C  Modified by Michael Kernaghan from an original program for
C  three dimensions written by Asher Peres.
C  The forty vectors were originally derived by Peres.
C  UWO NVE CYBER 2000 FORTRAN2

      PROGRAM  GHZFTN
      PARAMETER (N=36)
      INTEGER P(N,N), W(N), X(N), Y(N), Z(N), C(N)
      INTEGER L(N), OC(N,N), XX(N), YY(N), ZZ(N), WW(N)
      INTEGER NQUAD,NG, LAST, NQ, FOO, BOO, COO, LVL
      INTEGER H, I, J, K, II, JJ, KK, GG
      INTEGER V(8,36), M(36)
      DATA M/36*0/
      DATA V/

C   1 - 7
     1 0,1,0,0,0,0,0,0, 0,0,1,0,0,0,0,0,
     2 0,0,0,1,0,0,0,0, 0,0,0,0,1,0,0,0, 0,0,0,0,0,1,0,0,
     3 0,0,0,0,0,0,1,0, 0,0,0,0,0,0,0,1,
C   8 - 14
     4 1,1,1,1,0,0,0,0, 1,1,-1,-1,0,0,0,0, 1,-1,1,-1,0,0,0,0,
     5 1,-1,-1,1,0,0,0,0, 0,0,0,0,1,1,-1,-1,
     6 0,0,0,0,1,-1,1,-1, 0,0,0,0,1,-1,-1,1,
C   15 - 22
     4 1,1,0,0,1,1,0,0, 1,1,0,0,-1,-1,0,0, 1,-1,0,0,1,-1,0,0,
     5 1,-1,0,0,-1,1,0,0, 0,0,1,1,0,0,1,1, 0,0,1,1,0,0,-1,-1,
     6 0,0,1,-1,0,0,-1,1
C   23 - 30
     4 1,0,1,0,1,0,1,0, 1,0,1,0,-1,0,-1,0, 1,0,-1,0,1,0,-1,0,
     5 1,0,-1,0,-1,0,1,0, 0,1,0,1,0,1,0,1, 0,1,0,1,0,-1,0,-1,
     6 0,1,0,-1,0,1,0,-1,
C   31 - 37
     7 1,0,0,1,0,1,-1,0, 1,0,0,-1,0,1,1,0, 1,0,0,1,0,-1,1,0,
     8 1,0,0,-1,0,-1,-1,0, 0,1,1,0,-1,0,0,1, 0,1,-1,0,1,0,0,1,
     9 0,-1,1,0,1,0,0,1, 0,-1,-1,0,-1,0,0,1/

      OPEN (8,FILE='result')
      DO 100 I=1,N
      DO 100 J=I+1,N
      K=0
      P(I,J)=0
      DO 110 H=1,8
110   K=K+V(H,I)*V(H,J)
      IF (K.NE.0) GOTO 100
      P(I,J)=1
      P(J,I)=1
100   CONTINUE
      P(36,36)=0
      WRITE (*,'(36I2)') P
      DO 10 I=1,N
      C(I)=9
10    CONTINUE
```

```
11    FOO=1
      NQUAD=0
      BOO=0
      COO=0
      DO 12 I=1,N
      DO 12 J=I+1,N
      DO 12 K=J+1,N
      DO 12 G=K+1,N
      DO 12 II=G+1,N
      DO 12 JJ=II+1,N
      DO 12 KK=JJ+1,N
      DO 12 GG=KK+1,N
      BOO=(P(I,J)+P(I,K)+P(I,G)+P(I,II)+P(I,JJ)+P(I,KK)
     + +P(I,GG)+P(J,K)+P(J,G)+P(J,II)+P(J,JJ)+P(J,KK)
     + +P(J,GG)+P(K,G)+P(K,II))
      COO=(P(K,JJ)+P(K,GG)+P(G,II)+P(G,JJ)+P(G,KK)
     + +P(G,GG)+P(II,JJ)+P(II,KK)+P(II,GG)+P(JJ,KK)
     + +P(JJ,GG)+P(KK,GG))
      IF ((BOO+COO).NE.27) GOTO 12
      NQUAD=NQUAD+1
      X(NQUAD)=I
      Y(NQUAD)=J
      Z(NQUAD)=K
      W(NQUAD)=G
      XX(NQUAD)=II
      YY(NQUAD)=JJ
      ZZ(NQUAD)=KK
      WW(NQUAD)=GG
12    CONTINUE
      LVL=0
14    DO 15 NG=1,N
      IF (C(NG).EQ.9) THEN
      C(NG)=1
      GOTO 16
      ENDIF
15    CONTINUE
      WRITE (8,'(36I2)') C
C     Graph is colourable
C     Display orthonormal frames
      WRITE(8,'(40I3)') X
      WRITE(8,'(40I3)') Y
      WRITE(8,'(40I3)') Z
      WRITE(8,'(40I3)') W
      WRITE(8,'(40I3)') XX
      WRITE(8,'(40I3)') YY
      WRITE(8,'(40I3)') ZZ
      WRITE(8,'(40I3)') WW
      STOP
C     *Done*

16    LVL=LVL+1
      LAST=1
      L(LVL)=NG
      DO 17 J=1,N
```

```
17    OC(LVL,J)=C(J)
18    DO 19 J=1,N
      IF (P(NG,J).EQ.1) C(J)=0
19    CONTINUE
20    DO 21 NQ=1,NQUAD
      IF (C(X(NQ))+C(Y(NQ))+C(Z(NQ))+C(W(NQ))+C(XX(NQ))
     + +C(YY(NQ))+C(ZZ(NQ))+C(WW(NQ)).EQ.0) GOTO 22
21    CONTINUE
      GOTO 25
22    IF (LVL+LAST.GT.0) GOTO 23
C     colouring fails
      WRITE (8,'(''This set has no consistent colouring'')')
C     display orthonormal frames
      WRITE(8,'(40I2)') X
      WRITE(8,'(40I2)') Y
      WRITE(8,'(40I2)') Z
      WRITE(8,'(40I2)') W
      WRITE(8,'(40I2)') XX
      WRITE(8,'(40I2)') YY
      WRITE(8,'(40I2)') ZZ
      WRITE(8,'(40I2)') WW
      STOP
C     *Done*

23    DO 24 J=1,N
24    C(J)=OC(LVL,J)
      C(L(LVL))=0
      LAST=0
      LVL=LVL-1
      GOTO 20
25    DO 26 NQ=1,NQUAD
      IF (C(X(NQ))+C(Y(NQ))+C(Z(NQ))+C(W(NQ))+C(XX(NQ))
     + +C(YY(NQ))+C(ZZ(NQ))+C(WW(NQ)).EQ.9) GOTO 27
26    CONTINUE
      GOTO 14
27    IF (C(X(NQ)).EQ.9) THEN
      C(X(NQ))=1
      NG=X(NQ)
      GOTO 18
      ENDIF
      IF (C(Y(NQ)).EQ.9) THEN
      C(Y(NQ))=1
      NG=Y(NQ)
      GOTO 18
      ENDIF
      IF (C(Z(NQ)).EQ.9) THEN
      C(Z(NQ))=1
      NG=Z(NQ)
      GOTO 18
      ENDIF
      IF (C(W(NQ)).EQ.9) THEN
      C(W(NQ))=1
      NG=W(NQ)
      GOTO 18
```

```
      ENDIF
      IF (C(XX(NQ)).EQ.9) THEN
      C(XX(NQ))=1
      NG=XX(NQ)
      GOTO 18
      ENDIF
      IF (C(YY(NQ)).EQ.9) THEN
      C(YY(NQ))=1
      NG=YY(NQ)
      GOTO 18
      ENDIF
      IF (C(ZZ(NQ)).EQ.9) THEN
      C(ZZ(NQ))=1
      NG=ZZ(NQ)
      GOTO 18
      ENDIF
      IF (C(WW(NQ)).EQ.9) THEN
      C(WW(NQ))=1
      NG=WW(NQ)
      GOTO 18
      ENDIF
      END
C End GHZFTN
```

B.1 Computer analysis of GHZ eigenstates

The following is a sample output from a run of the program GHZFTN. In this example the program has in its data set the 36 projectors in eight dimensions that correspond with a subset of the common eigenstates of the three GHZ operators. The result reveals that the projection operators corresponding with these 36 vectors cannot be assigned values independently of the circumstances of their eventual measurement.

The output first indicates the graph is not Kochen-Specker colourable and then displays as columns the rays which compose the eigenbases among them. Here there are eleven eigenbases. If we suppose that we may attribute values to the rays we must satisfy the condition that the sum of zeroes and ones we attribute is unity for each eigenbasis. That is, each eight member eigenbasis may have just one 'one' and must have seven 'zeros'. For the sample output given here this rule can not be satisfied.

```
GHZFTN (Data=36_min)

This graph has no consistent colouring

Table of mutually orthogonal rays forming eigenbases
(each column is an 'octad')

 1  1  2  4  8  9 10 16 16 22 29
 2  3  3  5 11 11 11 17 18 25 30        8 x 11 = 88
 4  5  6  6 12 12 13 19 20 27 31
 7  7  7  7 13 14 14 21 21 28 32
29 22 15  8 30 22 15 29 22 29 33
30 23 16  9 32 23 16 30 24 32 34
31 24 17 10 34 26 19 34 26 33 35
32 25 18 11 35 27 20 36 28 34 36

Table of the frequency of each ray.

1: 2      8: 2     15: 2     22: 4     29: 4
2: 2      9: 2     16: 4     23: 2     30: 4
3: 2     10: 2     17: 2     24: 2     31: 2
4: 2     11: 4     18: 2     25: 2     32: 4
5: 2     12: 2     19: 2     26: 2     33: 2
6: 2     13: 2     20: 2     27: 2     34: 4
7: 4     14: 2     21: 2     28: 2     35: 2
                                       36: 2
 --        --        --        --        --
 16   +    16   +    16   +    16   +    24   =   88
```

The 36 8-dimensional rays forming the uncolourable set are listed in Table A.1.

The next example displays output of the program given a data set which *is* Kochen-Specker colourable. Instead of declaring the colourability the program supplies a list of value attributions which may successfully be applied to the set. In this example ray one, ten, twenty-six, and thirty-five may take value one and the remaining rays value zero without contradiction. In the example above in which no colouring is available there exists no such list of value attributions. For a successful colouring there may exist many different successful lists.

```
1 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 ...
... 0 1 0 0 0 0 0 0 0 0 0 1 0 0

 1  1  4  8  9 15 15 22 22 23 30
 2  3  5 11 11 16 17 23 25 24 31
 4  5  6 12 12 18 19 24 27 26 32
 7  7  7 13 14 21 21 25 28 29 33
30 22  8 31 22 30 22 26 30 31 34
31 23  9 33 23 31 24 27 33 32 35
32 24 10 35 26 35 26 28 34 36 36
33 25 11 36 27 37 28 29 35 37 37
```

B.2 Converting coordinate sets to graphs

The following is an algorithm for converting coordinates of vectors into adjacency matrices for graphs. The adjacency matrix is an array of ones and zeroes where a one occurs at (row i, column j) if i and j are orthogonal and a zero otherwise. The graph is read from the matrix by allocating a vertex for each element of the array. An edge $e(i,j)$ exists if i and j are orthogonal. Orthogonality is determined by taking an inner product and the result is recorded in an array which serves as an adjacency matrix.

```
C Quadder.f: Outputs adjacency matrix and 4-cliques
C            from coordinates.

      PROGRAM QUADDER
      PARAMETER (N=24)
      INTEGER P(N,N), W(N), X(N), Y(N), Z(N), C(N)
      INTEGER Q(N), R(N), S(N), T(N),  L(N), OC(N,N)
      INTEGER A, B, D, FOO, E, F, G, H
      OPEN (8,FILE='INPUT.QD')
      OPEN (9,FILE='OUTPUT.QD')
      DO 7 I=1,N
      DO 7 J=1,N
  7   P(I,J)=0

C     read N coordinate inputs (a,b,c,d) and
C     create an adjacency matrix
C *** Initialize arrays
      DO 8 E=1,N
      Q(E)=0
      R(E)=0
      S(E)=0
```

```
      T(E)=0
      DO 8 F=1,N
    8 P(E,F)=0
C *** Read input into arrays Q, R, S, T
      DO 9 M=1,N
      READ (8,'(4I3)',END=10) A, B, D, H
      Q(M)=A
      R(M)=B
      S(M)=D
      T(M)=H
    9 CONTINUE
   10 FOO=0
C *** Check orthoganality and write to matrix P
      DO 11 I=1,N
      DO 11 J=I+1,N
      IF ((Q(I)*Q(J))+(R(I)*R(J))+(S(I)*S(J))+(T(I)*T(J))
     + .EQ.0) THEN
      P(I,J)=1
      P(J,I)=1
      ENDIF
   11 FOO=1
C     Find quads of orthogonal rays
      DO 12 I=1,N
      DO 12 J=I+1,N
      DO 12 K=J+1,N
      DO 12 G=K+1,N
      IF (P(I,J)+P(I,K)+P(I,G)+P(J,K)+P(J,G)+P(K,G)
     + .NE.6) GOTO 12
      NQUAD=NQUAD+1
      X(NQUAD)=I
      Y(NQUAD)=J
      Z(NQUAD)=K
      W(NQUAD)=G
 12   CONTINUE
      WRITE (9,'(''adjacency matrix:'')')
      WRITE (9,'(24I2)') P
      WRITE (9,'(''    '')')
      WRITE (9,'(''Quads:'')')
      WRITE (9,'(39I2))') X
      WRITE (9,'(39I2))') Y
      WRITE (9,'(39I2))') Z
      WRITE (9,'(39I2))') W
      STOP
      END
C     *Done*
C  End QUADDER
```

Appendix C

Figures

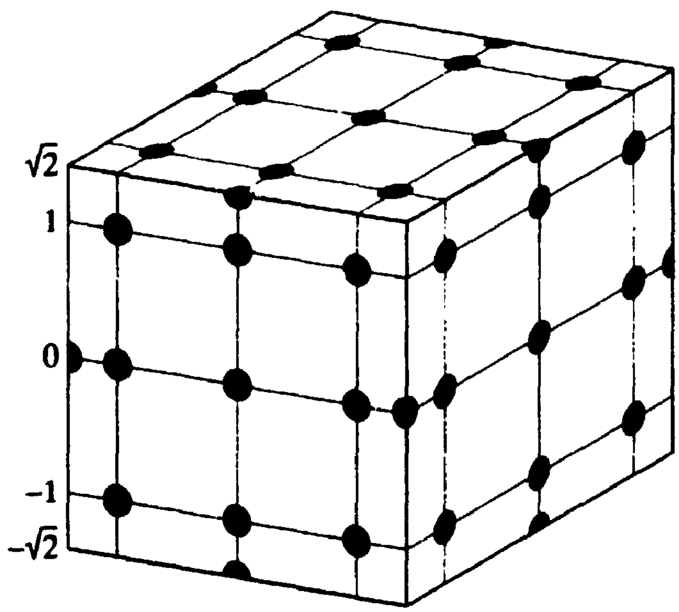

Figure C.1: Peres's cubic representation of his 33 directions, reprinted with the permission of the author.

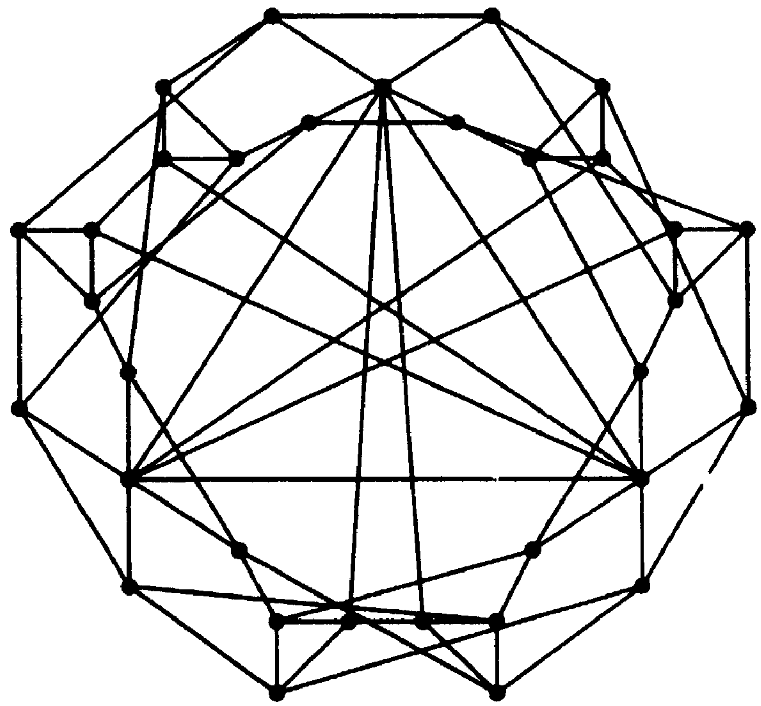

Figure C.2: The BKS-graph for Peres's 33 directions.

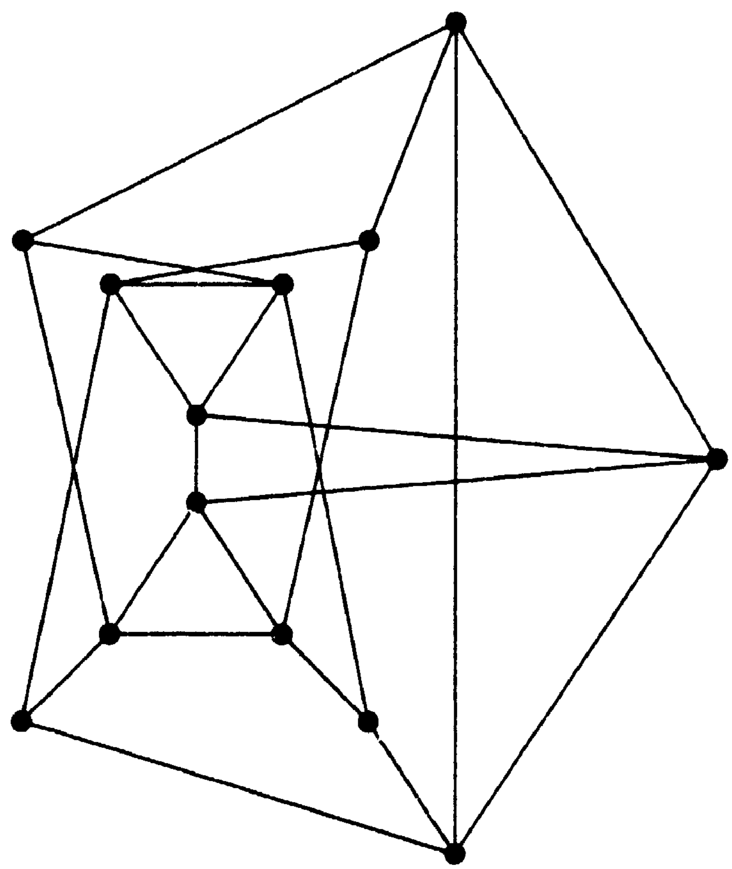

Figure C.3: The Penrose BKS-graph generator redrawn to display the composition from six 'nugget'-graphs.

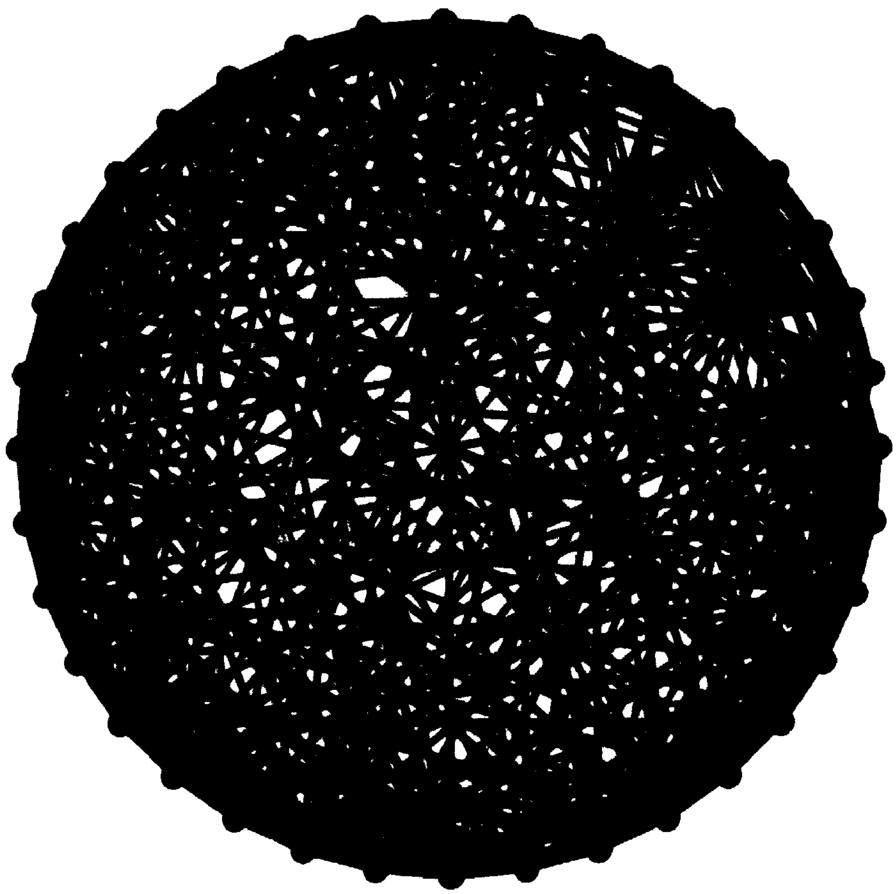

Figure C.4: A 36 ray subensemble of the 40 ray GHZ ensemble. There are 1,280 such subensembles. Each is an equivalent BKS-graph for 8 dimensions.

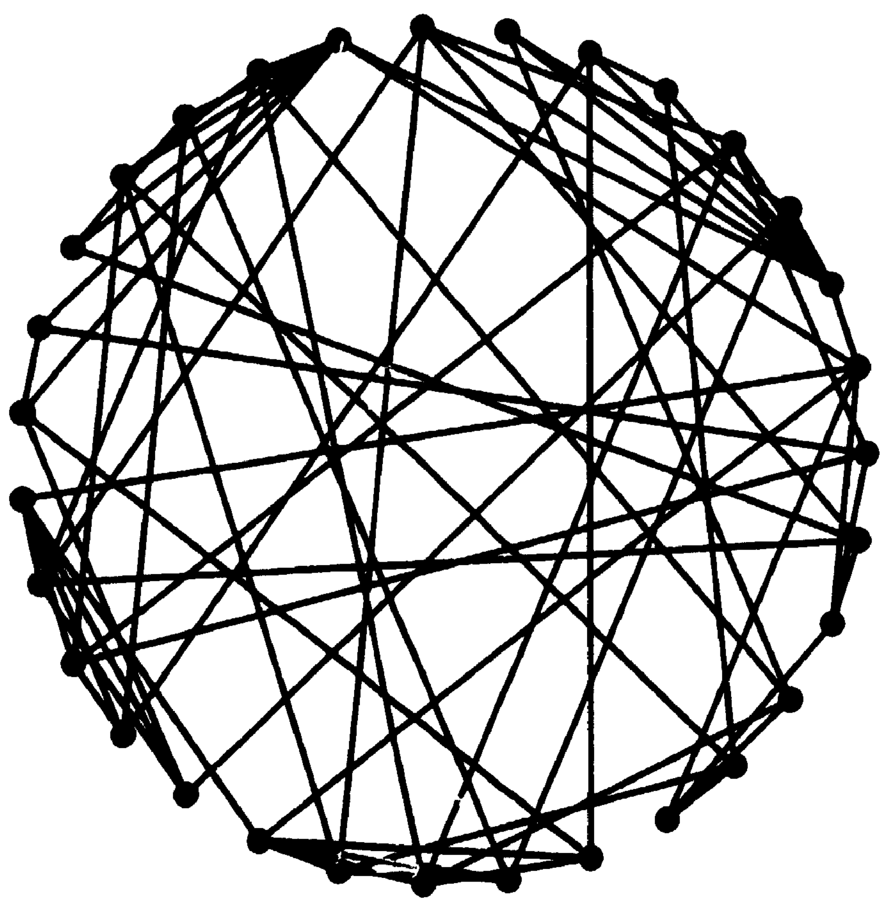

Figure C.5: A BKS-graph of Kochen and Conway's 31 ray uncolourable set. This is a 'circular' embedding of the vertices. A 'radial' embedding better displays the 3-cycles of the graph.

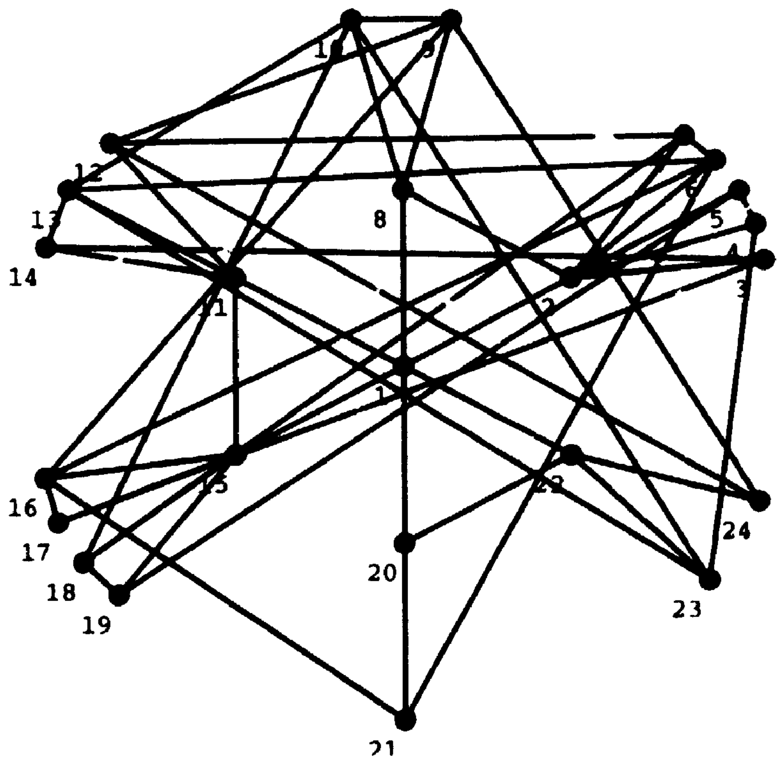

Figure C.6: The state dependent 24 vertex minimalization of the BKS-graph for Kochen and Conway's 31 rays, set in a radial embedding. If vertex 3 is green the graph is uncolourable.

Appendix D

Special Symbols

$\mathcal{H}$	Hilbert space	$\mathbf{R}^n$	n-dimensional real Hilbert space
$\mathbf{P}$	Projection operator	**A**,**B**	Vectors (or matrices)
H	Hamiltonian	$\vartheta()$	Valuation mapping
σ_x^1	Pauli spin operator	$\otimes$	Tensor Product
A	Operator	Tr	Trace
$\mathcal{B}$	Borel sets	E^α	α-dimensional Euclidean space
Z_2	Boolean Algebra	**B**()	Partial Boolean algebra
$\mathbf{S}^2$	The real sphere	$[\psi]$	Majorana representation
μ	Measure	Δ	Interval of the real line
$\Re$	The real numbers	W	Density operator
I	Identity operator	0	Zero operator
$\mathcal{O}$	Set of observables	$\mathcal{R}$	Riemann sphere

Table D.1: Table of mathematical symbols adopted in the text.

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